Unexpected convergence of lattice Boltzmann schemes

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In [1], we have studied experimentally the curious convergence of the D1Q3 multiple-relaxation time lattice Boltzmann scheme with one conserved variable when using the acoustic scaling. The same phenomenon is present for a wide variety of one, two or three-dimensional schemes.

We present the phenomenon in this abstract for the D2Q9 scheme. The ratio $\lambda \equiv \Delta x / \Delta t$ between the space step Δx and the time step Δt is kept fixed. With one conservation law, the momentum $J \equiv \sum_j v_j \, f_j$ relaxes towards $J^{\rm eq} = 0$ with a relation of the type

$$J^* = J + s \, (J^{\rm eq} - J) \, . \qquad (1)$$

We expect an asymptotic equation of diffusion type

$$\partial_t \rho \, - \, \kappa \, \Delta \rho \, = \, {\rm O}(\Delta x^2)$$

with the Taylor expansion method for asymptotic analysis with fixed relaxation coefficients, and in particular the coefficient s of the relation (1). Moreover, the diffusivity κ is given by the relation

$$\kappa \, = \, \frac{4 + \alpha}{6} \Big(\frac{1}{s} - \frac{1}{2} \Big) \, \lambda \, \Delta x \, ,$$

where α is a parameter of the equilibrium step of the lattice Boltzmann scheme.

On the other hand, if the diffusivity κ is fixed, the relaxation coefficient s is now given by a relation of the type

$$s \, = \, \frac{4 + \alpha}{6 \, \kappa} \, \lambda \, \Delta x \, + \, {\rm O}(\Delta x^2)$$

and is no longer fixed if the space step Δx tends to zero. The derivation of the equivalent partial differential equation has to be reconsidered. We obtain therefore an acoustic model with zeroth order dissipation:

$$\frac{\partial \rho}{\partial t} + {\rm div} J^* = {\rm O}(\Delta x) \, , \qquad \frac{\partial J^*_\alpha}{\partial t} + \frac{\lambda^2}{6} \, (4 + \alpha) \, \frac{\partial \rho}{\partial x_\alpha} + \frac{\lambda^2}{6 \, \kappa} \, (4 + \alpha) \, J^*_\alpha = {\rm O}(\Delta x) \, .$$

In the conference on Discrete Simulation of Fluid Dynamics, we will present a complete analytical and numerical experimental study for determining the asymptotic partial differential equations for various lattice Boltzmann schemes intended to approximate diffusive or fluid dynamic problems.

References

[1] Boghosian B.M., Dubois F., Graille B., Lallemand P. and Tekitek M.M., *Curious convergence properties of lattice Boltzmann schemes for diffusion with acoustic scaling*, Communications in Computational Physics, submitted, 2016.