Développement de Chapman-Enskog d'un schéma de Boltzmann sur réseau à l'ordre trois

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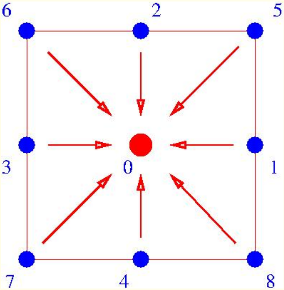
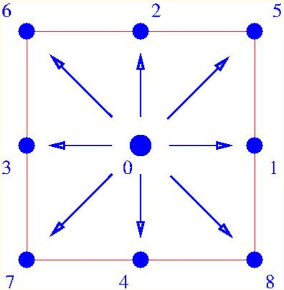
Lattice Boltzmann scheme



advection

collision



advection

Outlook

- Multiple Relaxation Times Lattice Boltzmann scheme
- Compact expression
- A general expansion result at third order
- Proof for Chapman-Enskog
- Proof for Taylor expansion method
- Link with the previous expansion at third-order (DCDS-A, 2009)
- Survey

Multiple Relaxation Times Lattice Boltzmann scheme

Two steps for one time iteration

(i) Nonlinear relaxation

the particle distribution f is modified locally

into a new distribution f^*

(ii) Linear advection

method of characteristics when it is exact !

Compact description of the lattice Boltzmann scheme:

$$f_j(x,\, t + \Delta t) \,=\, f_j^*(x - v_j\, \Delta t,\, t)\,, \qquad v_j \in \mathcal{V}\,, \quad x \in \mathcal{L}^0\,.$$

d'Humières matrix M for moments: $m_k \equiv M_{k\ell}\, f_\ell\,, \quad m \equiv \begin{pmatrix} W \\ Y \end{pmatrix}$

Conserved variables W ; after relaxation, $W^* = W$

Non-Conserved moments Y

equilibrium value: $Y^{\rm eq} = \Phi(W)$

after relaxation: $Y^* = Y + S\,(Y^{\rm eq} - Y), \quad m^* = \begin{pmatrix} W \\ Y^* \end{pmatrix}$

Important hypotheses

the discrete function $f(x, t)$ for

x vertex of the lattice

t discrete time

is supposed to be the restriction of a very regular function

denoted in the same way $f(x, t, \Delta t, s_k, \cdots)$

but defined for

x a point of the continuous space $\mathbb{R}^d$

t continuous time

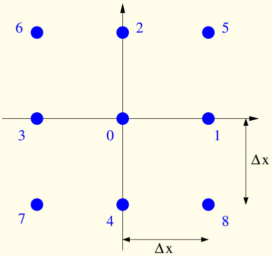
Δt the time step

s_k the relaxation parameters

The numerical velocity $\lambda \equiv \frac{\Delta x}{\Delta t}$ is fixed

The relaxation parameters s_k are fixed

Example of D2Q9



$$\lambda = \frac{\Delta x}{\Delta t}$$

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 0 & -\lambda & 0 & \lambda & -\lambda & -\lambda & \lambda \\ 0 & 0 & \lambda & 0 & -\lambda & \lambda & \lambda & -\lambda & -\lambda \\ -4\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 \\ 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 \\ 0 & -2\lambda^3 & 0 & 2\lambda^3 & 0 & \lambda^3 & -\lambda^3 & -\lambda^3 & \lambda^3 \\ 0 & 0 & -2\lambda^3 & 0 & 2\lambda^3 & \lambda^3 & \lambda^3 & -\lambda^3 & -\lambda^3 \\ 4\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 \end{bmatrix} \begin{matrix} \rho \\ J_x \\ J_y \\ \varepsilon \\ XX \\ XY \\ q_x \\ q_y \\ h \end{matrix}$$

Advection operator in the basis of moments

Momentum-velocity operator matrix $\Lambda \equiv M \, \mathrm{diag}\big(\sum_\alpha v^\alpha \, \partial_\alpha \big) \, M^{-1}$

$1 \leq \alpha \leq d =$ space dimension

Block decomposition $\Lambda \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Example of Λ operator matrix for the thermal D2Q9 scheme

$$\left[\begin{array}{ccc|cccccc}
0 & \partial_x & \partial_y & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2\lambda^2}{3}\,\partial_x & 0 & 0 & \frac{1}{6}\,\partial_x & \frac{1}{2}\,\partial_x & \partial_y & 0 & 0 & 0 \\
\frac{2\lambda^2}{3}\,\partial_y & 0 & 0 & \frac{1}{6}\,\partial_y & -\frac{1}{2}\,\partial_y & \partial_x & 0 & 0 & 0 \\
\hline
0 & \lambda^2\,\partial_x & \lambda^2\,\partial_y & 0 & 0 & 0 & \partial_x & \partial_y & 0 \\
0 & \frac{\lambda^2}{3}\,\partial_x & -\frac{\lambda^2}{3}\,\partial_y & 0 & 0 & 0 & -\frac{1}{3}\,\partial_x & \frac{1}{3}\,\partial_y & 0 \\
0 & \frac{2}{3}\lambda^2\,\partial_y & \frac{2}{3}\lambda^2\,\partial_x & 0 & 0 & 0 & \frac{1}{3}\,\partial_y & \frac{1}{3}\,\partial_x & 0 \\
0 & 0 & 0 & \frac{\lambda^2}{3}\,\partial_x & -\lambda^2\,\partial_x & \lambda^2\,\partial_y & 0 & 0 & \frac{1}{3}\,\partial_x \\
0 & 0 & 0 & \frac{\lambda^2}{3}\,\partial_y & \lambda^2\,\partial_y & \lambda^2\,\partial_x & 0 & 0 & \frac{1}{3}\,\partial_y \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda^2\,\partial_x & \lambda^2\,\partial_y & 0
\end{array} \right] .$$

Compact expression of the lattice Boltzmann scheme

Proposition $m(x, t+\varepsilon) = \exp(-\varepsilon\,\Lambda)\; m^*(x,\, t)$

$$
\begin{aligned}
m_k(x, t+\varepsilon) &= \sum_j M_{kj}\, f_j^*(x - v_j\,\varepsilon,\, t) \\
&= \sum_{j\,\ell} M_{kj}\,(M^{-1})_{j\ell}\; m_\ell^*(x - v_j\,\varepsilon,\, t) \\
&= \sum_{j\,\ell} M_{kj}\,(M^{-1})_{j\ell}\; \sum_{n=0}^{\infty} \frac{1}{n!}\Big(-\varepsilon \sum_\alpha v_j^\alpha\,\partial_\alpha\Big)^n\, m_\ell^*(x,\, t) \\
&= \sum_\ell \sum_{n=0}^{\infty} \frac{1}{n!} \sum_j M_{kj}\,\Big(-\varepsilon \sum_\alpha v_j^\alpha\,\partial_\alpha\Big)^n\,(M^{-1})_{j\ell}\; m_\ell^*(x,\, t) \\
&= \sum_\ell \Big[\sum_{n=0}^{\infty} \frac{1}{n!}\,\big(-\varepsilon\,\Lambda\big)^n_{k\ell}\Big]\; m_\ell^*(x,\, t) \\
&= \sum_\ell \exp(-\varepsilon\,\Lambda)_{k\ell}\; m_\ell^*(x,\, t) \\
&= \Big(\exp(-\varepsilon\,\Lambda)\; m^*(x,\, t)\Big)_k
\end{aligned}
$$

Chapman-Enskog framework

Introduce a small parameter ε. For numerical schemes, $\varepsilon = \Delta t$

$$f = f^{\rm eq} + \varepsilon\, f^1 + \varepsilon^2\, f^2 + {\rm O}(\varepsilon^3)$$

Chapman-Enskog hypothesis : the perturbation terms f^ℓ are only function of the equilibrium $f = f^{\rm eq}$

apply the d'Humières matrix:

$$M\,f = M\,f^{\rm eq} + \varepsilon\, M\,f^1 + \varepsilon^2\, M\,f^2 + {\rm O}(\varepsilon^3)$$

take the first conserved component : $W = W + 0$

take the second nonconserved component :

$$Y = Y^{\rm eq} + \varepsilon\,(M\,f^1)_Y + \varepsilon^2\,(M\,f^2)_Y + {\rm O}(\varepsilon^3)$$

The perturbation terms $\varepsilon^\ell\,(M\,f^\ell)_Y$ depend only on the conserved moments W.

Then $Y = \Phi(W) + S^{-1}\,(\varepsilon\,\Psi_1(W) + \varepsilon^2\,\Psi_2(W)) + {\rm O}(\varepsilon^3)$

Suppose also a multi-scale approach for the time dynamics:

$$\partial_t = \partial_{t_1} + \varepsilon\,\partial_{t_2} + \varepsilon^2\,\partial_{t_3} + {\rm O}(\varepsilon^3)$$

Chapman-Enskog expansion

The conserved quantities W follow a multi-time dynamics :

$$\partial_{t_1} W + \Gamma_1(W) = 0$$
$$\partial_{t_2} W + \Gamma_2(W) = 0$$
$$\partial_{t_3} W + \Gamma_3(W) = 0$$

The differential operators

$\Gamma_1(W)$, $\Psi_1(W)$, $\Gamma_2(W)$, $\Psi_2(W)$ and $\Gamma_3(W)$

of this expansion are recursively determined

coefficients at first order

$$\Gamma_1 = A\,W + B\,\Phi(W)$$

coefficients at second order

$$\Psi_1 = \mathrm{d}\Phi(W).\Gamma_1 - (C\,W + D\,\Phi(W))$$

$$\Gamma_2 = B\,\Sigma\,\Psi_1$$

Hénon matrix $\Sigma \equiv S^{-1} - \frac{1}{2}\,\mathrm{I}$

coefficients at third order

$$\Psi_2 = \Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + \mathrm{d}\Phi(W).\Gamma_2 - D\,\Sigma\,\Psi_1$$

$$\Gamma_3 = B\,\Sigma\,\Psi_2 + \frac{1}{12}\,B_2\,\Psi_1 - \frac{1}{6}\,B\,\mathrm{d}\Psi_1.\Gamma_1$$

Taylor *vs* Chapman-Enskog expansion methods

Asymptotic hypothesis: emerging partial differential equations

$$\partial_t W + \Gamma_1 + \Delta t \, \Gamma_2 + \Delta t^2 \, \Gamma_3 = \mathrm{O}(\Delta t^3)$$

Γ_j: vector obtained after j space derivations
of the conserved moments W
and the equilibrium vector $\Phi(W)$.

Non-Conserved moments:

$$Y = \Phi(W) + S^{-1} \left(\Delta t \, \Psi_1 + \Delta t^2 \, \Psi_2\right) \mathrm{O}(\Delta t^2)$$

Φ_j analogous to Γ_j but not with the same dimension!

