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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 8 Functions of several variables

- Some examples

An affine function: $\alpha(x, y) = ax + by + c$, a quadratic function: $q(x, y) = x^2 - y^2$, a power-exponential function: $h(x, y) = x^y$ and a rational fraction: $r(x, y) = \frac{x^2 y^2}{x^2+y^2}$ if $(x, y) \neq (0, 0)$, $r(0, 0) = 0$.

- Domain

A function f from $\mathbb{R}^2$ with real values associates to each pair (x, y) of real numbers one and only one number $f(x, y)$ if (x, y) belongs to the domain D. If $(x, y) \notin D$, then the number $f(x, y)$ does not exists.

For the previous examples, we have $D_\alpha = \mathbb{R}^2$, $D_q = \mathbb{R}^2$, $D_h =]0, +\infty[\times \mathbb{R}$ and $D_r = \mathbb{R}^2$.

- Partial functions

A function with two variables defines (at least) a double infinity of functions of a single variable. On one hand, with b given in $\mathbb{R}$, we have the function $x \longmapsto f(x, b)$ of the first variable. On the other hand, with $a \in \mathbb{R}$, we can introduce the function $y \longmapsto f(a, y)$ of the second variable.

- Partial derivatives

We suppose given a function $\mathbb{R}^2 \supset D \ni (x, y) \longmapsto f(x, y) \in \mathbb{R}$ of two variables and a point (a, b) that belongs to the domain of f. We say that f admits a partial derivative at the point (a, b) according to the first variable, noted $\frac{\partial f}{\partial x}(a, b)$, if and only if the partial function $x \longmapsto f(x, b)$ is derivable at the point a; we have $\frac{\partial f}{\partial x}(a, b) = \lim_{t \longrightarrow 0} \frac{1}{t}\big[f(a+t, b) - f(a, b)\big]$.

Similarly, we say that f admits a partial derivative at the point (a, b) relative to the second variable, noted $\frac{\partial f}{\partial y}(a, b)$, if and only if the partial function $y \longmapsto f(a, y)$ is derivable at the point b. In that case, $\frac{\partial f}{\partial y}(a, b) = \lim_{\theta \longrightarrow a} \frac{1}{\theta}\big[f(a, b+\theta) - f(a, b)\big]$.

For the functions proposed in the introduction, we have $\frac{\partial \alpha}{\partial x} = a$, $\frac{\partial \alpha}{\partial y} = b$, $\frac{\partial q}{\partial x} = 2x$, $\frac{\partial q}{\partial y} = -2y$, $\frac{\partial h}{\partial x} = \frac{y}{x} h$, $\frac{\partial h}{\partial y} = (\log x)\, h$, $\frac{\partial r}{\partial x} = \frac{2xy^4}{(x^2+y^2)^2}$, $\frac{\partial r}{\partial y} = \frac{2x^4 y}{(x^2+y^2)^2}$.

- Continuity

The function f is continuous at the point (a, b) if and only if the function $\varphi(u, v)$ defined by $\varphi(u, v) = f(a+u, b+v) - f(a, b)$ tends to zero if the point (u, v) tends to the origin $(0, 0)$.

François Dubois, 06 december 2022.

The functions α, q and r introduced previously are continuous at the point $(0,0)$.
If $f : D \longrightarrow \mathbb{R}$ is continuous for each point $(a,b) \in D$, we say that f is continuous in the domain D.
If $f : D \longrightarrow \mathbb{R}$ is continuous in D and if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function of a single variable, then the composite function $(g \circ f)(x,y) \equiv g(f(x,y))$ is a continuous function in the domain D.

- Differentiability

We suppose given a function of two variables $\mathbb{R}^2 \supset D \ni (x,y) \longmapsto f(x,y) \in \mathbb{R}$ and a point $(a,b) \in D$. We say that f is differentiable at the point (a,b) if the function f est "close" to an affine function in the vicinity of the point (a,b). More precisely, f is differentiable at the point (a,b) if and only if there exits two numbers α et β and a function φ of two variables (u,v) that tends to zero when (u,v) tends to the origin $(0,0)$, such that we have the expansion $f(a+u, b+v) = f(a,b) + \alpha\, u + \beta\, v + \sqrt{u^2+v^2}\, \varphi(u,v)$.
If f is differentiable at the point (a,b), it has also partial derivatives at the point. We have the relations $\frac{\partial f}{\partial x}(a,b) = \alpha$ and $\frac{\partial f}{\partial y}(a,b) = \beta$.

