

Harmonic quasi-isometric maps II : negatively curved manifolds

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Abstract

We prove that a quasi-isometric map, and more generally a coarse embedding, between pinched Hadamard manifolds is within bounded distance from a unique harmonic map.

1 Introduction

The aim of this article, which is a sequel to [4], is the following theorem.

Theorem 1.1. *Let $f : X \rightarrow Y$ be a quasi-isometric map between two pinched Hadamard manifolds. Then there exists a unique harmonic map $h : X \rightarrow Y$ which stays within bounded distance from f i.e. such that*

$$\sup_{x \in X} d(h(x), f(x)) < \infty .$$

We first recall a few definitions. A pinched Hadamard manifold X is a complete simply-connected Riemannian manifold of dimension at least 2 whose sectional curvature is pinched between two negative constants: $-b^2 \leq K_X \leq -a^2 < 0$. A map $f : X \rightarrow Y$ between two metric spaces X and Y is said to be *quasi-isometric* if there exist constants $c \geq 1$ and $C \geq 0$ such that f is (c, C) -*quasi-isometric*. This means that one has

$$c^{-1} d(x, x') - C \leq d(f(x), f(x')) \leq c d(x, x') + C \quad (1.1)$$

for all x, x' in X . A \mathcal{C}^2 map $h : X \rightarrow Y$ between two Riemannian manifolds X and Y is said to be *harmonic* if it satisfies the elliptic nonlinear partial differential equation $\text{tr}(D^2h) = 0$ where D^2h is the second covariant derivative of h .

Partial results towards the existence statement were obtained in [31], [41], [17], [27], [5]. A major breakthrough was achieved by Markovic who solved the Schoen conjecture, i.e. the case where $X = Y$ is the hyperbolic

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plane $\mathbb{H}_{\mathbb{R}}^2$, and by Lemm–Markovic who proved the existence for the case $X = Y = \mathbb{H}_{\mathbb{R}}^k$ in [29], [28] and [23]. The existence when both X and Y are rank one symmetric spaces, which was conjectured by Li and Wang in [25, Introduction], was proved in our paper [4]. We refer to [4, Section 1.2] for more motivations and a precise historical perspective on this result.

As explained in [11], the harmonic map h is not always a diffeomorphism even when f is a diffeomorphism.

Partial results towards the uniqueness statement were obtained by Li and Tam in [24], and by Li and Wang in [25]. All these papers were dealing with rank one symmetric spaces.

Note that Theorem 1.1 was conjectured by Markovic during a 2016 Summer School in Grenoble. According to our knowledge, Theorem 1.1 is new even in the case where both X and Y are assumed to be surfaces.

The strategy of the proof of the existence follows the lines of the proof in [4]. As in [4], we replace the quasi-isometric map f by a C^∞ -map whose first two covariant derivatives are bounded. But we need to modify the barycenter argument we used in [4] for this smoothing step. See Subsection 2.2.1 for more details on this step. As in [4], we then introduce the harmonic maps h_R that coincide with f on a sphere of X with large radius R and we need a uniform bound for the distances between the maps h_R and f . The heart of our argument is in Chapter 3 which contains the boundary estimates, and in Chapter 4 which contains the interior estimates, for $d(h_R, f)$. The proof of these interior estimates is based on a new simplification of an idea by Markovic in [28]. Indeed we will introduce a point x where $d(h_R(x), f(x))$ is maximal and focus on a subset U_{ℓ_0} of a sphere $S(x, \ell_0)$ whose definition (4.10) is much simpler than in [28] or [4]. This simplification is the key point which allows us to extend the arguments of [4] to pinched Hadamard manifolds. In this proof we use a uniform control on the harmonic measures on all the spheres of X , which is given in Proposition 4.9. We refer to Section 4.1 for more details on our strategy of proof of the existence.

In order to prove the uniqueness, we need to introduce Gromov-Hausdorff limits of the pointed metric spaces X and Y with respect to base points going to infinity and therefore to deal with C^2 -Riemannian manifolds with C^1 -metrics. This will be done in Chapter 5. We refer to Section 5.1 for more details on our strategy of proof of the uniqueness.

In Chapter 7, we extend Theorem 1.1 to coarse embeddings (see Definition 6.2 and Theorem 7.1). The proof is similar but relies on the existence of a boundary map for coarse embeddings. We also show that Theorem 1.1 can not be extended to Lipschitz maps (Example 7.3)

Chapter 6 is dedicated to the existence of this boundary map which, for a coarse embedding, is well-defined outside a set of zero Hausdorff dimension (Theorem 6.5). The existence of such a boundary map seems to be new.

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2 Smoothing

In this chapter, we recall a few basic facts on Hadamard manifolds, and we explain how to replace our quasi-isometric map f by a C^∞ map whose first two covariant derivatives are bounded.

2.1 The geometry of Hadamard manifolds

We first recall basic estimates on Hadamard manifolds for triangles, for images of triangles under quasi-isometric maps, and for the Hessian of the distance function.

All the Riemannian manifolds will be assumed to be connected. We will denote by d their distance function.

A Hadamard manifold is a complete simply connected Riemannian manifold X of dimension $k \geq 2$ whose curvature is non positive $K_X \leq 0$. For instance, the Euclidean space \mathbb{R}^k is a Hadamard manifold with zero curvature $K_X = 0$, and the real hyperbolic space $\mathbb{H}_{\mathbb{R}}^k$ is a Hadamard manifold with constant curvature $K_X = -1$. We will say that X is pinched if there exist constants $a, b > 0$ such that

$$-b^2 \leq K_X \leq -a^2 < 0.$$

For instance, the non-compact rank one symmetric spaces are pinched Hadamard manifolds.

Let x_0, x_1, x_2 be three points on a Hadamard manifold X . The *Gromov product* of the points x_1 and x_2 seen from x_0 is defined as

$$(x_1|x_2)_{x_0} := (d(x_0, x_1) + d(x_0, x_2) - d(x_1, x_2))/2. \quad (2.1)$$

We recall the basic comparison lemma which is one of the motivations for introducing the Gromov product.

Lemma 2.1. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq -a^2 < 0$. Let T be a geodesic triangle in X with vertices x_0, x_1, x_2 , and let θ_0 be the angle of T at the vertex x_0 .*

a) *One has $(x_0|x_2)_{x_1} \geq d(x_0, x_1) \sin^2(\theta_0/2)$.*

b) *One has $\theta_0 \leq 4e^{-a(x_1|x_2)_{x_0}}$.*

c) *Moreover, if $\min((x_0|x_1)_{x_2}, (x_0|x_2)_{x_1}) \geq b^{-1}$, one has $\theta_0 \geq e^{-b(x_1|x_2)_{x_0}}$.*

Proof. This is classical. See for instance [4, Lemma 2.1]. \square

We now recall the effect of a quasi-isometric map on the Gromov product.

Lemma 2.2. *Let X, Y be Hadamard manifolds with $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$, and let $f : X \rightarrow Y$ be a (c, C) -quasi-isometric map. There exists $A = A(a, b, c, C) > 0$ such that, for all x_0, x_1, x_2 in X , one has*

$$c^{-1}(x_1|x_2)_{x_0} - A \leq (f(x_1)|f(x_2))_{f(x_0)} \leq c(x_1|x_2)_{x_0} + A. \quad (2.2)$$

Proof. This is a general property of quasi-isometric maps between Gromov δ -hyperbolic spaces which is due to M. Burger. See [13, Prop. 5.15]. \square

When x_0 is a point in a Riemannian manifold X , we denote by d_{x_0} the distance function defined by $d_{x_0}(x) = d(x_0, x)$ for x in X . We denote by $d_{x_0}^2$ the square of this function. When $F : X \rightarrow \mathbb{R}$ is a C^2 function, we denote by DF its differential and by D^2F its second covariant derivative.

Lemma 2.3. *Let X be a Hadamard manifold and $x_0 \in X$.*

Assume that $-b^2 \leq K_X \leq -a^2 \leq 0$. The Hessian of the distance function d_{x_0} satisfies on $X \setminus \{x_0\}$

$$a \coth(a d_{x_0}) g_0 \leq D^2 d_{x_0} \leq b \coth(b d_{x_0}) g_0, \quad (2.3)$$

where $g_0 := g_X - Dd_{x_0} \otimes Dd_{x_0}$ and g_X is the Riemannian metric on X .

When $a = 0$ the left-hand side of (2.3) must be interpreted as $d_{x_0}^{-1} g_0$.

Proof. This is classical. See for instance [4, Lemma 2.3] \square

2.2 Smoothing rough Lipschitz maps

The following proposition will allow us to assume in Theorem 1.1 that the quasi-isometric map f we start with is C^∞ with bounded derivative and bounded second covariant derivative.

2.2.1 Rough Lipschitz maps

A map $f : X \rightarrow Y$ between two metric spaces X and Y is said to be *rough Lipschitz* if there exist constants $c \geq 1$ and $C \geq 0$ such that, for all x, x' in X , one has

$$d(f(x), f(x')) \leq c d(x, x') + C. \quad (2.4)$$

Proposition 2.4. *Let X, Y be two Hadamard manifolds with bounded curvatures $-b^2 \leq K_X \leq 0$ and $-b^2 \leq K_Y \leq 0$. Let $f : X \rightarrow Y$ be a rough Lipschitz map. Then there exists a C^∞ map $\tilde{f} : X \rightarrow Y$ within bounded distance from f and whose first two covariant derivatives $D\tilde{f}$ and $D^2\tilde{f}$ are bounded on X .*

We denote $k = \dim X$ and $k' = \dim Y$. We will first construct in 2.2.2 a regularized map $\tilde{f} : X \rightarrow Y$ which is Lipschitz continuous. This first construction is the same as for rank one symmetric spaces in [4, Proposition 2.4]. This construction will not allow us to control the second covariant derivative, hence we will have to combine this first construction with an iterative smoothing process in local charts that we will explain in 2.2.3.

2.2.2 Lipschitz continuity

The first part of the proof of Proposition 2.4 relies on the following classical lemma (see [20, Section 2]).

Lemma 2.5. *Let Y be a Hadamard manifold. a) Let μ be a positive finite Borel measure on Y supported by a ball $B(y_0, R)$. The function Q_μ on Y defined by*

$$Q_\mu(y) = \int_Y d(y, w)^2 d\mu(w)$$

has a unique minimum y_μ in Y called the center of mass of μ . This center of mass y_μ belongs to the ball $B(y_0, R)$.

b) Let μ_1, μ_2 be two positive finite Borel measures on Y . Assume that

- (i) $\mu_1(Y) \geq m$ and $\mu_2(Y) \geq m$ for some $m > 0$,*
- (ii) both μ_1 and μ_2 are supported on the same ball $B(y_0, R)$,*
- (iii) the norm of $\|\mu_1 - \mu_2\| \leq \varepsilon$.*

Then, the distance between their centers of mass y_{μ_1} and y_{μ_2} is bounded by

$$d(y_{\mu_1}, y_{\mu_2}) \leq 4\varepsilon R/m. \quad (2.5)$$

Proof of Lemma 2.5. a) Since the space Y is a proper space, i.e. its balls are compact, the function Q_μ is proper and admits a minimum y_μ . Since Y has non-positive curvature the median inequality holds, namely one has for all y, y_1, y_2, y_3 in Y where y_3 is the midpoint of y_1 and y_2 :

$$\frac{1}{2}d(y_1, y_2)^2 \leq d(y, y_1)^2 + d(y, y_2)^2 - 2d(y, y_3)^2. \quad (2.6)$$

Integrating (2.6) with respect to μ , one checks that the function Q_μ satisfies the following uniform convexity property : if y_3 is the midpoint of y_1 and y_2

$$\frac{m}{2}d(y_1, y_2)^2 \leq Q_\mu(y_1) + Q_\mu(y_2) - 2Q_\mu(y_3).$$

Applying this inequality with $y_1 = y_\mu$ and $y_2 = y$, one gets for each y in Y

$$\frac{m}{2}d(y_\mu, y)^2 \leq Q_\mu(y) - Q_\mu(y_\mu), \quad (2.7)$$

so that y_μ is the unique minimum of Q_μ .

We now check that $y_\mu \in B(y_0, R)$. By the median inequality (2.6), the ball $B(y_0, R)$ is convex, every point y in Y admits a unique nearest point y' on $B(y_0, R)$, and this point y' also satisfies the inequality

$$d(y', w) \leq d(y, w) \text{ for all } w \text{ in } B(y_0, R).$$

Therefore, one has $Q_\mu(y') \leq Q_\mu(y)$. This proves that the center of mass y_μ belongs to the ball $B(y_0, R)$.

b) Applying twice Inequality (2.7), one gets the two inequalities

$$\begin{aligned} \frac{m}{2} d(y_{\mu_1}, y_{\mu_2})^2 &\leq Q_{\mu_1}(y_{\mu_2}) - Q_{\mu_1}(y_{\mu_1}), \\ \frac{m}{2} d(y_{\mu_1}, y_{\mu_2})^2 &\leq Q_{\mu_2}(y_{\mu_1}) - Q_{\mu_2}(y_{\mu_2}). \end{aligned}$$

Summing these two inequalities yields

$$\begin{aligned} m d(y_{\mu_1}, y_{\mu_2})^2 &\leq (Q_{\mu_1} - Q_{\mu_2})(y_{\mu_2}) - (Q_{\mu_1} - Q_{\mu_2})(y_{\mu_1}) \\ &\leq \varepsilon \sup_{w \in B(y_0, R)} |d(y_{\mu_1}, w)^2 - d(y_{\mu_2}, w)^2| \\ &\leq 4\varepsilon R d(y_{\mu_1}, y_{\mu_2}), \end{aligned}$$

which proves (2.5). \square

We now choose a non-negative C^∞ function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ with support included in $] -1, 1[$, which is equal to 1 on a neighborhood of $[-\frac{1}{2}, \frac{1}{2}]$ and whose first derivative is bounded by $|\chi'| \leq 4$.

Proof of Proposition 2.4. First step: Lipschitz continuity. We now explain this first construction. We can assume $b = 1$. Since a rough Lipschitz map $f : X \rightarrow Y$ is always within bounded distance from a Borel measurable map, we can assume that f itself is Borel measurable. For x in X , we introduce the positive finite measure μ_x on Y such that the equality

$$\mu_x(\varphi) = \int_X \varphi(f(z)) \chi(d(x, z)) \, d\text{vol}_X(z)$$

holds for any positive function φ on Y . The measure μ_x is the image by f of a measure supported in the ball $B(x, 1)$. We define $\tilde{f}(x) \in Y$ to be the center of mass of this measure μ_x . Lemma 2.5.a tells us that the map $x \rightarrow \tilde{f}(x)$ is well-defined. The Lipschitz continuity of \tilde{f} will follow from Lemma 2.5.b applied to two measures $\mu_1 := \mu_{x_1}$ and $\mu_2 := \mu_{x_2}$ with x_1, x_2 in X . Let us check that the three assumptions in Lemma 2.5.b are satisfied.

(i) Because of the pinching of the curvature on X , the Bishop volume estimates tell us that there exist positive constants $0 < m_0 < M_0$ such that one has, for all x :

$$m_0 \leq \text{vol}(B(x, \frac{1}{2})) \leq \mu_x(Y) \leq \text{vol}(B(x, 1)) \leq M_0.$$

(ii) When x_1, x_2 are two points of X with $d(x_1, x_2) \leq 1$, the bound (2.4) ensures that both μ_{x_1} and μ_{x_2} are supported on the ball $B(f(x_1), 2c + C)$.

(iii) The norm of the difference of these measures satisfies

$$\begin{aligned} \|\mu_{x_1} - \mu_{x_2}\| &\leq M_0 \sup_{z \in X} |\chi(d(x_1, z)) - \chi(d(x_2, z))| \\ &\leq 4M_0 d(x_1, x_2). \end{aligned}$$

Thus Lemma 2.5 applies and yields a bound on the Lipschitz constant of \tilde{f} , namely

$$\text{Lip}(\tilde{f}) := \sup_{x_1 \neq x_2} \frac{d(\tilde{f}(x_1), \tilde{f}(x_2))}{d(x_1, x_2)} \leq \frac{16(2c + C)M_0}{m_0}. \quad \square$$

2.2.3 Bound on the second derivative

The second step of the proof of Proposition 2.4 relies on three lemmas. The first lemma provides a nice system of charts on Y .

Lemma 2.6. *Let Y be a Hadamard manifold with $-b^2 \leq K_Y \leq 0$ and $k' = \dim Y$. There exist constants $r_0 = r_0(k', b) > 0$ and $c_0 = c_0(k', b) > 1$ such that, for each y in Y , there exists a \mathcal{C}^∞ chart Φ_y for the open ball*

$$\Phi_y : \mathring{B}(y, r_0) \xrightarrow{\sim} U_y \subset \mathbb{R}^{k'} \quad \text{with} \quad \Phi_y(y) = 0 \quad (2.8)$$

and such that

$$\|D\Phi_y\| \leq c_0, \quad \|D\Phi_y^{-1}\| \leq c_0, \quad \|D^2\Phi_y\| \leq c_0, \quad \|D^2\Phi_y^{-1}\| \leq c_0. \quad (2.9)$$

In particular, one has for all $r < r_0$

$$\Phi_y(B(y, c_0^{-1}r)) \subset B(0, r) \quad \text{and} \quad B(0, c_0^{-1}r) \subset \Phi_y(B(y, r)). \quad (2.10)$$

We have endowed $\mathbb{R}^{k'}$ with the standard Euclidean structure.

Proof of Lemma 2.6. This is classical. One can for instance choose the so-called almost linear coordinates, as in [19, Section 2] or [32, Section 3]. They are defined in the following way. We fix an orthonormal basis $(e_i)_{1 \leq i \leq k'}$ for the tangent space $T_y Y$ and introduce the points $y_i := \exp_y(-e_i)$ in Y . The map Φ_y is defined by the formula

$$\Phi_y(z) = (d(z, y_1) - 1, \dots, d(z, y_{k'}) - 1),$$

where z belongs to a sufficiently small ball $\mathring{B}(y, r_0)$. See [19, p. 43 and 58] for a detailed proof. \square

There exist better systems of coordinates, the so-called harmonic coordinates. We will not need them in this chapter, but we will need them in Chapter 5 to prove uniqueness (see Lemma 5.2).

The second lemma explains how to modify a Lipschitz map g inside a tiny ball $B(x, r)$ of X so that the new map $g_{x,r}$ is constant on the ball $B(x, \frac{r}{2})$ and the first two derivatives of $g_{x,r}$ are controlled by those of g . We recall that $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative \mathcal{C}^∞ function with support included in $] -1, 1[$, which is equal to 1 on a neighborhood of $[-\frac{1}{2}, \frac{1}{2}]$ and which is 4-Lipschitz, i.e. $|\chi'| \leq 4$.

