Construction and analysis of lattice Boltzmann schemes

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Heidelberger Akademie der Wissenschaften
Monday 04 March 2024

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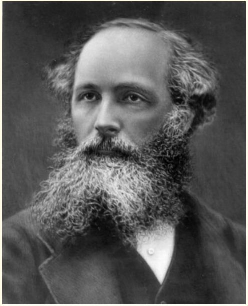
Maxwell - Boltzmann equilibrium (1872)

velocity distribution at thermodynamic equilibrium

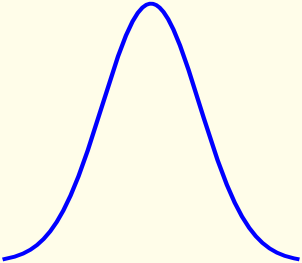
$$f^{\mathrm{eq}}(v) = \rho \left(\frac{\beta}{2\pi}\right)^{\frac{3}{2}} \exp\left(-\frac{\beta}{2} \mid v - u \mid^2\right), \ \beta = \frac{1}{k_B T}$$

the temperature T is a parameter

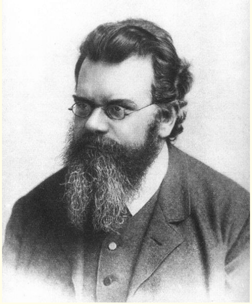
to describe the variance of the velocity distribution



James Clerk Maxwell
(1831 - 1879)



Carl Friedrich Gauss
(1777 - 1855)



Ludwig Boltzmann
(1844 - 1906)

Boltzmann equation (1872)

Kinetic theory of gases of Maxwell and Boltzmann

$\mathrm{d}m = f(t,\, x,\, v)\, \mathrm{d}x\, \mathrm{d}v$

the gas mass $\mathrm{d}m$ is described by a velocity distribution $f(t,\, x,\, v)$

Time evolution : Boltzmann equation

$$\frac{\partial f}{\partial t} + v.\nabla_x f = Q(f)\,, \quad x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ t > 0$$

free advection with velocity v — left hand side $\frac{\partial f}{\partial t} + v.\nabla_x f$

collisions inside the gas — right hand side $Q(f)$

First moments of the distribution f

mass $\rho = \int_{\mathbb{R}^3} f(v)\, \mathrm{d}v$

momentum $\rho\, u = \int_{\mathbb{R}^3} v\, f(v)\, \mathrm{d}v$

energy $\rho\, E = \int_{\mathbb{R}^3} \frac{1}{2}\, |v|^2\, f(v)\, \mathrm{d}v$

Chapman Enskog expansion (1915)

$$f(v) = f^{\mathrm{eq}}(v) + \varepsilon f^1(v) + \dots$$

formal derivation of the Navier Stokes equations

Sydney Chapman (1888 - 1970) www.npg.org.uk

David Enskog (1884 - 1947) www.mech.kth.se

“BGK” approximation (1954)

$\frac{\partial f}{\partial t} + v.\nabla_x f = -\frac{1}{\tau}(f - f^{\mathrm{eq}})$ collision time $\tau \approx \varepsilon$

Prabhu Lal Bhatnagar
(1912 - 1976)

math.iisc.ernet.in

Eugene Gross
(1926 - 1991)

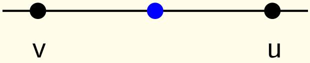
alchetron.com

Max Krook
(1913 - 1985)

wikipedia

Discrete velocities (1957)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = v^2 - u^2, \quad \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = u^2 - v^2$$



ro.wikipedia.org

Torsten Carleman (1892-1949)

Problèmes mathématiques dans la théorie cinétique des gaz
Mittag-Leffler Institute, Stockholm, 1957

D1Q2

$$\frac{\partial f_i}{\partial t} + v_i.\nabla_x f_i = Q_i(f_0, f_1, \ldots, f_{q-1}), \ 0 \leq i < q$$

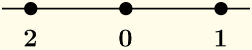
James Broadwell
(1921-2018)

www.nae.edu

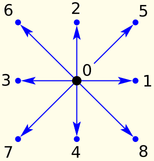
Renée Gatignol
(born in 1939)

Henri Cabannes
(1923 - 2016)

ljll.math.upmc.fr



D1Q3



D2Q9

Outline

- Boltzmann equation with discrete velocities
- Alternate directions
- Analysis for D1Q3
- Lattice Boltzmann schemes
- Isothermal Navier Stokes
- Thermal Navier Stokes
- Conclusion

Conclusion

ingredients for Multiple Relaxation Times lattice Boltzmann schemes

- choice of moments $m = M\,f$
- equilibrium vector function $\Phi(W)$
- relaxation matrix S

analysis with a generalization of Chapman Enskog expansion 😊

inverse problem for Navier Stokes $\Phi(W) = ?, \;\; S = ?$

isothermal Navier Stokes

- D3Q27 has a discrepancy for isothermal Navier Stokes 😡
- D3Q27-2 available for isothermal Navier Stokes 😊

thermal Navier Stokes

- we must impose $\gamma \equiv \frac{c_p}{c_v} = 2$ (2d), $\gamma = \frac{5}{3}$ (3d)
  and a Prandtl number satisfying $Pr = 1$ 😨
- D3Q27-2 available for thermal Navier Stokes 😄

stability has not been studied in this contribution 😨

numerical experiments are welcomed!

FD, “Equivalent partial differential equations of a lattice Boltzmann scheme”, Computers and Mathematics with Applications, vol. 55, p. 1441-1449, 2008.

FD, “Nonlinear fourth order Taylor expansion of lattice Boltzmann schemes”, Asymptotic Analysis, vol. 127, p. 297-337, 2022.

FD, zenodo deposit of the “abcd-ns” software, version v0, doi:10.5281/zenodo.6685127, june 2022.

FD and Pierre Lallemand, “On single distribution lattice Boltzmann schemes for the approximation of Navier Stokes equations”, Communications in Computational Physics, vol. 34, p. 613-671, 2023.

FD, Bruce M. Boghosian, Pierre Lallemand, “General fourth-order Chapman-Enskog expansion of lattice Boltzmann schemes”, Computers & Fluids, vol. 266, 106036, 2023.

Thank you for your attention!

www.akademienunion.de