le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 9 Curvilinear integral

- Introduction to curvilinear integrals

We suppose given a curve Γ in space $\mathbb{R}^2$: $[0,\,1] \ni t \longmapsto M(t) = (X(t),\,Y(t)) \in \mathbb{R}^2$. The curvilinear abscissa is the length of the curve. We have $\mathrm{d}s = \| \,\mathrm{d}M \,\| = \sqrt{(X'(t))^2 + (X'(t))^2}\,\mathrm{d}t$.
A first example is the arch of parabola. We have in this case $y = Y(x) = \frac{1}{2}\frac{x^2}{a}$ and we suppose $0 \leq x \leq a$ with $a > 0$. Then $\mathrm{d}s = \sqrt{1 + (Y'(x))^2}\,\mathrm{d}x = \sqrt{1 + (x/a)^2}\,\mathrm{d}x$ in this case.
An arch of circle admits a representation typically given by the relation
$[\theta_{\min},\,\theta_{\max}] \ni \theta \longmapsto (X(\theta),\,Y(\theta)) = (R\cos\theta,\,R\sin\theta)$. In that case, we have $\mathrm{d}s = R\,\mathrm{d}\theta$.
We suppose also given a function f from $\mathbb{R}^2$ and taking its values in $\mathbb{R}$: $f(M) \in \mathbb{R}$ if $M \in \mathbb{R}^2$. The question is to define the curvilinear integral $I = \int_\Gamma f(M)\,\mathrm{d}s$. This integral depends on the curve Γ and on the function f. We observe that if $f(M) \equiv 1$, then we have $\int_0^L \mathrm{d}s = L$, the length of the curve Γ.

- Parameterization

If we use the conventional settings $[0,\,1] \ni t \longmapsto M(t) = (X(t),\,Y(t)) \in \mathbb{R}^2$ for defining the curve Γ, the value $f(M)$ when M belongs to the curve Γ is equal to $f(X(t),\,Y(t))$ and we set $\int_\Gamma f(M(s))\,\mathrm{d}s = \int_0^1 f(X(t),\,Y(t))\,\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$.
For example, with the previous parabola $y = \frac{x^2}{2a}$, we have $X(x) = x$ and $Y(x) = \frac{x^2}{2a}$. Associated with the polynomial function $f(x,\,y) = x$, we obtain the explicitation of the curvilinear integral: $\int_\Gamma f(M(s))\,\mathrm{d}s = \int_0^a x\sqrt{1 + \frac{x^2}{a^2}}\,\mathrm{d}x$. After some lines of elementary calculus, this integral is equal to $\frac{a^2}{3}\,(2\sqrt{2} - 1)$.
An other example with the same function $f(x,\,y) = x$ and the half circle of radius R centered at the origin and in the half plane $\{y > 0\}$. We have $\int_\Gamma x\,\mathrm{d}s = \int_0^\pi R\cos\theta\,R\,\mathrm{d}\theta = 0$. With the half circle in the half plane $\{x > 0\}$, it comes $\int_\Gamma x\,\mathrm{d}s = \int_{-\pi/2}^{\pi/2} R\cos\theta\,R\,\mathrm{d}\theta = 2R^2$.

- The curvilinear integral does not depend on the parameterization

With the choice $[0,\,1] \ni t \longmapsto M(t) = (X(t),\,Y(t)) \in \mathbb{R}^2$ done previously, we have
$I = \int_\Gamma f(M(s))\,\mathrm{d}s = \int_0^1 f(X(t),\,Y(t))\,\frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t$. If we consider now an other parameterization of the same curve, $[\alpha,\,\beta] \ni \theta \longmapsto \widetilde{M}(\theta) = (\widetilde{X}(\theta),\,\widetilde{Y}(\theta)) \in \mathbb{R}^2$, we have the change of variable $\theta = K(t)$ such that $\widetilde{M}(\theta) = \widetilde{M}(K(t)) = M(t)$. The associated expression for the curvilinear integral takes the form $J = \int_\Gamma f(M(s))\,\mathrm{d}s = \int_\alpha^\beta f(\widetilde{X}(\theta),\,\widetilde{Y}(\theta))\,\frac{\mathrm{d}s}{\mathrm{d}\theta}\,\mathrm{d}\theta$. This expression is coherent

François Dubois, 13 december 2022.

with the previous one: we have $I = J$. More precisely, after the change of variable $\theta = K(t)$ in the integral J, we have $J = \int_\alpha^\beta f(\widetilde{X}(\theta), \widetilde{Y}(\theta)) \frac{ds}{d\theta} \, d\theta = \int_0^1 f(X(t), Y(t)) \frac{ds}{d\theta} \frac{d\theta}{dt} \, dt = I$.

- Circulation of a vector field

A vector field Φ is a vector valued function, defined for $(x, y) \in \mathbb{R}^2$ by its coordinates. We have $\Phi(x, y) = (\Phi_x(x, y), \Phi_y(x, y))$. The circulation γ of the vector field Φ along the curve Γ is by definition the curvilinear integral $\gamma = \int_\Gamma (\Phi(M), \tau(M)) \, ds$ with $\tau(M) = \frac{dM}{ds}$ is the tangent unitary vector along the curve Γ. Observe that the expression $(\Phi(M), \tau(M))$ is the scalar product $\Phi_x \tau_x + \Phi_y \tau_y$. Then we have also $\gamma = \int_\Gamma (\Phi(M), \tau(M)) \, ds = \int_\Gamma (\Phi_x \, dX + \Phi_y \, dY)$ because $\tau \, ds = dM = (dX, dY)$.

