le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 13 Surface integral

- Local parameterization

We give ourselves four real numbers $a < b$, $c < d$ and a space of parameters $(u, v) \in \mathbb{R}^2$ such that $a \le u \le b$ and $c \le v \le d$. We define the parametrized sheet Σ in space $\mathbb{R}^3$ by a continuously differentiable function Φ fom the rectangle $\widehat{\Sigma} \equiv [a, b] \times [c, d]$ and taking its values in $\mathbb{R}^3$. This application Φ is called the "local map". The parameterized sheet Σ is defined by the relation $\Sigma = \Phi(\widehat{\Sigma})$. A point $M(u, v)$ of the parametric sheet has coordinates x, y et z that are regular functions of the parameters u and v: $x = X(u, v)$, $y = Y(u, v)$ et $z = Z(u, v)$.

The fundamental example is a plane parallelogram. The equation of the plane is for example of the form $z = \alpha x + \beta y + \gamma$. For $a \le u \le b$ and $c \le v \le d$, we set $x = u$, $y = v$ and $z = \alpha u + \beta v + \gamma$.

A second example concerns a surface with equation $z = f(x, y)$. It is similar to the previous case except that the affine function $f(x, y) = \alpha x + \beta y + \gamma$ is replaced by a function f of two variables while remaining fairly regular.

The next example is the sphere centered at the origin O and of radius $R > 0$. We use the spherical coordinates of the three-dimensional space. We project the current point M of the sphere onto the $x\mathrm{O}y$ plane at a point m. We have: $z = R \cos\theta$ and $\mathrm{O}m = R \sin\theta$. It comes then $x = \mathrm{O}m \cos\varphi = R \sin\theta \cos\varphi$ and $y = \mathrm{O}m \sin\varphi = R \sin\theta \sin\varphi$.

- Tangent plane

We suppose the function Φ différentiable at the point (u, v):
$\Phi(u + \delta u, v + \delta v) = \Phi(u, v) + \frac{\partial \Phi}{\partial u}(u, v)\, \delta u + \frac{\partial \Phi}{\partial v}(u, v)\, \delta v + \| (u, v) \| \, \varepsilon(\delta u, \delta v)$. The two tangent vectors $\frac{\partial \Phi}{\partial u}(u, v)$ and $\frac{\partial \Phi}{\partial v}(u, v)$ are two vectors of space $\mathbb{R}^3$. We suppose that the family $(\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v})$ is non-degenerate and the point $M = \Phi(u, v)$ is a “regular point” of the surface. The tangent plane to the parametrized sheet Σ at the point $M = \Phi(u, v)$ is the affine plane that passes through the point $M(u, v)$ with an associated vector plane that has as its basis the family of two vectors $\frac{\partial \Phi}{\partial u}(u, v)$ and $\frac{\partial \Phi}{\partial v}(u, v)$.

In the case of a surface of the form $z = f(x, y)$, the two parameters are the abscissa x and the the ordinate y. We have $\frac{\partial \Phi}{\partial x} = (1, 0, \frac{\partial f}{\partial x})^{\mathrm{t}}$ and $\frac{\partial \Phi}{\partial y} = (0, 1, \frac{\partial f}{\partial y})^{\mathrm{t}}$. They always form a free family whatever the function f.

For a sphere of radius $R > 0$ and centered at the origin, we introduce the spherical coordinates, so the two polar angles θ and φ such that $x = R \sin\theta \cos\varphi$, $y = R \sin\theta \sin\varphi$ and $z = R \cos\theta$. The moving reference frame $(e_r, e_\theta, e_\varphi)$ is defined by the relations
$e_r(\theta, \varphi) = \sin\theta\,(\cos\varphi\, e_1 + \sin\varphi\, e_2) + \cos\theta\, e_3$, $e_\theta(\theta, \varphi) = \cos\theta\,(\cos\varphi\, e_1 + \sin\varphi\, e_2) - \sin\theta\, e_3$ and $e_\varphi(\varphi) = -\sin\varphi\, e_1 + \cos\varphi\, e_2$. In the case of the sphere of radius R, we have
$\mathrm{d}M = R\,(e_\theta\ \mathrm{d}\theta + e_\varphi \sin\theta\ \mathrm{d}\varphi)$; we deduce $\frac{\partial M}{\partial \theta} = R\, e_\theta$ and $\frac{\partial M}{\partial \varphi} = R \sin\theta\, e_\varphi$.

