Cubic differentials and hyperbolic convex sets

Yves Benoist and Dominique Hulin

Abstract

We give a parameterization of the moduli space of marked Gromov hyperbolic convex domains of $\mathbb{R}\mathrm{P}^2$ by the space of bounded holomorphic cubic differentials on the disk.

1 Introduction

A domain Ω in the real projective space $\mathbb{R}\mathrm{P}^n$ is properly convex when it reads as a bounded convex open set in a suitable affine chart. The Hilbert distance on Ω is then defined by

$$d_\Omega(x,y) = |\log[x,y,a,b]|,$$

where $[x,y,a,b]$ denotes the cross ratio of the quadruple (x,y,a,b), and a,b are the points of intersection of the line (xy) and the boundary $\partial\Omega$ of Ω.

We are interested in those properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^n$ for which the Hilbert geometry (Ω, d_Ω) is Gromov hyperbolic (see Gromov–Ghys–De la Harpe [13], or Definition 4.1). Benoist [1] gave a necessary and sufficient condition for a properly convex domain to be Gromov hyperbolic in terms of the regularity of its boundary, refining previous results by Karlsson and Noskov [15]. Another characterization, in terms of the area of its ideal triangles, is due to Colbois–Vernicos–Verovic [9].

In this paper we will restrict ourselves to properly convex domains of the projective plane $\mathbb{R}\mathrm{P}^2$. First, we give a new necessary and sufficient condition for Gromov hyperbolicity. Namely, a properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^2$ is Gromov hyperbolic if and only if the volume of the metric balls in (Ω, d_Ω) have a uniform exponential growth rate (Theorem 5.2). Second, we exhibit a natural bijection between the moduli space of marked Gromov hyperbolic properly convex domains of $\mathbb{R}\mathrm{P}^2$ and the space of bounded holomorphic cubic differentials on the disk (Theorem 6.6).

These two results rely on the study of the affine metric h_Ω, which is the complete Riemannian metric induced on Ω by the affine sphere asymptotic to the boundary of the convex cone $C \subset \mathbb{R}^3$ above the domain Ω. It was proved in [6] by Calabi that (in any dimension) the Ricci curvature of the affine metric on a properly convex domain of $\mathbb{R}\mathrm{P}^n$ is non positive. The key point of this article is that a properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^2$ is Gromov hyperbolic if and only if the curvature of its affine metric is bounded above by a negative constant (Corollary 4.10).

The paper is organized as follows. In Section 2 we briefly recall the definition of an affine sphere, and of the affine metric h_Ω on a properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^n$.

When $n = 2$, the uniformization theorem ensures that this affine metric h_Ω is conformal to a flat metric h_e. By studying the curvature equation, which relates the curvature of h_Ω to the conformal factor between h_Ω and h_e, we prove in Section 3 that the curvature of (Ω, h_Ω) is non positive, and that this curvature is strictly negative unless it is identically zero (Proposition 3.3). We then infer, with the help of Benzécri's cocompactness theorem 4.6, that a properly convex domain of $\mathbb{RP}^2$ is Gromov hyperbolic if and only if the curvature of its affine metric h_Ω is bounded above by a negative constant (Corollary 4.10). These arguments also prove that a Gromov hyperbolic domain (Ω, h_Ω) equipped with its affine metric is conformally quasi-isometric to the hyperbolic disk $(\mathbb{D}, h_0)$ (Proposition 4.9).

Using the Bishop-Günther volume estimates, and again Benzécri's cocompactness theorem, we characterize in Section 5 the Gromov hyperbolic properly convex domains of $\mathbb{RP}^2$ in terms of their growth profile (Theorem 5.2 and Lemma 5.5).

The definition of the Pick form of an (oriented) properly convex domain $\Omega \subset \mathbb{RP}^2$ is recalled in Lemma 3.2. We mentionned earlier that, when Ω is Gromov hyperbolic, then (Ω, h_Ω) is uniformized by the disk. In this case, we may thus read the Pick form of Ω on the disk, and obtain a bounded holomorphic cubic differential on $\mathbb{D}$ (Lemma 6.4). This construction leads in the final Section 6 to the construction of a natural bijection between the moduli space of marked Gromov hyperbolic properly convex domains of $\mathbb{RP}^2$, and the space of bounded holomorphic cubic differentials on the disk (Theorem 6.6).

A previous result in this spirit was first obtained by Loftin [20] and Labourie [16] (see also CP Wang [28]), who provided a bijection between the set of all properly convex structures on a given oriented compact surface S with negative Euler characteristic, and the set of pairs (J, U) where J is a complex structure on S and U is a cubic differential on (S, J).

This construction was later extended in Benoist–Hulin [2] to parameterize the set of convex projective structures with finite volume on a given oriented surface S with non-abelian fundamental group.

Heuristically, we may think of the moduli space of marked Gromov hyperbolic properly convex domains of $\mathbb{RP}^2$ as an analog of the universal Teichmüller space. Our parameterization is then analogous to the parameterization of the universal Teichmüller space by bounded holomorphic quadratic differentials conjectured by R. Schoen in [24]. See Gardiner–Harvey [12] and Sugawa [25] for surveys on the universal Teichmüller space. See also Bonsante–Schlenker [4] and Tam–Wan [26] for progress towards this conjecture.

2 Affine spheres

In this section, we briefly recall the definition and a few facts concerning affine spheres. For a more thorough treatment and references, we refer to [2].

Let $M \subset \mathbb{R}^{n+1}$ be an immersed hypersurface and $E = M \times \mathbb{R}^{n+1}$ be the trivial vector bundle of rank $n + 1$ over M. The standard affine (flat) connection on $\mathbb{R}^{n+1}$ induces a flat connection

∇ on E. Each choice of a transverse vector field $\xi : M \to \mathbb{R}^{n+1}$ induces a decomposition $E = TM \oplus L_\xi$, where L_ξ stands for the trivial line bundle over M spanned by ξ.

From now on, we assume that $M \subset \mathbb{R}^{n+1}$ is locally strictly convex, that is M can locally be written in an affine chart as the graph of a function with positive definite hessian.

Definition 2.1 *The hypersurface $M \subset \mathbb{R}^{n+1}$ is an affine sphere with center the origin and affine curvature -1 if the three following conditions are satisfied :*

- *the vector field $\xi : m \in M \to Om \in \mathbb{R}^{n+1}$ is transverse to M*

- *the flat connection ∇ reads on the decomposition $E = TM \oplus L_\xi$ as*

$$\begin{cases} \nabla_X Y = D_X Y + h(X,Y)\,\xi \in TM \oplus L_\xi \\ \nabla_X \xi = X \qquad\qquad\quad\, \in TM \oplus L_\xi \end{cases} \tag{2.1}$$

  where h is a positive definite 2-form on TM

- *for any h-orthonormal frame $(Y_1, \cdots, Y_n)$ of TM, one has $|\det(Y_1, \cdots, Y_n, \xi)| = 1$*

Observe that D is then a torsion free connection on TM : it is the *Blashke connection*. The Riemannian metric h on TM is the *affine metric* on M. All these notions are preserved by the group $\mathrm{SL}_{n+1}^{\pm}\mathbb{R}$ of real matrices with determinant ± 1.