Lemma 2.7. *Let X and Y be two Hadamard manifolds with bounded curvatures $-b^2 \leq K_X \leq 0$, $-b^2 \leq K_Y \leq 0$. Let $r_0 > 0$ and $c_0 \geq 1$ be as in Lemma 2.6. Let $g : X \rightarrow Y$ be a Lipschitz map, x be a point in X , $y = g(x)$ and let $0 < r < r_0$. Assume that*

$$\text{Lip}(g) < \frac{r_0}{c_0^2 r}. \quad (2.11)$$

Then the following formulas define a Lipschitz map $g_{r,x} : X \rightarrow Y$

$$\begin{aligned} g_{r,x}(z) &= g(x) && \text{when } d(z,x) \leq \frac{r}{2}, \\ &= \Phi_y^{-1} \left((1 - \chi(\frac{d(z,x)}{r})) \Phi_y(g(z)) \right) && \text{when } \frac{r}{2} \leq d(z,x) \leq r, \\ &= g(z) && \text{when } d(z,x) \geq r. \end{aligned}$$

One has the inequality between the Lipschitz constants on the ball $B(x,r)$:

$$\text{Lip}_{B(x,r)}(g_{r,x}) \leq 5c_0^2 \text{Lip}_{B(x,r)}(g). \quad (2.12)$$

In particular, one has

$$\text{Lip}(g_{r,x}) \leq 5c_0^2 \text{Lip}(g). \quad (2.13)$$

Moreover, if g is \mathcal{C}^2 in a neighborhood of a point z in X , then $g_{r,x}$ is also \mathcal{C}^2 in this neighborhood and one has

$$\|D^2 g_{r,x}(z)\| \leq \left(\|D^2 g(z)\| + \text{Lip}_{B(x,r)}(g)^2 + 1 \right) M_r, \quad (2.14)$$

where the constant $M_r \geq 1$ depends only on r , b , k , k' and χ .

Proof of Lemma 2.7. Condition (2.11) ensures that, for any point z in the ball $B(x,r)$, the image $g(z)$ belongs to the ball $\mathring{B}(y, c_0^{-2}r_0)$. Therefore, by (2.10), the vector $\Phi_y(g(z))$ belongs to the ball $\mathring{B}(0, c_0^{-1}r_0) \subset \mathbb{R}^{k'}$. When we multiply this vector by the scalar $1 - \chi(\cdot)$, the new vector is still in the same ball. This is why, using again (2.10), the element $g_{r,x}(z)$ is well-defined and belongs to $B(y, r_0)$.

The upper bound (2.12) follows from the chain rule. Indeed, when z is a point in $B(x,r)$ where g is differentiable, the bound (2.9) yields

$$\begin{aligned} \|Dg_{r,x}(z)\| &\leq c_0 \left(\frac{4}{r} \|\Phi_y(g(z))\| + \|D(\Phi_y \circ g)(z)\| \right) \\ &\leq 5c_0 \text{Lip}_{B(x,r)}(\Phi_y \circ g) \leq 5c_0^2 \text{Lip}_{B(x,r)}(g). \end{aligned}$$

The upper bound (2.14) follows from similar and longer computations left to the reader, which also use the bounds (2.3) for $D^2 d_x$. \square

We will also need a third lemma. We recall that a subset X_0 of a metric space X is said to be r -separated if the distance between two distinct points of X_0 is at least r .

Lemma 2.8. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq 0$. Let $k = \dim X$ and $N_0 := 100^k$. There exists a radius $r_0 = r_0(k, b) > 0$ such that, for any $r < r_0$, every $\frac{r}{2}$ -separated subset X_0 of X can be decomposed as a union of at most N_0 subsets which are $2r$ -separated.*

Proof of Lemma 2.8. The bound on the curvature of X and the Bishop volume estimates ensure that we can choose $r_0 > 0$ such that

$$\text{vol}B(x, 4r) \leq N_0 \text{vol}B(x, \frac{r}{4}) \quad \text{for all } r < r_0 \text{ and } x \text{ in } X. \quad (2.15)$$

This r_0 works. Indeed, let X_1, X_2, \dots, X_{N_0} be a sequence of disjoint $2r$ -separated subsets of X_0 with X_1 maximal in X_0 , X_2 maximal in $X_0 \setminus X_1$, and so on. Every point x of X_0 must be in one of the X_i 's with $i \leq N_0$ because, if it is not the case, each X_i contains a point in $B(x, 2r)$, contradicting (2.15). \square

Proof of Proposition 2.4. Second step: bound on $D^2\tilde{f}$. According to the first step of this proof, we can now assume that the map $f : X \rightarrow Y$ is c -Lipschitz with $c \geq 1$.

We can choose a new radius $r_0 = r_0(k, k', b)$ that satisfies both conclusions of Lemma 2.8 for X and of Lemma 2.6 for Y . We will use freely the notations of these two lemmas. Now let

$$r_1 = \frac{r_0}{5^{N_0} c_0^{2N_0+2} c}$$

and pick a maximal $\frac{r_1}{4}$ -separated subset X_0 of X . Thanks to Lemma 2.8, we write this set X_0 as a union

$$X_0 = X_1 \cup \dots \cup X_{N_0}$$

of N_0 subsets X_i which are $2r_1$ -separated.

In order to construct \tilde{f} from f , we will use a finite iterative process based on Lemma 2.7. Starting with $f_0 = f$, we construct by induction a finite sequence of maps f_i for $i \leq N_0$ and we set $\tilde{f} := f_{N_0}$. Using the notations of Lemma 2.7, the map f_i is defined from f_{i-1} by letting

$$\begin{aligned} f_i(z) &= (f_{i-1})_{r_1, x}(z) && \text{when } d(z, x) \leq r_1 \text{ for some } x \text{ in } X_{i+1}, \\ &= f_{i-1}(z) && \text{otherwise} \end{aligned}$$

so that the Lipschitz constants of these maps satisfy

$$\text{Lip}(f_i) \leq 5c_0^2 \text{Lip}(f_{i-1}) \leq 5^i c_0^{2i} c. \quad (2.16)$$

Indeed, once f_i is known to be well defined and to satisfy (2.16), it also satisfies the bound (2.11) : $\text{Lip}(f_i) < \frac{r_0}{c_0^2 r_1}$. Therefore Lemma 2.7 ensures that f_{i+1} is well defined and, using (2.12), that f_{i+1} also satisfies (2.16) :

$$\text{Lip}(f_{i+1}) \leq 5c_0^2 \text{Lip}(f_i) \leq 5^{i+1} c_0^{2(i+1)} c.$$

Let $\Lambda := M_{r_1} + 25c_0^4 + 1$. By (2.14) and (2.16), one has for any $i \leq N_0$ and z in X :

$$\|D^2 f_i(z)\| + \text{Lip}(f_i)^2 + 1 \leq \Lambda (\|D^2 f_{i-1}(z)\| + \text{Lip}(f_{i-1})^2 + 1) . \quad (2.17)$$

Since X_0 is a maximal $\frac{r_1}{4}$ -separated subset of X , every z in X belongs to at least one ball $\mathring{B}(x, \frac{r_1}{2})$ where x is in one of the sets X_{i_0} . But then the function f_{i_0} is constant in a neighborhood of z . Therefore, using (2.16) and applying $(N_0 - i_0)$ times the bound (2.17) one deduces that \tilde{f} is a \mathcal{C}^2 -map that satisfies the uniform upper bound

$$\|D^2 \tilde{f}(z)\| \leq ((5^{i_0} c_0^{2i_0} c)^2 + 1) \Lambda^{N_0 - i_0} \leq \Lambda^{N_0} c^2. \quad \square$$

3 Harmonic maps

In this chapter we begin the proof of the existence part in Theorem 1.1. We first recall basic facts satisfied by harmonic maps. We explain why a standard compactness argument reduces this existence part to proving a uniform upper bound on the distance between f and the harmonic map h_R which is equal to f on the sphere $S(O, R)$. Then we provide this upper bound near this sphere $S(O, R)$.

3.1 Harmonic functions and the distance function

We recall basic facts on the Laplace operator on Hadamard manifolds.

The Laplace-Beltrami operator Δ on a Riemannian manifold X is defined as the trace of the Hessian. In local coordinates, the Laplacian of a function φ is

$$\Delta \varphi = \text{tr}(D^2 \varphi) = \frac{1}{v} \sum_{i,j} \frac{\partial}{\partial x_i} (v g_X^{ij} \frac{\partial}{\partial x_j} \varphi) \quad (3.1)$$

where $v = \sqrt{\det(g_X^{ij})}$ is the volume density. The function φ is said to be harmonic if $\Delta \varphi = 0$ and subharmonic if $\Delta \varphi \geq 0$.

We will need the following basic lemma.

Lemma 3.1. *Let X be a Hadamard manifold with $K_X \leq -a^2 \leq 0$ and x_0 be a point in X . Then, the function d_{x_0} is subharmonic. More precisely, the distribution $\Delta d_{x_0} - a$ is non-negative.*

Proof. This is [4, Lemma 2.5]. □

3.2 Harmonic maps and the distance function

In this section, we recall two useful facts satisfied by a harmonic map h : the subharmonicity of the functions $d_{y_0} \circ h$, and Cheng's estimate for the differential Dh .

Definition 3.2. *Let $h : X \rightarrow Y$ be a C^2 map between two Riemannian manifolds. The tension field of h is the trace of the second covariant derivative $\tau(h) := \text{tr}D^2h$. The map h is said to be harmonic if $\tau(h) = 0$.*

Note that the tension field $\tau(h)$ is a Y -valued vector field on X , i.e. it is a section of the pulled-back of the tangent bundle $TY \rightarrow Y$ under the map $h : X \rightarrow Y$.

For instance, an isometric immersion with minimal image is always harmonic. The problem of the existence, regularity and uniqueness of harmonic maps under various boundary conditions is a very classical topic (see [10], [35], [19], [9], [40], [38] or [26]). In particular, when Y is simply connected and has non positive curvature, a harmonic map is always C^∞ i.e. it is indefinitely differentiable, and is a minimum of the energy functional among maps that agree with h outside a compact subset of X .

Lemma 3.3. *Let $h : X \rightarrow Y$ be a harmonic C^∞ map between Riemannian manifolds. Let $y_0 \in Y$ and let $\rho_h : X \rightarrow \mathbb{R}$ be the function $\rho_h := d_{y_0} \circ h$. If Y is Hadamard, the continuous function ρ_h is subharmonic on X .*

Proof. See [4, Lemma 3.2]. □

Another crucial property of harmonic maps is the following bound for their differential due to Cheng.

Lemma 3.4. *Let X, Y be two Hadamard manifolds with $-b^2 \leq K_X \leq 0$. Let $k = \dim X$, z be a point in X , $r > 0$ and let $h : B(z, r) \rightarrow Y$ be a harmonic C^∞ map such that the image $h(B(z, r))$ lies in a ball of radius R_0 . Then one has the bound*

$$\|Dh(z)\| \leq 2^5 k \frac{1+br}{r} R_0 .$$

In the applications, we will use this inequality with $r = b^{-1}$.

Proof. This is a simplified version of [8, Formula 2.9]. □

3.3 Existence of harmonic maps

In this section we prove Theorem 1.1, taking for granted Proposition 3.5 below.

Let X and Y be two Hadamard manifolds whose curvatures are pinched $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$. Let $f : X \rightarrow Y$ be a (c, C) -quasi-isometric \mathcal{C}^∞ map whose first two covariant derivatives are bounded.

We fix a point O in X . For $R > 0$, we denote by $B_R := B(O, R)$ the closed ball in X with center O and radius R and by ∂B_R the sphere that bounds B_R . Since the manifold Y is a Hadamard manifold, according to Hamilton in [16] (see also Schoen and Uhlenbeck in [36] and [37]), there exists a unique harmonic map $h_R : B_R \rightarrow Y$, which is \mathcal{C}^∞ on the closed ball B_R and satisfies the Dirichlet condition $h_R = f$ on the sphere ∂B_R . We denote by

$$d(h_R, f) = \sup_{x \in B(O, R)} d(h_R(x), f(x))$$

the distance between these two maps. The main step for proving existence in Theorem 1.1 is the following uniform estimate.

Proposition 3.5. *There exists a constant $\rho \geq 1$ such that, for any $R \geq 1$, one has $d(h_R, f) \leq \rho$.*

The constant ρ is a function of a, b, c, C, k and k' . More precisely, when f satisfies (4.1), ρ only needs to satisfy Conditions (4.6), (4.7) and (4.8).

We briefly recall the classical argument used to deduce Theorem 1.1 from this Proposition.

Proof of Theorem 1.1. As explained in Proposition 2.4, we may assume that the (c, C) -quasi-isometric map f is \mathcal{C}^∞ with bounded first two covariant derivatives. Pick an unbounded increasing sequence of radii R_n and let $h_{R_n} : B_{R_n} \rightarrow Y$ be the harmonic \mathcal{C}^∞ map that agrees with f on the sphere ∂B_{R_n} . Proposition 3.5 ensures that the sequence of maps (h_{R_n}) is locally uniformly bounded. Using the Cheng Lemma 3.4 it follows that the first derivatives are also locally uniformly bounded. The Ascoli-Arzelà theorem implies that, after extracting a subsequence, the sequence (h_{R_n}) converges uniformly on every ball B_S towards a continuous map $h : X \rightarrow Y$. Using the Schauder's estimates, one also gets a uniform bound for the $\mathcal{C}^{2, \alpha}$ -norms of h_{R_n} on B_S . If needed, note that these classical estimates will be recalled in Formulas (5.29), (5.32) and (5.33) in Section 5.6. Therefore, using the Ascoli-Arzelà theorem again, the sequence (h_{R_n}) converges in the \mathcal{C}^2 -norm and the limit map h is \mathcal{C}^2 and harmonic. By construction, this limit harmonic map h stays within bounded distance from the quasi-isometric map f . \square

Remark 3.6. By the uniqueness part of our Theorem 1.1 that we will prove in Chapter 5, the harmonic map h which stays within bounded distance from f is unique. Hence the above argument also proves that the whole family of harmonic maps h_R converges to h uniformly on the compact subsets of X when R goes to infinity.

3.4 Boundary estimate

In this section, we begin the proof of Proposition 3.5 : we bound the distance between h_R and f near the sphere ∂B_R .

Proposition 3.7. *Let X, Y be Hadamard manifolds and $k = \dim X$. Assume moreover that $K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq 0$. Let $c \geq 1$ and $f : X \rightarrow Y$ be a C^∞ map with $\|Df(x)\| \leq c$ and $\|D^2f(x)\| \leq bc^2$. Let $O \in X$, $R > 0$ and set $B_R := B(O, R)$.*

Let $h_R : B_R \rightarrow Y$ be the harmonic C^∞ map whose restriction to the sphere ∂B_R is equal to f . Then, one has for every x in B_R :

$$d(h_R(x), f(x)) \leq \frac{3kbc^2}{a} d(x, \partial B_R). \quad (3.2)$$

An important feature of this upper bound is that it does not depend on the radius R , provided the distance $d(x, \partial B_R)$ remains bounded. This is why we call (3.2) the *boundary estimate*. The proof relies on an idea of Jost in [19, Section 4].

Proof. This proposition is already in [4, Proposition 3.8]. We give here a slightly shorter proof. Let x be a point in B_R and y be a point in Y chosen so that $d(y, f(B_R)) \geq b^{-1}$ and

$$d_y(h_R(x)) - d_y(f(x)) = d(f(x), h_R(x)). \quad (3.3)$$

This point y is far away on the geodesic ray starting at $h_R(x)$ and containing $f(x)$. Let φ be the C^∞ function on the ball B_R defined by

$$\varphi(z) := d_y(h_R(z)) - d_y(f(z)) - \frac{3kbc^2}{a}(R - d_O(z)) \quad \text{for all } z \text{ in } B_R. \quad (3.4)$$

This function is the sum of three functions $\varphi = \varphi_1 + \varphi_2 + \varphi_3$.

The first function $\varphi_1 : z \mapsto d_y(h_R(z))$ is subharmonic on B_R i.e. one has $\Delta\varphi_1 \geq 0$. This follows from Lemma 3.3 and the harmonicity of the map h_R .

The second function $\varphi_2 : z \mapsto -d_y(f(z))$ has a bounded Laplacian, namely $|\Delta\varphi_2| \leq 3kbc^2$. Indeed, since y is far away, Formula (2.3) yields the bound $\|D^2d_y\| \leq 2b$ on $f(B_R)$ so that

$$|\Delta\varphi_2| = |\Delta(d_y \circ f)| \leq k\|D^2d_y\|\|Df\|^2 + k\|Dd_y\|\|D^2f\| \leq 3kbc^2.$$

The third function $\varphi_3 : z \mapsto -\frac{3kbc^2}{a}(R - d_O(z))$ has a Laplacian bounded below $\Delta\varphi_3 \geq 3kbc^2$. This follows from Lemma 3.1 which says that $\Delta d_O \geq a$.

Hence the function φ is subharmonic : $\Delta\varphi \geq 0$. Since φ is zero on ∂B_R , one gets $\varphi(x) \leq 0$ as required. \square

4 Interior estimate

In this chapter we complete the proof of Proposition 3.5.

4.1 Strategy

We first explain more precisely the notations and the assumptions that we will use in the whole chapter.

Let X and Y be Hadamard manifolds whose curvatures are pinched $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$. We start with a C^∞ quasi-isometric map $f : X \rightarrow Y$ whose first and second covariant derivatives are bounded. We fix constants $c \geq 1$ and $C > 0$ such that one has, for all x, x' in X :

$$\|Df(x)\| \leq c, \quad \|D^2f(x)\| \leq bc^2 \quad \text{and} \quad (4.1)$$

$$c^{-1}d(x, x') - C \leq d(f(x), f(x')) \leq cd(x, x'). \quad (4.2)$$

Note that the additive constant C on the right-hand side term of (1.1) has been removed since the derivative of f is now bounded by c .

4.1.1 Choosing the radius ℓ_0

We fix a point O in X . We introduce a fixed radius ℓ_0 depending only on a, b, k, k', c and C . This radius ℓ_0 is only required to satisfy the following three inequalities (4.3), (4.4) and (4.5) that will be needed later on.

The first condition we impose on the radius ℓ_0 is

$$b\ell_0 > 1. \quad (4.3)$$

The second condition we impose on the radius ℓ_0 is

$$\ell_0 > \frac{(A + b^{-1})c}{\sin^2(\varepsilon_0/2)} \quad \text{where} \quad \varepsilon_0 := (3c^2M)^{-N}, \quad (4.4)$$

where A is the constant given by Lemma 2.2, and M, N are the constants given by Proposition 4.9.

The third condition we impose on the radius ℓ_0 is

$$16e^{\frac{aC}{2}} e^{-\frac{a\ell_0}{4c}} < \theta_0 \quad \text{where} \quad \theta_0 := e^{-bA} (\varepsilon_0/4)^{\frac{bc}{a}}. \quad (4.5)$$

4.1.2 Assuming ρ to be large

We want to prove Proposition 3.5. For $R > 0$, recall that $h_R : B(O, R) \rightarrow Y$ is the harmonic C^∞ map whose restriction to the sphere $\partial B(O, R)$ is equal to f . We let

$$\rho := \sup_{x \in B(O, R)} d(h_R(x), f(x)).$$

We argue by contradiction. If this supremum ρ is not uniformly bounded with respect to R , we can fix a radius R such that ρ satisfies the following three inequalities (4.6), (4.7) and (4.8) that we will use later on.