- Vector product

Let u and v be two vectors in an euclidian space of dimension 3. The vector product $u \times v$ satisfies the following properties.
(i) The vector product $u \times v$ is orthogonal to the vectors u and v: $(u \times v, u) = (u \times v, v) = 0$.
(ii) If the vectors u and v are collinear, the vector product $u \times v$ is zero.
(iii) If $P(u, v)$ is the parallelogram generated by the vectors u and v:
$P(u, v) = \{x \in \mathbb{R}^3, \exists\, \theta, \xi, 0 \le \theta \le 1, 0 \le \xi \le 1, x = \theta\, u + \xi\, v\}$, then the area of this parallelogram is equal to the norm of the vector product $u \times v$: $|P(u, v)| = \|u \times v\|$. Moreover, $\|u \times v\| \le \|u\|\,\|v\|$.
(iv) We give ourselves a direct orthonormal basis (e_1, e_2, e_3) and the components of the two vectors u and v: $u = \sum_{j=1}^{3} u_j\, e_j$ and $v = \sum_{k=1}^{3} v_k\, e_k$. The vector product $u \times v$ is expressed in the basis (e_1, e_2, e_3) *via* the relation: $u \times v = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} e_3 + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} e_1 + \begin{vmatrix} u_3 & v_3 \\ u_1 & v_1 \end{vmatrix} e_2$.
(v) It is possible to prove that $w = u \times v$ is a biliear function of the two vectors u and v:
$(\alpha\, u + \beta\, u') \times v = \alpha\,(u \times v) + \beta\,(u' \times v)$ and $u \times (\alpha\, v + \beta\, v') = \alpha\,(u \times v) + \beta\,(u \times v')$, whatever the choice of vectors u, u', v, v' and whatever the choice of numbers α and β.
(vi) If $u \times v \neq 0$, the family $(u, v, u \times v)$ is a direct basis of $\mathbb{R}^3$: the change matrix of basis between a direct orthonormal basis (e_1, e_2, e_3) and the family $(u, v, u \times v)$ is strictly positive.
(vii) Be careful, the vector product is not associative: in general, $u \times (v \times w) \neq (u \times v) \times w$.

- Normal vector

We assume that the point $M = \Phi(u, v)$ is a regular point of the surface, *i.e.* that the family $(\frac{\partial \Phi}{\partial u}(u, v), \frac{\partial \Phi}{\partial v}(u, v))$ is free. Then the vector product $\frac{\partial \Phi}{\partial u}(u, v) \times \frac{\partial \Phi}{\partial v}(u, v)$ is not zero. The norm $\| \frac{\partial \Phi}{\partial u}(u, v) \times \frac{\partial \Phi}{\partial v}(u, v) \|$ of this vector product is equal to the surface of the parallelogram constructed with the two tangent vectors $\frac{\partial \Phi}{\partial u}(u, v)$ and $\frac{\partial \Phi}{\partial v}(u, v)$. We define the normal vector $n(u, v)$ as the unit vector constructed from this vector product:
$n(u, v) = \frac{1}{\| \frac{\partial \Phi}{\partial u}(u, v) \times \frac{\partial \Phi}{\partial v}(u, v) \|}\, \frac{\partial \Phi}{\partial u}(u, v) \times \frac{\partial \Phi}{\partial v}(u, v)$.
It is a vector orthogonal to the tangent plane. Hence the name "normal" or "normal vector".
For a surface of the form $z = f(x, y)$, we have $\frac{\partial \Phi}{\partial x} \times \frac{\partial \Phi}{\partial y} = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)^{\mathrm{t}}$ and
$n(x, y) = \frac{1}{\sqrt{1 + (\frac{\partial \Phi}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}}\, (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)^{\mathrm{t}}$.
For the sphere of radius R, we have the relation $\frac{\partial M}{\partial \theta} \times \frac{\partial M}{\partial \varphi} = R^2 \sin\theta\, e_r$. We deduce
$\| \frac{\partial M}{\partial \theta} \times \frac{\partial M}{\partial \varphi} \| = R^2 \sin\theta$ and $n(\theta, \varphi) = e_r(\theta, \varphi)$.