Thanks to the following theorem due to Cheng–Yau and An-Min Li, affine spheres provide a powerful tool for studying properly convex domains of $\mathbb{R}\mathrm{P}^n$. Recall that an open subset $\Omega \subset \mathbb{R}\mathrm{P}^n$ is *properly convex* when it reads, in a suitable affine chart, as a bounded convex domain of $\mathbb{R}^n$.

Theorem 2.2 (Cheng–Yau [7] and [8], An-Min Li [17])

1. *Let Ω be a properly convex domain of $\mathbb{R}\mathrm{P}^n$, and $C \subset \mathbb{R}^{n+1}$ be one of the two open convex cones above Ω.*

   (a) *There exists a unique embedded affine sphere $M \subset \mathbb{R}^{n+1}$ with center the origin and affine curvature -1, which is asymptotic to the boundary of the cone C*

   (b) *In particular, the canonical projection $p : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}\mathrm{P}^n$ induces a diffeomorphism between M and Ω*

   (c) *Moreover, the affine metric h on this affine sphere M is complete*

2. *Every affine sphere $M \subset \mathbb{R}^{n+1}$ with center the origin and affine curvature -1, which is complete for the affine metric, is asymptotic to the boundary of a cone above a properly convex subset of $\mathbb{R}\mathrm{P}^n$.*

This result allows us to systematically identify each properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^n$ with the corresponding affine sphere $M \subset \mathbb{R}^{n+1}$ with center the origin and affine curvature -1. The affine metric h of M reads on Ω through the canonical projection $p : M \to \Omega$, and yields a complete Riemannian metric h_Ω on Ω which we also call the *affine metric* of the domain Ω.

3

These affine spheres are related to the solutions of a real Monge–Ampère equation (see Cheng–Yau [7], or [2] Theorem 2.5). In dimension 2, the a priori estimates which were used by Cheng and Yau to solve the Monge–Ampère equation yield the following, that will be used in the next section.

Definition 2.3 *Define the Benzécri space $\mathcal{E}$ as the set of pairs (x, Ω), where $\Omega \subset \mathbb{RP}^2$ is a properly convex domain and x is a point in Ω.*

Proposition 2.4 *The Riemannian curvature $K_{h_\Omega}(x)$ of the affine metric h_Ω of Ω at the point x depends continuously on the pair $(x, \Omega) \in \mathcal{E}$.*

Proof Immediate consequence of [2, Corollary 3.3]. □

3   Curvature of 2-dimensional affine spheres

In this section, we provide estimates for the Riemannian curvature of 2-dimensional affine spheres equipped with their affine metric.

3.1   The Pick form

We first recall the relationship between the Pick form and the curvature of the affine metric.

Let $M \subset \mathbb{R}^{n+1}$ be an affine sphere with affine curvature -1. Denote by D^h the Levi-Civita connection of the affine metric h on M, and set $D = D^h + A$, where D is the Blaschke connection.

Definition 3.1 *The* Pick tensor *of M is the 3-tensor defined on TM by*

$$C(X, Y, Z) = h(A(X)Y, Z),$$

where X, Y, Z are tangent vectors.

In any dimension, the tensors A and C satisfy certain symmetry conditions, and the full Riemannian curvature tensor of h can be computed in terms of the tensor A (see Labourie [16], or [2, Lemmas 4.3 and 4.4]). In dimension 2, these symmetries of C have a pleasant interpretation in terms of the conformal structure of (M, h) (see Lemma 3.2 below).

Let $M \subset \mathbb{R}^3$ be a 2-dimensional affine sphere with affine curvature -1. Choose a (local) orientation of M, and local isothermal coordinates (x, y) for the affine metric h on M, so that $z = x + iy$ is a complex parameter for the complex manifold (M, J) associated to (M, h). Write $h = e^{2w}h_e$, where $h_e = dx^2 + dy^2$ denotes the Euclidean metric on $\mathbb{R}^2$. Then :

Lemma 3.2   *1. The Pick tensor C is the real part of a (unique) holomorphic cubic differential $U = f(z)\,dz^3$ on (M, J), which is called the* Pick form.

4

2. The Riemannian curvature K_h of the affine metric h reads as

$$K_h = -1 + 2e^{-6w} |f|^2 .$$

In particular, $K_h \geq -1$.

Proof 1. See for example [2, Lemma 4.8].

2. Since the connection ∇ is flat, it follows from the definition of the curvature that

$$K_h = -1 - h([A(X_1), A(X_2)] X_2, X_1) ,$$

where (X_1, X_2) is any h-orthonormal basis of TM (a proof is given in [2, Lemma 4.3]). It remains to express the second term on the right-hand side in terms of f and w. Recall that $C(X, Y, Z) = h(A(X)Y, Z)$. Define similarly a section A_e of $T^*M \otimes \mathrm{End}\,TM$ associated to the Euclidean metric h_e by letting $C(X, Y, Z) = h_e(A_e(X)Y, Z)$. Writing the holomorphic function f as $f = P + iQ$, the Pick tensor $C = \mathrm{Re}\,U$ reads as

$$C = P \, dx^3 - P \, dx \, dy^2 - Q \, dx^2 \, dy + Q \, dy^3 .$$

Thus, in the canonical basis (e_1, e_2) of $\mathbb{R}^2$, this yields

$$A_e(e_1) = \begin{pmatrix} P & -Q \\ -Q & -P \end{pmatrix} , \quad A_e(e_2) = \begin{pmatrix} -Q & -P \\ -P & Q \end{pmatrix} \text{ hence } [A_e(e_1), A_e(e_2)] = 2\,|f|^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The result follows by homogeneity, since $h = e^{2w} h_e$, $A = e^{-2w} A_e$, and one can choose $X_i = e^{-w} e_i$ $(i = 1, 2)$. $\qquad\square$

3.2 Curvature estimates

We have just seen that the Riemannian curvature K_h of the affine metric on a 2-dimensional affine sphere has -1 as a lower bound. We now seek an upper bound for K_h.

Proposition 3.3 *Let $M \subset \mathbb{R}^3$ be a 2-dimensional affine sphere with affine curvature -1, and assume the affine metric h to be complete.*

1. *The curvature K_h of the affine metric h satisfies $-1 \leq K_h \leq 0$.*

2. *If the curvature K_h vanishes at one point, then it is identically zero. In this case, M is the affine sphere above a projective triangle.*

Remark 3.4 *– Part 1 is due to Calabi in [6]. He proves more generally that the Ricci curvature of the affine metric of an n-dimensional complete affine sphere with affine curvature -1 satisfies*

$$-(n-1)\,h \leq \mathrm{Ricci}_h \leq 0 .$$

5

However we provide below a simple proof of Part 1, since this proof will be used for Part 2. – Thanks to Theorem 2.2, what we actually prove here is that the curvature K_{h_Ω} of the affine metric on a properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^2$ satisfies

$$-1 \leq K_{h_\Omega} < 0\,,$$

unless Ω is a projective triangle – in which case its affine metric is flat.