The first condition we impose on the radius ρ is

$$a\rho > 8kbc^2\ell_0. \quad (4.6)$$

The second condition we impose on the radius ρ is

$$\frac{2^7(a\rho)^2}{\sinh(a\rho/2)} < \theta_0. \quad (4.7)$$

The third condition we impose on the radius ρ is

$$\rho > 4c\ell_0M(2^{10}e^{b\ell_0}k)^N \quad (4.8)$$

where M, N are the constants given by Proposition 4.9.

We denote by x a point of $B(O, R)$ where the supremum (4.1.2) is achieved:

$$d(h_R(x), f(x)) = \rho.$$

According to the boundary estimate in Proposition 3.7, Condition (4.6) yields

$$d(x, \partial B(O, R)) \geq \frac{a\rho}{3kbc^2} \geq 2\ell_0.$$

Combined with Condition (4.3), this ensures that the ball $B(x, \ell_0)$ with center x and radius ℓ_0 satisfies the inclusion $B(x, \ell_0) \subset B(O, R - b^{-1})$. This inclusion will allow us to apply the Cheng's lemma 3.4 at each point z of the ball $B(x, \ell_0)$.

4.1.3 Getting a contradiction

We will focus on the restrictions of both maps f and h_R to this ball $B(x, \ell_0)$. We introduce the point $y := f(x)$. For y_1, y_2 in $Y \setminus \{y\}$, we denote by $\theta_y(y_1, y_2)$ the angle at y of the geodesic triangle with vertices y, y_1, y_2 . For z on the sphere $S(x, \ell_0)$, we will analyze the triangle inequality :

$$\theta_y(f(z), h_R(x)) \leq \theta_y(f(z), h_R(z)) + \theta_y(h_R(z), h_R(x)) \quad (4.9)$$

and prove that on a subset U_{ℓ_0} of the sphere, each term on the right-hand side is small (Lemmas 4.5 and 4.6) while the measure of U_{ℓ_0} is large enough (Lemma 4.4) to ensure that the left-hand side is not that small (Lemma 4.8), giving rise to the contradiction. These arguments rely on uniform lower and upper bounds for the harmonic measures on the spheres of X that will be given in Proposition 4.9.

We denote by ρ_h the function on $B(x, \ell_0)$ given by $\rho_h(z) = d(y, h_R(z))$ where again $y = f(x)$. By Lemma 3.3, this function is subharmonic.

Definition 4.1. *The subset U_{ℓ_0} of the sphere $S(x, \ell_0)$ is given by*

$$U_{\ell_0} = \left\{ z \in S(x, \ell_0) \mid \rho_h(z) \geq \rho - \frac{\ell_0}{2c} \right\}. \quad (4.10)$$

4.2 Measure estimate

We first observe that one can control the size of $\rho_h(z)$ and of $Dh_R(z)$ on the ball $B(x, \ell_0)$. We then derive a lower bound for the measure of U_{ℓ_0} .

Lemma 4.2. *For z in $B(x, \ell_0)$, one has*

$$\rho_h(z) \leq \rho + c\ell_0.$$

Proof. The triangle inequality and (4.2) give, for any z in $B(x, \ell_0)$:

$$\rho_h(z) \leq d(h_R(z), f(z)) + d(f(z), y) \leq \rho + c\ell_0. \quad \square$$

Lemma 4.3. *For z in $B(x, \ell_0)$, one has*

$$\|Dh_R(z)\| \leq 2^8 kb\rho.$$

Proof. For all z, z' in $B(O, R)$ with $d(z, z') \leq b^{-1}$, the triangle inequality and (4.2) yield

$$\begin{aligned} d(h_R(z), h_R(z')) &\leq d(h_R(z), f(z)) + d(f(z), f(z')) + d(f(z'), h_R(z')) \\ &\leq \rho + b^{-1}c + \rho \leq 2\rho + c\ell_0 \leq 3\rho. \end{aligned}$$

For these last two inequalities, we used Conditions (4.3) and (4.6). Applying the Cheng's lemma 3.4 with $R_0 = 3\rho$ and $r = b^{-1}$, one then gets for all z in $B(O, R - b^{-1})$ the bound $\|Dh_R(z)\| \leq 2^8 kb\rho$. \square

We now give a lower bound for the measure of U_{ℓ_0} .

Lemma 4.4. *Let $\sigma = \sigma_{x, \ell_0}$ be the harmonic measure on the sphere $S(x, \ell_0)$ at the center point x . Then one has*

$$\sigma(U_{\ell_0}) \geq \frac{1}{3c^2}. \quad (4.11)$$

Proof. By Lemma 3.3, the function ρ_h is subharmonic on the ball $B(x, \ell_0)$. Hence this function ρ_h is not larger than the harmonic function on the ball with same boundary values on the sphere $S(x, \ell_0)$. Comparing these functions at the center x , one gets

$$\int_{S(x, \ell_0)} (\rho_h(z) - \rho) d\sigma(z) \geq 0. \quad (4.12)$$

By Lemma 4.2, the function ρ_h is bounded by $\rho + c\ell_0$. Hence Equation (4.12) and the definition of U_{ℓ_0} implies

$$c\ell_0 \sigma(U_{\ell_0}) - \frac{\ell_0}{2c} (1 - \sigma(U_{\ell_0})) \geq 0$$

so that $\sigma(U_{\ell_0}) \geq \frac{1}{3c^2}$. \square

4.3 Upper bound for $\theta_y(f(z), h_R(z))$

For all z in U_{ℓ_0} , we give an upper bound for the angle between $f(z)$ and $h_R(z)$ seen from the point $y = f(x)$.

Lemma 4.5. *For z in U_{ℓ_0} , one has*

$$\theta_y(f(z), h_R(z)) \leq 4e^{\frac{aC}{2}} e^{-\frac{a\ell_0}{4c}}. \quad (4.13)$$

Proof. For z in U_{ℓ_0} , we consider the triangle with vertices y , $f(z)$ and $h_R(z)$. Its side lengths satisfy

$$d(h_R(z), f(z)) \leq \rho, \quad d(y, f(z)) \geq \frac{\ell_0}{c} - C, \quad d(y, h_R(z)) \geq \rho - \frac{\ell_0}{2c},$$

where we used successively the definition of ρ , the quasi-isometry lower bound (4.2) and the definition of U_{ℓ_0} . Hence, one gets the following lower bound for the Gromov product

$$(f(z)|h_R(z))_y \geq \frac{\ell_0}{4c} - \frac{C}{2}.$$

Since $K_Y \leq -a^2$, Lemma 2.1 now yields

$$\theta_y(f(z), h_R(z)) \leq 4e^{\frac{aC}{2}} e^{-\frac{a\ell_0}{4c}}. \quad \square$$

4.4 Upper bound for $\theta_y(h_R(z), h_R(x))$

For all z in $S(x, \ell_0)$, we give an upper bound for the angle between $h_R(z)$ and $h_R(x)$ seen from the point $y = f(x)$.

Lemma 4.6. *For all z in the sphere $S(x, \ell_0)$, one has*

$$\theta_y(h_R(z), h_R(x)) \leq \frac{2^5 (a\rho)^2}{\sinh(a\rho/2)}. \quad (4.14)$$

The proof will rely on the following lemma which also ensures that this angle $\theta_y(h_R(z), h_R(x))$ is well defined.

Lemma 4.7. *For all z in the ball $B(x, \ell_0)$, one has $\rho_h(z) \geq \rho/2$.*

Proof of Lemma 4.7. Assume by contradiction that there exists a point z_1 in the ball $B(x, \ell_0)$ such that $\rho_h(z_1) = \rho/2$. Set $r_1 := d(x, z_1)$. One has $0 < r_1 \leq \ell_0$. According to Lemma 4.3, one can bound the differential of h_R on the ball $B(x, \ell_0)$, namely

$$\sup_{B(x, \ell_0)} \|Dh_R\| \leq 2^8 kb\rho.$$

Hence one has

$$\rho_h(z) \leq \frac{3\rho}{4} \text{ for all } z \text{ in } S(x, r_1) \cap B(z_1, \frac{1}{2^{10}kb}).$$

By comparison with the hyperbolic plane with curvature $-b^2$, this intersection contains the trace on the sphere $S(x, r_1)$ of a cone C_α with vertex x and angle α as soon as $\sin \frac{\alpha}{2} \leq \frac{\sinh(2^{-11}/k)}{\sinh(br_1)}$. For instance we will choose for α the angle $\alpha := e^{-b\ell_0} 2^{-10}/k$.

Let $\sigma' = \sigma_{x, r_1}$ be the harmonic measure on the sphere $S(x, r_1)$ for the center point x . Using the subharmonicity of the function ρ_h as in the proof of Lemma 3.3, one gets the inequality

$$\int_{S(x, r_1)} (\rho_h(z) - \rho) d\sigma'(z) \geq 0. \quad (4.15)$$

By Lemma 4.2, the function ρ_h is bounded by $\rho + c\ell_0$. Using the bound $\rho_h(z) \leq \frac{3}{4}\rho$ when z is in the cone C_α , Equation (4.15) now implies that

$$c\ell_0 - \frac{\rho}{4} \sigma'(C_\alpha) \geq 0.$$

Using the uniform lower bounds for the harmonic measures on the spheres of X in Proposition 4.9, one gets

$$\rho \leq 4c\ell_0 M \alpha^{-N} = 4c\ell_0 M (2^{10} e^{b\ell_0} k)^N,$$

which contradicts Condition (4.8). \square

Proof of Lemma 4.6. Let us first sketch the proof. Let z be a point on the sphere $S(x, \ell_0)$. We denote by $t \mapsto z_t$, for $0 \leq t \leq \ell_0$, the geodesic segment between x and z . By Lemma 4.7, the curve $t \mapsto h_R(z_t)$ lies outside of the ball $B(y, \rho/2)$ and by Cheng's bound on $\|Dh_R(z_t)\|$ one controls the length of this curve.

We now detail the argument. We denote by $(\rho(y'), v(y')) \in]0, \infty[\times T_y^1 Y$ the polar exponential coordinates centered at y . For a point y' in $Y \setminus \{y\}$, they are defined by the equality $y' = \exp_y(\rho(y')v_\rho(y'))$. Since $K_Y \leq -a^2$ the Alexandrov comparison theorem for infinitesimal triangles and the Gauss lemma ([12, 2.93]) yield

$$\sinh(a\rho(y')) \|Dv(y')\| \leq a.$$

Writing $v_h := v \circ h_R$, we thus have for any z' in $B(x, \ell_0)$:

$$\sinh(a\rho_h(z')) \|Dv_h(z')\| \leq a \|Dh_R(z')\|.$$

Hence, Lemma 4.7 yields the inequality

$$\begin{aligned} \theta_y(h_R(z), h_R(x)) &\leq \ell_0 \sup_{0 \leq t \leq \ell_0} \|Dv_h(z_t)\| \\ &\leq \frac{a\ell_0}{\sinh(a\rho/2)} \sup_{0 \leq t \leq \ell_0} \|Dh_R(z_t)\|. \end{aligned}$$

Therefore, using Lemma 4.3 and Condition (4.6), one gets

$$\theta_y(h_R(z), h_R(x)) \leq \frac{2^8 k b \rho a \ell_0}{\sinh(a\rho/2)} \leq \frac{2^5 (a\rho)^2}{\sinh(a\rho/2)}. \quad \square$$

4.5 Lower bound for $\theta_y(f(z), h_R(x))$

We find a point z in U_{ℓ_0} for which the angle between $f(z)$ and $h(x)$ seen from $y = f(x)$ has an explicit lower bound.

Lemma 4.8. *There exist two points z_1, z_2 in U_{ℓ_0} such that*

$$\theta_y(f(z_1), f(z_2)) \geq \theta_0,$$

where θ_0 is the angle given by (4.5).

Proof of Lemma 4.8. Let $\sigma_0 := \frac{1}{3c^2}$. According to Lemma 4.4, one has $\sigma(U_{\ell_0}) \geq \sigma_0 > 0$. Thus, using the uniform upper bounds for the harmonic measures on the spheres of X in Proposition 4.9, one can find z_1, z_2 in U_{ℓ_0} such that the angle $\theta_x(z_1, z_2)$ between z_1 and z_2 seen from x satisfies

$$\sigma_0 \leq M \theta_x(z_1, z_2)^{\frac{1}{N}}.$$

This can be rewritten as

$$\theta_x(z_1, z_2) \geq \varepsilon_0, \quad (4.16)$$

where ε_0 is the angle introduced in (4.4) by the equality $\sigma_0 = M \varepsilon_0^{\frac{1}{N}}$. Therefore, using Lemma 2.1.a and Condition (4.4), we get the following lower bound on the Gromov products

$$\min((x|z_1)_{z_2}, (x|z_2)_{z_1}) \geq \ell_0 \sin^2(\varepsilon_0/2) \geq (A + b^{-1})c.$$

Using then Lemma 2.2, one gets

$$\min((y|f(z_1))_{f(z_2)}, (y|f(z_2))_{f(z_1)}) \geq b^{-1}. \quad (4.17)$$

This inequality (4.17) allows us to apply Lemma 2.1.c, which gives

$$\theta_y(f(z_1), f(z_2)) \geq e^{-b(f(z_1)|f(z_2))_y}.$$

Therefore, by Lemma 2.2, one has

$$\theta_y(f(z_1), f(z_2)) \geq e^{-bA} e^{-bc(z_1|z_2)_x}.$$

Using Lemma 2.1.b and Condition (4.16), one gets

$$\begin{aligned} \theta_y(f(z_1), f(z_2)) &\geq e^{-bA} (\theta_x(z_1, z_2)/4)^{\frac{bc}{a}} \\ &\geq e^{-bA} (\varepsilon_0/4)^{\frac{bc}{a}} = \theta_0, \end{aligned}$$

according to the definition (4.5) of θ_0 . □

End of the proof of Proposition 3.5. Using Lemmas 4.5 and 4.6 and the triangle inequality (4.9), one gets for any two points $z_i = z_1$ or z_2 in U_{ℓ_0} :

$$\begin{aligned}\theta_y(f(z_i), h_R(x)) &\leq 4e^{\frac{aC}{2}} e^{-\frac{a\ell_0}{4c}} + \frac{2^5(a\rho)^2}{\sinh(a\rho/2)} \\ &< \frac{1}{2}\theta_0 \quad \text{by Conditions (4.5) and (4.7)}.\end{aligned}$$

Therefore, using again a triangle inequality, one has

$$\theta_y(f(z_1), f(z_2)) < \theta_0,$$

which contradicts Lemma 4.8. \square

4.6 Harmonic measures

The following proposition gives the uniform lower and upper bounds for the harmonic measure on a sphere for the center which were used in the proof of Lemmas 4.7 and 4.8.

Proposition 4.9. *Let $0 < a < b$ and $k \geq 2$ be an integer. There exist positive constants M, N depending only on a, b, k such that for every k -dimensional Hadamard manifold X with pinched curvature $-b^2 \leq K_X \leq -a^2$, for every point x in X , every radius $r > 0$ and every angle $\theta \in [0, \pi]$ one has*

$$\frac{1}{M} \theta^N \leq \sigma_{x,r}(C_{x,\theta}) \leq M \theta^{\frac{1}{N}} \quad (4.18)$$

where $\sigma_{x,r}$ denotes the harmonic measure on the sphere $S(x, r)$ at the point x and where $C_{x,\theta}$ stands for any cone with vertex x and angle θ .

We recall that, by definition, $\sigma_{x,r}$ is the unique probability measure on the sphere $S(x, r)$ such that, for every continuous function h on the ball $B(x, r)$ which is harmonic in the interior $\mathring{B}(x, r)$, one has the equality

$$h(x) = \int_{S(x,r)} h(z) d\sigma_{x,r}(z).$$

A proof of Proposition 4.9 is given in [3]. It relies on various technical tools of the potential theory on pinched Hadamard manifolds : the Harnack inequality, the barrier functions constructed by Anderson and Schoen in [2] and upper and lower bounds for the Green functions due to Ancona in [1]. Related estimates are available like the one by Kifer–Ledrappier in [21, Theorem 3.1 and 4.1] where (4.18) is proven for the sphere at infinity or by Ledrappier–Lim in [22, Proposition 3.9] where the Hölder regularity of the Martin kernel is proven.

5 Uniqueness of harmonic maps

In this chapter we prove the uniqueness part in Theorem 1.1.

5.1 Strategy

In other words we will prove the following proposition.

Proposition 5.1. *Let X, Y be two pinched Hadamard manifolds and let $h_0, h_1 : X \rightarrow Y$ be two quasi-isometric harmonic maps that stay within bounded distance of one another:*

$$\sup_{x \in X} d(h_0(x), h_1(x)) < \infty.$$

Then one has $h_0 = h_1$.

When $X = Y = \mathbb{H}^2$, this proposition was first proven by Li and Tam in [24]. When both X and Y admit a cocompact group of isometries, this proposition was then proven by Li and Wang in [25, Theorem 2.3]. The aim of this chapter is to explain how to get rid of these extra assumptions.

Note that the assumption that the h_i are quasi-isometric is useful. Indeed there does exist non constant bounded harmonic functions on X . Note that there also exist bounded harmonic maps with open images. Here is a very simple example. Let $0 < \lambda < 1$. The map h_λ from the Poincaré unit disk \mathbb{D} of \mathbb{C} into itself given by $z \mapsto \lambda z$ is harmonic. More generally, for any harmonic map $h : \mathbb{D} \rightarrow \mathbb{D}$, the map $h_\lambda : \mathbb{D} \rightarrow \mathbb{D} : z \mapsto h(\lambda z)$ is a harmonic map with bounded image.

Before going into the technical details, we first explain the strategy of the proof of this uniqueness.

Strategy of proof of Proposition 5.1. We recall that the distance function $x \mapsto d(h_0(x), h_1(x))$ is a subharmonic function on X and that, by the maximum principle, a subharmonic function that achieves its maximum value is constant. Unfortunately since X is non-compact we can not a priori ensure that this bounded function achieves its maximum. This is why we will use a recentering argument. This recentering argument will force us to deal with Riemannian manifolds which are not \mathcal{C}^∞ (see Section 5.4).

We assume, by contradiction, that $h_0 \neq h_1$ and we choose a sequence of points p_n in X for which the distances

$$d(h_0(p_n), h_1(p_n)) \text{ converge to } \delta := \sup_{x \in X} d(h_0(x), h_1(x)) > 0 \quad (5.1)$$

and we set $q_n := h_0(p_n)$.