General nonlinear Taylor expansion method

Coefficients at first order

$$\Gamma_1 = A\,W + B\,\Phi(W)$$

Coefficients at second order

$$\Psi_1 = \mathrm{d}\Phi(W).\Gamma_1 - (C\,W + D\,\Phi(W))$$

$$\Gamma_2 = B\,\Sigma\,\Psi_1$$ with Σ the Hénon matrix: $\Sigma \equiv S^{-1} - \frac{1}{2}\,\mathrm{I}$

Coefficients at third order

$$\Psi_2 = \mathrm{d}\Phi(W).\Gamma_2 + \Sigma\;\mathrm{d}\Psi_1.\Gamma_1 - D\,\Sigma\,\Psi_1$$

$$\Gamma_3 = B\,\Sigma\,\Psi_2 + \frac{1}{6}\,B\;\mathrm{d}\Psi_1.\Gamma_1 - \frac{1}{12}\,B_2\,\Psi_1$$

We find the same formulas than with Chapman-Enskog:

the two expansions are equivalent!

Chapman-Enskog expansion : order zero

$$m(t+\varepsilon) = \exp(-\varepsilon\,\Lambda)\, m^*$$

$$m + \mathrm{O}(\varepsilon) = m^* + \mathrm{O}(\varepsilon)$$

first component : $W + \mathrm{O}(\varepsilon) = W^* + \mathrm{O}(\varepsilon)$

no more information because $W^* = W$

second component : $Y + \mathrm{O}(\varepsilon) = Y^* + \mathrm{O}(\varepsilon)$

relaxation: $Y^* = Y + S\,(\Phi(W) - Y)$

The matrix S is supposed fixed

then $Y = \Phi(W) + \mathrm{O}(\varepsilon)$

and $Y^* = \Phi(W) + \mathrm{O}(\varepsilon)$

Chapman-Enskog expansion : order one

$$m(t+\varepsilon) = \exp(-\varepsilon\,\Lambda)\, m^*$$

$$m + \varepsilon\,\partial_t m + \mathrm{O}(\varepsilon^2) = m^* - \varepsilon\,\Lambda\, m^* + \mathrm{O}(\varepsilon^2)$$

with $\partial_t = \partial_{t_1} + \varepsilon\,\partial_{t_2} + \varepsilon^2\,\partial_{t_3} + \mathrm{O}(\varepsilon^3)$

then $\quad m + \varepsilon\,\partial_{t_1} m + \mathrm{O}(\varepsilon^2) = m^* - \varepsilon\,\Lambda\, m^* + \mathrm{O}(\varepsilon^2)$

first component

$$W + \varepsilon\,\partial_{t_1} W + \mathrm{O}(\varepsilon^2) = W^* - \varepsilon\,(A\,W + B\,Y^*) + \mathrm{O}(\varepsilon^2)$$

with $\quad W^* = W \quad$ and $\quad Y^* = \Phi(W) + \mathrm{O}(\varepsilon)$

then $\quad \partial_{t_1} W = -(A\,W + B\,\Phi(W))$

$$\partial_{t_1} W + \Gamma_1(W) = 0 \quad \text{with} \quad \Gamma_1 = A\,W + B\,\Phi(W)$$

Chapman-Enskog expansion : order one bis

second component of $m + \varepsilon\, \partial_{t_1} m + \mathrm{O}(\varepsilon^2) = m^* - \varepsilon\, \Lambda\, m^* + \mathrm{O}(\varepsilon^2)$

$$Y + \varepsilon\, \partial_{t_1} Y + \mathrm{O}(\varepsilon^2) = Y^* - \varepsilon\, (C\, W + D\, Y^*) + \mathrm{O}(\varepsilon^2)$$

known: $\quad Y^* = Y + S\,(\Phi(W) - Y)$

$Y = \Phi(W) + \mathrm{O}(\varepsilon)$ and $Y^* = \Phi(W) + \mathrm{O}(\varepsilon)$

$$S\,(Y - \Phi(W)) = Y - Y^*$$

$$= -\varepsilon\, \partial_{t_1} (\Phi(W) + \mathrm{O}(\varepsilon)) - \varepsilon\, (C\, W + D\, (\Phi(W) + \mathrm{O}(\varepsilon)) + \mathrm{O}(\varepsilon^2)$$

$$= \varepsilon\, \big[-\, \mathrm{d}\Phi(W).\partial_{t_1} W - (C\, W + D\, \Phi(W))\big] + \mathrm{O}(\varepsilon^2)$$

$$= \varepsilon\, \big[\mathrm{d}\Phi(W).\Gamma_1 - (C\, W + D\, \Phi(W))\big] + \mathrm{O}(\varepsilon^2)$$

Then $\quad Y = \Phi(W) + \varepsilon\, S^{-1}\, \Psi_1(W) + \mathrm{O}(\varepsilon^2)$

$$\Psi_1 = \mathrm{d}\Phi(W).\Gamma_1 - (C\, W + D\, \Phi(W))$$

$$Y^* = \Phi(W) + \varepsilon\, (S^{-1} - \mathrm{I})\, \Psi_1(W) + \mathrm{O}(\varepsilon^2)$$

Chapman-Enskog expansion : Hénon's matrix

$$\Sigma = S^{-1} - \tfrac{1}{2}\,\mathrm{I}$$

$$Y = \Phi(W) + \varepsilon\, S^{-1}\, \Psi_1(W) + \mathrm{O}(\varepsilon^2)$$

Then $\quad Y = \Phi(W) + \varepsilon\left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\,\Psi_1(W) + \mathrm{O}(\varepsilon^2)$

$$Y^* = \Phi(W) + \varepsilon\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\,\Psi_1(W) + \mathrm{O}(\varepsilon^2)$$

$$\Psi_1 = \mathrm{d}\Phi(W).\Gamma_1 - (C\,W + D\,\Phi(W))$$

Chapman-Enskog expansion : order two

$$m + \varepsilon\,\partial_t m + \tfrac{1}{2}\,\varepsilon^2\,\partial_t^2 m + \mathrm{O}(\varepsilon^3) = m^* - \varepsilon\,\Lambda\,m^* + \tfrac{1}{2}\,\varepsilon^2\,\Lambda^2\,m^* + \mathrm{O}(\varepsilon^3)$$

then

$$m + \varepsilon\,(\partial_{t_1} + \varepsilon\,\partial_{t_2})m + \tfrac{1}{2}\,\varepsilon^2\,(\partial_{t_1} + \mathrm{O}(\varepsilon))^2 m + \mathrm{O}(\varepsilon^3)$$
$$= m^* - \varepsilon\,\Lambda\,m^* + \tfrac{1}{2}\,\varepsilon^2\,\Lambda^2\,m^* + \mathrm{O}(\varepsilon^3)$$

first component

$$W + \varepsilon\,(\partial_{t_1} + \varepsilon\,\partial_{t_2})W + \tfrac{1}{2}\,\varepsilon^2\,\partial_{t_1}^2 W + \mathrm{O}(\varepsilon^3)$$
$$= W - \varepsilon\,(A\,W + B\,Y^*) + \tfrac{1}{2}\,\varepsilon^2\,(A_2\,W + B_2\,Y^*) + \mathrm{O}(\varepsilon^3)$$
$$\text{with}\quad Y^* = \Phi(W) + \varepsilon\,(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1) + \mathrm{O}(\varepsilon^2)$$

second order term

$$\partial_{t_2} W + \tfrac{1}{2}\,\partial_{t_1}^2 W = -B\,(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1) + \tfrac{1}{2}\,(A_2 W + B_2\,\Phi)$$

Chapman-Enskog expansion : order two *(ii)*

$$\partial_{t_2} W + \tfrac{1}{2}\, \partial_{t_1}^2 W = -B\, \Sigma\, \Psi_1 + \tfrac{1}{2}\, B\, \Psi_1 + \tfrac{1}{2}\, (A_2 W + B_2\, \Phi)$$

with $\quad \partial_{t_1}^2 W = -\partial_{t_1} (\Gamma_1(W)) = -\partial_{t_1} (A\, W + B\, \Phi(W))$

$$= A\, \Gamma_1 + B \,\mathrm{d}\Phi(W).\Gamma_1 = A\, (A\, W + B\, \Phi) + B\, \mathrm{d}\Phi(W).\Gamma_1$$

$$\partial_{t_2} W + \tfrac{1}{2}\, (A^2\, W + A\, B\, \Phi) + \tfrac{1}{2}\, B\, \mathrm{d}\Phi(W).\Gamma_1 = -B\, \Sigma\, \Psi_1 + \tfrac{1}{2}\, B\, \Psi_1 + \tfrac{1}{2}\, (A_2 W + B_2\, \Phi)$$

with $A_2 = A^2 + B\, C, \quad B_2 = A\, B + B\, D$

$$\partial_{t_2} W + B\, \Sigma\, \Psi_1 = -\tfrac{1}{2}\, (A^2\, W + A\, B\, \Phi) - \tfrac{1}{2}\, B\, \mathrm{d}\Phi(W).\Gamma_1 + \tfrac{1}{2}\, B\, \Psi_1 + \tfrac{1}{2}\, (A^2 + B\, C)\, W + \tfrac{1}{2}\, (A\, B + B\, D)\, \Phi$$

$$= -\tfrac{1}{2}\, B\, \mathrm{d}\Phi(W).\Gamma_1 + \tfrac{1}{2}\, B\, (\mathrm{d}\Phi(W).\Gamma_1 - C\, W - D\, \Phi) + \tfrac{1}{2}\, B\, C\, W + \tfrac{1}{2}\, B\, D\, \Phi$$

$$= 0$$

$\partial_{t_2} W + \Gamma_2(W) = 0 \quad$ with $\quad \Gamma_2(W) = B\, \Sigma\, \Psi_1(W)$