- Scaled surface

To approximate a curve Γ, we give ourselves points on the curve and we approach the curve by the sequence of strings stretched between a point and its neighbor. Through two different points

there always passes one and only one line segment. Thus we obtain a continuous approximation of the curve Γ.
To approximate a surface, it is more delicate. Indeed, we give ourselves an integer $n \geq 1$ and we discretize first the rectangle $[a, b] \times [c, d]$ in the parameter space with small rectangles of the type $[a+ih, a+(i+1)h] \times [c+jk, c+(j+1)k]$ with $h = \frac{b-a}{n}$ and $k = \frac{b-a}{n}$. The image by the local map Φ is a curvilinear quadrangle whose four corners we denote by
$M_{ij} = \Phi(a+ih, c+jk)$, $M_{i+1,j} = \Phi(a+(i+1)h, c+jk)$,
$M_{i+1,j+1} = \Phi(a+(i+1)h, c+(j+1)k)$ and $M_{i,j+1} = \Phi(a+ih, c+(j+1)k)$. These four points are close enough if n is large enough belong to the surface Σ but are not coplanar in general. We propose to approximate the surface quadrilateral
$Q_{ij} = \Phi([a+ih, a+(i+1)h] \times [c+jk, c+(j+1)k])$ by a plane parallelogram $\widetilde{Q_{ij}}$ localized on the tangent plane to the surface Σ at the point M_{ij}. Precisely $\widetilde{Q_{ij}}$ is the parallelogram passing through the point M_{ij} and directed by two tangent vectors at the point M_{ij}, *i.e.* $h\frac{\partial\Phi}{\partial u}(a+ih, c+jk)$ and $k\frac{\partial\Phi}{\partial v}(a+ih, c+jk)$. We have
$\widetilde{Q_{ij}} = M_{ij} + \{\xi\, h\frac{\partial\Phi}{\partial u}(a+ih, c+jk) + \eta\, k\frac{\partial\Phi}{\partial v}(a+ih, c+jk), 0 \leq \xi, \eta \leq 1\}$. The area $|\widetilde{Q_{ij}}|$ of this parallelogram is given by the norm of the vector product of the two tangent vectors: $|\widetilde{Q_{ij}}| = hk \parallel (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(a+ih, c+jk) \parallel$.
Such a plane parallelogram is, for n large enough, *i.e.* h and k small enough, a good approximation of the surface parallelogram Q_{ij}. When we join all these parallelograms for $0 \leq i, j < n$, we obtain an approximation $\Sigma_n = \cup_{0\leq i,j<n} \widetilde{Q_{ij}}$ fairly accurate surface Σ, but it has the defect of being discontinuous at the interfaces. Hence the expression "scaled surface".

- Surface of a parametrized sheet

We first define the surface $|\Sigma_n|$ of the scaled surface Σ_n associated with the rectangle cutout $\widehat{\Sigma} = [a, b] \times [c, d]$ of the parameters into $n \times n$ small rectangles: $|\Sigma_n| = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} |\widetilde{Q_{ij}}|$. Given the surface of a piece of scale $\widetilde{Q_{ij}}$, we have $|\Sigma_n| = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \parallel (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(a+ih, c+jk) \parallel hk$. We then make the integer n tend to infinity. The double sum converges to the double integral on the rectangle $\widehat{\Sigma}$ of the function $(u, v) \longmapsto \parallel (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(u, v) \parallel$. We deduce an expression for the surface of the parameterized sheet: $|\Sigma| = \iint_{\widehat{\Sigma}} \parallel (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(u, v) \parallel \, \mathrm{d}u\, \mathrm{d}v$. We define the element of surface $\mathrm{d}\sigma$ by the relation $\mathrm{d}\sigma = \parallel \frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v} \parallel \, \mathrm{d}u\, \mathrm{d}v$ and then we write in a deceptively simple way: $|\Sigma| = \iint_{\widehat{\Sigma}} \mathrm{d}\sigma$. The surface element $\mathrm{d}\sigma$ does not depend on the chosen parameterization and the relation $|\Sigma| = \iint_{\widehat{\Sigma}} \mathrm{d}\sigma$ is intrinsic.
The metric term $\parallel (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v}) \parallel$ is to be compared to the length when calculating the length of an curve Γ: $|\Gamma| = \int_a^b \parallel (\frac{\mathrm{d}M}{\mathrm{d}t}(t) \parallel \, \mathrm{d}t = \int_a^b \mathrm{d}s$. For a sphere Σ of radius R, we have seen that $n = e_r$ and $\frac{\partial M}{\partial\theta} \times \frac{\partial M}{\partial\varphi} = R^2 \sin\theta\, e_r$. We can therefore write the surface element
$\mathrm{d}\sigma = \parallel \frac{\partial M}{\partial\theta} \times \frac{\partial M}{\partial\varphi} \parallel \, \mathrm{d}\theta\, \mathrm{d}\varphi = R^2 \sin\theta \, \mathrm{d}\theta\, \mathrm{d}\varphi$. Then we have for the sphere Σ such that $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$, $|\Sigma| = \int_0^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi\, R^2 \sin\theta = 4\pi R^2$.