The pinching conditions on X and Y ensure that, after extracting a subsequence, the two pointed metric spaces (X, p_n) and (Y, q_n) converge in the Gromov–Hausdorff topology to two pointed metric spaces (X_∞, p_∞) and (Y_∞, q_∞) which are \mathcal{C}^2 Hadamard manifolds with \mathcal{C}^1 Riemannian metrics satisfying the same pinching conditions (Proposition 5.14). Moreover,

extracting again a subsequence, the harmonic map h_0 (resp. h_1) seen as a sequence of maps between the pointed Hadamard manifolds (X, p_n) and (Y, q_n) converges locally uniformly to a map $h_{0,\infty}$ (resp $h_{1,\infty}$) between the pointed C^2 Hadamard manifolds (X_∞, p_∞) and (Y_∞, q_∞) . These harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ are still harmonic quasi-isometric maps (Lemma 5.15).

The limit distance function $x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ is a subharmonic function on X_∞ that now achieves its maximum $\delta > 0$ at the point p_∞ . Hence, by the maximum principle, this distance function is constant and equal to δ (Lemma 5.16). Generalizing [25, Lemma 2.2], we will see in Corollary 5.19 that this equidistance property implies that both $h_{0,\infty}$ and $h_{1,\infty}$ take their values in a geodesic of Y_∞ . This contradicts the fact that $h_{0,\infty}$ and $h_{1,\infty}$ are quasi-isometric maps, and concludes this strategy of proof. \square

In the following sections of Chapter 5, we fill in the details of the proof.

5.2 Harmonic coordinates

We first introduce the so-called harmonic coordinates, which improve the quasilinear coordinates introduced in Lemma 2.6.

We refer to [15] or [19] for more details.

The harmonic coordinates have been introduced by DeTurk and Kazdan and extensively used by Cheeger, Jost, Karcher, Petersen... to prove various compactness results for compact Riemannian manifolds. Beyond being harmonic, the main advantage of these coordinates is that, for every $\alpha \in]0, 1[$, they are uniformly bounded in $C^{2,\alpha}$ -norm, i.e. they are uniformly bounded in C^2 -norm and one also has a uniform control of the α -Hölder norm of their second covariant derivatives. Moreover, one has a uniform control on the size of the balls on which these harmonic charts are defined. This is what the following lemma tells us.

We endow \mathbb{R}^k with the standard Euclidean structure.

Lemma 5.2. *Let X be a k -dimensional Hadamard manifold with bounded curvature $-1 \leq K_X \leq 0$. Let $0 < \alpha < 1$. There exist two constants $r_0 = r_0(k) > 0$ and $c_0 = c_0(k, \alpha) > 0$ such that, for every x in X , there exists a C^∞ -diffeomorphism*

$$\Psi_x : \mathring{B}(x, r_0) \xrightarrow{\sim} U_x \subset \mathbb{R}^k \text{ with } \Psi_x(x) = 0, \quad (5.2)$$

$$\|D\Psi_x\| \leq c_0, \quad \|D\Psi_x^{-1}\| \leq c_0, \quad \|D^2\Psi_x\| \leq c_0, \quad \|D^2\Psi_x^{-1}\| \leq c_0 \quad (5.3)$$

and such that each component z_1, \dots, z_k of Ψ_x is a harmonic function.

In particular, one has for all $r < r_0$:

$$\Psi_x(B(x, c_0^{-1}r)) \subset B(0, r) \text{ and } B(0, c_0^{-1}r) \subset \Psi_x(B(x, r)). \quad (5.4)$$

(iii) The second covariant derivatives of all Ψ_x are also uniformly α -Hölder :

$$\|D^2\Psi_x\|_{C^\alpha} \leq c_0. \quad (5.5)$$

This α -Hölder semi-norm $\|D^2\Psi_x\|_{C^\alpha}$ is defined as follows. Using the vector fields $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_k}$ on $\mathring{B}(x, r_0)$ associated to our coordinate system $\Psi_x = (z_1, \dots, z_k)$, we reinterpret the tensor $D^2\Psi_x$ as a family of vector valued functions on $\mathring{B}(x, r_0)$. Indeed, we set

$$T_x^{ij}(z) = D^2\Psi_x(z)\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) \in \mathbb{R}^k, \text{ for } i, j \text{ in } \{1, \dots, k\},$$

and the bound (5.5) means that

$$\|D^2\Psi_x\|_{C^\alpha} := \max_{i,j} \sup_{z, z'} \frac{\|T_x^{ij}(z) - T_x^{ij}(z')\|}{d(z, z')^\alpha} \leq c_0. \quad (5.6)$$

These uniform bounds (5.3) and (5.5) have three consequences.

First, in the harmonic coordinate systems $\Psi_x = (z_1, \dots, z_k)$, the Christoffel coefficients Γ_{ij}^ℓ are uniformly bounded in C^α -norm. Indeed, these coefficients $(\Gamma_{ij}^\ell)_{1 \leq \ell \leq k}$ are the components of the vector $-D^2\Psi_x\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) \in \mathbb{R}^k$.

Second, on their domain of definition, the transition functions

$$\Psi_{x'} \circ \Psi_x^{-1} \text{ are uniformly bounded in the } C^{2,\alpha}\text{-norm.} \quad (5.7)$$

Third, in the coordinate systems $\Psi_x = (z_1, \dots, z_k)$, the coefficients of the metric tensor

$$g_{ij} := g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) \text{ are uniformly bounded in the } C^{1,\alpha}\text{-norm.} \quad (5.8)$$

Proof of Lemma 5.2. See [19, p. 62 and 65] or [32, Section 4]. \square

5.3 Gromov-Hausdorff convergence

In this section, we recall the definition of Gromov–Hausdorff convergence for pointed metric spaces and some of its key properties. We refer to [7] for more details.

5.3.1 Definition

When X is a metric space, we will denote by d or d_X the distance on X . We denote by $B(x, R)$ the closed ball with center x and radius R , and by $\mathring{B}(x, R)$ the open ball. We recall that a metric space X is *proper* if all its balls are compact or, equivalently, if X is complete and for all $R > 0$ and $\varepsilon > 0$ every ball of radius R can be covered by finitely many balls with radius ε .

We also recall the notion of Gromov–Hausdorff distance between two (isometry class of proper) pointed metric spaces.

Definition 5.3. *The Gromov–Hausdorff distance between two pointed metric spaces (X, p) and (Y, q) is the infimum of the $\varepsilon > 0$ for which there exists a subset \mathcal{R} of $X \times Y$, called a correspondence, such that :*

- (i) *the correspondence \mathcal{R} contains the pair (p, q) ,*
- (ii) *for all x in the ball $B(p, \varepsilon^{-1})$, there exists y in Y with (x, y) in \mathcal{R} ,*
- (iii) *for all y in the ball $B(q, \varepsilon^{-1})$, there exists x in X with (x, y) in \mathcal{R} ,*
- (iv) *for all (x, y) and (x', y') in \mathcal{R} , one has $|d(x, x') - d(y, y')| \leq \varepsilon$.*

Heuristically, this correspondence \mathcal{R} must be thought as an ε -rough isometry between these two balls with radius ε^{-1} .

Based on this definition, a sequence (X_n, p_n) of pointed metric spaces converges to a pointed metric space (X_∞, p_∞) if, for all $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, there exists a map $f_n : B(p_n, \varepsilon^{-1}) \rightarrow X_\infty$ such that

- (α) $d(f_n(p_n), p_\infty) \leq \varepsilon$,
- (β) $|d(f_n(x), f_n(x')) - d(x, x')| \leq \varepsilon$, for all x, x' in $B(p_n, \varepsilon^{-1})$,
- (γ) the ε -neighborhood of $f_n(B(p_n, \varepsilon^{-1}))$ contains the ball $B(p_\infty, \varepsilon^{-1} - \varepsilon)$.

This definition 5.3 is only useful for complete metric spaces. Indeed, the Gromov–Hausdorff topology does not distinguish between a metric space and its completion. It does not distinguish either between two pointed metric spaces that are isometric : the Gromov–Hausdorff distance is a distance on the set of isometry classes of proper pointed metric spaces. See [7, Theorem 8.1.7]

The following equivalent definition of Gromov–Hausdorff convergence is useful when one wants to get rid of the ambiguity coming from the group of isometries of (X_∞, p_∞) .

Fact 5.4. *Let (X_n, p_n) , for $n \geq 1$, and (X_∞, p_∞) be pointed proper metric spaces. The sequence (X_n, p_n) converges to (X_∞, p_∞) if and only if there exists a complete metric space Z containing isometrically all the metric spaces X_n and X_∞ as disjoint closed subsets, and such that*

- (a) *the sequence of points p_n converges to p_∞ in Z ,*
- (b) *the sequence of closed subsets X_n converges to X_∞ for the Hausdorff topology.*

Statement (b) means that

- every point z of X_∞ is the limit of a sequence $(x_n)_{n \geq 1}$ with $x_n \in X_n$,
- every cluster point $z \in Z$ of a sequence $(x_n)_{n \geq 1}$ with $x_n \in X_n$ belongs to X_∞ .

Sketch of proof of Fact 5.4. Assume that the sequence (X_n, p_n) converges to (X_∞, p_∞) . We want to construct the metric space Z . We can choose a sequence $\varepsilon_n \searrow 0$, and correspondences \mathcal{R}_n on $X_n \times X_\infty$ as in Definition 5.3 with $p = p_n$, $q = p_\infty$ and $\varepsilon = \varepsilon_n$. This allows us to construct, for every $n \geq 1$, a metric space Y_n which is the disjoint union of X_n and X_∞ , which

contains isometrically both X_n and X_∞ and such that the distance between two points x in X_n and y in X_∞ is given by

$$d_{Y_n}(x, y) = \inf\{d_{X_n}(x, x') + \varepsilon + d_{X_\infty}(y', y)\}, \quad (5.9)$$

where the infimum is over all the pairs (x', y') which belong to \mathcal{R}_n .

The space Z is defined as the disjoint union of all the X_n and of X_∞ . The distance on Z is given on each union $Y_n := X_n \cup X_\infty$ by (5.9) and the distance between points x in X_m and z in X_n with $m \neq n$ is given by

$$d_Z(x, z) = \inf\{d_{Y_m}(x, y) + d_{Y_n}(y, z)\}, \quad (5.10)$$

where the infimum is over all the points y in X_∞ .

Then (a) follows from (i) and (b) follows from (ii), (iii) and (iv). \square

The choice of such isometric embeddings of all X_n and X_∞ in a fixed metric space Z will be called a realization of the Gromov-Hausdorff convergence. Such a realization is not unique. It is useful since it allows us to define the notion of a converging sequence of points x_n in X_n to a limit x_∞ in X_∞ by the condition $d_Z(x_n, x_\infty) \xrightarrow{n \rightarrow \infty} 0$.

5.3.2 Compactness criterion

A fundamental tool in this topic is the following compactness result for *uniformly proper* pointed metric spaces due to Cheeger-Gromov :

Fact 5.5. *Let $(X_n, p_n)_{n \geq 1}$ be a sequence of pointed proper metric spaces. Suppose that, for all $R > 0$ and $\varepsilon > 0$, there exists an integer $N = N(R, \varepsilon)$ such that, for all $n \geq 1$, the ball $B(p_n, R)$ of X_n can be covered by N balls with radius ε . Then there exists a subsequence of (X_n, p_n) which converges to a proper pointed metric space (X_∞, p_∞) .*

For a proof see [7, Theorem 8.1.10].

The following lemma gives us a compactness property for sequences of Lipschitz functions between pointed metric spaces.

Lemma 5.6. *Let $(X_n, p_n)_{n \geq 1}$ and $(Y_n, q_n)_{n \geq 1}$ be sequences of pointed proper metric spaces which converge respectively to proper pointed metric spaces (X_∞, p_∞) and (Y_∞, q_∞) . As in Fact 5.4, we choose metric spaces Z_X and Z_Y which realize these Gromov-Hausdorff convergences as Hausdorff convergences.*

Let $c > 1$ and let $(f_n : X_n \rightarrow Y_n)_{n \geq 1}$ be a sequence of c -Lipschitz maps such that $f_n(p_n) = q_n$. Then there exists a c -Lipschitz map $f_\infty : X_\infty \rightarrow Y_\infty$ such that, after extracting a subsequence, the sequence of maps f_n converges to f_∞ . This means that for each sequence $x_n \in X_n$ which converges to $x_\infty \in X_\infty$, the sequence $f_n(x_n) \in Y_n$ converges to $f_\infty(x_\infty) \in Y_\infty$.

Proof. This follows from basic topology arguments.

First step. We first choose a point x_∞ in X_∞ and a sequence x_n in X_n converging to x_∞ . Since the metric space Z_Y is proper and the sequence $f_n(x_n)$ is bounded in Z_Y we can assume after extracting a subsequence that the sequence $f_n(x_n)$ converges to a point $y_\infty \in Y_\infty$. Since the f_n are c -Lipschitz, this limit y_∞ does not depend on the choice of the sequence x_n converging to x_∞ . We define $f_\infty(x_\infty) := y_\infty$.

Second step. We choose a countable dense subset $S_\infty \subset X_\infty$ and use Cantor's diagonal argument to ensure that the first step is valid simultaneously for all points x_∞ in S_∞ .

Last step. One checks that the limit map $f_\infty : S_\infty \rightarrow Y_\infty$ is c -Lipschitz. Hence it extends uniquely as a c -Lipschitz map $f_\infty : X_\infty \rightarrow Y_\infty$ and the sequence f_n converges locally uniformly to f_∞ . \square

5.3.3 Length spaces and Alexandrov spaces

We recall a few well-known definitions (see [7]).

A *length space* is a complete metric space for which the distance δ between two points is the infimum of the length of the curves joining them. When X is proper, any two points at distance δ can be joined by a curve of length δ . Such a curve is called a *geodesic segment*.

Let $K \leq 0$. A *CAT(K)-space* or *CAT-space with curvature at most K* is a length space in which any geodesic triangle (P, Q, R) is thinner than a comparison triangle $(\bar{P}, \bar{Q}, \bar{R})$ in the plane \bar{X} of constant curvature K . Let us explain what this means. A *comparison triangle* is a triangle in \bar{X} with the same side lengths. For every point P' on the geodesic segment $[P, Q]$ we denote by \bar{P}' the corresponding point on the geodesic segment $[\bar{P}, \bar{Q}]$ i.e. the point such that $d(P, P') = d(\bar{P}, \bar{P}')$. *Thinner* means that one always has $d(P', R) \leq d(\bar{P}', \bar{R})$. Note that a CAT(0)-space is always simply connected (See [6, Corollary II.1.5]). We also recall that in a proper CAT(0)-space, any two points can be joined by a geodesic and that this geodesic is unique.

Similarly, a *metric space with curvature at least K* is a length space in which any geodesic triangle (P, Q, R) is thicker than a comparison triangle $(\bar{P}, \bar{Q}, \bar{R})$ in the plane \bar{X} of constant curvature K . *Thicker* means that one always has $d(P', R) \geq d(\bar{P}', \bar{R})$.

The following proposition tells us that these properties are closed for the Gromov–Hausdorff topology.

Fact 5.7. *Let $(X_n, p_n)_{n \geq 1}$ and (X_∞, p_∞) be pointed proper metric spaces. Let $K \leq 0$. Assume that the sequence (X_n, p_n) converges to (X_∞, p_∞) .*

- (i) *If the X_n 's are length spaces, then X_∞ is also a length space.*
- (ii) *If the X_n 's are CAT(K) spaces, then X_∞ is also a CAT(K) space.*
- (iii) *If moreover the X_n 's have curvature at least K , then X_∞ too.*

Proof. (i) See [7, Theorem 8.1.9].

(ii) See [6, Corollary II.3.10].

(iii) See [7, Theorem 10.7.1]. □

5.4 Hadamard manifolds with \mathcal{C}^1 metrics

In this section we focus on \mathcal{C}^2 Hadamard manifolds when the Riemannian metric is only assumed to be \mathcal{C}^1 . These Hadamard manifolds will occur in Section 5.5 as Gromov-Hausdorff limits of pinched \mathcal{C}^∞ Hadamard manifolds.

5.4.1 Definition

We need first to clarify the definitions. We will deal with \mathcal{C}^2 manifolds X . This means that X has a system of charts $x \mapsto (x_1, \dots, x_k)$ into \mathbb{R}^k for which the transition functions are of class \mathcal{C}^2 . These manifolds will be endowed with a \mathcal{C}^1 Riemannian metric g . This means that in any \mathcal{C}^2 chart, the functions $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are continuously differentiable.

In general, on such a Riemannian manifold, there might exist two different geodesics which are tangent at the same point (see [18] for an example with a $\mathcal{C}^{1,\alpha}$ -Riemannian metric). The following lemma tells us that this kind of examples will not occur here since we are dealing only with CAT(0)-spaces whose curvature is bounded below. Note that, since the metric tensor is not assumed to be twice differentiable, the expression “curvature bounded below” refers to the definitions in Section 5.3.

Definition 5.8. *By a \mathcal{C}^2 Hadamard manifold with a \mathcal{C}^1 metric, we mean a \mathcal{C}^2 manifold endowed with a \mathcal{C}^1 Riemannian metric which is CAT(0) and complete.*

5.4.2 Exponential map

Lemma 5.9. *Let X be a \mathcal{C}^2 Hadamard manifold with a \mathcal{C}^1 metric of bounded curvature.*

a) *For all x in X and v in $T_x X$ there is a unique geodesic $t \mapsto \exp_x(tv)$ starting from x at speed v . This geodesic is of class \mathcal{C}^2 .*

b) *This exponential map induces an homeomorphism $\Psi : TX \xrightarrow{\sim} X \times X$ given by, $\Psi(x, v) = (x, \exp_x(v))$ for x in X and v in $T_x X$.*

Proof. This lemma looks very familiar. But, since the Christoffel coefficients might not be Lipschitz continuous, we cannot apply Cauchy–Lipschitz theorem on Existence and Uniqueness of solutions of differential equations.

a) Since the Christoffel coefficients are continuous, we can apply Peano–Arzela theorem. It tells us that there exists at least one geodesic of class \mathcal{C}^2 starting from x at speed v . Uniqueness follows from the lower bound on the curvature.

b) Since X is CAT(0), the map Ψ is a bijection. Since a uniform limit of geodesic on X is also a geodesic, the map Ψ is continuous. This map Ψ is also proper, therefore it is an homeomorphism. \square

5.4.3 Geodesic interpolation of h_0 and h_1

In the sequel of this section we prove a few technical properties of the interpolation h_t of two equidistant Lipschitz maps h_0 and h_1 with values in a Hadamard manifold (lemmas 5.10). In Section 5.8, we will apply this lemma to two equidistant harmonic maps h_0 and h_1 obtained by a limit process. This lemma 5.10 will be used to compare the energy of h_0 and h_1 with the energy of some small perturbations of h_0 and h_1 . However, in this section 5.4, we do not need to assume h_0 and h_1 to be harmonic. Here are the precise assumptions and notations for Lemma 5.10.