Before going into the proof of this proposition we note that, since the affine metric is assumed to be complete, M is simply connected (Theorem 2.2). Choose an orientation on M and let J denote the conformal class of h – thus J is a complex structure on M. Then (M, J) is uniformized either by the disk or by the plane. In other words, the Riemann surface (M, J) associated to (M, h) is isomorphic to $\mathbb{D}_a = \{z \in \mathbb{C} \,|\, |z| < a\}$, with either $a = 1$ or $a = \infty$.

Choose an isomorphism $\varphi : (M, J) \to \mathbb{D}_a$ of Riemann surfaces. In this global coordinate system for M, the Pick form reads on $\mathbb{D}_a$ as $U = f(z)\, dz^3$, while the affine metric h reads as $h = e^{2w} h_e$, where $h_e = dx^2 + dy^2$ denotes the Euclidean metric on $\mathbb{D}_a$ ($z = x + iy$).

Proof of Proposition 3.3

1. We proved in Lemma 3.2 that the Riemannian curvature of the affine metric satisfies $K_h \geq -1$. It thus remains to prove that K_h is non positive. We work in the coordinate system $\varphi : M \to \mathbb{D}_a$. Since the Euclidean metric h_e is flat, the curvature equation for $h = e^{2w} h_e$ reads as

$$\Delta_h w = K_h\,.$$

Here Δ_h denotes the Riemannian Laplace operator relative to h, with the sign convention that $\Delta_h = -\mathrm{Trace}\, D^h \nabla_h$. Hence, it follows from Lemma 3.2 that

$$\Delta_h w = -1 + 2e^{-6w} |f|^2\,. \tag{3.1}$$

We want to prove that $2e^{-6w} |f|^2 \leq 1$. We may thus assume that the holomorphic function f is not identically zero. Define, on the domain $\mathcal{O} = \{f \neq 0\} \subset \mathbb{D}_a$, a function τ by letting

$$2\tau := 6w - 2\log|f| - \log 2$$

so that $e^{-2\tau} = 2e^{-6w} |f|^2$. Our aim is to prove that τ is non negative. We observe that the equality $\Delta_h \tau = 3\, \Delta_h w$ holds on $\mathcal{O}$, since the function $\log|f|$ is harmonic on this domain.

- Assume first that τ reaches its minimum value at some point $x_0 \in \mathbb{D}_a$. The result follows from Equation (3.1), since

$$0 \geq \Delta_h \tau(x_0) = 3\, \Delta_h w(x_0) = 3\left(-1 + e^{-2\tau(x_0)}\right).$$

- In case τ does not reach its minimum $\inf \tau \in [-\infty, \infty[$, we set $F = (1 + e^\tau)^{-1}$. The function F is non negative on $\mathbb{D}_a$, and vanishes on $\{f = 0\}$. It is bounded above and does not reach its maximum. Since the Riemannian manifold $(\mathbb{D}_a, h)$ is complete and its curvature is

bounded below (by -1), the generalized maximum principle of Yau [29] applies to F, and yields a sequence $x_n \in \mathbb{D}_a$ of points with $F(x_n) \to \sup F$ (that is, with $\tau(x_n) \to \inf \tau$) and $\lim_{n\to\infty} \Delta_h F(x_n) \in [0, \infty]$. A simple computation gives

$$\Delta_h F = \frac{-e^\tau}{(1 + e^\tau)^2} \Delta_h \tau + |\nabla_h F|^2 \left(e^\tau - e^{-\tau}\right).$$

Plugging Equation (3.1) into this equation yields

$$\Delta_h F = 3 \frac{e^\tau}{(1 + e^\tau)^2} \left(1 - e^{-2\tau}\right) + |\nabla_h F|^2 \left(e^\tau - e^{-\tau}\right).$$

Assume that $\inf \tau \in [-\infty, 0[$, and express this identity at the points x_n. The first term on the right-hand side has a limit in $[-\infty, 0[$ when n goes to infinity, while the second term is non positive when n is large enough. This gives a contradiction, since the sequence was chosen so that $\lim_{n\to\infty} \Delta_h F(x_n) \in [0, \infty]$.

2. We have proved so far that the function τ is non negative on $\mathcal{O}$. We now prove that either τ does not vanish, or that it is identically zero. It follows from (3.1) and the definition of τ that

$$-\Delta_h \tau = 3 \left(1 - e^{-2\tau}\right) \leq 6 \tau.$$

Let $\Delta_e = -\left(\partial_x^2 + \partial_y^2\right)$ denote the Euclidean Laplace operator on $\mathbb{D}_a$. The conformal invariance of the Laplace operator ensures that, for any $p \in \mathcal{O}$, there exists a constant $c > 0$ and a neighborhood of p in $\mathcal{O}$ where

$$(-\Delta_e \tau) \leq c \tau$$

holds. The fact that either $\tau > 0$ or $\tau \equiv 0$ will thus be an immediate consequence of the following Lemma.

Lemma 3.5 *Let $\tau : \mathcal{U} \subset \mathbb{R}^2 \to [0, \infty[$ be a smooth non negative function defined on an open subset of $\mathbb{R}^2$. Assume that τ satisfies $(-\Delta_e \tau) \leq c \tau$, where c is a positive constant. Then, the zero set $\{\tau = 0\}$ is open.*

Remark 3.6 *The same statement actually holds true in $\mathbb{R}^n$ ($n \geq 1$). The proof is similar.*

Proof Assume that the domain $\mathcal{U}$ contains the closed ball $\bar{B}(0, R)$, and that $\tau(0) = 0$. We let

$$M(r) := \int_0^{2\pi} \tau(r\, e^{i\theta}) \frac{d\theta}{2\pi}$$

denote the mean value of the function τ on the circle with center 0 and radius r, where $0 \leq r \leq R$. Then, the Green's representation formula for $\tau(0)$ (see Hörmander [14, p.119]) reads as

$$\tau(0) = M(r) + \frac{1}{2\pi} \int_{B(0,r)} \Delta_e \tau(y) \log \frac{r}{|y|}\, dy$$

7

(recall our sign convention for Δ_e). Since τ vanishes at the origin, our hypothesis on $\Delta_e \tau$ yields

$$M(r) \leq \frac{c}{2\pi} \int_{B(0,r)} \tau(y) \log \frac{r}{|y|} \, dy \, ,$$

hence

$$M(r) \leq \frac{c \, R^2}{4} \sup_{[0,R]} M(t)$$

for every $0 \leq r \leq R$. When R is small enough (namely when $c \, R^2 < 4$), this proves that τ is identically zero on the ball $\bar{B}(0, R)$. $\qquad \square$

Proof of Proposition 3.3 (continued) We have just proved that, when the Riemannian curvature of a complete 2-dimensional affine sphere (M, h) with affine curvature -1 vanishes at one point, then (M, h) is flat. In this case, it follows from Li–Penn [18] or Magid–Ryan [21] that M is the affine sphere above a projective triangle. $\qquad \square$

4 Curvature of Gromov hyperbolic domains of $\mathbb{RP}^2$

In this section, we refine the previous curvature estimates and prove that the curvature of a Gromov hyperbolic domain of $\mathbb{RP}^2$, equipped with its affine metric, is pinched between two negative constants.