Chapman-Enskog expansion : order two bis

$$m + \varepsilon\,(\partial_{t_1} + \varepsilon\,\partial_{t_2})m + \tfrac{1}{2}\,\varepsilon^2\,\partial_{t_1}^2\,m + \mathrm{O}(\varepsilon^3)$$
$$= m^* - \varepsilon\,\Lambda\,m^* + \tfrac{1}{2}\,\varepsilon^2\,\Lambda^2\,m^* + \mathrm{O}(\varepsilon^3)$$

second component

$$Y + \varepsilon\,\partial_{t_1} Y + \varepsilon^2\,\partial_{t_2} Y + \tfrac{1}{2}\,\varepsilon^2\,\partial_{t_1}^2\,Y + \mathrm{O}(\varepsilon^3)$$
$$= Y^* - \varepsilon\,(C\,W + D\,Y^*) + \tfrac{1}{2}\,\varepsilon^2\,(C_2\,W + D_2\,Y^*) + \mathrm{O}(\varepsilon^3)$$

$$S\,(Y - \Phi(W)) = Y - Y^*$$
$$= -\varepsilon\,\partial_{t_1} Y - \varepsilon^2\left(\partial_{t_2} Y + \tfrac{1}{2}\,\partial_{t_1}^2\,Y\right) - \varepsilon\,(C\,W + D\,Y^*)$$
$$+\tfrac{1}{2}\,\varepsilon^2\,(C_2\,W + D_2\,Y^*) + \mathrm{O}(\varepsilon^3)$$

with $Y = \Phi(W) + \varepsilon\left(\Sigma\,\Psi_1 + \tfrac{1}{2}\,\Psi_1\right) + \mathrm{O}(\varepsilon^2)$

$$= -\varepsilon\,\partial_{t_1}\Phi(W) - \varepsilon^2\left(\partial_{t_1}(\Sigma\,\Psi_1 + \tfrac{1}{2}\,\Psi_1) + \partial_{t_2}\Phi(W) + \tfrac{1}{2}\,\partial_{t_1}^2\Phi(W)\right)$$
$$-\varepsilon\,(C\,W + D\,Y^*) + \tfrac{1}{2}\,\varepsilon^2\,(C_2\,W + D_2\,Y^*) + \mathrm{O}(\varepsilon^3)$$

Chapman-Enskog expansion : order two bis *(ii)*

$$S\,(Y-\Phi(W)) = -\varepsilon\,\partial_{t_1}\Phi(W)$$
$$-\varepsilon^2\left(\partial_{t_1}(\Sigma\,\Psi_1+\tfrac{1}{2}\,\Psi_1)+\partial_{t_2}\Phi(W)+\tfrac{1}{2}\,\partial_{t_1}^2\Phi(W)\right)$$
$$-\varepsilon\,(C\,W+D\,Y^*)+\tfrac{1}{2}\,\varepsilon^2\,(C_2\,W+D_2\,Y^*)+\mathrm{O}(\varepsilon^3)$$
$$\text{with}\quad Y^*=\Phi(W)+\varepsilon\left(\Sigma\,\Psi_1-\tfrac{1}{2}\,\Psi_1\right)+\mathrm{O}(\varepsilon^2)$$

we have by definition $S\,(Y-\Phi(W))=\varepsilon\,\Psi_1+\varepsilon^2\,\Psi_2+\mathrm{O}(\varepsilon^3)$

second order term

$$\Psi_2=-\Sigma\,\partial_{t_1}\Psi_1-\tfrac{1}{2}\,\partial_{t_1}\Psi_1-\partial_{t_2}\Phi(W)-\tfrac{1}{2}\,\partial_{t_1}^2\Phi(W)$$
$$-D\left(\Sigma\,\Psi_1-\tfrac{1}{2}\,\Psi_1\right)+\tfrac{1}{2}\,C_2\,W+\tfrac{1}{2}D_2\,\Phi(W)$$
$$\text{with}\quad \partial_{t_1}\Psi_1=\mathrm{d}\Psi_1.\partial_{t_1}W=-\,\mathrm{d}\Psi_1.\Gamma_1$$
$$\text{and}\quad \partial_{t_1}\Psi_1=\partial_{t_1}\left(\mathrm{d}\Phi.\Gamma_1-C\,W-D\,\Phi(W)\right)$$
$$=\partial_{t_1}(\,\mathrm{d}\Phi.\Gamma_1)-C\,\partial_{t_1}W-D\,\mathrm{d}\Phi.\partial_{t_1}W$$
$$=\partial_{t_1}(\,\mathrm{d}\Phi.\Gamma_1)+C\,\Gamma_1+D\,\mathrm{d}\Phi.\Gamma_1$$

$$\Psi_2=\Sigma\,\mathrm{d}\Psi_1.\Gamma_1-\tfrac{1}{2}\left(\partial_{t_1}(\,\mathrm{d}\Phi.\Gamma_1)+C\,\Gamma_1+D\,\mathrm{d}\Phi.\Gamma_1\right)-\partial_{t_2}\Phi(W)$$
$$-\tfrac{1}{2}\,\partial_{t_1}^2\Phi(W)-D\left(\Sigma\,\Psi_1-\tfrac{1}{2}\,\Psi_1\right)+\tfrac{1}{2}\,C_2\,W+\tfrac{1}{2}D_2\,\Phi(W)$$

Chapman-Enskog expansion : order two bis *(iii)*

$$\Psi_2 = \Sigma\, \mathrm{d}\Psi_1.\Gamma_1 - \tfrac{1}{2}\left(\partial_{t_1}(\mathrm{d}\Phi.\Gamma_1) + C\,\Gamma_1 + D\,\mathrm{d}\Phi.\Gamma_1\right) - \partial_{t_2}\Phi(W)$$
$$-\tfrac{1}{2}\,\partial_{t_1}^2\Phi(W) - D\left(\Sigma\,\Psi_1 - \tfrac{1}{2}\,\Psi_1\right) + \tfrac{1}{2}\,C_2\,W + \tfrac{1}{2}D_2\,\Phi(W)$$
$$\text{with}\ \ \partial_{t_2}\Phi(W) = \mathrm{d}\Phi(W).\partial_{t_2}W = -\Phi(W).\Gamma_2$$
$$\partial_{t_1}^2\Phi(W) = \partial_{t_1}\left(\partial_{t_1}\Phi(W)\right) = \partial_{t_1}(\mathrm{d}\Phi.\partial_{t_1}W) = -\partial_{t_1}(\mathrm{d}\Phi.\Gamma_1)$$

$$\Psi_2 = \Sigma\, \mathrm{d}\Psi_1.\Gamma_1 - \tfrac{1}{2}\left(\partial_{t_1}(\mathrm{d}\Phi.\Gamma_1) + C\,\Gamma_1 + D\,\mathrm{d}\Phi.\Gamma_1\right) + \Phi(W).\Gamma_2$$
$$+\tfrac{1}{2}\,\partial_{t_1}(\mathrm{d}\Phi.\Gamma_1) - D\,\Sigma\,\Psi_1 + \tfrac{1}{2}\,D\,\Psi_1 + \tfrac{1}{2}\,C_2\,W + \tfrac{1}{2}D_2\,\Phi(W)$$
$$\text{with}\ \ C_2 = C\,A + B\,D, \qquad D_2 = C\,B + D^2$$

$$\Psi_2 = \Sigma\, \mathrm{d}\Psi_1.\Gamma_1 - \tfrac{1}{2}\left(C\,\Gamma_1 + D\,\mathrm{d}\Phi.\Gamma_1\right) + \Phi(W).\Gamma_2 - D\,\Sigma\,\Psi_1 + \tfrac{1}{2}\,D\,\Psi_1$$
$$+\tfrac{1}{2}\ C\,(A\,W + B\,\Phi) + \tfrac{1}{2}\,D\,(C\,W + B\,\Phi)$$
$$\text{with}\ \ C\,W + B\,\Phi = \mathrm{d}\Phi.\Gamma_1 - \Psi_1$$

$$\Psi_2 = \Sigma\, \mathrm{d}\Psi_1.\Gamma_1 - \tfrac{1}{2}\,D\,\mathrm{d}\Phi.\Gamma_1 + \Phi(W).\Gamma_2 - D\,\Sigma\,\Psi_1 + \tfrac{1}{2}\,D\,\Psi_1$$
$$+\tfrac{1}{2}\,D\,(\mathrm{d}\Phi.\Gamma_1 - \Psi_1)$$

$$\Psi_2 = \Sigma\, \mathrm{d}\Psi_1.\Gamma_1 + \mathrm{d}\Phi(W).\Gamma_2 - D\,\Sigma\,\Psi_1$$

Chapman-Enskog expansion : order three

$$m + \varepsilon\, \partial_t m + \tfrac{1}{2}\, \varepsilon^2\, \partial_t^2 m + \tfrac{1}{6}\, \varepsilon^3\, \partial_t^3 m + \mathrm{O}(\varepsilon^4)$$
$$= m^* - \varepsilon\, \Lambda\, m^* + \tfrac{1}{2}\, \varepsilon^2\, \Lambda^2\, m^* - \tfrac{1}{6}\, \varepsilon^3\, \Lambda^3\, m^* + \mathrm{O}(\varepsilon^4)$$

then

$$m + \varepsilon\, (\partial_{t_1} + \varepsilon\, \partial_{t_2} + \varepsilon^2\, \partial_{t_3}) m + \tfrac{1}{2}\, \varepsilon^2\, (\partial_{t_1} + \varepsilon\, \partial_{t_2} + \mathrm{O}(\varepsilon^2))^2 m$$
$$+ \tfrac{1}{6}\, \varepsilon^3\, (\partial_{t_1} + \mathrm{O}(\varepsilon))^3 m + \mathrm{O}(\varepsilon^4) = m^* - \varepsilon\, \Lambda\, m^*$$
$$+ \tfrac{1}{2}\, \varepsilon^2\, \Lambda^2\, m^* - \tfrac{1}{6}\, \varepsilon^3\, \Lambda^3\, m^* + \mathrm{O}(\varepsilon^4)$$

first component

$$W + \varepsilon\, (\partial_{t_1} + \varepsilon\, \partial_{t_2} + \varepsilon^2\, \partial_{t_3}) W + \tfrac{1}{2}\, \varepsilon^2\, (\partial_{t_1}^2 + \varepsilon\, \partial_{t_1}\, \partial_{t_2} + \varepsilon\, \partial_{t_2}\, \partial_{t_1}) W$$
$$+ \tfrac{1}{6}\, \varepsilon^3\, \partial_{t_1}^3 W + \mathrm{O}(\varepsilon^4) = W - \varepsilon\, (A\, W + B\, Y^*)$$
$$+ \tfrac{1}{2}\, \varepsilon^2\, (A_2\, W + B_2\, Y^*) - \tfrac{1}{6}\, \varepsilon^3\, (A_3\, W + B_3\, Y^*) + \mathrm{O}(\varepsilon^4)$$

with $\quad Y^* = \Phi(W) + \varepsilon\, (\Sigma\, \Psi_1 - \tfrac{1}{2} \Psi_1) + \varepsilon^2\, (\Sigma\, \Psi_2 - \tfrac{1}{2} \Psi_2) + \mathrm{O}(\varepsilon^3)$

