General fourth order Chapman-Enskog expansion of lattice Boltzmann schemes

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In order to derive the equivalent partial differential equations of a lattice Boltzmann scheme, a classical Chapman Enskog expansion [1, 2] is popular in the lattive Boltzmann community. A main drawback of this approach is the fact that multiscale expansions are used without a clear mathematical signification of the various variables and functions. Independently of this framework, we have proposed in [3] the Taylor expansion method to obtain formally the equivalent partial differential equations. The infinitesimal variable is simply the time step Δt, proportional to the space step Δx with the acoustic scaling.
We consider a general lattice Boltzmann scheme with q discrete velocities: $f_j(x, t + \Delta t) = f_j^*(x - v_j \Delta t, t)$ for $0 \leq j < q$. In the multiple time relaxation framework of d'Humières [4], moments m are introduced with the help of an invertible matrix M: $m_k = \sum_j M_{k\ell} f_j$. We decompose the moments m in the following way: $m \equiv (W \;\; Y)^{\rm t}$. The conserved variables W are not modified after relaxation: $W^* = W$. The microscopic variables Y are changed in a nonlinear way by the relaxation process: $Y^* = Y + S\,(\Phi(W) - Y)$. The matrix S is invertible, and often chosen as diagonal. It is supposed to be fixed in the asymptotic process presented hereafter. The equilibrium values $Y^{\rm eq} = \Phi(W)$ are given smooth functions of the conserved variables. When Y^* is evaluated, and the distribution f^* after relaxation is simply given according to $f^* = M^{-1} m^*$.
The momentum-velocity operator matrix Λ is the advection operator of the lattice seen in the space of moments: $\Lambda_{k\ell} = \sum_{j,\alpha} M_{kj}\, v_j^\alpha\, \partial_\alpha\, (M^{-1})_{j\ell}$ for $0 \leq k, \ell < q$. Then we have an exponential form of the discrete iteration of the lattice Boltzmann scheme: $m(x, t + \Delta t) = \exp(-\Delta t\, \Lambda)\; m^*(x, t)$. With this general framework, the Chapman-Enskog formalism [1, 2] suppose that $\Delta t \equiv \varepsilon$. We first expand the nonconserved moments as differential nonlinear function of the conserved variables: $Y = \Phi(W) + \varepsilon\, \Psi_1(W) + \varepsilon^2\, \Psi_2(W) + \varepsilon^3\, \Psi_3(W) + {\rm O}(\varepsilon^4)$ and we suppose that a multi-scale approach is present for the time dynamics: $\partial_t = \partial_{t_1} + \varepsilon\, \partial_{t_2} + \varepsilon^2\, \partial_{t_3} + \varepsilon^3\, \partial_{t_4} + {\rm O}(\varepsilon^4)$. Then we prove that the conserved quantities W follow the following multi-time dynamics : $\partial_{t_k} W + \Gamma_k(W) = 0$ for $1 \leq k \leq 4$. The differential operators $\Gamma_i(W)$ and $\Psi_j(W)$ of this expansion are recursively determined as a function of the data. Our result express that they are exactly the ones derived with the Taylor expansion method [5] at fourth order.

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