4.1 Gromov hyperbolicity

We first recall the definition of Gromov hyperbolicity. Then, we consider the natural action of $\mathrm{SL}_3\mathbb{R}$ on $\mathbb{RP}^2$ and give alternative definitions of Gromov hyperbolicity for properly convex domains of $\mathbb{RP}^2$ (Corollaries 4.5 and 4.7).

Definition 4.1 *A geodesic metric space (E, d) is* Gromov hyperbolic *if there exists a constant $\delta > 0$ such that, for any geodesic triangle (xyz) in E, each edge $[xy]$ is at distance at most δ of the union $[xz] \cup [zy]$ of the two others. In this case, the metric space E is said to be δ-hyperbolic.*

Let $\Omega \subset \mathbb{RP}^2$ be a properly convex domain. The Hilbert distance on Ω is defined by

$$d_\Omega(x, y) = |\log[x, y, a, b]| \, ,$$

where a and b are the points where the line xy intersects the boundary of Ω, and $[x, y, a, b]$ denotes the cross-ratio of the quadruple (x, y, a, b). The Hilbert distance derives from the Finsler metric defined for any point $x \in \Omega$ and any vector $X \in T_x\Omega$ as

$$||X||_{F,\Omega} = \left(\frac{1}{||x - a||} + \frac{1}{||x - b||} \right) ||X|| \, ,$$

where a and b are the points of intersection of $\partial\Omega$ with the line defined by $(x; X)$, and $|| \, ||$ is any Euclidean norm on an affine chart with $\overline{\Omega} \subset \mathbb{R}^n$ (see eg the survey by Vernicos [27]). Observe finally that any automorphism $g \in \mathrm{SL}_3\mathbb{R}$ of the projective plane obviously induces an isometry $g : (\Omega, d_\Omega) \to (g(\Omega), d_{g(\Omega)})$.

8

Definition 4.2 *A properly convex domain* $\Omega \subset \mathbb{R}\mathrm{P}^2$ *is said to be* δ-hyperbolic *when the metric space* (Ω, d_Ω), *where* d_Ω *denotes the Hilbert distance on* Ω, *is* δ-hyperbolic.

Notation 4.3 We introduce
– the set X of properly convex domains of $\mathbb{R}\mathrm{P}^2$
– the subset $X_\delta \subset X$ which consists of the domains $\Omega \in X$ which are δ-hyperbolic ($\delta > 0$)

Proposition 4.4 (Benoist [1]) *Let* $\delta > 0$. *Then* X_δ *is a* $\mathrm{SL}_3\mathbb{R}$-*invariant closed subset of* X *for the Hausdorff topology on* X.

A consequence is the following criterion for Gromov hyperbolicity.

Corollary 4.5 [1, Corollaires 2.9 and 2.13] *Let* $\Omega \in X$ *be a properly convex domain. Then* Ω *is not Gromov hyperbolic if and only if the orbit closure* $\overline{\mathrm{SL}_3\mathbb{R} \cdot \Omega}$ *in* X *contains the projective triangle* $\mathcal{T}$.

We introduced earlier the Benzécri space $\mathcal{E}$, which is the set of pairs (x, Ω) where $\Omega \subset \mathbb{R}\mathrm{P}^2$ is a properly convex domain and x is a point in Ω (Definition 2.3). The following fundamental compactness theorem is due to Benzécri. It will be used below to give an alternative definition of Gromov hyperbolicity for properly convex domains of $\mathbb{R}\mathrm{P}^2$ (Corollary 4.7) and will be used again in Proposition 4.8 to refine, in the case of δ-hyperbolic domains, the curvature estimates for the affine metric we obtained in Proposition 3.3.

Theorem 4.6 (Benzécri [3]) *The natural action of* $\mathrm{SL}_3\mathbb{R}$ *on the space* $\mathcal{E}$, *equipped with the Hausdorff topology, is cocompact.*

This compactness result allowed us to prove in [2] that the ratio of the Finsler metric and the affine metric on any properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^2$ is uniformly bounded. More precisely, there exists a uniform constant $c > 0$ such that, for any $\Omega \in X$, the following holds

$$1/c \, h_\Omega \leq \| \ \|_{F,\Omega} \leq c \, h_\Omega \,. \tag{4.1}$$

Since Gromov hyperbolicity is a property which is invariant by quasi-isometries (Gromov–Ghys–de la Harpe [13, chap.5 §2]), we infer immediately that :

Corollary 4.7 *Let* $\Omega \subset \mathbb{R}\mathrm{P}^2$ *be a properly convex domain. Then* Ω *is Gromov hyperbolic (that is, for the Hilbert metric* d_Ω) *if and only if the Riemannian surface* (Ω, h_Ω) *is Gromov hyperbolic, where* h_Ω *is the affine metric on* Ω.

4.2 Further curvature estimates

Recall that the curvature K_{h_Ω} of the affine metric h_Ω on a properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^2$ satisfies $-1 \leq K_{h_\Omega} \leq 0$ (Proposition 3.3 and Remark 3.4). We now prove, for δ-hyperbolic properly convex domains, the following uniform pinching result.

Proposition 4.8 *Let $\delta > 0$. There exists a constant $k_\delta < 0$ such that the curvature K_{h_Ω} of the affine metric on any δ-hyperbolic properly convex domain $\Omega \subset \mathbb{RP}^2$ satisfies*

$$-1 \leq K_{h_\Omega} \leq k_\delta < 0 \,.$$

Proof We proceed by contradiction and assume that there exists $\delta > 0$, a sequence $\Omega_n \in X_\delta$ of δ-hyperbolic properly convex domains of $\mathbb{RP}^2$, and a sequence of points $x_n \in \Omega_n$ such that the curvature $K_{h_{\Omega_n}}(x_n)$ of the affine metric h_{Ω_n} at the point x_n goes to zero when $n \to \infty$. Using Benzécri's compactness Theorem 4.6, we may assume the sequence (x_n, Ω_n) to converge in $\mathcal{E}$ to (x, Ω) for the Hausdorff topology.

Proposition 2.4 ensures that the curvature K_{h_Ω} of the affine metric of Ω vanishes at the point x, and it follows from Proposition 3.3 that the Riemannian manifold (Ω, h_Ω) is flat. Since the affine metric h_Ω is complete, (Ω, h_Ω) is isometric to $(\mathbb{R}^2, \text{can})$. We deduce from Corollary 4.7 that the limit domain Ω is not Gromov hyperbolic : a contradiction to Proposition 4.4. $\square$

We infer below that, when $\Omega \in X$ is Gromov hyperbolic, then (Ω, h_Ω) is uniformized by the disk. More precisely, let $\Omega \subset \mathbb{RP}^2$ be a properly convex domain and h_Ω be the affine metric on Ω. Choose an orientation on Ω, and let (as in paragraph 3.2) (Ω, J) denote the complex manifold associated to (Ω, h_Ω).