Chapman-Enskog expansion : order three *(ii)*

$$W + \varepsilon\,(\partial_{t_1} + \varepsilon\,\partial_{t_2} + \varepsilon^2\,\partial_{t_3})W + \tfrac{1}{2}\,\varepsilon^2\,(\partial_{t_1}^2 + \varepsilon\,\partial_{t_1}\,\partial_{t_2} + \varepsilon\,\partial_{t_2}\,\partial_{t_1})W$$
$$+\tfrac{1}{6}\,\varepsilon^3\,\partial_{t_1}^3 W + \mathrm{O}(\varepsilon^4) = W - \varepsilon\,A\,W$$
$$-\varepsilon\,B\,\big(\Phi(W) + \varepsilon\,(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1) + \varepsilon^2\,(\Sigma\,\Psi_2 - \tfrac{1}{2}\Psi_2)\big) + \tfrac{1}{2}\,\varepsilon^2\,A_2\,W$$
$$+\tfrac{1}{2}\,\varepsilon^2\,B_2\,\big(\Phi(W) + \varepsilon\,(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1)\big) - \tfrac{1}{6}\,\varepsilon^3\,(A_3\,W + B_3\,\Phi) + \mathrm{O}(\varepsilon^4)$$

third order term

$$\partial_{t_3} W + \tfrac{1}{2}\,(\partial_{t_1}\partial_{t_2} W + \partial_{t_2}\partial_{t_1} W) + \tfrac{1}{6}\,\partial_{t_1}^3 W$$
$$= -B\,(\Sigma\,\Psi_2 - \tfrac{1}{2}\Psi_2) + \tfrac{1}{2}\,B_2\,(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1) - \tfrac{1}{6}\,(A_3\,W + B_3\,\Phi)$$

with

$$\partial_{t_1}\partial_{t_2} W = \partial_{t_1}(-B\,\Sigma\,\Psi_1) = -B\,\Sigma\,\mathrm{d}\Psi_1.\partial_{t_1} W = B\,\Sigma\,\mathrm{d}\Psi_1.\Gamma_1$$
$$\partial_{t_2}\partial_{t_1} W = \partial_{t_2}(-A\,W - B\,\Phi) = A\,\Gamma_2 - B\,\mathrm{d}\Phi.\partial_{t_2} W$$
$$= A\,B\,\Sigma\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_2$$

$$\partial_{t_3} W + \tfrac{1}{2}\,B\,\Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + \tfrac{1}{2}\,(A\,B\,\Sigma\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_2) + \tfrac{1}{6}\,\partial_{t_1}^3 W$$
$$= -B\,(\Sigma\,\Psi_2 - \tfrac{1}{2}\Psi_2) + \tfrac{1}{2}\,B_2\,(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1) - \tfrac{1}{6}\,(A_3\,W + B_3\,\Phi)$$

Chapman-Enskog expansion : order three *(iii)*

$$\partial_{t_3} W + \tfrac{1}{2}\left(B\,\Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + A\,B\,\Sigma\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_2\right) + \tfrac{1}{6}\,\partial_{t_1}^3 W$$
$$= -B\left(\Sigma\,\Psi_2 - \tfrac{1}{2}\Psi_2\right) + \tfrac{1}{2}\,B_2\left(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1\right) - \tfrac{1}{6}\left(A_3\,W + B_3\,\Phi\right)$$

with

$$\partial_{t_1}^3 W = \partial_{t_1}\left(A\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_1\right) = \partial_{t_1}\left(A\left(A\,W + B\,\Phi\right) + B\,\mathrm{d}\Phi.\Gamma_1\right)$$
$$= -A\left(A\,\Gamma_1 - B\,\mathrm{d}\Phi.\Gamma_1\right) - B\,\mathrm{d}(\mathrm{d}\Phi.\Gamma_1).\Gamma_1$$
$$= -A^2\,\Gamma_1 - A\,B\,\mathrm{d}\Phi.\Gamma_1 - B\,\mathrm{d}(\mathrm{d}\Phi.\Gamma_1).\Gamma_1$$

and $\quad A_3 = A_2\,A + B_2\,C, \qquad B_3 = A_2\,B + B_2\,D$

$$A_3\,W + B_3\,\Phi = A_2\left(A\,W + B\,\Phi\right) + B_2\left(C\,W + D\,\Phi\right)$$
$$= \left(A^2 + B\,C\right)\Gamma_1 + \left(A\,B + B\,D\right)\left(\mathrm{d}\Phi.\Gamma_1 - \Psi_1\right)$$
$$= A\left(A\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_1\right) + B\left(C\,\Gamma_1 + D\,\mathrm{d}\Phi.\Gamma_1\right) - B_2\,\Psi_1$$
$$= A\left(A\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_1\right) + B\left(\mathrm{d}(\mathrm{d}\Phi.\Gamma_1).\Gamma_1 - \mathrm{d}\Psi_1.\Gamma_1\right) - B_2\,\Psi_1$$
$$= -\partial_{t_1}^3 W - B\,\mathrm{d}\Psi_1.\Gamma_1 - B_2\,\Psi_1$$

Chapman-Enskog expansion : order three *(iv)*

$$\partial_{t_3} W + \tfrac{1}{2}\left(B\,\Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + A\,B\,\Sigma\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_2\right) = -B\,\Sigma\,\Psi_2 + \tfrac{1}{2}\,B\,\Psi_2 + \tfrac{1}{2}\,B_2\left(\Sigma\,\Psi_1 - \tfrac{1}{2}\Psi_1\right) + \tfrac{1}{6}\left(B\,\mathrm{d}\Psi_1.\Gamma_1 + B_2\,\Psi_1\right)$$

with $\Psi_2 = \Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + \mathrm{d}\Phi.\Gamma_2 - D\,\Sigma\,\Psi_1$

$$\partial_{t_3} W + \tfrac{1}{2}\left(B\,\Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + A\,B\,\Sigma\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_2\right) = -B\,\Sigma\,\Psi_2 + \tfrac{1}{2}\,B\left(\Sigma\,\mathrm{d}\Psi_1.\Gamma_1 + \mathrm{d}\Phi.\Gamma_2 - D\,\Sigma\,\Psi_1\right) + \tfrac{1}{2}\,B_2\,\Sigma\,\Psi_1 - \left(\tfrac{1}{4} - \tfrac{1}{6}\right) B_2\,\Psi_1 + \tfrac{1}{6}\,B\,\mathrm{d}\Psi_1.\Gamma_1$$

then $\partial_{t_3} W + B\,\Sigma\,\Psi_2 + \frac{1}{12}\,B_2\,\Psi_1 - \frac{1}{6}\,B\,\mathrm{d}\Psi_1.\Gamma_1 = 0$

$$\Gamma_3 = B\,\Sigma\,\Psi_2 + \tfrac{1}{12}\,B_2\,\Psi_1 - \tfrac{1}{6}\,B\,\mathrm{d}\Psi_1.\Gamma_1$$

Taylor second-order expansion

$$m + \Delta t \, \partial_t m + \tfrac{1}{2} \, \Delta t^2 \, \partial_t^2 m + \mathrm{O}(\Delta t^3) = \\ = m^* - \Delta t \, \Lambda \, m^* + \tfrac{1}{2} \, \Delta t^2 \, \Lambda^2 \, m^* + \mathrm{O}(\Delta t^3)$$

We replace the vector m by its two components W and Y

$$W + \Delta t \, \partial_t W + \tfrac{1}{2} \, \Delta t^2 \, \partial_t^2 W + \mathrm{O}(\Delta t^3) = \\ = W - \Delta t \, (A \, W + B \, Y^*) + \tfrac{1}{2} \, \Delta t^2 \, (A_2 \, W + B_2 \, Y^*) + \mathrm{O}(\Delta t^3)$$

$$Y + \Delta t \, \partial_t Y + \tfrac{1}{2} \, \Delta t^2 \, \partial_t^2 Y + \mathrm{O}(\Delta t^3) = \\ = Y^* - \Delta t \, (C \, W + D \, Y^*) + \tfrac{1}{2} \, \Delta t^2 \, (C_2 \, W + D_2 \, Y^*) + \mathrm{O}(\Delta t^3)$$

At zero-order: $Y - Y^* = \mathrm{O}(\Delta t)$ and $Y^* \equiv Y + S \, (\Phi(W) - Y)$

The matrix S is supposed fixed

then $Y = \Phi(W) + \mathrm{O}(\Delta t)$ and $Y^* = \Phi(W) + \mathrm{O}(\Delta t)$

Taylor second-order expansion *(ii)*

$Y^* = \Phi(W) + \mathrm{O}(\Delta t)$

Second-order partial differential equations:

$$\partial_t W + \tfrac{1}{2}\,\Delta t\,\partial_t^2 W + \mathrm{O}(\Delta t^2) = \\ = -(A\,W + B\,Y^*) + \tfrac{1}{2}\,\Delta t\,(A_2\,W + B_2\,Y^*) + \mathrm{O}(\Delta t^2)$$

at first-order:

$$\partial_t W + \mathrm{O}(\Delta t) = -A\,W - B\,\Phi(W) + \mathrm{O}(\Delta t)$$

then $\quad \partial_t W = -\Gamma_1 \quad$ with $\quad \Gamma_1 = A\,W + B\,\Phi(W)$

then $\;\partial_t Y = \mathrm{d}\Phi(W).\partial_t W + \mathrm{O}(\Delta t) = -\,\mathrm{d}\Phi(W).\Gamma_1 + \mathrm{O}(\Delta t)$

$$Y - Y^* = -\Delta t\,\partial_t Y - \Delta t\,(C\,W + D\,Y^*) + \mathrm{O}(\Delta t^2)$$

then $\quad S\,(Y - \Phi) = \Delta t\,(\,\mathrm{d}\Phi(W).\Gamma_1 - C\,W - D\,\Phi) + \mathrm{O}(\Delta t^2)$

$$Y = \Phi + \Delta t\,S^{-1}\,(\,\mathrm{d}\Phi(W).\Gamma_1 - C\,W - D\,\Phi) + \mathrm{O}(\Delta t^2)$$

and $\quad \Psi_1 = \,\mathrm{d}\Phi(W).\Gamma_1 - (C\,W + D\,\Phi)$

Taylor second-order expansion *(iii)*

$$\partial_t W = -\Gamma_1 + \mathrm{O}(\Delta t) = -(A\,W + B\,\Phi(W)) + \mathrm{O}(\Delta t)$$

