Nonparameterized 'Entropy Fix' for Roe's Method

François Dubois*
Aerospatiale Espace & Defense,
78133 Les Mureaux, France
and
Guillaume Mehlman†
Ecole Polytechnique,
Centre de Mathématiques Appliquées,
91128 Palaiseau, France

Introduction

ROE'S approximate Riemann solver is very popular and enables easy upwinding for general computational fluid dynamics (CFD) problems. The main drawback with this method is that nonphysical expansion shocks can occur in the vicinity of sonic points. We recall that Roe's method[1] for the general hyperbolic system of conservation laws

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=0 \qquad (1)$$

consists in replacing the exact solution of local Riemann problems by the solution of the approximate linear hyperbolic problem whose flux function is defined by

$$F(U_k,\ U_{k+1},\ U)=F(U_k)+A(U_k,U_{k+1})\cdot(U-U_k)$$

Received June 13, 1991; presented at the AIAA 10th Computational Fluid Dynamics Conference, Honolulu, HI, June 24–27, 1991; revision received Sept. 6, 1991; accepted for publication Sept. 12, 1991.

*Chief, Département Mathématiques Appliquées et Calcul Scientifique. Member AIAA.

†Research Scientist. Member AIAA.

between the grid points U_k and U_{k+1}. The matrix $A(U_l,\ U_r)$ is called a Roe-type linearization and is required to have the following properties:

1) $F(U_r)-F(U_l)=A(U_l,\ U_r)\cdot(U_r-U_l)$
2) $A(U,U)=\mathrm{d}F(U)$
3) $A(U_l,U_r)$ has real eigenvalues and a complete set of eigenvectors.

In the sequel, we assume that system (1) is hyperbolic and admits a Roe-type linearization. We also assume that U is an m-column vector and that the flux function $F(U)$ is a vector-valued function of m components. Let $r_i(U)$ and $\lambda_i(U)$ denote the eigenvectors and associated eigenvalues of the jacobian $\mathrm{d}F(U)$. Similarly, let $R_i(U_l,U_r)$ and λ_i^R denote the eigenvectors and associated eigenvalues of the matrix $A(U_l,U_r)$.

There are several objections to the spreading devices classically used[2,3] in order to cope with nonphysical solutions. In both previous examples, the underlying idea is to give an a priori representation of the solution. We present a new approach based on a nonlinear modification of the flux function.

Definition of the Modified Flux Function

Since problems occur at sonic points, we decide to modify F^R only at sonic points. Let w_j denote the characteristic variables

$$U-U_l=\sum_{j=1}^{m} w_j R_j(U_l,U_r)$$

In particular, we designate by α_j the characteristic variables associated with the jump U_r-U_l. We define m intermediate states:

$$U_0=U_l\ ,\ \ldots\ ,\ U_j=U_{j-1}+\alpha_j R_j(U_l,U_r)\ ,\ \ldots\ ,\ U_m=U_r$$

Let S be the set of sonic indices

$$S=\{j,\ \lambda_j(U_{j-1})<0<\lambda_j(U_j)\}$$

We introduce the following modified flux function parameterized by U_l and U_r:

$$F^{DM}(U_l,U_r,U)=F(U_l)+\sum_{i=1}^{m} g_i(w_i)R_i(U_l,U_r)$$

where the g_is are parameterized by the states $(U)_{j=1,\ldots,m}$ and are defined for $\alpha_i>0$ according to

$$\text{if}\quad i\notin S,\quad \forall w,\quad g_i(w)=\lambda_i^R\cdot w$$

$$\text{if}\quad i\in S,\quad g_i(w)=\begin{cases}p_i(w), & 0<w<\alpha_i\\ \lambda_i^R\cdot w, & w<0 \text{ or } w>\alpha_i\end{cases}$$

and where p_i is the unique Hermite polynomial of degree 3 defined by the conditions:

$$g_i(0)=0,\quad g_i(\alpha_i)=\lambda_i^R\cdot\alpha_i,\quad g_i'(0)=\lambda_i(U_{i-1}),\quad g_i'(\alpha_i)=\lambda_i(U_i)$$

Note that $\lambda_i(U_{i-1})$ and $\lambda_i(U_i)$ are the true eigenvalues of the physical flux at the intermediate states U_i given by the Roe-matrix $A(U_l,U_r)$. Away from sonic points, F^{DM} coincides with the linearized Roe flux F^R. If the initial flux F in Eq. (1) is at least of class C^1, and if the matrix $A(U_l,U_r)$ is continuous with respect to U_l and U_r, then the modified flux F^{DM} is a continuous function of all three arguments.