Proposition 4.9 *Assume that the properly convex set $\Omega \subset \mathbb{RP}^2$ is Gromov hyperbolic.*

1. *The Riemann surface (Ω, J) is isomorphic to the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.*

2. *Let $h_{\Omega,0}$ denote the (conformal) hyperbolic metric on (Ω, J). Then, the affine metric reads as $h_\Omega = e^{2v} h_{\Omega,0}$ where the conformal factor v is bounded.*

Proof 1. It has been proved by Osserman [23] that a simply-connected Riemannian surface (Ω, h_Ω) with curvature bounded above by a negative constant is conformally equivalent to the disk. Hence the result follows from Proposition 4.8.

2. Both the affine metric h_Ω and the hyperbolic metric $h_{\Omega,0}$ are complete, and with curvature pinched between two negative constants. This makes the second assertion a special case of Yau's version of the Schwarz Lemma [30]. See eg [2, Lemma 5.2] for a proof. $\square$

Before concluding this section, we may observe that the previous discussion yields the following criterion for Gromov hyperbolicity.

Corollary 4.10 *A properly convex domain $\Omega \subset \mathbb{RP}^2$ is Gromov hyperbolic if and only if the curvature of its affine metric is bounded above by a negative constant.*

Proof One implication is Proposition 4.8. For the converse, observe that (Ω, d_Ω) is quasi-isometric to the simply-connected Riemannian manifold (Ω, h_Ω), which is complete and with curvature bounded above by a negative constant, hence is Gromov hyperbolic. $\square$

5 Gromov hyperbolicity and growth profile

In this section, we use the previous curvature estimates to prove that a properly convex domain of $\mathbb{R}\mathrm{P}^2$ is Gromov hyperbolic if and only if its affine metric, or equivalently its Finsler metric, has uniform exponential growth.

We may characterize the properly convex domains of $\mathbb{R}\mathrm{P}^2$ which are Gromov hyperbolic in terms of the growth profile of their Finsler metric. Let $\Omega \subset \mathbb{R}\mathrm{P}^2$ be a properly convex domain. Recall that the Hilbert distance d_Ω derives from a Finsler metric $|| \ ||_{F,\Omega}$ (see paragraph 4.1). To this Finsler metric $|| \ ||_{F,\Omega}$, we associate the Buseman measure $\mu_{F,\Omega}$ on Ω, which is defined as follows. Choose an affine chart such that $\overline{\Omega} \subset \mathbb{R}^2$, and endow the affine plane $\mathbb{R}^2$ with a Lebesgue measure m. In this chart, the measure $\mu_{F,\Omega}$ has density $(1/m(B_{F,\Omega}(x,1)))$ with respect to the Lebesgue measure m, where $B_{F,\Omega}(x,1) = \{X \in T_x\Omega, \ ||X||_{F,\Omega} < 1\}$ denotes the unit ball at the point x for the Finsler metric (see eg Marquis [22] or Vernicos [27]).

Definition 5.1 *Let $\Omega \subset \mathbb{R}\mathrm{P}^2$ be a properly convex domain. The* Finsler growth profile γ_Ω *of Ω is the function defined, for $R > 0$, as*

$$\gamma_{F,\Omega}(R) = \inf_{x \in \Omega} \mu_{F,\Omega}(B_{F,\Omega}(x,R)) \,.$$

The following theorem gives a characterization, in terms of their Finsler growth profile, of those properly convex domains of $\mathbb{R}\mathrm{P}^2$ which are Gromov hyperbolic.

Theorem 5.2 *Let $\Omega \subset \mathbb{R}\mathrm{P}^2$ be a properly convex domain. Then*

1. *either there exists a positive constant a such $\gamma_{F,\Omega}(R) \geq e^{aR}$ for any $R \geq 1$*

2. *or there exists a positive constant b such that $\gamma_{F,\Omega}(R) \leq b\,R^2$ for any $R > 0$*

Moreover the domain Ω is Gromov hyperbolic if and only if the balls in $(\Omega, || \ ||_{F,\Omega})$ have a uniform exponential growth, that is, in case 1.

This theorem will easily follow from a similar alternative concerning the affine growth profile (Lemma 5.5), that we define below.

Definition 5.3 *Let $\Omega \subset \mathbb{R}\mathrm{P}^2$ be a properly convex domain. The* affine growth profile γ_{h_Ω} *of Ω is the function defined for $R > 0$ by*

$$\gamma_{h_\Omega}(R) = \inf_{x \in \Omega} \mathrm{vol}_{h_\Omega}(B_{h_\Omega}(x,R)) \,,$$

where vol_{h_Ω} and $B_{h_\Omega}(x,R)$ denote, respectively, the Riemannian measure and the Riemannian ball with center x and radius R with respect to the affine metric h_Ω.

Example 5.4 *The triangle $\mathcal{T} \subset \mathbb{R}\mathrm{P}^2$, when equipped with its affine metric, is isometric to $\mathbb{R}^2$. Hence its affine growth profile is*

$$\gamma_{h_\mathcal{T}}(R) = \pi\,R^2 \,.$$

Lemma 5.5 *Let $\Omega \subset \mathbb{RP}^2$ be a properly convex domain.*

1. *Assume that Ω is Gromov hyperbolic. Then, there exists a positive constant $a > 0$ such that $\gamma_{h_\Omega}(R) \geq e^{aR}$ for $R \geq 1$.*

2. *Assume that Ω is not Gromov hyperbolic. Then, $\gamma_{h_\Omega}(R) = \pi R^2$ holds for $R > 0$.*

Proof of Lemma 5.5 1. Let $\Omega \subset \mathbb{RP}^2$ be δ-hyperbolic ($\delta > 0$). It follows from Theorem 2.2 and Proposition 4.8 that (Ω, h_Ω) is a complete simply-connected Riemannian surface with curvature $K_h \leq k_\delta < 0$ bounded above by a negative constant. The Bishop–Günther volume estimates (see for example [11, 3.98]) assert that, for any $x \in \Omega$ and $R > 0$, the volume $\mathrm{vol}_{h_\Omega}(B_{h_\Omega}(x, R))$ is larger than the volume of the ball of radius R in the 2-dimensional simply-connected model manifold with constant negative curvature k_δ. Hence the result.

2. Let $\Omega \subset \mathbb{RP}^2$ be any properly convex domain. Since the curvature of h_Ω is everywhere non positive (Proposition 3.3), the Bishop–Günther volume estimates read, for any $x \in \Omega$ and $R > 0$, as $\mathrm{vol}_{h_\Omega}(B_{h_\Omega}(x, R)) \geq \pi R^2$. Thus $\gamma_{h_\Omega}(R) \geq \pi R^2$.