Then $\partial_t^2 W = \partial_t\big(-\Gamma_1 + \mathrm{O}(\Delta t)\big) \;= -\,\mathrm{d}\Gamma_1.\partial_t W + \mathrm{O}(\Delta t)$

$$= A\,\Gamma_1 + B\;\mathrm{d}\Phi.\partial_t W + \mathrm{O}(\Delta t)$$

$$= A\,\Gamma_1 + B\;\mathrm{d}\Phi.\Gamma_1 + \mathrm{O}(\Delta t)$$

Second-order partial differential equations:

$$\partial_t W + \tfrac{1}{2}\,\Delta t\,\partial_t^2 W + \mathrm{O}(\Delta t^2) =$$

$$= -(A\,W + B\,Y^*) + \tfrac{1}{2}\,\Delta t\,(A_2\,W + B_2\,Y^*) + \mathrm{O}(\Delta t^2)$$

$$\partial_t W = -A\,W - B\,\big(\Phi + (\Sigma - \tfrac{1}{2}\mathrm{I})\,\Delta t\,\Psi_1\big) - \tfrac{1}{2}\,\Delta t\,(A\,\Gamma_1 + B\;\mathrm{d}\Phi.\Gamma_1)$$

$$+\tfrac{1}{2}\,\Delta t\,\big((A^2 + B\,C)\,W + (A\,B + B\,D)\,\Phi\big) + \mathrm{O}(\Delta t^2)$$

$$= -A\,W - B\,\Phi + \Delta t\,\Big(-B\,\Sigma\,\Psi_1 + \tfrac{1}{2}B\,\big(\,\mathrm{d}\Phi.\Gamma_1 - C\,W - D\,\Phi\big)$$

$$-\tfrac{1}{2}(A\,(A\,W + B\,\Phi)) - \tfrac{1}{2}\,B\;\mathrm{d}\Phi.\Gamma_1 + \tfrac{1}{2}\,(A^2 + B\,C)\,W$$

$$+\tfrac{1}{2}\,(A\,B + B\,D)\,\Phi\Big) + \mathrm{O}(\Delta t^2)$$

$\partial_t W = -\Gamma_1 - \Delta t\,B\,\Sigma\,\Psi_1 + \mathrm{O}(\Delta t^2)$ and $\Gamma_2 = B\,\Sigma\,\Psi_1$

Link with the original Taylor expansion method (2007)

First-order $\partial_t W = -\Gamma_1 + \mathrm{O}(\Delta t) = -A\, W - B\, \Phi(W) + \mathrm{O}(\Delta t)$

id est $\partial_t W_i + \Lambda_{ij}^{\beta}\, \partial_\beta W_j + \Lambda_{i\ell}^{\beta} \partial_\beta \Phi(W)_\ell = \mathrm{O}(\Delta t)$

or $\partial_t W_i + \Lambda_{i\ell}^{\beta} \partial_\beta m_\ell^{\mathrm{eq}} = \mathrm{O}(\Delta t)$

with the notation $m_k^{\mathrm{eq}} \equiv \Phi(W)_k$

Defect of conservation $\theta_k \equiv \partial_t m_k^{\mathrm{eq}} + \sum_{\ell\beta} \Lambda_{k\ell}^{\beta}\, \partial_\beta m_\ell^{\mathrm{eq}}$

vector of conservation defect

$$
\begin{aligned}
\theta &= \partial_t \Phi(W) + \Lambda . m^{\mathrm{eq}} \\
&= \mathrm{d}\Phi . (-\Gamma_1 - \Delta t\, \Gamma_2 + \cdots) + (C\, W + D\, \Phi(W)) \\
&= (-\mathrm{d}\Phi . \Gamma_1 + \mathrm{O}(\Delta t)) + (C\, W + D\, \Phi(W)) \\
&= -\Psi_1 + \mathrm{O}(\Delta t)
\end{aligned}
$$

$\theta = -\Psi_1 + \mathrm{O}(\Delta t)$

Second-order term

$$
(B\, \Sigma\, \Psi_1)_i = \sum_k B_{ik}\, \sigma_k\, (-\theta_k + \mathrm{O}(\Delta t)) = \sum_k \Lambda_{ik}^{\beta}\, \sigma_k\, \partial_\beta \theta_k + \mathrm{O}(\Delta t)
$$

and $\partial_t W_i + \Lambda_{i\ell}^{\beta} \partial_\beta m_\ell^{\mathrm{eq}} = \Delta t \sum_k \Lambda_{ik}^{\beta}\, \sigma_k\, \partial_\beta \theta_k + \mathrm{O}(\Delta t^2)$

Taylor third-order expansion

$$m + \Delta t\, \partial_t m + \tfrac{1}{2}\, \Delta t^2\, \partial_t^2 m + \tfrac{1}{6}\, \Delta t^3\, \partial_t^3 m + \mathrm{O}(\Delta t^4) =$$
$$= m^* - \Delta t\, \Lambda\, m^* + \tfrac{1}{2}\, \Delta t^2\, \Lambda^2\, m^* + \tfrac{1}{6}\, \Delta t^3\, \Lambda^3\, m^* + \mathrm{O}(\Delta t^4)$$

We replace the vector m by its two components W and Y

$$W + \Delta t\, \partial_t W + \tfrac{1}{2}\, \Delta t^2\, \partial_t^2 W + \tfrac{1}{6}\, \Delta t^3\, \partial_t^3 W + \mathrm{O}(\Delta t^4) =$$
$$= W - \Delta t\, (A\, W + B\, Y^*) + \tfrac{1}{2}\, \Delta t^2\, (A_2\, W + B_2\, Y^*)$$
$$- \tfrac{1}{6}\, \Delta t^3\, (A_3\, W + B_3\, Y^*) + \mathrm{O}(\Delta t^4)$$

then $\partial_t W = -A\, W - B\, Y^* + \tfrac{1}{2}\, \Delta t\, (A_2\, W + B_2\, Y^* - \partial_t^2 W)$
$- \tfrac{1}{6}\, \Delta t^2\, (A_3\, W + B_3\, Y^* - \partial_t^3 W) + \mathrm{O}(\Delta t^3)$

$$Y + \Delta t\, \partial_t Y + \tfrac{1}{2}\, \Delta t^2\, \partial_t^2 Y + \mathrm{O}(\Delta t^3) =$$
$$= Y^* - \Delta t\, (C\, W + D\, Y^*) + \tfrac{1}{2}\, \Delta t^2\, (C_2\, W + D_2\, Y^*) + \mathrm{O}(\Delta t^3)$$

and

$$Y - Y^* \equiv S\, (Y - \Phi)$$
$$= -\Delta t\, (C\, W + D\, Y^* + \partial_t Y) + \tfrac{1}{2} \Delta t^2\, (C_2\, W + D_2\, Y^* - \partial_t^2 Y) + \mathrm{O}(\Delta t^3)$$

Computation of the coefficient Ψ_2

$$S\,(Y - \Phi) = -\Delta t\,(C\,W + D\,Y^* + \partial_t Y) + \tfrac{1}{2}\,\Delta t^2\,(C_2\,W + D_2\,Y^* - \partial_t^2 Y) + \mathrm{O}(\Delta t^3)$$

with [see Annex -1- page (iii)]

$$C\,W + D\,Y^* = -\Psi_1 + \mathrm{d}\Phi.\Gamma_1 + \Delta t\,D\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2)$$

$$C_2\,W + D_2\,Y^* = \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1 - \Psi_1).\Gamma_1 - D\,\Psi_1 + \mathrm{O}(\Delta t)$$

and [see Annex -2- pages (iii) and (iv)]

$$\partial_t Y = -\,\mathrm{d}\Phi.\Gamma_1 - \Delta t\,\Big(\mathrm{d}\Phi.\Gamma_2 + \big(\Sigma + \tfrac{1}{2}\,\mathrm{I}\big)\,\mathrm{d}\Psi_1.\,\Gamma_1\Big) + \mathrm{O}(\Delta t^2)$$

$$\partial_t^2 Y = \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1).\Gamma_1 + \mathrm{O}(\Delta t)$$

Then

$$S\,(Y - \Phi) = \Delta t\,\Psi_1 + \Delta t^2\,\Big(-\,D\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{d}\Phi.\Gamma_2 + \big(\Sigma + \tfrac{1}{2}\,\mathrm{I}\big)\,\mathrm{d}\Psi_1.\,\Gamma_1 - \tfrac{1}{2}\,\big(\mathrm{d}\Psi_1.\,\Gamma_1 + D\,\Psi_1\big)\Big) + \mathrm{O}(\Delta t^3)$$

$$= \Delta t\,\Psi_1 + \Delta t^2\,\big(\Sigma\,\mathrm{d}\Psi_1.\,\Gamma_1 + \mathrm{d}\Phi.\Gamma_2 - D\,\Sigma\,\Psi_1\big) + \mathrm{O}(\Delta t^3)$$

$$\Psi_2 = \Sigma\,\mathrm{d}\Psi_1.\,\Gamma_1 + \mathrm{d}\Phi.\Gamma_2 - D\,\Sigma\,\Psi_1$$

Computation of the coefficient Γ_3

$$\partial_t W = -A\, W - B\, Y^* + \tfrac{1}{2}\, \Delta t \,(A_2\, W + B_2\, Y^* - \partial_t^2 W) \\ -\tfrac{1}{6}\, \Delta t^2 \,(A_3\, W + B_3\, Y^* + \partial_t^3 W) + \mathrm{O}(\Delta t^3)$$

with [see Annex -1- pages (i) and (ii)]

$$A\, W + B\, Y^* = A\, W + B\, \Phi + \Delta t\, B \left(\Sigma - \tfrac{1}{2}\, \mathrm{I}\right) \Psi_1 \\ + \Delta t^2\, B \left(\Sigma - \tfrac{1}{2}\, \mathrm{I}\right) \Psi_2 + \mathrm{O}(\Delta t^3)$$

$$A_2\, W + B_2\, Y^* = A\, \Gamma_1 + B\,(\, \mathrm{d}\Phi.\Gamma_1 - \Psi_1) + \Delta t\, B_2 \left(\Sigma - \tfrac{1}{2}\, \mathrm{I}\right) \Psi_1 \\ + \mathrm{O}(\Delta t^2)$$

$$A_3\, W + B_3\, Y^* = \partial^2\Gamma_1.\Gamma_1 - B\, \mathrm{d}\Psi_1.\Gamma_1 - B_2\, \Psi_1 + \mathrm{O}(\Delta t)$$

and [see Annex -2- pages (i) and (ii)]

$$\partial_t^2 W = \mathrm{d}\Gamma_1.\, \Gamma_1 + \Delta t\,(\, \mathrm{d}\Gamma_1.\, \Gamma_2 + \mathrm{d}\Gamma_2.\, \Gamma_1) + \mathrm{O}(\Delta t^2)$$

$$\partial_t^3 W = -\partial^2\Gamma_1.\, \Gamma_1 + \mathrm{O}(\Delta t)$$

$$\partial_t W = -\Gamma_1 - \Delta t\, B\, \Sigma\, \Psi_1$$

$$+\Delta t^2 \left(-B \left(\Sigma - \tfrac{1}{2}\, \mathrm{I}\right) \Psi_2 + \tfrac{1}{2} \left(B_2 \left(\Sigma - \tfrac{1}{2}\, \mathrm{I}\right) \Psi_1 - \mathrm{d}\Gamma_1.\, \Gamma_2 - \mathrm{d}\Gamma_2.\, \Gamma_1\right) \right. \\ \left. + \tfrac{1}{6}\,(B\, \mathrm{d}\Psi_1.\Gamma_1 + B_2\, \Psi_1)\right) + \mathrm{O}(\Delta t^3)$$

