le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Student presentation 7 (course 13)

- Variational formulation for an elliptic problem

The domain Ω is a bounded part in $\mathbb{R}^2$. We denote by $\Gamma \equiv \partial\Omega$ its boundary. It is supposed to be regular.

If v and w are two regular scalar functions defined on the set $\Gamma \equiv \partial\Omega$ and n the external boundary to Γ, we recall that $\frac{\partial v}{\partial n} \equiv \nabla v \,.\, n \equiv \sum_j \frac{\partial v}{\partial x_j} n_j$.

a) Give some examples of such a domain. Precise the geometrical nature of the boundary Γ and give some information about the external normal n.

Let f be a regular given scalar function defined on the set $\Omega \cup \Gamma$. We consider the following problem: search a scalar funtion u defined Ω such that $-\Delta u = f$ in Ω and $u = 0$ on the boundary Γ. We recall that $\Delta u \equiv \sum_j \frac{\partial^2 u}{\partial x_j^2}$. This problem is called the homogeneous Dirichlet problem for the Poisson equation.

b) Show that $-\int_\Omega \Delta v \, w \, \mathrm{d}x = \int_\Omega \nabla v \,.\, \nabla w \, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial v}{\partial n} \, w \, \mathrm{d}\gamma$.

Let u and v be two functions that are both solution of the problem $-\Delta \zeta = f$ in Ω and $\zeta = 0$ on Γ.

c) What is the system of equations satisfied by the difference $\varphi \equiv u - v$?

d) Deduce from the previous questions that for an arbitrary funtion w identically equal to zero on the boundary Γ, we have $\int_\Omega \nabla \varphi \,.\, \nabla w \, \mathrm{d}x = 0$.

e) Deduce from the previous questions that the function φ is idetically null and that the Dirichlet problem $-\Delta u = f$ in Ω and $u = 0$ on Γ admits at most one regular solution.

François Dubois, october 2022.