Now assume that Ω is not Gromov hyperbolic. Corollary 4.5 provides a sequence $g_n \in \mathrm{SL}_3\mathbb{R}$ such that the sequence of images $g_n\Omega \in X$ converges to the projective triangle $\mathcal{T}$ in the Hausdorff topology. Fix a point $x_0 \in \mathcal{T}$, and let $x_n := g_n^{-1}(x_0) \in \Omega$, so that the sequence $g_n(x_n, \Omega)$ converges to $(x_0, \mathcal{T})$ in the Benzécri space $\mathcal{E}$. As recalled earlier ([2, Corollary 3.3], see Proposition 2.4), the affine metric h_Ω at the point $x \in \Omega$ depends continuously on $(x, \Omega) \in \mathcal{E}$. Hence, for any fixed $R > 0$, the sequence

$$\mathrm{vol}_{h_\Omega}(B_{h_\Omega}(x_n, R)) = \mathrm{vol}_{h_{g_n(\Omega)}}(B_{h_{g_n(\Omega)}}(x_0, R))$$

converges to $\mathrm{vol}_{h_\mathcal{T}}(B_{h_\mathcal{T}}(x_0, R)) = \pi R^2$. This yields the reverse inequality $\gamma_{h_\Omega}(R) \leq \pi R^2$. $\quad\square$

Proof of Theorem 5.2 As recalled earlier the affine metric and the Finsler metric on a properly convex domain $\Omega \subset \mathbb{RP}^2$ are uniformly equivalent. More precisely, there exists a uniform constant $c > 0$ such that, for any $\Omega \in X$

$$1/c\, h_\Omega \leq \|\ \|_{F,\Omega} \leq c\, h_\Omega \tag{4.1}$$

holds. Hence this assertion is an immediate consequence of Lemma 5.5. $\quad\square$

Remark There are several natural ways of defining a notion of volume on a Finsler manifold. However for two such choices – corresponding to the measures μ_1 and μ_2 on the Finsler manifold M – there will exist a constant $\alpha > 0$ (actually depending only on the dimension of M) such that

$$1/\alpha\, \mu_1 \leq \mu_2 \leq \alpha\, \mu_1$$

(see Burago–Burago–Ivanov [5, 5.5.3]). In particular, the ratio of two growth profiles on a properly convex domain $\Omega \subset \mathbb{RP}^2$, corresponding to two different choices of a volume on the Finsler manifold $(\Omega, \|\ \|_{F,\Omega})$, will be bounded. Thus the above criterion for Gromov hyperbolicity will hold true for any of these choices.

6 Parameterization of marked hyperbolic domains of $\mathbb{R}\mathrm{P}^2$

In this section, we exhibit a natural bijection between the moduli space of marked Gromov hyperbolic convex domains of $\mathbb{R}\mathrm{P}^2$ and the space of bounded holomorphic cubic differentials on the disk.

6.1 Uniformization of marked hyperbolic domains

Let $\Omega \subset \mathbb{R}\mathrm{P}^2$ be a Gromov hyperbolic properly convex domain. We have seen in Proposition 4.9 that the affine metric h_Ω is conformal to a (unique) complete hyperbolic metric $h_{\Omega,0}$. The Riemannian surface $(\Omega, h_{\Omega,0})$ is thus isometric to the hyperbolic disk $(\mathbb{D}, h_0)$.

Two isometries $(\Omega, h_{\Omega,0}) \xrightarrow{\sim} (\mathbb{D}, h_0)$ differ by the action of an isometry of $(\mathbb{D}, h_0)$. Recall that an isometry of $(\mathbb{D}, h_0)$ extends continuously to the closed disk $\overline{\mathbb{D}} \subset \mathbb{C}$, and is determined (whether orientation-preserving or not) by the distinct images in $\partial\mathbb{D} \subset \mathbb{C}$ of three distinct points of the boundary $\partial\mathbb{D}$. Working with marked convex domains (Definition 6.2) will allow us to get rid of the group $\mathrm{Isom}(\mathbb{D}, h_0)$.

Lemma 6.1 *Let $\Omega \subset \mathbb{R}\mathrm{P}^2$ be a Gromov hyperbolic properly convex domain. Any isometry $\varphi : (\Omega, h_{\Omega,0}) \to (\mathbb{D}, h_0)$ induces a bijection $\partial\varphi : \partial\Omega \to \partial\mathbb{D}$ between the boundaries $\partial\Omega \subset \mathbb{R}\mathrm{P}^2$ and $\partial\mathbb{D} \subset \mathbb{C}$ of these sets.*

Proof Since the domain Ω is assumed to be Gromov hyperbolic, we know that Ω is strictly convex (see Benoist [1]). Hence the geodesics for the Hilbert metric d_Ω are segments (de la Harpe [10, Proposition 2]), and the boundary $\partial\Omega$ identifies with the set of equivalence classes of geodesic rays in (Ω, d_Ω) – two geodesic rays being equivalent when within a finite distance of each other. Similarly, the boundary $\partial\mathbb{D}$ identifies with the set of equivalence classes of geodesic rays in $(\mathbb{D}, h_0)$.

Let $\varphi : (\Omega, h_{\Omega,0}) \to (\mathbb{D}, h_0)$ be an isometry. Then $\varphi : (\Omega, d_\Omega) \to (\mathbb{D}, h_0)$ is a bijective quasi-isometry between two Gromov hyperbolic geodesic metric spaces which are proper (namely, their closed balls are compact). The lemma now follows from Gromov–Ghys–de la Harpe [13, Chapitre 5 - Théorème 25]. $\square$

Definition 6.2 – A marked *properly convex domain in the projective plane* is a quadruple $(\Omega; x_1, x_2, x_3)$, *where $\Omega \subset \mathbb{R}\mathrm{P}^2$ is a properly convex domain and (x_1, x_2, x_3) are three distinct points on the boundary $\partial\Omega$.*

We systematically equip the marked domain $(\Omega; x_1, x_2, x_3)$ with the orientation for which the triple of points (x_1, x_2, x_3) is positively ordered on $\partial\Omega$, and let (Ω, J) denote the Riemann surface underlying h_Ω as in §3.2.
– We also mark the unit disk $\mathbb{D}$ by choosing eg the (positive) triple of points $(1, i, -1)$ in $\partial\mathbb{D}$.

Corollary 6.3 *Let $(\Omega; x_1, x_2, x_3)$ be a marked Gromov hyperbolic properly convex domain of $\mathbb{R}\mathrm{P}^2$. Then, there exists a unique uniformizing map $\varphi : (\Omega, J) \to \mathbb{D}$ which is marking preserving, namely such that $\partial\varphi(x_1) = 1$, $\partial\varphi(x_2) = i$ and $\partial\varphi(x_3) = -1$.*

Proof Immediate consequence of Lemma 6.1 and the previous discussion. $\square$

6.2 Construction of the bijection

We associate to any marked Gromov hyperbolic properly convex domain of $\mathbb{R}\mathrm{P}^2$ a bounded holomorphic cubic differential on the unit disk $\mathbb{D}$.

Let $(\Omega; x_1, x_2, x_3)$ be a marked properly convex domain of $\mathbb{R}\mathrm{P}^2$, where Ω is Gromov hyperbolic. Theorem 2.2 identifies Ω with the corresponding affine sphere M. Both the affine metric h and the Pick tensor C of M read on Ω to give the affine metric h_Ω and the Pick tensor C_Ω of Ω.