Computation of the coefficient Γ_3 *(ii)*

$$\partial_t W + \Gamma_1 + \Delta t\, \Gamma_2 = \Delta t^2 \Big(- B \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Psi_2 + \tfrac{1}{2} \big(B_2 \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Psi_1 - \mathrm{d}\Gamma_1 .\, \Gamma_2 - \mathrm{d}\Gamma_2 .\, \Gamma_1\big) + \tfrac{1}{6} \big(B\ \mathrm{d}\Psi_1 .\Gamma_1 + B_2\, \Psi_1\big)\Big) + \mathrm{O}(\Delta t^3)$$

and the coefficient of Δt^2 is given by

$$\Gamma_3 = B\,\Sigma\,\Psi_2 - \tfrac{1}{2}\, B\,\Psi_2 + \tfrac{1}{2}\,(A\,B + B\,D)\,\Sigma\,\Psi_1 + \big(\tfrac{1}{4} - \tfrac{1}{6}\big)\, B_2\,\Psi_1 - \tfrac{1}{2}\,(A\,\Gamma_2 + B\ \mathrm{d}\Phi.\Gamma_2) - \tfrac{1}{2}\, B\,\Sigma\ \mathrm{d}\Psi_1 .\,\Gamma_1 - \tfrac{1}{6}\, B\ \mathrm{d}\Psi_1 .\Gamma_1$$

because $\Gamma_2 = B\,\Sigma\,\Psi_1$

$$= B\,\Sigma\,\Psi_2 - \tfrac{1}{2}\, B\,(D\,\Sigma\,\Psi_1 - \mathrm{d}\Phi.\Gamma_2 - \Sigma\ \mathrm{d}\Psi_1 .\,\Gamma_1) + \tfrac{1}{2}\, B\,D\,\Sigma\,\Psi_1 \quad \tfrac{1}{12}\, B_2\,\Psi_1 - \tfrac{1}{2}\, B\ \mathrm{d}\Phi.\Gamma_2 - \tfrac{1}{2}\, B\,\Sigma\ \mathrm{d}\Psi_1 .\,\Gamma_1 - \tfrac{1}{6}\, B\ \mathrm{d}\Psi_1 .\Gamma_1$$

Finally,

$$\Gamma_3 = B\,\Sigma\,\Psi_2 + \tfrac{1}{12}\, B_2\,\Psi_1 - \tfrac{1}{6}\, B\ \mathrm{d}\Psi_1 .\,\Gamma_1$$

Link with the previous formal expansion at third-order

Defect of conservation $\qquad \theta_k \equiv \partial_t m_k^{\rm eq} + \sum_{\ell\beta} \Lambda_{k\ell}^{\beta} \, \partial_\beta m_\ell^{\rm eq}$

vector of conservation defect $\qquad \theta = \partial_t \Phi(W) + \Lambda . m^{\rm eq}$

then $\qquad \theta = -\Psi_1 - \Delta t \; {\rm d}\Phi . \Gamma_2 + {\rm O}(\Delta t)$

and $\qquad \partial_t \theta = -\partial_t \Psi_1 + {\rm O}(\Delta t) \;\; = {\rm d}\Psi_1 . \, \Gamma_1 + {\rm O}(\Delta t)$

We have the following calculus:

$$
\begin{aligned}
\Gamma_2 + \Delta t \, \Gamma_3 &= B \, \Sigma \, \Psi_1 + \Delta t \, \big(B \, \Sigma \, \Psi_2 + \tfrac{1}{12} \, B_2 \, \Psi_1 - \tfrac{1}{6} \, B \; {\rm d}\Psi_1 . \, \Gamma_1 \big) \\
&= B \, \Sigma \, \big(-\theta - \Delta t \; {\rm d}\Phi . \Gamma_2 + {\rm O}(\Delta t^2) \big) + \Delta t \, B \, \Sigma \big(\Sigma \; {\rm d}\Psi_1 . \Gamma_1 + {\rm d}\Phi . \Gamma_2 \\
&\qquad - D \, \Sigma \, \Psi_1 \big) + \tfrac{1}{12} \, \Delta t \, B_2 \, \Psi_1 - \tfrac{1}{6} \, \Delta t \, B \; {\rm d}\Psi_1 . \, \Gamma_1 \\
&= -B \, \Sigma \, \theta + \Delta t \, \Big(- \big(B \, \Sigma \, D \, \Sigma \, - \tfrac{1}{12} \, B_2 \big) \, \Psi_1 \\
&\qquad + B \, \big(\Sigma^2 - \tfrac{1}{6} \big) \; {\rm d}\Psi_1 . \, \Gamma_1 \Big) + {\rm O}(\Delta t^2) \\
&= -B \, \Sigma \, \theta + \Delta t \, \Big(\big(B \, \Sigma \, D \, \Sigma \, - \tfrac{1}{12} \, B_2 \big) \, \theta \\
&\qquad + B \, \big(\Sigma^2 - \tfrac{1}{6} \big) \, \partial_t \theta \Big) + {\rm O}(\Delta t^2)
\end{aligned}
$$

Link with the previous formal expansion at third-order *(ii)*

$$\Gamma_2 + \Delta t\, \Gamma_3 = -B\,\Sigma\,\theta + \Delta t\,\Big(\big(B\,\Sigma\,D\,\Sigma\, - \tfrac{1}{12}\,B_2\big)\,\theta + B\,\big(\Sigma^2 - \tfrac{1}{6}\big)\,\partial_t\theta\,\Big) + \mathrm{O}(\Delta t^2)$$

Third-order partial equivalent equations:

$$\partial_t W + \Lambda\, m^{\mathrm{eq}} - \Delta t\, B\,\Sigma\,\theta + \Delta t^2\,\Big(\big(B\,\Sigma\,D\,\Sigma\, - \tfrac{1}{12}\,B_2\big)\,\theta + B\,\big(\Sigma^2 - \tfrac{1}{6}\big)\,\partial_t\theta\,\Big) = \mathrm{O}(\Delta t^3)$$

Explicitation with the cartesian components:

$$\big(B\,\Sigma\,D\,\Sigma\,\theta\big)_i = \Lambda^{\beta}_{ik}\,\sigma_k\,\big(\Lambda^{\gamma}_{k\ell}\big)\,\sigma_\ell\;\partial_\beta\partial_\gamma\theta_\ell = \Lambda^{\beta}_{ik}\,\sigma_k\,\Lambda^{\gamma}_{k\ell}\,\sigma_\ell\;\partial_\beta\partial_\gamma\theta_\ell$$

$$\big(B_2\,\theta\big)_i = \big(\Lambda^{\beta}_{ik}\,\partial_\beta\big)\,\big(\Lambda^{\gamma}_{k\ell}\,\partial_\gamma\big)\,\theta_\ell = \Lambda^{\beta}_{ik}\,\Lambda^{\gamma}_{k\ell}\;\partial_\beta\partial_\gamma\theta_\ell$$

$$\Big(\big(B\,\Sigma\,D\,\Sigma\, - \tfrac{1}{12}\,B_2\big)\,\theta\Big)_i = \Lambda^{\beta}_{ik}\,\Lambda^{\gamma}_{k\ell}\,\big(\sigma_k\,\sigma_\ell - \tfrac{1}{12}\big)\,\partial_\beta\partial_\gamma\theta_\ell$$

$$\Big(B\,\big(\Sigma^2 - \tfrac{1}{6}\big)\,\partial_t\theta\Big)_i = \Lambda^{\beta}_{ik}\,\big(\sigma_k^2\, - \tfrac{1}{6}\big)\,\partial_\beta\partial_t\theta_k$$

Other form of the third-order partial equivalent equations:

$$\partial_t W_i + \Lambda^{\alpha}_{ik}\,\partial_\alpha\, m^{\mathrm{eq}}_k - \Delta t\,\Lambda^{\beta}_{ik}\,\sigma_k\,\partial_\beta\,\theta_k + \Delta t^2\Big(\Lambda^{\beta}_{ik}\,\Lambda^{\gamma}_{k\ell}\,\big(\sigma_k\,\sigma_\ell - \tfrac{1}{12}\big)\,\partial_\beta\partial_\gamma\theta_\ell + \Lambda^{\beta}_{ik}\,\big(\sigma_k^2\, - \tfrac{1}{6}\big)\,\partial_\beta\partial_t\theta_k\Big) = \mathrm{O}(\Delta t^3)$$

Link with the previous formal expansion at third-order *(iii)*

$$\partial_t W_i + \Lambda_{ik}^{\beta}\, \partial_\beta\, m_k^{\rm eq} - \Delta t\, \Lambda_{ik}^{\beta}\, \sigma_k\, \partial_\beta\, \theta_k$$
$$+\Delta t^2 \Big(\Lambda_{ik}^{\beta}\, \Lambda_{k\ell}^{\gamma} \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell + \Lambda_{ik}^{\beta} \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k \Big) = {\rm O}(\Delta t^3)$$

Classical lattice Boltzmann scheme : $M_{0j} = 1$ and $M_{\alpha j} = v_j^\alpha$

then $\Lambda_{0\ell}^{\alpha} = M_{0j}\, v_j^\alpha\, (M^{-1})_{j\ell} = M_{\alpha j}\, (M^{-1})_{j\ell} = \delta_{\alpha\ell}$

and
$$\Lambda_{0k}^{\beta}\, \Lambda_{k\ell}^{\gamma} \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell = \delta_{\beta k}\, \Lambda_{k\ell}^{\gamma} \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell$$
$$= \Lambda_{\beta\ell}^{\gamma} \big(\sigma_\beta\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell$$

Thermal case: $\sigma_\beta \neq 0$ and
$$\Lambda_{0k}^{\beta}\, \Lambda_{k\ell}^{\gamma} \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell = \Lambda_{\beta\ell}^{\gamma} \big(\sigma_\beta\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell$$

Fluid case; the momentum is conserved: $\sigma_\beta = 0$. Then
$$\Lambda_{0k}^{\beta}\, \Lambda_{k\ell}^{\gamma} \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell = -\tfrac{1}{12}\, \Lambda_{\beta\ell}^{\gamma}\, \partial_\beta \partial_\gamma \theta_\ell$$

$$\Lambda_{0k}^{\beta} \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \delta_{\beta k} \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \big(\sigma_\beta^2 - \tfrac{1}{6}\big)\, \partial_t (\partial_\beta \theta_\beta)$$