For example with the circle $X(\theta) = R \cos\theta, Y(\theta) = R \sin\theta$, we have $\tau(\theta) = (-\sin\theta, \cos\theta)$. For the circulation $\gamma = \int_\Gamma (\Phi(M), \tau(M)) \, ds$ of the vector field $\Phi(x, y) = (-y, x)$ along this circle, we first observe that we have $(\Phi(M), \tau(M)) = R$, then $\gamma = \int_0^{2\pi} R \, ds = 2\pi R^2$.

- When the vector field is the gradient of a potential

The vector field Φ can be written $\Phi = \nabla\psi$ for some scalar function ψ if we have $\Phi_x = \frac{\partial\psi}{\partial x}$ and $\Phi_y = \frac{\partial\psi}{\partial y}$. Then the circulation of this vector field depends only on the extremities of the curve Γ. With $M(0) = A$ and $M(1) = B$, we have $\gamma = \int_\Gamma (\nabla\psi(M), \tau(M)) \, ds = \psi(B) - \psi(A)$.

The proof consists to evaluate the scalar product (Φ, dM). We have
$(\Phi, dM) = \big(\frac{\partial\psi}{\partial x} \frac{dX}{dt} + \frac{\partial\psi}{\partial y} \frac{dY}{dt}\big) \, dt = \frac{d}{dt}[\psi(X(t), Y(t))] \, dt$. After integration we obtain
$\int_A^B (\Phi, dM) = \int_A^B \frac{d}{dt}[\psi(X(t), Y(t))] \, dt = \psi(B) - \psi(A)$.

- Flux of a vector field in two space dimensions

We recall that for a plane curve, we have chosen in a previous chapter a tangent vector $\tau = \frac{dM}{ds} = \big(\frac{dx}{ds}, \frac{dy}{ds}\big)$ and a normal vector n such that $n_x = \tau_y$ and $n_y = -\tau_x$. Then the orthonormal basis (n, τ) is a direct basis.

The flux φ of the vector field Φ along the curve Γ is defined by $\varphi = \int_\Gamma (\Phi(M), n) \, ds$ and this curvilinear integral can also be written $\varphi = \int_\Gamma \big(\Phi_x(M) \, \tau_y(M) - \Phi_y(M) \, \tau_x(M)\big) \, ds$ or more simply $\varphi = \int_\Gamma (\Phi_x \, dY - \Phi_y \, dX)$.

For example the flux of the field $\Phi(x, y) = (x, y)$ along the whole circle of radius R centered at the origin is equal to $\int_\Gamma (\Phi, n) \, ds = 2\pi R^2$. The flux of the same vector field along the parabola of equation $y = \frac{x^2}{2a}$ with the constraint $0 \le x \le a$ is equal to $\frac{a^2}{6}$.

Exercices

- Along an arch of parabola

In the affine Euclidian plane, we wonsider the parabola of equation $y = x^2$ and the points $A(-1, 1)$ and $B(2, 4)$ on this parabola.

Compute the curvilear integral $I = \int_A^B (xy\,\mathrm{d}x + (x+y)\,\mathrm{d}y)$. $\left[\frac{69}{4}\right]$

- Along an half circle

Let Γ be the half circle in the affine Euclidian plane of radius $R > 0$, centered at the origin and included in the half plane $y > 0$. Let n be the unity normal vector pointing in the direction opposite to the origin.

a) Illustrate these geometrical data with a drawing.

b) How express the curvilinear alscissa s along the half circle Γ?

c) Compute the curvilinear integral $I = \int_\Gamma x n_x\,\mathrm{d}s$. $\left[\frac{1}{2}\pi R^2\right]$

d) Compute the curvilinear integral $J = \int_\Gamma [(x-y)\,\mathrm{d}x + (x+y)\,\mathrm{d}y]$. $\left[\pi R^2\right]$

- Along a complete circle

The letter C names the circle of radius equal to one centered at the origin. We suppose that it is oriented in the direct sense.

Compute the integral $I = \int_C [(x-y^3)\,\mathrm{d}x + x^3\,\mathrm{d}y]$. $\left[\frac{3}{2}\pi\right]$

- Along an other arch of parabola

Let $a > 0$ a number and $[0, a] \ni x \longmapsto Y(x) = \frac{1}{2}\frac{x^2}{a}$ an arch Γ of parabola. We introduce the scalar field $\psi(x, y) = \frac{1}{2}(x^2 + y^2)$ and the vector field $\Phi = \nabla\psi$.

a) What are the components of the vector field Φ?

b) Compute directly the circulation $\gamma = \int_\Gamma (\Phi(M), \tau(M))\,\mathrm{d}s$ of this vector field.

c) Using a result proposed in the course, recover this result. $\left[\frac{5}{8}a^2\right]$