A result of convergence for a one-dimensional two-velocities lattice Boltzmann scheme *

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We have presented an introduction of lattice Boltzmann schemes, with the first ideas of cellular automata on a square lattice [1], their extension to triangular lattices [2] and a spectacular numerical result proposed in [3]. The major characteristic of these discrete flows is the presence of Monte-Carlo noise. Then cellular automata have been replaced by various approximations of Boltzmann equation with discrete velocities. The simplest example [4] contains only two velocities.

The one-dimensional model with two velocities [5] is denoted by D1Q2 in the terminology of lattice Boltzmann schemes. It introduces a given volocity a, an equilibrium function $\mathbb{R} \ni u \longmapsto \Phi(u) \in \mathbb{R}$ and a small positive parameter ε:

$$\partial_t u + \partial_x v = 0\,,\;\; \partial_t v + a^2\, \partial_x u = \frac{1}{\varepsilon}\big(\Phi(u) - v\big)\,. \tag{1}$$

A formal Chapman-Enskog expansion at first order relative to ε conducts to a second order equivalent partial differential equation:

$$\partial_t u + \partial_x \Phi(u) - \varepsilon\, \partial_x\big((a^2 - \Phi'(u)^2)\, \partial_x u\big) = \mathrm{O}(\varepsilon^2)\,. \tag{2}$$

A rigorous proof of convergence is established in [6]. The discretisation with finite volume schemes leads to a convergent approach and this has been established in [7, 8].

For the system (1), the lattice Boltzmann scheme first consider the ordinary differential equation $\partial_t v = \frac{1}{\varepsilon}\big(\Phi(u) - v\big)$ and implement an explicit first order scheme during this collision step:

$$v_j^* = v_j^n + \frac{\Delta t}{\varepsilon}\left(\Phi(u_j^n) - v_j^n\right). \tag{3}$$

The parameter $s \equiv \frac{\Delta t}{\varepsilon}$ is directly introduced as a given number in the numerical simulation. After this collision step, the density of particles $f_\pm^*$ are naturally associated to a digonalized form of the system (1) and we have

$$\big(f_+^*\big)_j = \tfrac{1}{2}\left(u + \tfrac{v^*}{a}\right)_j \;\text{ and }\; \big(f_-^*\big)_j = \tfrac{1}{2}\left(u - \tfrac{v^*}{a}\right)_j.$$

Then the propagation of the particles during one time step is written with an upwind scheme associated to a Courant number always identical to 1:

$$\big(f_+\big)_j^{n+1} = \big(f_+^*\big)_{j-1}\,,\;\; \big(f_-\big)_j^{n+1} = \big(f_-^*\big)_{j+1}\,. \tag{4}$$

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Finally the moments u and v at the new time step follow the simple relations

$$u_j^{n+1} = \big(f_+\big)_j^{n+1} + \big(f_-\big)_j^{n+1} \quad\text{and}\quad v_j^{n+1} = a\,\big[\big(f_+\big)_j^{n+1} - \big(f_-\big)_j^{n+1}\big].$$

Consistency of the numerical scheme (3)(4) with the system (1) is satisfied when $a = \frac{\Delta x}{\Delta t}$. The previous D1Q2 scheme is generalized to a large number of DdQq stencils for d space dimensions and q discrete velocities. The principle is always to treat the collision with an explicit time scheme and the discrete advection with the exact scheme for a Courant number equal to unity. We refer to [9, 10, 11, 12, 13, 14] for major developments of lattice Boltzmann schemes.

The asymptotic analysis supposes typically that the ratio $\lambda \equiv \frac{\Delta x}{\Delta t}$ is fixed and that the relation parameter $s \equiv \frac{\Delta t}{\varepsilon}$ is also fixed. When the space and time steps tend to zero, the lattice Boltzmann scheme (3)(4) can be formally expanded and an equivalent partial differential equation is emerging. For the previous scheme, we obtain

$$\partial_t u + \partial_x \Phi(u) - \Delta t \left(\frac{1}{s} - \frac{1}{2}\right) \partial_x\big((a - \Phi'(u)^2)\,\partial_x u\big) = \mathrm{O}(\Delta t^2)\,. \tag{5}$$

This result was first obtained in [15] for cellular automata. It has been extended with the Taylor expansion method [16, 17, 18] to general nonlinear lattice Boltzmann schemes up to fourth order accuracy [19]. Observe in the relation (5) that for $s \simeq 2$, the asymptotic viscosity is drastically reduced in comparison with the expansion (2). In consequence, industrial applications at high Reynolds number are used in automotive industry [20] since 20 years and are in development for transonic aerodynamics [21].

Nevertheless, we have reported in [22, 23] an unexpected convergence previously observed in [17] for the heat equation when the time and space steps tend to zero with a fixed ratio $\lambda \equiv \frac{\Delta x}{\Delta t}$. The thermal diffusion coefficient evaluated with the Taylor expansion method $\mu \simeq \big(\frac{1}{s} - \frac{1}{2}\big)\,\lambda\,\Delta x$ remains constant and Δx tends to zero. Therefore, the parameter s tends also to zero and is no more fixed as supposed in the asymptotic expansion. The lattice Boltzmann equation remains stable, even if it is an explicit scheme with a ratio $\frac{\Delta t}{\Delta x^2}$ larger than 1. But it is no more consistent with the heat equation, and converges to a system of damped acoustics!

Finally, considering again the lattice Boltzmann scheme (3)(4), we can write it as a finite difference scheme:

$$\left\{\begin{array}{l} \frac{u_j^{n+1}-u_j^n}{\Delta t} + \frac{v_{j+1}^n - v_{j-1}^n}{2\,\Delta x} - \frac{1}{2}\,\frac{\Delta x^2}{\Delta t}\,\frac{u_{j+1}^n - 2\,u_j^n + u_{j-1}^n}{\Delta x^2} \\ \qquad\qquad + \frac{s}{2\,\lambda\,\Delta t}\,\big((\Phi(u_{j+1}^n) - v_{j+1}^n) - (\Phi(u_{j-1}^n) - v_{j-1}^n)\big) = 0 \\ \frac{v_j^{n+1}-v_j^n}{\Delta t} + \lambda^2\,\frac{u_{j+1}^n - u_{j-1}^n}{2\,\Delta x} - \frac{1}{2}\,\frac{\Delta x^2}{\Delta t}\,\frac{v_{j+1}^n - 2\,v_j^n + v_{j-1}^n}{\Delta x^2} \\ \qquad\qquad - \frac{s}{2\,\Delta t}\,\big((\Phi(u_{j+1}^n) - v_{j+1}^n) + (\Phi(u_{j-1}^n) - v_{j-1}^n)\big) = 0\,. \end{array}\right. \tag{6}$$

In [24], we have proven the following convergence theorem. When $\lambda \equiv \frac{\Delta x}{\Delta t}$ is fixed and if the parameter $0 < s \leq 1$ is also fixed, the D1Q2 lattice Boltzmann scheme (3)(4) or (6) converges to the unique entropy solution of the scalar conservation law $\partial_t u + \partial_x \Phi(u) = 0$. The proof uses classical mathematical methods [7, 8] for establishing the convergence: L^∞ stability, total variation estimates, and discrete entropy inequalities.

The lattice Boltzmann schemes have proven their efficiency for a wide number of applications like isothermal flows, compressible flows with heat transfer, non-ideal fluids, multiphase and multi-component flows, microscale gas flows, soft-matter flows, *etc.* Last but not least, stability is one of the main remaining open questions.

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