Quartic Parameters for Acoustic Applications of Lattice Boltzmann Scheme

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• The numerical scheme introduced by D. d'Humières [2] is a flexible framework to discretize mathematical models in physics and engineering. It introduces the choice of N ($= 4$ for three-dimensional acoustic applications) conserved momenta: $m_i \equiv W_i$ (for $0 \leq i \leq N-1$) and relaxation coefficients s_k that define the evolution of non-conserved momenta: $m_k^* = m_k + s_k \, (m_k^{\rm eq} - m_k)$. Moreover the time step Δt is a natural parameter for the evolution of particle populations: $f_j(x,\, t + \Delta t) = f_j^*(x - v_j \, \Delta t,\, t)$. When the equilibrium momenta $m_k^{\rm eq}$ are a linear function of the conserved variables W, we have shown in [4] that is is possible with the so-called Taylor expansion method [3] to derive N equivalent partial differential equations for the unknown vector W at first order in time and at any order of accuracy in space.

• The system of linearized conservation equations issued from the lattice Boltzmann scheme can be written under the form $A(\Delta t,\, \partial) \bullet W = 0$ with a compact notation: $A(\Delta t,\, \partial)$ is a 4×4 (for three-dimensional applications) matrix of differential operators of high order relative to the conservative variables W. We search the eigenmodes of operator $A(\Delta t, \partial)$, *id est* the eigenvalues $\lambda_j(\Delta t, \partial)$ and the eigenvectors $r_j(\Delta t, \partial)$ such

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that $A(\Delta t,\, \partial) \bullet r_j(\Delta t, \partial) \;=\; \lambda_j \, r_j(\Delta t, \partial)$. We introduce the diagonal matrix $\Lambda(\Delta t, \partial)$ composed by the eigenvalues $\lambda_j(\Delta t, \partial)$ and the square matrix $R(\Delta t, \partial)$ composed by the eigenvectors. Then the previous relation can be written under the synthetic form: $A(\Delta t,\, \partial) \bullet R(\Delta t, \partial) = R(\Delta t, \partial) \bullet \Lambda(\Delta t, \partial)$.

• Moreover, the operator $A(\Delta t, \partial)$ is a polynomial relatively to the variable Δt: $A(\Delta t,\, \partial) \equiv A_0(\partial) + \Delta t \, A_1(\partial) + \Delta t^2 \, A_2(\partial) + \Delta t^3 \, A_3(\partial) + \mathrm{O}\big(\Delta t^4\big)$. We can apply in this case the perturbation theory for linear operators (see *e.g.* [1] for an elementary introduction). First for $\Delta t = 0$, the operator $A_0(\partial)$ is exactly the perfect linear acoustic model and a system $R_0(\partial)$ of classical reference eigenvectors can be given. The two acoustic waves and the two shear waves are put in evidence with this diagonalization. The parameter Δt is supposed to be infinitesimal and we introduce an expansion of the eigenvalues with diagonal matrices $\Lambda_j(\partial)$: $\Lambda(\Delta t,\, \partial) \equiv \Lambda_0(\partial) + \Delta t \, \Lambda_1(\partial) + \Delta t^2 \, \Lambda_2(\partial) + \Delta t^3 \, \Lambda_3(\partial) + \mathrm{O}\big(\Delta t^4\big)$ and relative perturbations $Q_j(\partial)$ of the eigenvectors: $R(\Delta t, \partial) \equiv R_0(\partial) \bullet \big(\mathrm{Id} + \Delta t \, Q_1(\partial) + \Delta t^2 \, Q_2(\partial) + \Delta t^3 \, Q_3(\partial) + \mathrm{O}(\Delta t^4)\big)$. We insert the previous expansions inside the eigenmode condition and find step by step the expression of eigenvalues and eigenvectors at any order of accuracy.

• Doing this, we expand formally the eigenvalues in terms of the infinitesimal parameter. We can adjust the coefficients s_k of the d'Humières lattice Boltzmann scheme in order to enforce fourth order accuracy (quartic parameters presented in [4] in the case of shear waves). In this contribution, we show that the previous methodology is also applicable to acoustic situations with D2Q13 and D3Q27 schemes. For simpler schemes as D2Q9 and D3Q19, we show that the simulation of acoustic waves is improved when isotropy conditions are enforced. We will present various simulations to verify these ideas and will list a few pending questions.

References

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