The Pick tensor C_Ω writes as $C_\Omega = \mathrm{Re}\, U_\Omega$, where the Pick form U_Ω is a holomorphic cubic differential on the Riemann surface (Ω, J) associated with $(\Omega; x_1, x_2, x_3)$ (Lemma 3.2, Definition 6.2). Let $\varphi : (\Omega, J) \to \mathbb{D}$ be the unique uniformizing map which is marking preserving (Corollary 6.3). The Pick form U_Ω reads on $\mathbb{D}$ through φ as a holomorphic cubic differential

$$\varphi_*(U_\Omega) = f(z)\, dz^3\,, \quad |z| < 1\,.$$

Lemma-Definition 6.4 *The holomorphic cubic differential $\varphi_*(U_\Omega) = f(z)\, dz^3$ on $\mathbb{D}$ is bounded. Namely, the function*

$$z \in \mathbb{D} \to |f(z)|\, (1 - |z|)^3 \in [0, \infty[$$

is bounded.

Proof Associate to the Pick form U_Ω the Pick measure $\mu_P := |U_\Omega|^{2/3}$ on Ω, which reads on $\mathbb{D}$ through the chart φ as $\varphi_*(\mu_P) = |f(z)|^{2/3}\, |dz|^2$.

As a consequence of Benzécri's compactness theorem 4.6, we proved in [2, Lemma 5.7] the existence of a universal constant $C > 0$ such that the ratio $\Lambda := \mu_P/\mu_{h_\Omega}$ of the Pick measure by the Riemannian measure of the affine metric on any properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^2$ is bounded above by C.

Since Ω is Gromov hyperbolic, the affine metric reads on $\mathbb{D}$ through φ as $\varphi_*(h_\Omega) = e^{2v} h_0$, where $h_0 = \frac{4|dz^2|}{(1-|z|^2)^2}$ is the hyperbolic metric on the disk and the conformal factor v is bounded (Theorem 4.9). The result follows, since Λ reads on $\mathbb{D}$ as $\Lambda = |f(z)|^{2/3}(1 - |z|^2)^2\, \frac{e^{-2v}}{4}$. $\qquad\square$

Notation 6.5 *We introduce*
– the set Y_G of marked Gromov hyperbolic properly convex domains of $\mathbb{R}\mathrm{P}^2$
– the set $\mathcal{C}_b$ of bounded holomorphic cubic differentials on the disk

Theorem 6.6 *The map*

$$T : (\Omega; x_1, x_2, x_3) \in Y_G \to \varphi_*(U_\Omega) \in \mathcal{C}_b\,,$$

where (Ω, J) is the Riemann surface associated to $(\Omega; x_1, x_2, x_3)$, U_Ω is the Pick form on (Ω, J) and $\varphi : (\Omega, J) \to \mathbb{D}$ is the unique marking preserving uniformizing map, goes to the quotient under the natural action of $\mathrm{Aut}\mathbb{R}\mathrm{P}^2 = \mathrm{SL}_3\mathbb{R}$ on Y_G and induces a bijection

$$\tau : Y_G/\mathrm{SL}_3\mathbb{R} \to \mathcal{C}_b$$

between the moduli space of marked Gromov hyperbolic properly convex domains of $\mathbb{R}\mathrm{P}^2$, and the space of bounded holomorphic cubic differentials on the disk.

Proof Since affine spheres with center the origin and affine curvature -1 as well as their Blaschke connection and affine metric are preserved by the action of $\mathrm{SL}_3\mathbb{R}$, it follows from the definition of the Pick tensor (Definition 3.1) that the map T goes to the quotient under the action of $\mathrm{SL}_3\mathbb{R}$ on Y_G. In other words, the map τ is well-defined. To complete the proof of Theorem 6.6, we provide in the next final paragraph an inverse map for τ. We proceed as in Loftin [19], or [2, Section 6].

6.3 Construction of the inverse map for τ

Let U be a holomorphic cubic differential on the disk and $h = e^{2v}\, h_0$ be a conformal metric, where $h_0 = \frac{4|dz^2|}{(1-|z|^2)^2}$ denotes the hyperbolic metric on $\mathbb{D}$. We first investigate the conditions under which there will exist an immersion $j : \mathbb{D} \hookrightarrow \mathbb{R}^3$ of the disk as an affine sphere $j(\mathbb{D})$ with center the origin and affine curvature -1, and such that
  – the affine metric on $j(\mathbb{D}) \simeq \mathbb{D}$ is the conformal metric h
  – the Pick form on the (naturally oriented) affine sphere $j(\mathbb{D}) \simeq \mathbb{D}$ is the cubic differential U

Proposition 6.7 *Let $h = e^{2v}\, h_0$ be a conformal metric and $U = f(z)\, dz^3$ be a holomorphic cubic differential on the disk.*

1. *The pair (h, U) consists of the affine metric and Pick form of an immersion $j : \mathbb{D} \hookrightarrow \mathbb{R}^3$ as an affine sphere with center the origin and affine curvature -1 if and only if the conformal factor $v : \mathbb{D} \to \mathbb{R}$ is solution of Wang's equation*

$$\Delta_0 v = -e^{-2v} + 1 + k e^{-4v}\,, \tag{6.1}$$

where Δ_0 is the Laplace operator on $\mathbb{D}$ for the hyperbolic metric h_0 and

$$k(z) = \frac{1}{32}\,(1 - |z|^2)^6\,|f(z)|^2 \quad (z \in \mathbb{D})$$

2. *When Equation (6.1) is satisfied such an immersion $j : \mathbb{D} \hookrightarrow \mathbb{R}^3$, as an affine sphere with affine metric h and Pick form U, is unique up to $\mathrm{SL}_3^{\pm}\mathbb{R}$*

Proof See Loftin [19] or [2, Corollary 6.3]. Recall our sign convention $\Delta_0 = -\mathrm{Tr}\, D^{h_0} \nabla_{h_0}$. $\square$

It results from the discussion of the previous paragraph that we will be only interested in bounded holomorphic cubic differentials on the disk. In this case, we have the following existence and uniqueness result for the solutions of Equation (6.1).

Proposition 6.8 *Assume that $U = f(z)\, dz^3$ is a bounded holomorphic cubic differential on the disk. Then Wang's equation (6.1) admits a unique bounded solution.*

Proof Since the cubic differential U is assumed to be bounded, the function k is also bounded on $\mathbb{D}$. Moreover k is obviously non negative. The result follows from [2, Proposition 6.5] (see also Loftin [19]), since $(\mathbb{D}, h_0)$ is a complete Riemannian manifold with constant curvature. $\square$

We may now wrap up the proof of Theorem 6.6.

Proof of theorem 6.6 (continued) Let $(\Omega; x_1, x_2, x_3)$ be a marked Gromov hyperbolic properly convex domain of $\mathbb{R}P^2$. As explained in the previous paragraphs 6.1 and 6.2, the affine metric of Ω reads on $\mathbb{D}$ through the unique marking preserving uniformizing map $\varphi : (\Omega, J) \to \mathbb{D}$ as $\varphi_*(h_\Omega) = e^{2v}h_0$, where h_0 is the hyperbolic metric on the disk and the conformal factor v is bounded. And the Pick form on (Ω, J) reads as a bounded holomorphic cubic differential $\varphi_*(U_\Omega)$ on the disk.