- Surface integral

A function f defined on a parametric sheet Σ can also be written as a function of the parameters u and v: $\widehat{f}(u, v) = f(\Phi(u, v))$. For $n \geq 1$ we introduce the scaled surface Σ_n associated to a discretization $M_{ij} = \Phi(a+ih, c+jk)$ of $\widehat{\Sigma} = [a, b] \times [c, d]$. We can approximate the function

f on Σ_n by the stepped function equal to the constant $f(M_{ij})$ in each parallelogram $\widetilde{Q_{ij}}$. Given the value $\| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(a+ih,\, c+jk) \|$ of the surface of this parallelogram, we define the approximate integral I_n of the function f on the surface Σ by
$I_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(M_{ij}) \, \| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(a+ih,\, c+jk) \| \, hk$. If n tends to infinity and if the function function f is continuous to fix the ideas, the sequence I_n converges to the double integral $I = \iint_{\widehat{\Sigma}} f(\Phi(u,\, v)) \, \| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(u,\, v) \| \, \mathrm{d}u \, \mathrm{d}v$. We define the surface integral $\int_\Sigma f(M) \, \mathrm{d}\sigma$ by the relation $\int_\Sigma f(M) \, \mathrm{d}\sigma = \iint_{\widehat{\Sigma}} f(\Phi(u,\, v)) \, \| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(u,\, v) \| \, \mathrm{d}u \, \mathrm{d}v$. It does not depend on the parameterization.
In the case where $f(M) \equiv 1$, we do find the value $|\Sigma|$ for the area of the surface Σ: $\int_\Sigma \mathrm{d}\sigma = |\Sigma|$.

- Flow of a vector field

We give ourselves a vector field $\varphi : \mathbb{R}^3 \longmapsto \mathbb{R}^3$ continuous to fix the ideas. If $f(M) = (\Phi,\, n)$, scalar product of the field φ against the normal vector of the surface Σ, the corresponding surface integral defines the flux Φ of the vector field φ on the surface Σ: $\Phi = \int_\Sigma (\Phi,\, n) \, \mathrm{d}\sigma$.

Exercises

- On half-spheres

We denote by Σ the half-sphere centered at the origin, of radius $R > 0$ and defined also by the inequality $z \geq 0$.
a) In troduce a parameterization of this half-sphere with the spherical coordinate system r, θ and φ such that $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$ and $z = r \cos\theta$.
b) In what intervals vary the angles θ and φ ?
c) Propose an expression for the surface element $\mathrm{d}\sigma$ as a function of the variables of the problem.
d) Compute the integral $I = \int_\Sigma z \, \mathrm{d}\sigma$. $[\pi R^3]$
e) Go back to the questions b), c) and d) of this exercice raplacing on one hand the half-sphere Σ by the half-sphere $\widetilde{\Sigma}$ of radius $R > 0$, centered at the origin and defined by the inequality $x \geq 0$ and on the other hand the integral I by $J = \int_{\widetilde{\Sigma}} x \, \mathrm{d}\sigma$.
f) Why the questions d) and e) are related in a simple manner ?

- Surface of a truncated cone

We consider a truncated cone with a circular basis, a radius $R > 0$ and a height equal to $h > 0$.
a) Show that the half-angle θ at the summit satisfies to the relation $\tan\theta = \frac{R}{h}$.
b) Show that truncated cone can be parameterized with the relations $x = R \cos\varphi \, (1 - \frac{z}{h})$, $y = R \sin\varphi \, (1 - \frac{z}{h})$ and $z = z$, with $0 \leq \varphi \leq 2\pi$ and $0 \leq z \leq h$.
c) Compute the cartesian components of the vectors $\frac{\partial M}{\partial \varphi}$ and $\frac{\partial M}{\partial z}$.
d) Express the element of surface $\mathrm{d}\sigma$ as a function of the geometrical parameters R and h, of the coordinates φ and z along the truncated cone and of the product $\mathrm{d}\varphi \, \mathrm{d}z$.
e) Show that the surface S of this truncated cone is equal to $\pi R \sqrt{R^2 + h^2}$.

- Computation of a flux

We consider the half-sphere Σ with radius $R > 0$ centered at the origin and defined by the inequality $z \geq 0$. We denote by n the normal vector field pointing in a direction such that $n_z \geq 0$. We consider also the vector field $\psi(x, y, z) = (x, y, 0)$.

a) Compute the scalar product (ψ, n) on the half-sphere Σ. $[R \sin^2 \theta]$

b) Computer the flux $\Phi = \int_\Sigma (\psi.n)\, \mathrm{d}\sigma$ of the vector field ψ on the half-sphere Σ. $\left[\frac{4}{3}\pi R^3\right]$