- Theorem: differentiability implies continuity

When f is differentiable at the point $(a,b) \in D$, then it is continuous at the point.
Be careful! The existence of partial derivatives does not imply the differentiability! The function s defined by the conditions $s(x,y) = \frac{x^5}{(y-x^2)^2+x^8}$ if $(x,y) \neq (0,0)$ and $s(0,0) = 0$ admits partial derivatives $\frac{\partial s}{\partial x}(0,0)$ and $\frac{\partial s}{\partial y}(0,0)$ at the origin but the function s is not continuous at the point $(0,0)$.

- Remark concerning the notations

The differential $\mathrm{d}f(a,b)$ is a linear map defined by the relation $\mathrm{d}f(a,b).(u,v) = \frac{\partial f}{\partial x}(a,b)\, u + \frac{\partial f}{\partial y}(a,b)\, v$. Introduce the two coordinate functions $X(x,y) = x$ and $Y(x,y) = y$. Then we have $\mathrm{d}X(a,b).(u,v) = u$ and $\mathrm{d}Y(a,b).(u,v) = v$. In consequence, we can write $\mathrm{d}f(a,b).(u,v) = \frac{\partial f}{\partial x}(a,b)\, \mathrm{d}X(a,b).(u,v) + \frac{\partial f}{\partial y}(a,b)\, \mathrm{d}Y(a,b).(u,v)$. This relation between numbers is true for each $(u,v) \in \mathbb{R}^2$. Then we can write an equality between linear forms: $\mathrm{d}f(a,b) = \frac{\partial f}{\partial x}(a,b)\, \mathrm{d}X(a,b) + \frac{\partial f}{\partial y}(a,b)\, \mathrm{d}Y(a,b)$. We usually skip the reference to the argument (a,b) and we obtain the relation $\mathrm{d}f = \frac{\partial f}{\partial x}\, \mathrm{d}X + \frac{\partial f}{\partial y}\, \mathrm{d}Y$. With a litle purpose of notation, we replace X by x and Y by y. Then we have $\mathrm{d}f = \frac{\partial f}{\partial x}\, \mathrm{d}x + \frac{\partial f}{\partial y}\, \mathrm{d}y$, the usual way for computing differentials.

- Differentiation of composite functions: a first case.

We suppose given a function of two variables $\mathbb{R}^2 \supset D \ni (x,y) \longmapsto f(x,y) \in \mathbb{R}$ and two functions $\mathbb{R} \ni t \longmapsto X(t)$ and $\mathbb{R} \ni t \longmapsto Y(t)$ in such a way that for each t, we have the condition $(X(t), Y(t)) \in D$. Then the composite function $g(t) = f(X(t), Y(t))$ is well defined for each t.
If f is differentiable on the domain D and if the functions $t \longmapsto X(t)$ and $t \longmapsto Y(t)$ are derivables, then the function $t \longmapsto g(t)$ is derivable and we have the relation
$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial f}{\partial x}(X(t), Y(t))\, \frac{\mathrm{d}X}{\mathrm{d}t} + \frac{\partial f}{\partial y}(X(t), Y(t))\, \frac{\mathrm{d}Y}{\mathrm{d}t}$.

- Differentiation of composite functions: a second case.

We replace the functions X and Y of the previous section by the two functions $\mathbb{R}^2 \supset \Delta \ni (u, v) \longmapsto X(u, v) \in \mathbb{R}$ and $\mathbb{R}^2 \supset \Delta \ni (u, v) \longmapsto Y(u, v) \in \mathbb{R}$ of two variables. As previously, we suppose that for each $(u, v) \in \Delta$, we have $(X(u, v), Y(u, v)) \in D$. Then the composite function $g(u, v) = f(X(u, v), Y(u, v))$ is well defined for $(u, v) \in \Delta$.
If f is differentiable on D and if the functions X and Y are differentiable on Δ, then the composite function $g(u, v) = f(X(u, v), Y(u, v))$ is differentiable on Δ and the partial derivatives are evaluated with the relations $\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial X}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial u}$ and $\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial X}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial v}$.