Definition of the Modified Numerical Flux

Let $V_{l,r}$ be the unique entropy solution of the Riemann problem

$$\begin{cases}\dfrac{\partial V}{\partial t}+\dfrac{\partial F^{DM}(U_l,\ U_r,\ V)}{\partial x}=0\\[2mm] V(x,0)=\begin{cases}0, & x<0\\ U_r-U_l, & x>0\end{cases}\end{cases}$$

we define the numerical flux by

$$\Phi^{DM}(U_l, U_r) = F^{DM}[V_{l,r}(0,t)]$$

Algebraic Expression of the Numerical Flux

For $i \in S$, the Hermite interpolation polynomial $p_i(w)$ introduced previously is defined by

$$p_i(w) = aw^3 + bw^2 + cw$$

with

$$a = \frac{\lambda_i(U_i) + \lambda_i(U_{i-1}) - 2\lambda_i^R}{\alpha_i^2}$$

$$b = \frac{3\lambda_i^R - 2\lambda_i(U_{i-1}) - \lambda_i(U_i)}{\alpha_i}$$

$$c = \lambda_i(U_{i-1})$$

The modified numerical flux has the expression

$$\Phi^{DM}(U_l, U_r) = F(U_l) + \sum_{i \notin S,\ \lambda_i^R < 0} \lambda_i^R \alpha_i R_i(U_l, U_r) + \sum_{i \in S} p_i(\theta_i^*) R_i(U_l, U_r)$$

where

$$\theta_i^* = \frac{-\lambda_i(U_{i-1})\cdot\alpha_i}{3\lambda_i^R - 2\lambda_i(U_{i-1}) - \lambda_i(U_i) + \sqrt{[3\lambda_i^R - \lambda_i(U_i) - \lambda_i(U_{i-1})]^2 - \lambda_i(U_{i-1})\lambda_i(U_i)}}$$

is the argument of the unique extremum of g_i between 0 and α_i.

Remark

Note that when α_i is positive, $g_i(\theta_i^*)$ is the unique minimum of the polynomial p_i between 0 and α_i and we have

$$g_i(\theta_i^*) \le 0$$

$$g_i(\theta_i^*) \le \lambda_i^R \alpha_i$$

It is easy to see that our numerical flux can be written in a centered form that makes the added numerical viscosity explicit:

$$\Phi^{DM}(U_l, U_r) = \Phi^R(U_l, U_r) + \sum_{i \in S} \sup\,[g_i(\theta_i^*);\ g_i(\theta_i^*) - \lambda_i^R \alpha_i] R_i(U_l, U_r)$$

where $\Phi^R(U_l, U_r)$ is the classical Roe flux.[1]

Theorem: Convergence to the Unique Entropy Solution

Let f be a convex scalar flux and u^0 initial data in $L^\infty(R) \cap BV(R)$. The semidiscrete numerical scheme:

$$\frac{du_j}{dt} = -\frac{1}{h}[\Phi^{DM}(u_j, u_{j+1}) - \Phi^{DM}(u_{j-1}, u_j)]$$

with

$$u_j(0) = \frac{1}{h}\int_{(j-1/2)h}^{(j+1/2)h} u^0(x)\, dx$$

where h is the mesh step, converges to the unique entropy solution of Eq. (1) with initial data u^0 (see proof in Ref. 4).

Conclusions

We have proposed a nonparameterized approach to entropy enforcement for Roe-type schemes. It is based on the exact resolution of a Riemann problem associated with a Hermite interpolation of the physical flux. In the scalar convex case, we have proved convergence of the method of lines to the unique entropy solution. Numerical results[5] for the Euler equations extend the conclusions of the scalar case.

References

[1]Roe, P. L., "Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes," *Journal of Computational Physics*, Vol. 43, No. 2, 1981, pp. 357–372.

[2]Harten, A., and Hyman, J. M., "Self-Adjusting Grid Methods for One-Dimensional Hyperbolic Conservation Laws," *Journal of Computational Physics*, Vol. 50, No. 1, 1983, pp. 235–269.

[3]Roe, P. L., "Some Contributions to the Modeling of Discontinuous Flow," Modeling of Discontinuous Flow," *Lectures in Applied Mathematics*, edited by Engquist, Osher, and Sommerville, Vol. 22, American Mathematical Society, Providence, RI, 19085, pp. 163–193.

[4]Dubois, F., and Mehlman, G., "A Non-Parameterized Entropy Correction for Roe's Approximate Riemann Solver," CMAP Rept. 248, Ecole Polytechnique, Palaiseau, France, 1991.

[5]Mehlman, G., Thivet, F., Candel, S., and Dubois, F., "Computation of Hypersonic Flows with a Fully Coupled Implicit Solver and an Extension of the CVDV Model for Thermochemical Relaxation," GAMNI Hypersonic Workshop, Antibes, France, April 1991 (to be published).