Link with the previous formal expansion at third-order *(iv)*

$$\partial_t W_i + \Lambda_{ik}^{\beta}\, \partial_\beta\, m_k^{\rm eq} - \Delta t\, \Lambda_{ik}^{\beta}\, \sigma_k\, \partial_\beta\, \theta_k$$
$$+\Delta t^2 \Big(\Lambda_{ik}^{\beta}\, \Lambda_{k\ell}^{\gamma} \, \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell + \Lambda_{ik}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k \Big) = {\rm O}(\Delta t^3)$$

$$\Lambda_{0k}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \begin{cases} \Lambda_{\beta\ell}^{\gamma}\, \big(\sigma_\beta\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell & \text{thermics} \\ -\tfrac{1}{12}\, \Lambda_{\beta\ell}^{\gamma}\, \partial_\beta \partial_\gamma \theta_\ell & \text{fluid} \end{cases}$$

$$\Lambda_{0k}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \big(\sigma_\beta^2 - \tfrac{1}{6}\big)\, \partial_t (\partial_\beta \theta_\beta)$$

Thermal case: $\sigma_\beta \neq 0$ and

$$\Lambda_{0k}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \big(\sigma_\beta^2 - \tfrac{1}{6}\big)\, \partial_t (\partial_\beta \theta_\beta)$$

Fluid case; the momentum is conserved: $\theta_\beta = {\rm O}(\Delta t)$. Then

$$\Lambda_{0k}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = {\rm O}(\Delta t)$$

Link with the previous formal expansion at third-order *(v)*

$$\partial_t W_i + \Lambda^\beta_{ik}\, \partial_\beta\, m^{\rm eq}_k - \Delta t\, \Lambda^\beta_{ik}\, \sigma_k\, \partial_\beta\, \theta_k$$
$$+ \Delta t^2 \Big(\Lambda^\beta_{ik}\, \Lambda^\gamma_{k\ell}\, \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell + \Lambda^\beta_{ik}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k \Big) = {\rm O}(\Delta t^3)$$

Classical notations: $W_0 = \rho \equiv \sum_j f_j\,,\ \ W_\alpha = J_\alpha = \sum_j v_j^\alpha\, f_j$

$$\Lambda^\beta_{0k}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \begin{cases} \Lambda^\gamma_{\beta\ell}\, \big(\sigma_\beta\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell & \text{thermics} \\ -\tfrac{1}{12}\, \Lambda^\gamma_{\beta\ell}\, \partial_\beta \partial_\gamma \theta_\ell & \text{fluid} \end{cases}$$

$$\Lambda^\beta_{0k}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k = \begin{cases} \big(\sigma_\beta^2 - \tfrac{1}{6}\big)\, \partial_t (\partial_\beta \theta_\beta) & \text{thermics} \\ 0 & \text{fluid} \end{cases}$$

Mass conservation in the thermal case :

$$\partial_t \rho + \partial_\alpha J^{\rm eq}_\alpha - \Delta t\, \sigma_\alpha\, \partial_\alpha \theta_\alpha + \Delta t^2 \Big(\Lambda^\gamma_{\beta\ell}\, \big(\sigma_\beta\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell$$
$$+ \big(\sigma_\beta^2 - \tfrac{1}{6}\big)\, \partial_t (\partial_\beta \theta_\beta) \Big) = {\rm O}(\Delta t^3)$$

[relation (35) of *DCDS-A*, 2009].

Mass conservation in the fluid case :

$$\partial_t \rho + \partial_\alpha J_\alpha - \tfrac{1}{12}\, \Delta t^2\, \Lambda^\gamma_{\beta\ell}\, \partial_\beta \partial_\gamma \theta_\ell = {\rm O}(\Delta t^3)$$

[relation (40) of *DCDS-A*, 2009].

Link with the previous formal expansion at third-order *(vi)*

$$\partial_t W_i + \Lambda_{ik}^{\beta}\, \partial_\beta\, m_k^{\mathrm{eq}} - \Delta t\, \Lambda_{ik}^{\beta}\, \sigma_k\, \partial_\beta\, \theta_k$$
$$+\Delta t^2 \Big(\Lambda_{ik}^{\beta}\, \Lambda_{k\ell}^{\gamma}\, \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell + \Lambda_{ik}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k \Big) = \mathrm{O}(\Delta t^3)$$

Momentum conservation for the fluid case: $i = \alpha$

$$\partial_t J_\alpha + \Lambda_{\alpha k}^{\beta}\, \partial_\beta\, m_k^{\mathrm{eq}} - \Delta t\, \Lambda_{\alpha k}^{\beta}\, \sigma_k\, \partial_\beta\, \theta_k$$
$$+\Delta t^2 \Big(\Lambda_{\alpha k}^{\beta}\, \Lambda_{k\ell}^{\gamma}\, \big(\sigma_k\, \sigma_\ell - \tfrac{1}{12}\big)\, \partial_\beta \partial_\gamma \theta_\ell$$
$$+\Lambda_{\alpha k}^{\beta}\, \big(\sigma_k^2 - \tfrac{1}{6}\big)\, \partial_\beta \partial_t \theta_k \Big) = \mathrm{O}(\Delta t^3)$$

[relation (41) of *DCDS-A*, 2009].

Survey

- Compact iteration of lattice Boltzmann schemes
- Block decomposition of the moment-velocity operator matrix
- Asymptotic expansion of the equivalent partial differential equations and of the non conserved moments
- Explicitation of the coefficients up to order 3 with recursive formulas containing less than 3 terms
- Intensive use of differential calculus
- Validation of the nonlinear expansion for fluid flow and thermal problems up the the order 3
- Validation of the non linear expansion at second order for Navier Stokes flows with two en three space dimensions (Icmmes 2018, ICIAM 2019)
- New applications soon !

Merci de votre attention !

Annex -1-

$$Y = \Phi(W) + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\left(\Delta t\,\Psi_1 + \Delta t^2\,\Psi_2 + \Delta t^3\,\Psi_3\right) + \mathrm{O}(\Delta t^4)$$
$$Y^* = \Phi(W) + \left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\left(\Delta t\,\Psi_1 + \Delta t^2\,\Psi_2 + \Delta t^3\,\Psi_3\right) + \mathrm{O}(\Delta t^4)$$

$$A\,W + B\,Y^* =$$
$$= A\,W + B\Big(\Phi(W) + \left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\left(\Delta t\,\Psi_1 + \Delta t^2\,\Psi_2 + \Delta t^3\,\Psi_3\right)\Big) + \mathrm{O}(\Delta t^4)$$
$$A\,W + B\,Y^* = A\,W + B\,\Phi + \Delta t\,B\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_1 + \Delta t^2\,B\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_2 + \Delta t^3\,B\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_3 + \mathrm{O}(\Delta t^4)$$

$$A_2\,W + B_2\,Y^* = A_2\,W + B_2\Phi(W) + B_2\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Big(\Delta t\,\Psi_1 + \Delta t^2\,\Psi_2\Big) + \mathrm{O}(\Delta t^3)$$
$$= (A^2 + B\,C)\,W + (A\,B + B\,D)\,\Phi + \Delta t\,B_2\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_1 + \Delta t^2\,B_2\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_2 + \mathrm{O}(\Delta t^3)$$
$$A_2\,W + B_2\,Y^* = A\,\Gamma_1 + B\left(\mathrm{d}\Phi.\Gamma_1 - \Psi_1\right) + \Delta t\,B_2\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_1 + \Delta t^2\,B_2\left(\Sigma - \tfrac{1}{2}\,\mathrm{I}\right)\Psi_2 + \mathrm{O}(\Delta t^3)$$

Annex -1- *(ii)*

$$
\begin{aligned}
A_3\,W + B_3\,Y^* &= A_3\,W + B_3\big(\Phi(W) + \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\Delta t\,\Psi_1\big) + \mathrm{O}(\Delta t^2) \\
&= (A_2\,A + B_2\,C)\,W + (A_2\,B + B_2\,D)\,\Phi \\
&\qquad + \Delta t\,B_3\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2) \\
&= A_2\,\Gamma_1 + B_2\,(\,\mathrm{d}\Phi.\Gamma_1 - \Psi_1) + \Delta t\,B_3\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2) \\
&= (A^2 + B\,C)\,\Gamma_1 + (A\,B + B\,D)\,\mathrm{d}\Phi.\Gamma_1 - B_2\,\Psi_1 \\
&\qquad + \Delta t\,B_3\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2) \\
&= A\,(A\,\Gamma_1 + B\,\mathrm{d}\Phi.\Gamma_1) + B\,(C\,\Gamma_1 + D\,\mathrm{d}\Phi.\Gamma_1) - B_2\,\Psi_1 \\
&\qquad + \Delta t\,B_3\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2) \\
&\qquad\qquad \text{because}\quad C\,W + D\,\Phi = \mathrm{d}\Phi.\Gamma_1 - \Psi_1 \\
&\qquad\qquad \text{and}\quad C\,\Gamma_1 + D\,\mathrm{d}\Phi.\Gamma_1 = \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1 - \Psi_1).\Gamma_1 \\
&= A\,\mathrm{d}\Gamma_1.\Gamma_1 + B\,\mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1 - \Psi_1).\Gamma_1 - B_2\,\Psi_1 \\
&\qquad + \Delta t\,B_3\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2)
\end{aligned}
$$

$$
\begin{aligned}
A_3\,W + B_3\,Y^* &= \partial^2\Gamma_1.\Gamma_1 - B\,\mathrm{d}\Psi_1.\Gamma_1 - B_2\,\Psi_1 \\
&\qquad + \Delta t\,B_3\,\big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\,\Psi_1 + \mathrm{O}(\Delta t^2)
\end{aligned}
$$