Exercices

- Laplacian in polar coordinates

A point (x, y) of the affine Euclidian plane not located at the origin can be parametrized with the two dimensional polar coordinates (r, θ): $x = r\cos\theta$ and $y = r\sin\theta$. Let f be a two times continuously differentiable function of the pair (x, y) with real values; we have $f(x, y) \in \mathbb{R}$.
We introduce the Laplacian of f : $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ and independently the function g of the variables r and θ such that $g(r, \theta) = f(r\cos\theta, r\sin\theta)$.
a) From the relation $r^2 = x^2 + y^2$, show that the partial derivatives $\frac{\partial r}{\partial x}$ et $\frac{\partial r}{\partial y}$ are respectively equal to $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$.
b) Similarly, from the the relation $\tan\theta = \frac{y}{x}$, prove that we have $\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} = -\frac{1}{r}\sin\theta$ and $\frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{1}{r}\cos\theta$.
c) Compute $\frac{\partial g}{\partial r}$ and $\frac{\partial g}{\partial\theta}$ as functions of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
d) Deduce from the previous question that we have $\frac{\partial f}{\partial x} = \cos\theta\frac{\partial g}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial g}{\partial\theta}$ and $\frac{\partial f}{\partial y} = \sin\theta\frac{\partial g}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial g}{\partial\theta}$.
e) Using the four auxiliary fonctions $f_1(x, y) = \frac{x}{\sqrt{x^2+y^2}}$, $f_2(x, y) = \frac{-y}{x^2+y^2}$, $f_3(x, y) = \frac{y}{\sqrt{x^2+y^2}}$ and $f_4(x, y) = \frac{x}{x^2+y^2}$, establish the following relations $\frac{\partial}{\partial x}(\cos\theta) = \frac{1}{r}\sin^2\theta$, $\frac{\partial}{\partial x}\big(-\frac{1}{r}\sin\theta\big) = \frac{2}{r^2}\sin\theta\cos\theta$, $\frac{\partial}{\partial y}(\sin\theta) = \frac{1}{r}\cos^2\theta$ and $\frac{\partial}{\partial y}\big(\frac{1}{r}\cos\theta\big) = -\frac{2}{r^2}\sin\theta\cos\theta$.
f) Deduce from the relations obtained in the previous questions the expressions of the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ as fonctions of r, θ, $\frac{\partial g}{\partial r}$, $\frac{\partial g}{\partial\theta}$, $\frac{\partial^2 g}{\partial r^2}$, $\frac{\partial^2 g}{\partial r\partial\theta}$ and $\frac{\partial^2 g}{\partial\theta^2}$. Be careful, each result contains five terms!
g) Deduce from the previous question the identity $\Delta f(x, y) = \frac{1}{r}\frac{\partial}{\partial r}\big(r\frac{\partial g}{\partial r}\big) + \frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}$.

• Method of characteristics

We suppose given a real numer $a \in \mathbb{R}$ and a derivable function u_0 from $\mathbb{R}$ to $\mathbb{R}$. We search an unknown function $u(x, t)$ of two variables that satisfies on one hand to the advection equation $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$ for $x \in \mathbb{R}$ and $t > 0$ and on the other hand to the initial condition $u(x, 0) = u_0(x)$ for each $x \in \mathbb{R}$. Independently, for a fixed $y \in \mathbb{R}$, we set $v(t) = u(at + y, t)$.

a) Prove that if the function u is solution of the advection equation, then the derivative $\frac{\mathrm{d}v}{\mathrm{d}t}$ is equal to zero.

b) Deduce from the previous question that for each $y \in \mathbb{R}$ and each $t \geq 0$, we have the relation $u(at + y, t) = u_0(y)$.

c) Establish that every differentiable solution of the advection equation $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$ satisfying the initial condition $u(x, 0) = u_0(x)$ for each $x \in \mathbb{R}$ is necessarily of the form $u(x, t) = u_0(x - at)$, $x \in \mathbb{R}$, $t > 0$.

d) With an elementary calculus, show that the function u defined by $u(x, t) = u_0(x - at)$ is effectively a solution of the prolem composed on one hand by the advection equation $\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$ (with $x \in \mathbb{R}$ et $t > 0$) and on the other hand by the initial condition $u(x, 0) = u_0(x)$ (with $x \in \mathbb{R}$).

• Kernel of the heat equation

We suppose given $\sigma > 0$. For $x \in \mathbb{R}$ and $t > 0$ we set $\varphi(x, t) = \frac{1}{\sqrt{t}} \exp(-\frac{x^2}{4\sigma^2 t})$.

a) Propose an expression for the partial derivative $\frac{\partial \varphi}{\partial t}$.

b) Same question for $\frac{\partial \varphi}{\partial x}$.

c) Same question for $\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\big(\frac{\partial \varphi}{\partial x}\big)$.

d) Verify that the function φ is a solution of the heat equation in one space dimension: $\frac{\partial \varphi}{\partial t} - \sigma^2 \frac{\partial^2 \varphi}{\partial x^2} = 0$ for $x \in \mathbb{R}$ and $t > 0$.