Let X be a \mathcal{C}^2 Riemannian manifold with a \mathcal{C}^1 metric and Y be a \mathcal{C}^2 Hadamard manifold with \mathcal{C}^1 metric. Let $h_0, h_1 : X \rightarrow Y$ be two \mathcal{C}^1 maps such that one has

$$d(h_0(x), h_1(x)) = 1 \text{ for all } x \text{ in } X. \quad (5.11)$$

Since Y is a Hadamard manifold, there exists a unique map

$$\begin{aligned} h : [0, 1] \times X &\rightarrow Y \\ (t, x) &\mapsto h(t, x) = h_t(x) \end{aligned} \quad (5.12)$$

such that, for all x in X , the path $t \mapsto h_t(x)$ is the unit speed geodesic joining $h_0(x)$ and $h_1(x)$. This map h is called the *geodesic interpolation* of h_0 and h_1 . By convexity of the distance function, h is Lipschitz continuous. Therefore, by Rademacher's theorem, the map h is differentiable on a subset of full measure (with respect to the Riemannian measure on X). In particular, there exists a subset $X' \subset X$ of full measure such that, for all x in X' , the map h is differentiable at (x, t) for almost all t in $[0, 1]$. In particular, for all tangent vector $V \in T_x X$ at a point $x \in X'$, the following derivative

$$t \mapsto J_V(t) := D_x h_t(V) \in T_{h_t(x)} Y \quad (5.13)$$

is well-defined for almost all t in $[0, 1]$. Such a measurable vector field J_V on the geodesic $t \mapsto h_t(x)$ will be called a *Jacobi field*. We denote by

$$t \mapsto \tau_x(t) := \partial_t h_t(x) \in T_{h_t(x)} Y \quad (5.14)$$

the unit tangent vector to the geodesic $t \mapsto h_t(x)$.

Lemma 5.10. *We keep these assumptions and notations. Let x be a point in X' and $V \in T_x X$.*

a) *There exists a constant $\alpha_V \in \mathbb{R}$ such that*

$$\langle J_V(t), \tau_x(t) \rangle = \alpha_V, \text{ for all } t \text{ in } [0, 1] \text{ where } J_V(t) \text{ is defined.} \quad (5.15)$$

b) There exists a convex function $t \mapsto \varphi_V(t)$ on $[0, 1]$ such that

$$\varphi_V(t) = \|J_V(t)\|, \text{ for all } t \text{ in } [0, 1] \text{ where } J_V(t) \text{ is defined.} \quad (5.16)$$

c) The function $\psi_V := (\varphi_V^2 - \alpha_V^2)^{1/2}$ is also a convex function on $[0, 1]$.

Proof. When Y is a \mathcal{C}^∞ Hadamard manifold, the vector field J_V is a classical Jacobi field and this lemma is well known. Indeed, the function ψ_V is the norm of the orthogonal component K_V of the Jacobi field J_V , and Inequality (5.12) follows from the Jacobi equation satisfied by this Jacobi field K_V . We now explain how to adapt the classical proof when Y is only assumed to be a \mathcal{C}^2 Hadamard manifold with a \mathcal{C}^1 metric.

a) Since the path $t \mapsto h_t(x)$ is a unit speed geodesic, one has the equality $d(h_s(x), h_t(x)) = |t-s|$ for all s, t in $[0, 1]$. Differentiating this equality gives, when $J_V(s)$ and $J_V(t)$ are defined,

$$\langle J_V(s), \tau_x(s) \rangle = \langle J_V(t), \tau_x(t) \rangle.$$

Hence this scalar product is almost surely constant.

b) Let c be a \mathcal{C}^1 curve $c : [-\varepsilon_0, \varepsilon_0] \rightarrow X$ with $c(0) = x$ and $\partial_s c(0) = V$. Since the space Y is CAT(0), when $s > 0$, the functions

$$t \mapsto \varphi_s(t) := \frac{1}{s} d(h_t(c(0)), h_t(c(s)))$$

are convex on $[0, 1]$. Let $S_V := \{t \in [0, 1] \mid J_V(t) \text{ is defined}\}$. This set S_V has full measure and contains the endpoints 0 and 1. For all t in this set S_V , one can compute the limit of these functions $\lim_{s \rightarrow 0} \varphi_s(t) = \|J_V(t)\|$. Since these functions φ_s are convex, the limit $\varphi_V(t) := \lim_{s \rightarrow 0} \varphi_s(t)$ exists for all t in $[0, 1]$ and is a convex function.

c) We slightly change the parametrization of the geodesic interpolation : the function $k : (t, s) \mapsto k_t(s) := h_{t-s\alpha_V}(c(s))$ is well defined when $t - s\alpha_V$ is in $[0, 1]$, and the paths $t \mapsto k_t(s)$ are also unit speed geodesics. Hence, for almost all t in $[0, 1]$, the vector field

$$t \mapsto K_V(t) := \partial_s k_t(0) \in T_{k_t(0)} Y \quad (5.17)$$

is well-defined and one has the orthogonal decomposition

$$J_V(t) = K_V(t) + \alpha_V \tau_x(t).$$

In particular, one has the equality,

$$\psi_V(t) = \|K_V(t)\|. \quad (5.18)$$

The same argument as in b) with the Jacobi field K_V proves that the function ψ_V is also convex. \square

5.4.4 Geodesic interpolation in negative curvature

The following Lemma 5.11 improves Lemma 5.10 when the curvature of Y is uniformly negative. Indeed, it tells us that the norm $t \mapsto \psi_V(t)$ of the Jacobi field K_V is uniformly convex.

Lemma 5.11. *We keep the assumptions and notations of Lemma 5.10. Moreover we assume that Y is a $\text{CAT}(-a^2)$ -space with $a > 0$. Then the function ψ_V satisfies the following uniform convexity property,*

$$\psi_V(t) \leq \frac{\sinh(a(1-t))}{\sinh(a)}\psi_V(0) + \frac{\sinh(at)}{\sinh(a)}\psi_V(1) \quad \text{for all } t \text{ in } [0, 1]. \quad (5.19)$$

Remark 5.12. One can reformulate (5.19) as the following inequality between positive measures

$$\frac{d^2}{dt^2}\psi_V \geq a^2\psi_V.$$

Proof. This inequality (5.19) will follow from an upper bound for the norm of the Jacobi field $t \mapsto K_V(t)$ by the norm of a well chosen Jacobi field $t \mapsto \bar{K}(t)$ along a geodesic segment in the hyperbolic plane of curvature $-a^2$. Here are the details of the construction of this Jacobi field $t \mapsto \bar{K}(t)$.

Using a slight rescaling, we can assume without loss of generality that the geodesics $t \mapsto k_t(s)$ are defined for t in $[0, 1]$ and that the Jacobi field $K_V(t)$ is well defined for $t = 0$ and for $t = 1$. We choose $s > 0$. Later on we will let s go to 0. We set $P_t := k_t(0)$ and $Q_{s,t} := k_t(s)$, and we apply Reshetnyak Lemma 5.13 to the four points $P_0, P_1, Q_{s,1}, Q_{s,0}$. According to this lemma, there exists a convex quadrilateral \bar{C}_s in the hyperbolic plane \bar{Y} of curvature $-a^2$ with vertices $\bar{P}_0, \bar{P}_1, \bar{Q}_{s,1}, \bar{Q}_{s,0}$, and a 1-Lipschitz map $j : \bar{C}_s \rightarrow Y$ whose restriction to each of the four geodesic sides $\bar{P}_0\bar{P}_1, \bar{P}_1\bar{Q}_{s,1}, \bar{Q}_{s,1}\bar{Q}_{s,0}, \bar{Q}_{s,0}\bar{P}_0$ is an isometry onto each of the four geodesic segments $P_0P_1, P_1Q_{s,1}, Q_{s,1}Q_{s,0}, Q_{s,0}P_0$. Indeed, since $d(\bar{P}_0, \bar{P}_1) = 1$, we can assume that the two vertices \bar{P}_0 and \bar{P}_1 do not depend on s and that the quadrilateral \bar{C}_s is positively oriented.

Since the vectors $K_V(0)$ and $K_V(1)$ are orthogonal to the geodesic segment $t \mapsto k_t(0)$, by Lemma 5.9, each of the four successive angles θ_i (for $i = 1, \dots, 4$) between the four successive geodesic segments $P_0P_1, P_1Q_{s,1}, Q_{s,1}Q_{s,0}, Q_{s,0}P_0$ in Y is equal to $\frac{\pi}{2} + o(1)$, where $o(1)$ denotes a quantity that goes to 0 when s goes to 0. Since j is 1-Lipschitz, each of the corresponding four successive angles $\bar{\theta}_i$ between the four successive geodesic sides $\bar{P}_0\bar{P}_1, \bar{P}_1\bar{Q}_{s,1}, \bar{Q}_{s,1}\bar{Q}_{s,0}, \bar{Q}_{s,0}\bar{P}_0$ in the hyperbolic plane \bar{Y} is not smaller than θ_i . Since the sum of these four angles $\bar{\theta}_i$ is bounded above by 2π , each of these four angles $\bar{\theta}_i$ also satisfies when s goes to 0 :

$$\bar{\theta}_i = \frac{\pi}{2} + o(1). \quad (5.20)$$

Denote by $t \mapsto \bar{P}_t$ and $t \mapsto \bar{Q}_{s,t}$ the unit speed parametrizations of the sides $\bar{P}_0\bar{P}_1$ and $\bar{Q}_0\bar{Q}_1$. For t in $[0, 1]$, one has $j(\bar{P}_t) = P_t$ and $j(\bar{Q}_{s,t}) = Q_{s,t}$,

and also

$$d(P_t, Q_{s,t}) \leq d(\bar{P}_t, \bar{Q}_{s,t}) \quad (5.21)$$

with equality when $t = 0$ or 1 :

$$d(P_0, Q_{s,0}) = d(\bar{P}_0, \bar{Q}_{s,0}) \quad \text{and} \quad d(P_1, Q_{s,1}) = d(\bar{P}_1, \bar{Q}_{s,1}). \quad (5.22)$$

We now focus on these convex quadrilaterals \bar{C}_s in the hyperbolic plane \bar{Y} of curvature $-a^2$. We write $\bar{Q}_{s,t} = \exp_{\bar{P}_t}(s\bar{K}_{s,t})$ where $\bar{K}_{s,t}$ belongs to $T_{\bar{P}_t}\bar{Y}$. Since $K_V(0)$ and $K_V(1)$ are well defined, by (5.17), (5.18), (5.20) and (5.22), the limits

$$\bar{K}(0) = \lim_{s \rightarrow 0} \bar{K}_{s,0} \quad \text{and} \quad \bar{K}(1) = \lim_{s \rightarrow 0} \bar{K}_{s,1}$$

exist and satisfy

$$\|\bar{K}(0)\| = \psi_V(0) \quad \text{and} \quad \|\bar{K}(1)\| = \psi_V(1). \quad (5.23)$$

Therefore, the limits

$$\bar{K}(t) = \lim_{s \rightarrow 0} \bar{K}_{s,t}$$

exist for all t in $[0, 1]$. Moreover, by (5.17), (5.18) and (5.21), they satisfy the inequalities

$$\psi_V(t) \leq \|\bar{K}(t)\|. \quad (5.24)$$

Since the vector field $t \mapsto \bar{K}(t)$ is a Jacobi field along the geodesic segment $t \mapsto \bar{P}_t$, which is orthogonal to the tangent vector, its norm

$$\bar{\psi}(t) := \|\bar{K}(t)\|$$

satisfies the Jacobi differential equation

$$\frac{d^2}{dt^2} \bar{\psi} = a^2 \bar{\psi}.$$

Hence, one has the equality

$$\bar{\psi}(t) = \frac{\sinh(a(1-t))}{\sinh(a)} \bar{\psi}(0) + \frac{\sinh(at)}{\sinh(a)} \bar{\psi}(1) \quad \text{for all } t \text{ in } [0, 1]. \quad (5.25)$$

We now deduce Inequality (5.19) directly from (5.23), (5.24) and (5.25). \square

We have used the following existence result for a majorizing quadrilateral due to Reshetnyak in [34]. More precisely we have used the boundary of this majorizing quadrilateral \bar{C} .

Lemma 5.13. *Let Y be a $\text{CAT}(-a^2)$ metric space and \bar{Y} be the hyperbolic plane of curvature $-a^2$. Then, for every four points P_0, P_1, Q_1, Q_0 in Y there exists a convex quadrilateral \bar{C} in \bar{Y} with vertices $\bar{P}_0, \bar{P}_1, \bar{Q}_1, \bar{Q}_0$ and a 1-Lipschitz map $j : \bar{C} \rightarrow Y$ which is an isometry on each of the four geodesic sides of \bar{C} , and which sends each of these four vertices \bar{R}_i on the corresponding given point R_i in Y .*

5.5 Limits of Hadamard manifolds

In this section we describe the Gromov–Hausdorff limits of pinched Hadamard manifolds.

The following proposition is a variation on the Cheeger compactness theorem.

Proposition 5.14. *Let $(X_n, p_n)_{n \geq 1}$ be a sequence of k -dimensional pointed Hadamard manifolds with pinched curvature $-1 \leq K_{X_n} \leq -a^2 \leq 0$.*

a) There exists a subsequence of (X_n, p_n) which converges to a pointed proper CAT-space (X_∞, p_∞) with curvature between -1 and $-a^2$.

b) This space X_∞ has a structure of a \mathcal{C}^2 Hadamard manifold such that the distance on X_∞ comes from a \mathcal{C}^1 Riemannian metric.

The same proof shows that X_∞ has a structure of a $\mathcal{C}^{2,\alpha}$ Hadamard manifold with a $\mathcal{C}^{1,\alpha}$ Riemannian metric, for every $0 < \alpha < 1$. We will not use this improvement.

Even though this proposition follows from [33, Theorem 72 p. 311], we give a sketch of proof below.

Proof. a) The assumption on the curvature of X_n ensures that for each $R > 0$, one has uniform estimates for the volumes of balls with radius R in X_n : for all $n \geq 1$ and x in X_n , one has

$$\text{vol}B_{\mathbb{R}^k}(O, R) \leq \text{vol}B_{X_n}(x, R) \leq \text{vol}B_{\mathbb{H}^k}(O, R).$$

Therefore, for each $0 < \varepsilon < R$, there exists an integer $N = N(R, \varepsilon)$ such that every ball $B_{X_n}(p_n, R)$ of X_n can be covered by N balls of radius ε . Hence, according to Fact 5.5, there exists a subsequence of (X_n, p_n) which converges to a proper pointed metric space (X_∞, p_∞) . According to Fact 5.7, X_∞ is a CAT-space with curvature between -1 and $-a^2$.

b) It remains to check that X_∞ is a \mathcal{C}^2 manifold with a \mathcal{C}^1 Riemannian metric. We isometrically imbed the converging sequence (X_n, p_n) in a proper metric space Z as in Fact 5.4. We fix $r_0 > 0$ and $c_0 > 0$ as in Lemma 5.2 where we introduced the harmonic coordinates, and we choose a maximal $\frac{r_0}{2c_0}$ -separated subset S_∞ of X_∞ . For each x_∞ in S_∞ , we choose a sequence x_n of points in X_n that converges to x_∞ . By (5.3), the harmonic charts

$$\Psi_{x_n} : \mathring{B}(x_n, \frac{r_0}{c_0}) \rightarrow \mathbb{R}^k \tag{5.26}$$

are uniformly bi-lipschitz. More precisely, for all z, z' in $\mathring{B}(x_n, \frac{r_0}{c_0})$, one has

$$c_0^{-1}d(z, z') \leq \|\Psi_{x_n}(z) - \Psi_{x_n}(z')\| \leq c_0 d(z, z').$$

Hence after extracting a subsequence, this sequence of charts Ψ_{x_n} converges towards a bi-lipschitz map

$$\Psi_{x_\infty} : \mathring{B}(x_\infty, \frac{r_0}{c_0}) \rightarrow \mathbb{R}^k. \tag{5.27}$$

The extraction can be chosen simultaneously for all the points x_∞ in the countable set S_∞ . This collection of maps Ψ_{x_∞} endows X_∞ with a structure of a Lipschitz manifold.

We now want to prove that this manifold X_∞ is a \mathcal{C}^2 manifold. Indeed we will check that, for any x_∞ and x'_∞ in S_∞ , the transition functions $\Phi_{x'_\infty} \circ \Phi_{x_\infty}^{-1}$ are of class \mathcal{C}^2 . This just follows from the fact that these transition functions are uniform limit on compact sets of the transition functions $\Phi_{x'_n} \circ \Phi_{x_n}^{-1}$ which are, by (5.7), uniformly bounded in the $\mathcal{C}^{2,\alpha}$ -norm.

Finally, we check that the distance d on X_∞ comes from a \mathcal{C}^1 Riemannian metric on X_∞ . By (5.8), the Riemannian metrics $(g_n)_{ij}$ on X_n , seen as functions in the charts Ψ_{x_n} of X_n , are uniformly bounded in the $\mathcal{C}^{1,\alpha}$ -norm. Extracting again a subsequence, there exists a \mathcal{C}^1 Riemannian metric $(g_\infty)_{ij}$ in the charts Ψ_{x_∞} of X_∞ such that the sequence of metrics

$$(g_n)_{ij} \text{ converges to } (g_\infty)_{ij} \text{ in the } \mathcal{C}^1 \text{ topology.} \quad (5.28)$$

Let d_∞ be the distance on X_∞ associated with g_∞ . We check that $d_\infty = d$ on X_∞ . Let x'_∞ and x''_∞ be points in X_∞ . They are limits of points x'_n and x''_n in X_n . Let c_n be the geodesic segment joining x'_n to x''_n . Extracting once more a subsequence, the curves c_n converge uniformly to a curve joining x'_∞ and x''_∞ . This curve must be a geodesic for g_∞ . This proves that $d_\infty(x'_\infty, x''_\infty) = d(x'_\infty, x''_\infty)$. \square

5.6 Convergence of harmonic maps

We now explain how to obtain the limit harmonic maps.

We first notice that we can extend Definition 3.2 : A \mathcal{C}^2 map $h : X \rightarrow Y$ between two \mathcal{C}^2 Riemannian manifolds with \mathcal{C}^1 metrics X and Y is said to be *harmonic* if its tension field is zero, namely $\tau(h) := \text{tr} D^2 h = 0$. Indeed, the tension field of a \mathcal{C}^2 map h at a point x depends only on the 2-jet of h and on the 1-jet of the metrics of X and Y at the points x and $h(x)$. More precisely, writing h in a coordinate system $h : (x_1, \dots, x_k) \mapsto (h_1, \dots, h'_k)$, the equation $\text{tr} D^2 h = 0$ reads as

$$\Delta h_\lambda = - \sum_{ij\mu\nu} g^{ij} \Gamma_{\mu\nu}^\lambda \frac{\partial h_\mu}{\partial x_i} \frac{\partial h_\nu}{\partial x_j} \quad (\lambda \leq k') \quad (5.29)$$

where $\Gamma_{\mu\nu}^\lambda$ are the Christoffel coefficients on Y and where Δ is the Laplace operator on X defined as in (3.1) :

$$\Delta : \varphi \mapsto \frac{1}{v} \frac{\partial}{\partial x_i} \left(v g^{ij} \frac{\partial \varphi}{\partial x_j} \right) \quad (5.30)$$

where $v = \sqrt{\det(g_{ij})}$ denotes the volume density on X . See [19, Section 1.3] for more details.