Annex -1- *(iii)*

$$
\begin{aligned}
& C\, W + D\, Y^* = \\
& = C\, W + D\Big(\Phi(W) + \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big) \big(\Delta t\, \Psi_1 + \Delta t^2\, \Psi_2\big)\Big) + \mathrm{O}(\Delta t^3)
\end{aligned}
$$

$$
\begin{aligned}
C\, W + D\, Y^* = {} & \mathrm{d}\Phi.\Gamma_1 - \Psi_1 + \Delta t\, D\, \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Psi_1 \\
& + \Delta t^2\, D\, \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Psi_2 + \mathrm{O}(\Delta t^3)
\end{aligned}
$$

$$
\begin{aligned}
C_2\, W + D_2\, Y^* &= C_2\, W + D_2 \big(\Phi(W) + \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Delta t\, \Psi_1\big) + \mathrm{O}(\Delta t^2) \\
&= (C\, A + D\, C)\, W + (C\, B + D^2)\, \Phi(W) + \\
&\qquad + D_2\, \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Delta t\, \Psi_1 + \mathrm{O}(\Delta t^2) \\
&= C\, \Gamma_1 + D\, (\, \mathrm{d}\Phi.\Gamma_1 - \Psi_1) + D_2\, \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Delta t\, \Psi_1 + \mathrm{O}(\Delta t^2)
\end{aligned}
$$

$$
\begin{aligned}
& C_2\, W + D_2\, Y^* = \\
& \mathrm{d}(\, \mathrm{d}\Phi.\Gamma_1 - \Psi_1).\Gamma_1 - D\, \Psi_1 + D_2\, \big(\Sigma - \tfrac{1}{2}\,\mathrm{I}\big)\, \Delta t\, \Psi_1 + \mathrm{O}(\Delta t^2)
\end{aligned}
$$

Annex -1- *(iv)*

$$
\begin{aligned}
C_3\, W + D_3\, Y^* &= C_3\, W + D_3 \Phi(W) + \mathrm{O}(\Delta t) \\
&= (C_2\, A + D_2\, C)\, W + (C_2\, B + D_2\, D)\, \Phi(W) + \mathrm{O}(\Delta t) \\
&= C_2\, \Gamma_1 + D_2\, (\, \mathrm{d}\Phi . \Gamma_1 - \Psi_1) + \mathrm{O}(\Delta t) \\
&= (C\, A + D\, C)\, \Gamma_1 + (C\, B + D^2)\ \mathrm{d}\Phi . \Gamma_1 - D_2\, \Psi_1 + \mathrm{O}(\Delta t) \\
&= C\ \mathrm{d}\Gamma_1 . \Gamma_1 + D\ \mathrm{d}(\, \mathrm{d}\Phi . \Gamma_1 - \Psi_1) . \Gamma_1 - D_2\, \Psi_1 + \mathrm{O}(\Delta t) \\
&= \partial^2 (\Psi_1 + \gamma_1) . \Gamma_1 + D\ \mathrm{d}\Psi_1 . \Gamma_1 + D_2\, \Psi_1 + \mathrm{O}(\Delta t)
\end{aligned}
$$

$$
C_3\, W + D_3\, Y^* = \partial^2 (\, \mathrm{d}\Phi . \Gamma_1 - \Psi_1) . \Gamma_1 - D\ \mathrm{d}\Psi_1 . \Gamma_1 - D_2\, \Psi_1 + \mathrm{O}(\Delta t)
$$

Annex -2-

$$\partial_t W = -\Gamma_1 - \Delta t\, \Gamma_2 - \Delta t^2\, \Gamma_3 + \Delta t^3\, \Gamma_4 + \mathrm{O}(\Delta t^4)$$

Successive derivatives of the conserved moments

$$\begin{aligned}\partial_t^2 W &= \partial_t\big(-\Gamma_1 - \Delta t\, \Gamma_2 - \Delta t^2\, \Gamma_3\big) + \mathrm{O}(\Delta t^3)\\ &= -\,\mathrm{d}\big(\Gamma_1 + \Delta t\, \Gamma_2 + \Delta t^2\, \Gamma_3 + \mathrm{O}(\Delta t^3)\big).\, \partial_t W\\ &= \mathrm{d}\big(\Gamma_1 + \Delta t\, \Gamma_2 + \Delta t^2\, \Gamma_3\big).\, \big(\Gamma_1 + \Delta t\, \Gamma_2 + \Delta t^2\, \Gamma_3\big) + \mathrm{O}(\Delta t^3)\end{aligned}$$

$$\begin{aligned}\partial_t^2 W = {} & \mathrm{d}\Gamma_1.\,\Gamma_1 + \Delta t\,(\,\mathrm{d}\Gamma_1.\,\Gamma_2 + \mathrm{d}\Gamma_2.\,\Gamma_1)\\ & + \Delta t^2\,(\,\mathrm{d}\Gamma_1.\,\Gamma_3 + \mathrm{d}\Gamma_2.\,\Gamma_2 + \mathrm{d}\Gamma_3.\,\Gamma_1) + \mathrm{O}(\Delta t^3)\end{aligned}$$

$$\partial_t^2 W = A\,\Gamma_1 + B\ \mathrm{d}\Phi.\Gamma_1 + \Delta t\,(\,\mathrm{d}\Gamma_1.\,\Gamma_2 + \mathrm{d}\Gamma_2.\,\Gamma_1) + \mathrm{O}(\Delta t^2)$$

Annex -2- *(ii)*

then

$$
\begin{aligned}
\partial_t^3 W &= \partial_t(\partial_t^2 W) \\
&= \partial_t\big(A\,\Gamma_1 + B\ \mathrm{d}\Phi.\Gamma_1 + \Delta t\,\partial_t(\,\mathrm{d}\Gamma_2.\,\Gamma_1 + \,\mathrm{d}\Gamma_1.\,\Gamma_2)\big) + \mathrm{O}(\Delta t^2) \\
&= \partial_t\big(\,\mathrm{d}\Gamma_1.\,\Gamma_1 + \Delta t\,(\,\mathrm{d}\Gamma_1.\,\Gamma_2 + \,\mathrm{d}\Gamma_2.\,\Gamma_1)\big) + \mathrm{O}(\Delta t^2) \\
&= \,\mathrm{d}\big(\,\mathrm{d}\Gamma_1.\,\Gamma_1 + \Delta t\,(\,\mathrm{d}\Gamma_1.\,\Gamma_2 + \,\mathrm{d}\Gamma_2.\,\Gamma_1)\big).\,\partial_t W + \mathrm{O}(\Delta t^2) \\
&= \,\mathrm{d}(\,\mathrm{d}\Gamma_1.\,\Gamma_1).(-\Gamma_1 - \Delta t\,\Gamma_2) \\
&\qquad + \Delta t\,(\,\mathrm{d}\Gamma_1.\,\Gamma_2 + \,\mathrm{d}\Gamma_2.\,\Gamma_1).(-\Gamma_1) + \mathrm{O}(\Delta t^2) \\
&= -\,\mathrm{d}\big(\,\mathrm{d}\Gamma_1.\,\Gamma_1\big).\Gamma_1 - \Delta t\,\big[\,\mathrm{d}(A\,\Gamma_2 + B\ \mathrm{d}\Phi.\Gamma_2 \\
&\qquad + B\,\Sigma\ \mathrm{d}\Psi_1.\,\Gamma_1).\,\Gamma_1 + \,\mathrm{d}(A\,\Gamma_1 + B\ \mathrm{d}\Phi.\Gamma_1).\,\Gamma_2\big] + \mathrm{O}(\Delta t^2) \\
&= -\partial^2\Gamma_1.\,\Gamma_1 - \Delta t\,\big[A\ \mathrm{d}\Gamma_2.\,\Gamma_1 + B\ \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_2).\Gamma_1 \\
&\qquad + B\,\Sigma\,\partial^2\Psi_1.\,\Gamma_1 + A\ \mathrm{d}\Gamma_1.\,\Gamma_2 + B\ \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1).\Gamma_2\big] + \mathrm{O}(\Delta t^2)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t^3 W &= -\partial^2\Gamma_1.\,\Gamma_1 + \Delta t\,\big[A\,(\,\mathrm{d}\Gamma_1.\,\Gamma_2 + \,\mathrm{d}\Gamma_2.\,\Gamma_1) \\
&\quad + B\,(\,\mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1).\,\Gamma_2 + \,\mathrm{d}(\,\mathrm{d}\Phi.\Gamma_2).\,\Gamma_1)) + B\,\Sigma\,\partial^2\Psi_1.\,\Gamma_1\big] + \mathrm{O}(\Delta t^2)
\end{aligned}
$$

Annex -2- *(iii)*

$$Y = \Phi(W) + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\left(\Delta t\,\Psi_1 + \Delta t^2\,\Psi_2 + \mathrm{O}(\Delta t^3)\right)$$

Successive derivatives of the non-conserved moments

$$\begin{aligned}
\partial_t Y =\ & \mathrm{d}\Phi.\partial_t W + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\left(\Delta t\ \mathrm{d}\Psi_1.\,\partial_t W + \Delta t^2\ \mathrm{d}\Psi_2.\,\partial_t W\right) \\
& \qquad + \mathrm{O}(\Delta t^3) \\
=\ & \mathrm{d}\Phi.\left(-\Gamma_1 - \Delta t\,\Gamma_2 - \Delta t^2\,\Gamma_3\right) \\
& \quad + \Delta t\left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\ \mathrm{d}\Psi_1.\left(-\Gamma_1 - \Delta t\,\Gamma_2\right) \\
& \qquad - \Delta t^2\left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\ \mathrm{d}\Psi_2.\,\Gamma_1 + \mathrm{O}(\Delta t^3)
\end{aligned}$$

$$\begin{aligned}
\partial_t Y =\ & -\mathrm{d}\Phi.\Gamma_1 - \Delta t\left(\mathrm{d}\Phi.\Gamma_2 + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\ \mathrm{d}\Psi_1.\,\Gamma_1\right) \\
& - \Delta t^2\left(\mathrm{d}\Phi.\Gamma_3 + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\left(\mathrm{d}\Psi_1.\,\Gamma_2 + \mathrm{d}\Psi_2.\,\Gamma_1\right)\right) + \mathrm{O}(\Delta t^3)
\end{aligned}$$

Annex -2- *(iv)*

$$\partial_t Y = -\,\mathrm{d}\Phi.\Gamma_1 - \Delta t\left(\mathrm{d}\Phi.\Gamma_2 + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\,\mathrm{d}\Psi_1.\,\Gamma_1\right) + \mathrm{O}(\Delta t^2)$$

then

$$\begin{aligned}\partial_t^2 Y = &-\mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1).(-\Gamma_1 - \Delta t\,\Gamma_2)\\ &+\Delta t\left(\mathrm{d}(\,\mathrm{d}\Phi.\Gamma_2).\Gamma_1 + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\partial^2\Psi_1.\,\Gamma_1\right) + \mathrm{O}(\Delta t^2)\end{aligned}$$

$$\begin{aligned}\partial_t^2 Y = &\ \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1).\Gamma_1\\ &+\Delta t\left(\mathrm{d}(\,\mathrm{d}\Phi.\Gamma_1).\Gamma_2 + \mathrm{d}(\,\mathrm{d}\Phi.\Gamma_2).\Gamma_1 + \left(\Sigma + \tfrac{1}{2}\,\mathrm{I}\right)\partial^2\Psi_1.\,\Gamma_1\right)\\ &+\mathrm{O}(\Delta t^2)\end{aligned}$$