Conversely, let U be a bounded holomorphic cubic differential on the disk. Proposition 6.8 yields a unique bounded solution v of Equation (6.1), and Proposition 6.7 ensures that the pair (h, U), where $h = e^{2v}h_0$, consists of the affine metric and the Pick form of an immersion $j : \mathbb{D} \hookrightarrow \mathbb{R}^3$ of the disk as an affine sphere with center the origin and affine curvature -1.

The solution v is bounded. It follows first that the metric h is complete, hence that the affine sphere $j(\mathbb{D})$ is asymptotic to a cone above a properly convex domain $\Omega \subset \mathbb{R}P^2$ (Theorem 2.2). Second, the domain (Ω, h_Ω) equipped with its affine metric is isometric to $(\mathbb{D}, h)$, hence quasi-isometric to the hyperbolic disk $(\mathbb{D}, h_0)$ – which is Gromov hyperbolic. As mentionned earlier in in §4.1, Gromov hyperbolicity is invariant under quasi-isometries, hence (Ω, h_Ω) is also Gromov hyperbolic. This means that the domain Ω is Gromov hyperbolic (Corollary 4.7).

Let $p : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ denote the canonical projection. Since the map $p \circ j : (\mathbb{D}, h_0) \to (\Omega, d_\Omega)$ is a quasi-isometry, it follows from the proof of Lemma 6.1 that it induces a bijection $\partial(p \circ j)$ between the boundaries $\partial \mathbb{D}$ and $\partial \Omega$. We mark the domain Ω by choosing the triple (x_1, x_2, x_3), where $x_1 = \partial(p \circ j)(1)$, $x_2 = \partial(p \circ j)(i)$ and $x_3 = \partial(p \circ j)(-1)$.

Two immersions $j_1, j_2 : \mathbb{D} \hookrightarrow \mathbb{R}^3$ of the disk as an affine sphere, corresponding to the same pair (h, U), differ by an element of $\mathrm{SL}_3^{\pm}\mathbb{R}$ (Proposition 6.7 again). Thus the affine spheres $j_1(\mathbb{D})$ and $j_2(\mathbb{D})$ are asymptotic to cones over two properly convex domains $\Omega_1, \Omega_2 \subset \mathbb{R}P^2$ which differ by the action of an element of $\mathrm{SL}_3\mathbb{R}$, so that the above construction actually defines a map

$$\rho : \mathcal{C}_b \to Y_G/\mathrm{SL}_3\mathbb{R}.$$

It follows from the construction that ρ is an inverse map for τ, which concludes the proof. $\square$

Y. Benoist and D. Hulin
Université Paris-Sud
Laboratoire de Mathématiques – UMR 8628
Orsay 91405, France

yves.benoist@math.u-psud.fr
dominique.hulin@math.u-psud.fr

References

[1] Yves Benoist. Convexes hyperboliques et fonctions quasisymétriques. *Publ. Math. IHES*, (97):181–237, 2003.

[2] Yves Benoist and Dominique Hulin. Cubic differentials and finite volume convex projective surfaces. *Geom. Top.*, 2013. To appear.

[3] Jean-Paul Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[4] Francesco Bonsante and Jean-Marc Schlenker. Maximal surfaces and the universal Teichmüller space. *Invent. Math.*, 182:279–333, 2010.

[5] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*. GSM 33. 2001.

[6] Eugenio Calabi. Complete affine hyperspheres I. In *Symposia Mathematica, Vol. X*, pages 19–38. 1972.

[7] Shiu Yuen Cheng and Shing Tung Yau. On the regularity of the Monge-Ampère equation $\det(\partial^2 u/\partial x_i \partial s x_j) = F(x, u)$. *Comm. Pure Appl. Math.*, 30:41–68, 1977.

[8] Shiu Yuen Cheng and Shing Tung Yau. Complete affine hypersurfaces I. The completeness of affine metrics. *Comm. Pure Appl. Math.*, 39:839–866, 1986.

[9] B. Colbois, C. Vernicos, and P. Verovic. Area of ideal triangles and Gromov hyperbolicity in Hilbert geometry. *Illinois J. Math.*, 52:319–343, 2008.

[10] Pierre de la Harpe. On Hilbert's metric for simplices. In *Geometric group theory vol.1*, volume 181 of *London Math. Soc. Lecture Note Ser.*, pages 97–119. CUP, 1993.

[11] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer, 2004.

[12] Frederick P. Gardiner and William J. Harvey. Universal Teichmüller space. In *Handbook of complex analysis: geometric function theory, Vol. 1*, pages 457–492. North-Holland, 2002.

[13] Etienne Ghys and Pierre de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Birkhäuser, 1990.

[14] Lars Hörmander. *Notions of convexity*. Birkhäuser, 1994.

[15] Anders Karlsson and Guennadi A. Noskov. The Hilbert metric and Gromov hyperbolicity. *Enseign. Math.*, 48:73–89, 2002.

[16] François Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure Appl. Math. Q.*, 3:1057–1099, 2007.

[17] An Min Li. Calabi conjecture on hyperbolic affine hyperspheres II. *Math. Ann.*, 293:485–493, 1992.

[18] An Min Li and Gabi Penn. Uniqueness theorems in affine differential geometry II. *Results Math.*, 13:308–317, 1988.

[19] John Loftin. Applications of affine differential geometry to RP^2 surfaces. Ph.D. Thesis, http://andromeda.rutgers.edu/~loftin.

[20] John Loftin. Affine spheres and convex $\mathbb{R}P^n$ manifolds. *Amer. J. Math.*, 123:255–274, 2001.

[21] Martin A. Magid and Patrick J. Ryan. Flat affine spheres in $\mathbf{R}^3$. *Geom. Dedicata*, 33:277–288, 1990.

[22] Ludovic Marquis. Espaces des modules marqués des surfaces projectives proprement convexes de volume fini. *Geometry and Topology*, 14:2103–2149, 2010.

[23] Robert Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific J. Math.*, 7:1641–1647, 1957.

[24] Richard M. Schoen. The role of harmonic mappings in rigidity and deformation problems. L.N.M. 143, pages 179–200. 1993.

[25] Toshiyuki Sugawa. The universal Teichmüller space and related topics. In *Quasiconformal mappings and their applications*, pages 261–289. Narosa, 2007.

[26] Luen-Fai Tam and Tom Y. H. Wan. Quasi-conformal harmonic diffeomorphism and the universal Teichmüller space. *J. Differential Geom.*, 42:368–410, 1995.

[27] Constantin Vernicos. Introduction aux géométries de Hilbert. Sémin. Théor. Spectr. Géom. (23), pages 145–168. Univ. Grenoble I.

[28] Chang Ping Wang. Some examples of complete hyperbolic affine 2-spheres in R^3. LNM 1481, pages 271–280. Springer, 1991.

[29] Shing Tung Yau. Harmonic functions on complete Riemannian manifolds. *Comm. Pure Appl. Math.*, 28:201–228, 1975.

[30] Shing Tung Yau. A general Schwarz lemma for Kähler manifolds. *Amer. J. Math.*, 100:197–203, 1978.