Lemma 5.15. *Let $(X_n, p_n)_{n \geq 1}$ and $(Y_n, q_n)_{n \geq 1}$ be two sequences of equi-dimensional pointed Hadamard manifolds with curvature between -1 and 0 . Let $c, C > 0$ and let $h_n : X_n \rightarrow Y_n$ be a sequence of (c, C) -quasi-isometric harmonic maps such that $\sup_n d(h_n(p_n), q_n) < \infty$. After extracting a subsequence, the sequences of pointed metric spaces (X_n, p_n) and (Y_n, q_n) converge respectively to pointed \mathcal{C}^2 manifolds with \mathcal{C}^1 Riemannian metrics (X_∞, p_∞) and (Y_∞, q_∞) , and the sequence of maps h_n converges to a c -quasi-isometric map $h_\infty : X_\infty \rightarrow Y_\infty$. This map h_∞ is of class \mathcal{C}^2 and is harmonic.*

Proof. Since they are harmonic, the maps h_n are \mathcal{C}^∞ . Since these maps are also (c, C) -quasi-isometric, according to Cheng's Lemma 3.4, there exists some constant $C' > 0$ such that the maps h_n are C' -Lipschitz. The first two statements then follow from Proposition 5.14 and Lemma 5.6.

It remains to show that the limit map h_∞ is of class \mathcal{C}^2 and harmonic. The key point will be a uniform bound for the $\mathcal{C}^{2,\alpha}$ -norm of h_n in suitable harmonic coordinates. Let $k := \dim X_n$ and $k' := \dim Y_n$. Let x_∞ be a point in X_∞ and $y_\infty := h_\infty(x_\infty)$. Let x_n be a sequence in X_n converging to x_∞ and let $y_n := h_n(x_n)$.

We look at the maps h_n through the harmonic charts Ψ_{x_n} of X_n and Ψ_{y_n} of Y_n as in (5.26). By (5.27), these charts converge respectively to charts Ψ_{x_∞} of X_∞ and Ψ_{y_∞} of Y_∞ . By (5.28), in these charts, the Riemannian metrics of X_n and Y_n converge to the Riemannian metrics of X_∞ and Y_∞ in the $\mathcal{C}^{1,\alpha}$ -norm.

Let $0 < \alpha < 1$. When one writes Equation (5.29) for $h = h_n$ in these harmonic coordinates on a small open ball $\Omega := \mathring{B}(0, \frac{r_0}{c_0 C'})$ of \mathbb{R}^k that does not depend on n , one gets

$$\sum_{ij} g^{ij} \frac{\partial^2 h_\lambda}{\partial z_i \partial z_j} = - \sum_{ij\mu\nu} g^{ij} \Gamma_{\mu\nu}^\lambda \frac{\partial h_\mu}{\partial z_i} \frac{\partial h_\nu}{\partial z_j}. \quad (5.31)$$

The coefficients of this equation depend on n , but Lemma 5.2 ensures that they are uniformly bounded in the \mathcal{C}^α -norm. The Schauder estimates for functions u on Ω and compact sets K of Ω as in [33, Theorem 70 p. 303] thus tell us that

$$\|u\|_{\mathcal{C}^{1,\alpha},K} \leq M (\|\Delta u\|_{\mathcal{C}^0,\Omega} + \|u\|_{\mathcal{C}^\alpha,\Omega}) \quad (5.32)$$

$$\|u\|_{\mathcal{C}^{2,\alpha},K} \leq M (\|\Delta u\|_{\mathcal{C}^\alpha,\Omega} + \|u\|_{\mathcal{C}^\alpha,\Omega}) \quad (5.33)$$

for some constant $M = M(k, \Omega, K)$. Therefore, since the maps h_n are C' -Lipschitz, combining (5.29), (5.32) and (5.33) yields a uniform bound for the $\mathcal{C}^{2,\alpha}$ -norm of the maps h_n . Hence the Ascoli Lemma ensures that, after extracting a subsequence, the sequence of maps h_n converges towards a \mathcal{C}^2 map in the \mathcal{C}^2 topology. This proves that the limit map h_∞ is \mathcal{C}^2 and is harmonic. \square

5.7 Construction of the limit equidistant harmonic maps

We now explain why the limit harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ constructed in the strategy of Proposition 5.1 are equidistant.

We first sum up the construction of these limit maps.

We start with two Hadamard manifolds X, Y of bounded curvatures, and with two distinct quasi-isometric harmonic maps $h_0, h_1 : X \rightarrow Y$ such that $\delta := d(h_0, h_1)$ is finite and non-zero. We choose a sequence of points p_n in X such that $d(h_0(p_n), h_1(p_n))$ converges to δ and we set $q_{0,n} := h_0(p_n)$ and $q_{1,n} := h_1(p_n)$. We will frequently replace this sequence by subsequences without mentioning it. By Proposition 5.14, there exist two \mathcal{C}^2 Hadamard manifolds with \mathcal{C}^1 metrics (X_∞, p_∞) and $(Y_\infty, q_{0,\infty})$ which are the Gromov–Hausdorff limits of the pointed metric spaces (X, p_n) and $(Y, q_{0,n})$. These limit Hadamard manifolds also have bounded curvature. We denote by $q_{1,\infty}$ the limit in Y_∞ of the sequence $q_{1,n}$. By the Cheng Lemma 3.4, the harmonic quasi-isometric maps h_0 and h_1 are Lipschitz continuous. By Lemma 5.6, there exists a limit map $h_{0,\infty} : (X_\infty, p_\infty) \rightarrow (Y_\infty, q_{0,\infty})$ of the sequence of Lipschitz continuous maps $h_0 : (X, p_n) \rightarrow (Y, q_{0,n})$. There exists also a limit map $h_{1,\infty} : (X_\infty, p_\infty) \rightarrow (Y_\infty, q_{1,\infty})$ of the sequence of Lipschitz continuous maps $h_1 : (X, p_n) \rightarrow (Y, q_{1,n})$. By Lemma 5.15, these limit maps $h_{0,\infty}$ and $h_{1,\infty}$ are still harmonic quasi-isometric maps.

Lemma 5.16. *With the above notation, the two limit harmonic quasi-isometric maps $h_{0,\infty}, h_{1,\infty}$ are equidistant. More precisely, for all x in X_∞ , one has $d(h_{0,\infty}(x), h_{1,\infty}(x)) = \delta > 0$ where $\delta := d(h_0, h_1)$.*

We will apply this lemma to two pinched Hadamard manifolds X, Y . In this case, the limit \mathcal{C}^2 Hadamard manifolds X_∞, Y_∞ will also be pinched.

Proof. Let Δ_∞ be the Laplace operator on X_∞ defined as in (5.30). We first check that the function $\varphi_\infty : x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ is subharmonic on X_∞ . This means that $\Delta_\infty \varphi_\infty$ is a positive measure on X_∞ . Assume first that the Riemannian metric on Y_∞ is \mathcal{C}^∞ . In this case, φ_∞ is the composition of a harmonic map $h = (h_0, h_1) : X_\infty \rightarrow Y_\infty \times Y_\infty$ and of a convex \mathcal{C}^∞ -function $F = d : Y_\infty \times Y_\infty \rightarrow \mathbb{R}$, so that the function φ_∞ is subharmonic on X_∞ because of the formula

$$\Delta_\infty(F \circ h) = \sum_{1 \leq i \leq k} D^2 F(D_{e_i} h, D_{e_i} h) + \langle DF, \tau(h) \rangle,$$

where $(e_i)_{1 \leq i \leq k}$ is an orthonormal basis of the tangent space to X .

Since the Riemannian metric on Y might not be of class \mathcal{C}^∞ , we will use instead a limit argument. We fix a point x_∞ in X_∞ . In a chart (x_1, \dots, x_k) , the Laplace operator Δ_∞ of the Riemannian metric $(g_\infty)_{ij}$ of X_∞ reads as

$$\psi \mapsto \Delta_\infty \psi = \frac{1}{v_\infty} \frac{\partial}{\partial x_i} \left(v_\infty g_\infty^{ij} \frac{\partial \psi}{\partial x_j} \right), \quad (5.34)$$

where v_∞ is the volume density. We want to prove that for every \mathcal{C}^2 function $\psi \geq 0$ with compact support in a small neighborhood of x_∞ , one has

$$\int_{\mathbb{R}^k} \varphi_\infty \Delta_\infty \psi v_\infty dx \geq 0. \quad (5.35)$$

This function φ_∞ on the pointed metric space (X_∞, p_∞) is equal to the limit of the sequence of functions $\varphi_n : x \mapsto d(h_0(x), h_1(x))$ on the pointed metric spaces (X, p_n) , as defined in Lemma 5.6. Note that the dependence on n comes from the base point p_n which moves with n . We choose a sequence x_n in X_n converging to x_∞ . As in the proof of Lemma 5.15, we look at the functions φ_n through the harmonic charts Ψ_{x_n} of X_n . By (5.27), these charts converge to a chart Ψ_{x_∞} of X_∞ . By (5.28), in these charts (x_1, \dots, x_k) the Riemannian metric $(g_n)_{ij}$ of X_n converge to the Riemannian metric $(g_\infty)_{ij}$ of X_∞ in \mathcal{C}^1 topology.

Since, by the above argument, the functions φ_n are subharmonic for the metric $(g_n)_{ij}$, for every \mathcal{C}^2 function $\psi \geq 0$ with compact support in these charts, one has at each step n

$$\int \varphi_n \Delta_n \psi v_n dx \geq 0 \quad (5.36)$$

where Δ_n and v_n are the Laplace operator and the volume density of the metric $(g_n)_{ij}$. Letting n go to ∞ in (5.36) gives (5.35). This proves that the function φ_∞ is subharmonic.

By construction, this subharmonic function φ_∞ on X_∞ achieves its maximum $\delta > 0$ at the point p_∞ . By (5.34), the Laplace operator is an elliptic linear differential operator with continuous coefficients. Hence, by the strong maximum principle in [14, Theorem 8.19 p.198], this function φ_∞ is constant and equal to δ . \square

The aim of Sections 5.8 and 5.9 is to prove that such equidistant harmonic quasi-isometric maps $h_{0,\infty}$ and $h_{1,\infty}$ can not exist (Corollary 5.19) when Y_∞ is pinched. This will conclude the proof of Proposition 5.1.

5.8 Equidistant harmonic maps

We first study equidistant harmonic maps without any pinching assumption.

The following lemma 5.17 extends [25, Lemma 2.2] to the case where the source space X is only assumed to be a \mathcal{C}^2 -Hadamard manifold. We include a complete proof to deal with this weaker regularity assumption.

Lemma 5.17. *Let X, Y be two \mathcal{C}^2 Hadamard manifolds with \mathcal{C}^1 Riemannian metrics of bounded curvature. Let $h_0, h_1 : X \rightarrow Y$ be two harmonic maps*

such that the distance function $x \mapsto d(h_0(x), h_1(x))$ is constant. For t in $[0, 1]$, let h_t be the geodesic interpolation of h_0 and h_1 as in (5.12). Then for almost all x in X , t in $[0, 1]$ and V in $T_x X$, one has

$$\|Dh_0(V)\| = \|Dh_t(V)\| = \|Dh_1(V)\|. \quad (5.37)$$

Note that we can not conclude that Equality (5.37) is valid for all x and t since the interpolation h_t might not be of class C^1 .

We will use the following straightforward inequality for convex functions.

Lemma 5.18. *Let $t \mapsto \Phi_t$ be a non-negative convex function on $[0, 1]$. Then, for all t in $[0, \frac{1}{2}]$, one has*

$$\Phi_t + \Phi_{1-t} \leq \Phi_0 + \Phi_1 - 2t(\Phi_0 + \Phi_1 - 2\Phi_{1/2}). \quad (5.38)$$

Proof of Lemma 5.18. We just add the following two convexity inequalities $\Phi_t \leq (1-2t)\Phi_0 + 2t\Phi_{1/2}$ and $\Phi_{1-t} \leq (1-2t)\Phi_1 + 2t\Phi_{1/2}$. \square

Proof of Lemma 5.17. The idea is to construct two small perturbations f_ε and g_ε of the harmonic maps h_0 and h_1 with support in a compact set K of X , and to compare the sum of the energies of f_ε and g_ε with the sum of the energies of h_0 and h_1 .

Let $0 \leq \varepsilon \leq 1$. Here is the definition of the two maps $f_\varepsilon : X \rightarrow Y$ and $g_\varepsilon : X \rightarrow Y$. We fix a C^1 cut-off function $\eta : X \mapsto [0, 1]; x \mapsto \eta_x$ with compact support K , and we let for all x in X :

$$f_\varepsilon(x) := h_{\varepsilon\eta_x}(x) \quad \text{and} \quad g_\varepsilon(x) := h_{1-\varepsilon\eta_x}(x). \quad (5.39)$$

These functions are Lipschitz continuous, so that they are almost everywhere differentiable. In order to compute their differentials, we use the notations (5.13) and (5.14) : for all x in a subset $X' \subset X$ of full measure, for all V in $T_x X$, for almost all t in $[0, 1]$, we let

$$J_V(t) := D_x h_t(V) \quad \text{and} \quad \tau_x(t) := \partial_t h_t(x).$$

For such a tangent vector V , it follows from Lemma 5.10.b that there exists a convex function $t \mapsto \varphi_V(t)$ such that $\varphi_V(t) = \|J_V(t)\|$ for all t where the derivative $J_V(t)$ exists. By the chain rule, for almost all ε in $[0, 1]$, the differentials of f_ε and g_ε are given, for almost all x in X and all V in $T_x X$, by

$$Df_\varepsilon(V) = J_V(\varepsilon\eta_x) + \varepsilon V.\eta \tau_x(\varepsilon\eta_x), \quad (5.40)$$

$$Dg_\varepsilon(V) = J_V(1 - \varepsilon\eta_x) - \varepsilon V.\eta \tau_x(1 - \varepsilon\eta_x) \quad (5.41)$$

where $V.\eta = d\eta(V)$ is the derivative of the function η in the direction V .

According to Lemma 5.10.a, for almost all x in X and all V in $T_x X$, the scalar product $\langle J_V(t), \tau_x(t) \rangle$ is almost surely constant. Therefore, for almost

all ε in $[0, 1]$, x in X and V in the unit tangent bundle $T_x^1 X$, one has the equality

$$\|Df_\varepsilon(V)\|^2 + \|Dg_\varepsilon(V)\|^2 = \varphi_V(\varepsilon\eta_x)^2 + \varphi_V(1 - \varepsilon\eta_x)^2 + 2\varepsilon^2(V.\eta)^2. \quad (5.42)$$

We introduce the convex function $t \mapsto \Phi_t^V := \varphi_V(t)^2$. Using Inequality (5.38), one gets for almost all ε in $[0, 1]$, x in X and V in $T_x^1 X$ the bound

$$\|Df_\varepsilon(V)\|^2 + \|Dg_\varepsilon(V)\|^2 \leq \Phi_0^V + \Phi_1^V - 2\varepsilon\eta_x(\Phi_0^V + \Phi_1^V - 2\Phi_{1/2}^V) + 2\varepsilon^2(V.\eta)^2.$$

We recall that the energy over K of a Lipschitz map $h : X \rightarrow Y$ is

$$E_K(h) := \int_K \|D_x h\|^2 dx = \int_{T^1 K} \|Dh(V)\|^2 dV,$$

where dx is the Riemannian measure on X and dV the Riemannian measure on $T^1 X$. Integrating the previous inequality on the unit tangent bundle of K , one gets the following inequality relating the energy over K of f_ε , g_ε , h_0 and h_1 :

$$E_K(f_\varepsilon) + E_K(g_\varepsilon) - E_K(h_0) - E_K(h_1) \leq -\varepsilon \int_{T_K^1} A(V) dV + O(\varepsilon^2) \quad (5.43)$$

where A is the function on $T^1 X$ defined for almost all x in X and V in $T_x^1 X$, by

$$A(V) := 2\eta_x(\Phi_0^V + \Phi_1^V - 2\Phi_{1/2}^V).$$

Since the harmonic maps h_0 and h_1 are critical points for the energy functional, Inequality (5.43) implies that

$$\int_{T^1 K} A(V) dV \leq 0. \quad (5.44)$$

Since the function Φ^V is convex, the function A is non-negative. Therefore Inequality (5.44) implies that the function A is almost surely zero. Since the function η was arbitrary, this tells us that, for almost all V in $T^1 X$, one has

$$2\Phi_{1/2}^V = \Phi_0^V + \Phi_1^V.$$

Since Φ^V is the square of the convex function φ_V , this tells us that for almost all V in TX , the function φ_V is constant. This proves (5.37). \square

5.9 Equidistant harmonic maps in negative curvature

The following Corollary 5.19 improves the conclusion of Lemma 5.17 when the curvature of Y is uniformly negative.

Corollary 5.19. *Let $a > 0$. Let X, Y be two C^2 Hadamard manifolds with C^1 Riemannian metrics. Assume moreover that Y is $\text{CAT}(-a^2)$. Let $h_0, h_1 : X \rightarrow Y$ be two harmonic maps such that the distance function $x \mapsto d(h_0(x), h_1(x))$ is constant. Then either $h_0 = h_1$ or*

$$h_0 \text{ and } h_1 \text{ take their values in the same geodesic } \Gamma \text{ of } Y. \quad (5.45)$$

This means that, when $h_0 \neq h_1$, there exists a geodesic $t \rightarrow \gamma(t)$ in Y and two harmonic functions u_0, u_1 on X such that $h_0 = \gamma \circ u_0$, $h_1 = \gamma \circ u_1$ and the difference $u_1 - u_0$ is a bounded harmonic function on X .

Note that this case is ruled out when h_0 and h_1 are within bounded distance from a quasi-isometric map $f : X \rightarrow Y$ since X has dimension $k \geq 2$.

Proof of Corollary 5.19. We can assume that the distance between h_0 and h_1 is equal to 1. We recall a few notations that we have already used. For t in $[0, 1]$, let h_t be the geodesic interpolation of h_0 and h_1 . For x in X , let $\tau_x(t) := \partial_t h_t(x)$. Since the map $(t, x) \mapsto h_t(x)$ is Lipschitz continuous, the vector $J_V(t) := Dh_t(V)$ is well-defined for almost all t in $[0, 1]$, x in X and V in $T_x X$. For all such t, x, V , we set

$$\alpha_V(t) := \langle J_V(t), \tau_x(t) \rangle, \quad \varphi_V(t) := \|J_V(t)\|, \quad \psi_V(t) := (\varphi_V(t)^2 - \alpha_V(t)^2)^{1/2}.$$

By Lemmas 5.10.a and 5.17, one has the equalities

$$\alpha_V(0) = \alpha_V(t) = \alpha_V(1) \quad \text{and} \quad \varphi_V(0) = \varphi_V(t) = \varphi_V(1) \quad (5.46)$$

for almost all t in $[0, 1]$ and almost all V in TX , so that

$$\psi_V(0) = \psi_V(t) = \psi_V(1).$$

Comparing these equalities with the uniform convexity of the function ψ_V in (5.19), one infers that $\psi_V(t) = 0$. Hence, when $J_V(t)$ is defined, one has

$$J_V(t) = \alpha_V(0) \tau_x(t). \quad (5.47)$$

We now explain why (5.47) implies (5.45). It is enough to check that, for every C^1 curve

$$c : [0, 1] \rightarrow X; s \mapsto c_s$$

with speed at most $1/3$, the images

$$h_0(c_{[0,1]}) \text{ and } h_1(c_{[0,1]}) \text{ are both included in the geodesic } \Gamma \quad (5.48)$$

of Y containing both $h_0(c_0)$ and $h_1(c_0)$.

The idea is to construct an auxiliary curve C with zero derivative. Let $\beta : [0, 1] \rightarrow [-1/3, 1/3]$ be the function given by $s \mapsto \beta_s := \int_0^s \alpha_{c'_r}(0) dr$. For t_0 in $[1/3, 2/3]$, let C be the curve

$$C : [0, 1] \rightarrow Y; s \mapsto C(s) := h_{t_0 - \beta_s}(c_s).$$

Since the speed of c is bounded by $1/3$, the curve C is well-defined. By construction C is a Lipschitz continuous path, and by (5.46) and (5.47), for almost all s , its derivative is,

$$C'(s) = (\alpha_{c'_s}(t_0 - \beta_s) - \alpha_{c'_s}(0)) \tau_{c_s}(t_0 - \beta_s) = 0$$

Therefore, one has $C(s) = C(0)$ for all s in $[0, 1]$, that is

$$h_{t_0 - \beta_s}(c_s) = h_{t_0}(c_0).$$

Using this equality for two distinct values of t_0 , we deduce that the geodesic segments $h_{[0,1]}(c_0)$ and $h_{[0,1]}(c_s)$ meet in at least two points. This proves (5.48) and ends the proof of Corollary 5.19. \square

This also ends the proof of Proposition 5.1.

6 Boundary maps for weakly coarse embeddings

This chapter is independent of the previous ones. We prove that a weakly coarse embedding between pinched Hadamard manifolds admits a boundary map which is well-defined outside a set of zero Hausdorff dimension. We prove that the fibers of this boundary map also have zero Hausdorff dimension (Theorem 6.5). More precisely, we will prove quantitative versions of these facts (Propositions 6.13 and 6.15) that we will use in Chapter 7.

6.1 Weakly coarse embeddings

In this section, we introduce various classes of rough Lipschitz maps $f : X \rightarrow Y$ between pinched Hadamard manifolds generalizing the quasi-isometric maps.

Let X and Y be Hadamard manifolds with pinched sectional curvatures $-b^2 \leq K_X \leq -a^2 < 0$, $-b^2 \leq K_Y \leq -a^2 < 0$. Let $k = \dim X$, $k' = \dim Y$.

Definition 6.1. *Let $c > 0$. A map $f : X \rightarrow Y$ is rough c -Lipschitz if, for all $x, x' \in X$, with $d(x, x') \leq 1$ one has $d(f(x), f(x')) \leq c$.*

When $f : X \rightarrow Y$ is a rough c -Lipschitz map, one has for all x, x' in X :

$$d(f(x), f(x')) \leq c d(x, x') + c.$$

Definition 6.2. *A map $f : X \rightarrow Y$ is a coarse embedding if there exist two non-decreasing unbounded functions φ_1, φ_2 such that, for all $x, x' \in X$:*

$$\varphi_1(d(x, x')) \leq d(f(x), f(x')) \leq \varphi_2(d(x, x')). \quad (6.1)$$

Note that a map which is within bounded distance from a coarse embedding is also a coarse embedding. In Definition 6.2 one may always assume that the function φ_2 is an affine function, that is, f is rough Lipschitz. A quasi-isometric map is a special case of a coarse embedding, where φ_1 is also an affine function.

Definition 6.3. *A weakly coarse embedding $f : X \rightarrow Y$ is a rough Lipschitz map for which there exist $c_0, C_0 > 0$ such that, for all x, x' in X :*

$$d(f(x), f(x')) \leq c_0 \Rightarrow d(x, x') \leq C_0. \quad (6.2)$$

Equivalently, this means that there exist two non-decreasing non-negative and non-zero functions φ_1, φ_2 such that (6.1) holds. Of course, any coarse embedding $f : X \rightarrow Y$ is a weakly coarse embedding.

Example 6.4. *There exist many coarse and weakly coarse embeddings f from \mathbb{H}^2 to \mathbb{H}^3 . More precisely, for any non-decreasing 1-Lipschitz function $\varphi_1 : [0, \infty[\rightarrow [0, \infty[$ with $\varphi_1(0) = 0$ one can choose a 1-Lipschitz map f for which φ_1 is the best lower bound in (6.1).*

Proof. Indeed, one first constructs a unit-speed \mathcal{C}^1 curve $f_0 : \mathbb{R} \rightarrow \mathbb{H}^2$ such that $\varphi_1(t) = \min_{s \in \mathbb{R}} d(f_0(s+t), f_0(s))$ for every $t \geq 0$. We set $\mathbb{H}^1 := \mathbb{R}$ and, for $k \geq 1$, we embed each space \mathbb{H}^k as a totally geodesic hyperplane in \mathbb{H}^{k+1} and denote by $x \rightarrow n_x$ a unit normal vector field to \mathbb{H}^k in \mathbb{H}^{k+1} . We now define the Lipschitz map $f : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ as $f(\exp(tn_s)) := \exp(tn_{f_0(s)})$ for all s in \mathbb{H}^1 and $t \in \mathbb{R}$. \square

For any point $x_0 \in X$ and $r > 0$, we identify through the exponential map each sphere $S(x_0, r)$ with the unit tangent sphere

$$S_{x_0} := \{\xi \in T_{x_0}X \mid \|\xi\| = 1\}.$$

More precisely, when $\xi \in S_{x_0}$ is a unit tangent vector at the point x_0 , we denote by $r \mapsto \xi_r := \exp_{x_0}(r\xi)$ the corresponding unit-speed geodesic ray (so that $\xi_0 = x_0$).

We denote by $\bar{X} = X \cup \partial X$ the visual compactification of X . The boundary ∂X is the set of (equivalence classes of) rays in X . The map $\psi_{x_0} : \xi \mapsto \lim_{r \rightarrow \infty} \xi_r$ gives an homeomorphism of the unit tangent sphere S_{x_0} with the sphere at infinity ∂X . We say that a subset A of ∂X has *zero Hausdorff dimension* if, seen in S_{x_0} , it has zero Hausdorff dimension. One can check that this property does not depend on the choice of x_0 , because for any another point $x_1 \in X$, the homeomorphism $\psi_{x_1}^{-1} \circ \psi_{x_0}$ is bi-Hölder.

In this Chapter 6 we will prove the following theorem.

Theorem 6.5. *Let $f : X \rightarrow Y$ be a weakly coarse embedding between pinched Hadamard manifolds.*

- a) There exists a subset $A \subset \partial X$ of zero Hausdorff dimension such that, for all $\xi \in \partial X \setminus A$, the limit $\partial f(\xi) := \lim_{r \rightarrow \infty} f(\xi_r)$ exists in ∂Y .
- b) For every $\xi \in \partial X \setminus A$, the fiber $\{\eta \in \partial X \setminus A \mid \partial f(\eta) = \partial f(\xi)\}$ has zero Hausdorff dimension.

The map $\partial f : \partial X \setminus A \rightarrow \partial Y$ is called the boundary map of f .

The proof of Theorem 6.5 will last up to the end of this Chapter. The quantitative estimates (6.8) and (6.10) that we will obtain during this proof will be reused in Chapter 7.

6.2 Hausdorff dimension and Frostman measures

In this section we introduce classical notations and definitions from geometric measure theory.

Definition 6.6. Let $M, \nu > 0$. A Borel probability measure σ on a compact metric space S is said to be (M, ν) -Frostman if, for all $\xi \in S$ and all $r > 0$, one has the following bound for the measures of the balls :

$$\sigma(B(\xi, r)) \leq M r^\nu. \quad (6.3)$$

Proposition 4.9 tells us that all the harmonic measures $\sigma_{x,r}$ of a pinched Hadamard manifold are $(M, 1/N)$ Frostman, where the constants (M, N) do not depend on the center x nor the radius $r > 0$.

Let $\nu > 0$ and $\delta > 0$. For a subset $A \subset S$, we denote

$$H_\delta^\nu(A) = \inf \left\{ \sum_{i \geq 1} \text{diam}(U_i)^\nu \mid A \subset \bigcup_i U_i, \text{diam}(U_i) \leq \delta \right\}.$$

When $\delta = \infty$, we denote similarly

$$H_\infty^\nu(A) = \inf \left\{ \sum_{i \geq 1} \text{diam}(U_i)^\nu \mid A \subset \bigcup_i U_i \right\}. \quad (6.4)$$

We recall that the ν -dimensional Hausdorff measure of A is defined as

$$H^\nu(A) = \sup_{\delta > 0} H_\delta^\nu(A)$$

and the Hausdorff dimension of A is

$$\dim_H(A) = \inf \{ \nu > 0 \mid H^\nu(A) = 0 \}.$$

Observe that one also has

$$\dim_H(A) = \inf \{ \nu > 0 \mid H_\infty^\nu(A) = 0 \}. \quad (6.5)$$

The following easy lemma relates $H_\infty^\nu(A)$ with Frostman measures.

Lemma 6.7. Let σ be a (M, ν) -Frostman measure on a compact metric space S and $A \subset S$. Then one has $\sigma(A) \leq M H_\infty^\nu(A)$.

Proof. Observe that $\sigma(A) \leq \sum_{i \geq 1} \sigma(U_i) \leq M \sum_{i \geq 1} \text{diam}(U_i)^\nu$ for any covering (U_i) of A . □

6.3 Image of a large sphere

In this section we focus on those points of a sphere $S(x_0, r)$ whose images under a weakly coarse embedding are too close from a given point.

The following definition will play a key role in the proof of Theorem 6.5.

Definition 6.8. *Let $c, C_1, C_2 > 0$. A rough c -Lipschitz map $f : X \rightarrow Y$ satisfies property \mathcal{C}_{C_1, C_2} if, for all points $x_0 \in X$, $y_0 \in Y$ and all $r, s > 0$, the set*

$$A_{x_0, y_0, r, s} := \{\xi \in S_{x_0} \mid d(y_0, f(\xi_r)) \leq s\} \quad (6.6)$$

can be covered by at most $C_1 e^{bk's}$ balls of radius $C_2 e^{-ar}$, where $k' = \dim Y$. If such constants C_1, C_2 exist, we say that f satisfies property \mathcal{C} .

In this definition the unit sphere S_{x_0} is endowed with the distance induced by the Riemannian norm on $T_{x_0}X$.

The bound on the size of a covering of the set (6.6) will be very useful for Hausdorff dimension estimations. The precise value bk' for the exponential growth in Definition 6.8 is not very important. It is the one obtained in the next proposition and it merely avoids the introduction of another parameter.

Proposition 6.9. *Every weakly coarse embedding $f : X \rightarrow Y$ satisfies property \mathcal{C} .*

In particular, Propositions 6.13 and 6.15 below apply to all weakly coarse embeddings f .

We will use the Bishop volume estimates (see for example [12]) which compare the volume of balls in X and in the hyperbolic space \mathbb{H}^k .

Lemma 6.10. *Let X be a pinched Hadamard manifold with dimension k and sectional curvature $-b^2 \leq K_X \leq -a^2 < 0$. Then, for $R > 0$, one has*

$$a^{-k} \text{vol}(B_{\mathbb{H}^k}(O, aR)) \leq \text{vol}(B_X(x, R)) \leq b^{-k} \text{vol}(B_{\mathbb{H}^k}(O, bR)).$$

We will also need to bound angles by Gromov products as in Lemma 2.1.

Lemma 6.11. *Let Y be a Hadamard manifold with curvature $K_Y \leq -a^2 < 0$. Then, for all $y_0 \in Y$ and $y_1, y_2 \in Y \setminus \{y_0\}$ one has the bound*

$$\theta_{y_0}(y_1, y_2) \leq 4 e^{-a(y_1, y_2)_{y_0}},$$

where $\theta_{y_0}(y_1, y_2)$ is the angle at y_0 of the geodesic triangle (y_0, y_1, y_2) and $(y_1, y_2)_{y_0} := \frac{1}{2}(d(y_0, y_1) + d(y_0, y_2) - d(y_1, y_2))$ is the Gromov product.

Proof of Proposition 6.9. We will see that f satisfies property \mathcal{C}_{C_1, C_2} where the constants C_1, C_2 depend only on a, b, k' , and on c_0, C_0 from (6.2).

It follows from the volume estimates of Lemma 6.10 that there exists a constant $C_1 > 0$ such that for each ball $B(y_0, s) \subset Y$ ($s > 0$) and each covering of minimal cardinality of this ball by balls with radii $c_0/2$

$$B(y_0, s) \subset \cup_{i \in I} B(y_i, c_0/2),$$

this cardinality is at most $C_1 e^{bk's}$.

Since f is a (c_0, C_0) -weakly coarse embedding, for each $i \in I$, the inverse image of this ball $f^{-1}(B(y_i, c_0/2))$ is either empty or lies in a ball $B(x_i, C_0) \subset X$. By Lemma 6.11, the intersection $B(x, C_0) \cap S(x_0, r)$ lies in a cone with vertex x_0 and angle $\theta_r = C_2 e^{-ar}$. \square

Remark 6.12. Any map $\tilde{f} : X \rightarrow Y$ within bounded distance from a map $f : X \rightarrow Y$ satisfying Property \mathcal{C} also satisfies Property \mathcal{C} .

6.4 Construction of the boundary map

We now investigate the long-term behavior of the images of geodesic rays under a rough Lipschitz map satisfying Property \mathcal{C} .

Let X, Y be pinched Hadamard manifolds and $f : X \rightarrow Y$ be a rough Lipschitz map satisfying Property \mathcal{C} . The following proposition 6.13 tells us that, except for a set of rays of zero Hausdorff dimension, the image under f of a ray goes to infinity in Y at positive speed and this image converges to a point in ∂Y .

We need some notations. For $x_0 \in X$, let A_{x_0} be the set of rays whose image do not go to infinity at positive speed, namely

$$A_{x_0} := \{\xi \in S_{x_0} \mid \liminf_{n \rightarrow \infty} \frac{1}{n} d(f(x_0), f(\xi_n)) = 0\}.$$

One has $A_{x_0} = \bigcap_{\alpha > 0} A_{x_0, \alpha}$, where, for $\alpha > 0$, we denote

$$A_{x_0, \alpha} := \{\xi \in S_{x_0} \mid \liminf_{n \rightarrow \infty} \frac{1}{n} d(f(x_0), f(\xi_n)) < \alpha\}.$$

One has $A_{x_0, \alpha} \subset \bigcap_{n_0 \geq 1} A_{x_0, \alpha}(n_0)$, where, for $n_0 \geq 1$, one defines

$$A_{x_0, \alpha}(n_0) := \{\xi \in S_{x_0} \mid d(f(x_0), f(\xi_n)) \leq n\alpha \text{ for some } n \geq n_0\}.$$

With the definition (6.6), one has $A_{x_0, \alpha}(n_0) = \bigcup_{n \geq n_0} A_{x_0, f(x_0), n, n\alpha}$.

Proposition 6.13. *Let X, Y be pinched Hadamard manifolds with sectional curvatures $-b^2 \leq K \leq -a^2 < 0$. Let $c, C_1, C_2 > 0$ and $f : X \rightarrow Y$ be a rough c -Lipschitz map satisfying Property \mathcal{C}_{C_1, C_2} . Let $\alpha > 0$, $k' = \dim Y$ and $\nu_\alpha := \frac{bk'\alpha}{a}$. For $\nu > \nu_\alpha$, we introduce the constant $C_{3, \alpha, \nu} := \frac{C_1 C_2^\nu}{1 - e^{-a(\nu - \nu_\alpha)}}$. Then, the following holds for any $x_0 \in X$ and $n_0 \geq 1$.*

a) One has

$$H_\infty^\nu(A_{x_0, \alpha}(n_0)) \leq C_{3, \alpha, \nu} e^{-a(\nu - \nu_\alpha)n_0}. \quad (6.7)$$

b) For every (M, ν) -Frostman measure σ on S_{x_0} , one has

$$\sigma(A_{x_0, \alpha}(n_0)) \leq M C_{3, \alpha, \nu} e^{-a(\nu - \nu_\alpha)n_0}. \quad (6.8)$$

c) One has $\dim_H(A_{x_0, \alpha}) \leq \nu_\alpha$.

d) One has $\dim_H(A_{x_0}) = 0$.

e) For every $\xi \in S_{x_0} \setminus A_{x_0}$, the limit $\partial f(\xi) := \lim_{r \rightarrow \infty} f(\xi_r)$ exists in ∂Y .

The bound (6.8) can be interpreted as a large deviation inequality for the random path $f(\xi_t)$ when the ray ξ is chosen randomly with law σ . A key point is that the constants involved in (6.8) do not depend on the (M, ν) -Frostman measure σ . We will apply it later on to various harmonic measures $\sigma = \sigma_{x_0, r}$ on X .

Proof of Proposition 6.13. a) Since f satisfies Property C_{C_1, C_2} , one has

$$\begin{aligned} H_\infty^\nu(A_{x_0, \alpha}(n_0)) &\leq \sum_{n \geq n_0} H_\infty^\nu(A_{x_0, f(x_0), n, n\alpha}) \\ &\leq \sum_{n \geq n_0} C_1 e^{a\nu_\alpha n} C_2^\nu e^{-a\nu n} = C_{3, \alpha, \nu} e^{-a(\nu - \nu_\alpha)n_0}. \end{aligned}$$

b) This follows from a) and Lemma 6.7.

c) Letting n_0 go to infinity in (6.7), one gets $H_\infty^\nu(A_{x_0, \alpha}) = 0$ for all $\nu > \nu_\alpha$. Therefore, (6.5) yields that $\dim_H(A_{x_0, \alpha}) \leq \nu_\alpha$.

d) One has $\dim_H(A_{x_0}) \leq \inf_{\alpha > 0} \dim_H(A_{x_0, \alpha}) = 0$.

e) Since f is rough Lipschitz, one may assume that the parameters r are integers and apply the following lemma 6.14 to the sequence $y_n = f(\xi_n)$. \square

Lemma 6.14. *Let Y be a Hadamard manifold with sectional curvature $K_Y \leq -a^2 < 0$. Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in Y such that*

$$\sup_{n \geq 0} d(y_n, y_{n+1}) < \infty \text{ and } \liminf_{n \rightarrow \infty} \frac{1}{n} d(y_0, y_n) > 0.$$

Then, the limit $y_\infty := \lim_{n \rightarrow \infty} y_n$ exists in the visual boundary ∂Y .

Proof. Choose $c > 0$, $\alpha > 0$ and $n_0 \geq 1$ such that

$$d(y_n, y_{n+1}) \leq c \text{ and } d(y_0, y_n) \geq n\alpha \text{ for all } n \geq n_0.$$

By Lemma 6.11, the inequality $\theta_{y_0}(y_n, y_{n+1}) \leq 4e^{ac/2} e^{-a\alpha n}$ holds for any $n \geq n_0$. Since this series converges, there exists a geodesic ray $\gamma_+ \subset Y$ with origin y_0 such that $\lim_{n \rightarrow \infty} \theta_{y_0}(y_n, \gamma_+) = 0$. \square

Unlike quasi-isometric maps, coarse embedding may not have boundary values in every direction. See Example 6.4 where we could begin with a curve f_0 that spirals away in \mathbb{H}^2 .

6.5 The fibers of the boundary map

We now investigate the fibers of the boundary map ∂f of a rough Lipschitz map satisfying property \mathcal{C} .

The following proposition 6.15 tells us that the fibers of the boundary map have zero Hausdorff dimension.

We keep the notations of Section 6.4 and introduce more notations. As before, X, Y are pinched Hadamard manifolds and $f : X \rightarrow Y$ is a rough c -Lipschitz map satisfying Property \mathcal{C} . For $x_0 \in X$ and $\xi \in S_{x_0}$, let $B_{x_0}^\xi$ be the set of rays η that “do not go away from ξ at positive speed”, namely

$$B_{x_0}^\xi := \{\eta \in S_{x_0} \mid \lim_{n_0 \rightarrow \infty} \inf_{n, p \geq n_0} \frac{1}{n+p} d(f(\xi_n), f(\eta_p)) = 0\}.$$

One has $B_{x_0}^\xi = \bigcap_{\alpha > 0} B_{x_0, \alpha}^\xi$, where, for $\alpha > 0$, we set $\beta_\alpha := \frac{\alpha^2}{2\alpha+c}$ and let

$$B_{x_0, \alpha}^\xi := \{\eta \in S_{x_0} \mid \lim_{n_0 \rightarrow \infty} \inf_{n, p \geq n_0} \frac{1}{n+p} d(f(\xi_n), f(\eta_p)) < \beta_\alpha\}.$$

One has $B_{x_0, \alpha}^\xi \subset \bigcap_{n_0 \geq 1} B_{x_0, \alpha}^\xi(n_0)$, where we let for any $n_0 \geq 1$:

$$B_{x_0, \alpha}^\xi(n_0) := \{\eta \in S_{x_0} \mid d(f(\xi_n), f(\eta_p)) \leq (n+p)\beta_\alpha \text{ for some } n, p \geq n_0\}.$$

This specific value for β_α has been chosen in order to obtain the same exponent in (6.7) and in the following (6.9).

Proposition 6.15. *Let X, Y be pinched Hadamard manifolds with sectional curvatures $-b^2 \leq K \leq -a^2 < 0$. Let $c, C_1, C_2 > 0$ and $f : X \rightarrow Y$ be a rough c -Lipschitz map with Property \mathcal{C}_{C_1, C_2} . Let $\alpha > 0$, $k' = \dim Y$, $\nu_\alpha := \frac{bk'\alpha}{a}$ and $\beta_\alpha := \frac{\alpha^2}{2\alpha+c}$. For $\nu > \nu_\alpha$, we set $C_{4, \alpha, \nu} := \frac{C_1 C_2^\nu}{(1-e^{-bk'\beta_\alpha})(1-e^{-a(\nu-\nu_\alpha)})}$. Then, the following holds for any $x_0 \in X$ and $n_0 \geq 1$.*

a) For $\xi \in S_{x_0} \setminus A_{x_0, \alpha}(n_0)$, one has

$$H_\infty^\nu(B_{x_0, \alpha}^\xi(n_0)) \leq C_{4, \alpha, \nu} e^{-a(\nu-\nu_\alpha)n_0}. \quad (6.9)$$

b) For $\xi \in S_{x_0} \setminus A_{x_0, \alpha}(n_0)$ and any (M, ν) -Frostman measure σ on S_{x_0} , one has

$$\sigma(B_{x_0, \alpha}^\xi(n_0)) \leq M C_{4, \alpha, \nu} e^{-a(\nu-\nu_\alpha)n_0}. \quad (6.10)$$

c) For $\xi \in S_{x_0} \setminus A_{x_0, \alpha}$, one has $\dim_H(B_{x_0, \alpha}^\xi) \leq \nu_\alpha$.

d) For $\xi \in S_{x_0} \setminus A_{x_0}$, one has $\dim_H(B_{x_0}^\xi) = 0$.

e) Assume $n_0 \geq \frac{4e^{2ac}}{1-e^{-a\beta_\alpha}}$. For $\xi, \eta \in S_{x_0} \setminus A_{x_0, \alpha}(n_0)$ with $\eta \notin B_{x_0, \alpha}^\xi(n_0)$ and for all $n, p \geq \ell_0 := \frac{4n_0c}{\alpha}$, one has the lower bound for the angle

$$\theta_{f(x_0)}(f(\xi_n), f(\eta_p)) \geq \frac{1}{2} e^{-2n_0bc}. \quad (6.11)$$

f) For $\xi, \eta \in S_{x_0} \setminus A_{x_0}$, with $\eta \notin B_{x_0}^\xi$, one has $\partial f(\eta) \neq \partial f(\xi)$.

We begin with a technical covering lemma.

Lemma 6.16. *We keep the notations of Proposition 6.15. Fix $n_0 \geq 1$. For $\xi \in S_{x_0}$ and $p \geq n_0$, let*

$$B_{x_0, \alpha, p}^\xi(n_0) := \{\eta \in S_{x_0} \mid d(f(\xi_n), f(\eta_p)) \leq (n+p)\beta_\alpha \text{ for some } n \geq n_0\}.$$

If ξ is not in $A_{x_0, \alpha}(n_0)$, the set $B_{x_0, \alpha, p}^\xi(n_0)$ can be covered by at most $\frac{C_1 e^{bk'\alpha p}}{1 - e^{-bk'\beta_\alpha}}$ balls of radius $C_2 e^{-ap}$.

Proof of Lemma 6.16. Using the notation (6.6), we have the equality

$$B_{x_0, \alpha, p}^\xi(n_0) = \bigcup_{n \geq n_0} A_{x_0, f(\xi_n), p, (n+p)\beta_\alpha}.$$

The key point is that, since f is rough c -Lipschitz and $\xi \notin A_{x_0, \alpha}(n_0)$, this union is a finite union. Indeed, assume that an integer $n \geq n_0$ satisfies

$$d(f(\xi_n), f(\eta_p)) \leq (n+p)\beta_\alpha$$

for some $\eta \in S_{x_0}$. Since $d(f(x_0), f(\xi_n)) \geq n\alpha$ and $d(f(x_0), f(\eta_p)) \leq pc$, one must have

$$n\alpha - pc \leq (n+p)\beta_\alpha.$$

By our choice of β_α , this inequality is equivalent to

$$(n+p)\beta_\alpha \leq p\alpha.$$

Therefore, using Definition 6.8, one can cover the set $B_{x_0, \alpha, p}^\xi(n_0)$ by at most $C_1 \sum_n e^{bk'(n+p)\beta_\alpha}$ balls of radius $C_2 e^{-ap}$, where this sum runs over the integers $n \geq n_0$ such that $(n+p)\beta_\alpha \leq p\alpha$. Computing this sum, one deduces that the set $B_{x_0, \alpha, p}^\xi$ can be covered by at most $\frac{C_1 e^{bk'\alpha p}}{1 - e^{-bk'\beta_\alpha}}$ balls of radius $C_2 e^{-ap}$. \square

Proof of Proposition 6.15. a) Since $B_{x_0, \alpha}^\xi(n_0) = \bigcup_{p \geq n_0} B_{x_0, \alpha, p}^\xi(n_0)$, Lemma 6.16 yields

$$\begin{aligned} H_\infty^\nu(B_{x_0, \alpha}^\xi(n_0)) &\leq \sum_{p \geq n_0} H_\infty^\nu(B_{x_0, \alpha, p}^\xi(n_0)) \\ &\leq \sum_{p \geq n_0} \frac{C_1 e^{a\nu\alpha p}}{1 - e^{-\beta_\alpha bk'}} C_2^\nu e^{-a\nu p} = C_{4, \alpha, \nu} e^{-a(\nu - \nu_\alpha)n_0}. \end{aligned}$$

b) This follows from a) and Lemma 6.7.

c) Letting n_0 go to infinity in (6.9) one gets $H_\infty^\nu(B_{x_0, \alpha}^\xi) = 0$, for all $\nu > \nu_\alpha$. Therefore, using (6.5), it follows that $\dim_H(B_{x_0, \alpha}^\xi) \leq \nu_\alpha$.

d) One has $\dim_H(B_{x_0}^\xi) \leq \inf_{\alpha > 0} \dim_H(B_{x_0, \alpha}^\xi) = 0$.

e) This is a consequence of the following lemma 6.17 applied to the sequences $y_n = f(\xi_n)$ and $z_p = f(\eta_p)$.

f) This follows from e). \square

6.6 Two sequences going away from one another

The aim of this section is to prove the following lemma which provides, in a pinched Hadamard manifold, a lower bound for the angle between points in two sequences with bounded speed that “go away from one another at positive speed”.

Lemma 6.17. *Let Y be a Hadamard manifold with sectional curvature $-b^2 \leq K_Y \leq -a^2 < 0$. Let $c \geq \alpha \geq \beta > 0$ and $n_0 \geq \frac{4e^{2ac}}{1-e^{-a\beta}}$. Let $(y_n)_{n \in \mathbb{N}}$ and $(z_p)_{p \in \mathbb{N}}$ be two sequences of points in Y with $y_0 = z_0$, such that*

$$d(y_n, y_{n+1}) \leq c \text{ and } d(z_p, z_{p+1}) \leq c \text{ for any integers } n, p \geq 0, \quad (6.12)$$

$$d(y_0, y_n) \geq n\alpha, \quad d(y_0, z_p) \geq p\alpha \text{ and } d(y_n, z_p) \geq (n+p)\beta \text{ for } n, p \geq n_0. \quad (6.13)$$

Then, for any integer $n, p \geq \ell_0 := \frac{4n_0c}{\alpha}$, one has

$$\theta_{y_0}(y_n, z_p) \geq \frac{1}{2}e^{-2n_0bc}. \quad (6.14)$$

We will need two geometric lemmas.

We know that the orthogonal projection from a Hadamard manifold on a geodesic is a 1-Lipschitz map. The following lemma gives us a more precise information when the curvature is bounded from above.

Lemma 6.18. *Let Y be a Hadamard manifold with sectional curvature $K_Y \leq -a^2 < 0$. Let $\gamma \subset Y$ be a geodesic. Then, the orthogonal projection $\pi : Y \rightarrow \gamma$ is smooth and, for $y \in Y$, the norm of its differential satisfies*

$$\|D_y\pi\| \leq \frac{1}{\cosh(ad(y, \gamma))} \leq 2e^{-ad(y, \gamma)}.$$

Proof of Lemma 6.18. The proof relies on a Jacobi field estimate (see [12]).

Let $y \in Y \setminus \gamma$, let $\bar{y} = \pi(y) \in \gamma$ and $\ell = d(y, \gamma) = d(y, \bar{y})$. Denote by $c : s \in [0, \ell] \rightarrow c(s) \in Y$ the unit-speed parametrization of the geodesic segment $[\bar{y}, y]$ with $c(0) = \bar{y}$ and $c(\ell) = y$.

Let $v \in T_y Y$. We want to bound the ratio $\|D_y\pi(v)\|/\|v\|$. We may thus assume that v is orthogonal to $\text{Ker}D_y\pi$, i.e. that v is orthogonal to the geodesic c at the point y .

Choose a smooth curve $t \rightarrow y(t) \in Y$ with $y(0) = y$ and $y'(0) = v$, and let $\bar{y}(t) = \pi(y(t)) \in \gamma$. We can assume that, for all t , one has $d(y(t), \bar{y}(t)) = \ell$. For each parameter t , introduce the constant-speed geodesic $c_t : [0, \ell] \rightarrow Y$ such that $c_t(0) = \bar{y}(t)$ and $c_t(\ell) = y(t)$. By construction, each vector $u(t) := \frac{d}{ds}c_t(s)|_{s=0} \in T_{\bar{y}(t)}Y$ is normal to γ at the point $\bar{y}(t)$.

The map $(s, t) \rightarrow c_t(s)$ is a variation of geodesics, so that $J : s \in [0, \ell] \rightarrow \frac{d}{dt}c_t(s)|_{t=0} \in T_{c(s)}Y$ is a Jacobi field along the geodesic c . We have $J(0) = D_y\pi(v)$ and $J(\ell) = v$. Since both $J(0)$ and $J(\ell)$ are normal to c , it follows that J is a normal Jacobi field. Since γ is a geodesic and each $u(t)$

is normal to γ , we infer from the equality $J'(0) = u'(0)$ that $J'(0)$ is normal to γ , i.e. orthogonal to $J(0)$. The Jacobi field equation $J'' + R(c', J)c' = 0$ and the hypothesis on the curvature now yield

$$(\|J\|^2)'' = 2\|J'\|^2 - 2R(c', J, c', J) \geq 2(\|J'\|)^2 + 2a^2\|J\|^2$$

and therefore

$$\|J\|'' \geq a^2\|J\|.$$

Since $\|J\|'(0) = \frac{\langle J(0), J'(0) \rangle}{\|J(0)\|} = 0$, one deduces that $\|J(t)\| \geq \|J(0)\| \cosh(at)$, for all $t \geq 0$. In particular, one has $\|D_y \pi(v)\| \leq \|v\| / \cosh(a\ell)$. \square

The second lemma is an easy angle comparison lemma.

Lemma 6.19. *Let Y be a Hadamard manifold with sectional curvature $-b^2 \leq K_Y \leq 0$. Let $\gamma \subset Y$ be a geodesic, $y_0 \in \gamma$, $y \in Y$ and $\bar{y} = \pi(y)$ be the projection of y on γ . Assume that $d(y_0, \bar{y}) \leq R$ and $d(\bar{y}, y) \geq R$, then one has $\theta_{y_0}(y, \bar{y}) \geq \frac{1}{2}e^{-bR}$.*

Proof of Lemma 6.19. The angles of a triangle in $\mathbb{H}^2(-b^2)$ with same side-lengths are smaller than the angles of the triangle $(y_0 y \bar{y})$. It follows that $\theta_{y_0}(y, \bar{y}) \geq \varphi$, where φ is the angle of an isosceles right triangle in $\mathbb{H}^2(-b^2)$ with adjacent sides R , which is $\varphi = \arctan\left(\frac{1}{\cosh(bR)}\right) \geq \frac{1}{2}e^{-bR}$. \square

Proof of Lemma 6.17. Let γ_+ be a geodesic ray starting from $y_0 = z_0$. Denote by $\pi : Y \rightarrow \gamma$ the orthogonal projection on the geodesic γ that contains γ_+ . Identify $\gamma \sim \mathbb{R}$ so that $\gamma_+ \sim [0, \infty[$. Introduce, for $n, p \in \mathbb{N}$, the points $\bar{y}_n = \pi(y_n)$ and $\bar{z}_p = \pi(z_p)$, and the sub-intervals $I_n = [\bar{y}_n, \bar{y}_{n+1}]$ and $J_p = [\bar{z}_p, \bar{z}_{p+1}]$ of γ .

Let $R := 2n_0c$. We claim that

$$\min(\bar{y}_N, \bar{z}_P) \leq R \text{ for all } N, P \geq 0. \quad (6.15)$$

According to (6.12), one has the bound $\max(\bar{y}_{n_0}, \bar{z}_{n_0}) \leq n_0c$. Hence it is enough to check that the interval $\mathcal{I} := [\bar{y}_{n_0}, \bar{y}_N] \cap [\bar{z}_{n_0}, \bar{z}_P]$ has length at most $|\mathcal{I}| \leq n_0c$.

Let $q \in \mathcal{I}$. This point lies in some non-empty interval $I_n \cap J_p$ with $n, p \geq n_0$. Since the projection π is 1-Lipschitz, using (6.12) again yields $d(\bar{y}_n, \bar{z}_p) \leq 2c$. According to (6.13) one has $d(y_n, z_p) \geq \beta(n+p)$ so that

$$\text{either } d(y_n, \bar{y}_n) \geq n\beta - c \text{ or } d(z_p, \bar{z}_p) \geq p\beta - c,$$

and Lemma 6.18 now provides a bound for the length of one of the intervals I_n or J_p :

$$\text{either } |I_n| \leq 2c e^{2ac - na\beta} \text{ or } |J_p| \leq 2c e^{2ac - pa\beta}.$$

It follows that the interval \mathcal{I} is bounded by

$$\begin{aligned} |\mathcal{I}| &\leq \sum_{n \geq n_0} 2c e^{2ac - na\beta} + \sum_{p \geq n_0} 2c e^{2ac - pa\beta} \\ &\leq \frac{4c e^{2ac}}{1 - e^{-a\beta}} e^{-n_0 a\beta} \leq n_0 c. \end{aligned}$$

This proves our claim (6.15).

Now, let $n, p \geq \ell_0 := \frac{4n_0 c}{\alpha}$ so that, by (6.13), one has $d(y_0, y_n) \geq 2R$ and $d(y_0, z_p) \geq 2R$. The claim (6.15) tells us that

$$\text{either } d(y_0, \bar{y}_n) \leq R \text{ or } d(y_0, \bar{z}_p) \leq R.$$

Hence by Lemma 6.19, one has

$$\text{either } \theta_{y_0}(y_n, \gamma_+) \geq \frac{1}{2} e^{-bR} \text{ or } \theta_{y_0}(z_p, \gamma_+) \geq \frac{1}{2} e^{-bR}.$$

Since this is true for any ray γ_+ based at y_0 , one gets $\theta_{y_0}(y_n, z_p) \geq \frac{1}{2} e^{-bR}$. \square

Proof of Theorem 6.5. Point a) follows from Propositions 6.13.d and 6.13.e.

Point b) follows from Propositions 6.15.d and 6.15.f. \square

Remark 6.20. It follows from the proof that Theorem 6.5 also holds true for any rough Lipschitz map $f : X \rightarrow Y$ between pinched Hadamard manifolds that satisfies property \mathcal{C} .

7 Beyond quasi-isometric maps

The aim of this Chapter 7 is the following extension of Theorem 1.1 to all weakly coarse embeddings f , and in particular to all coarse embeddings f (see Definitions 6.2 and 6.3).

7.1 Weakly coarse embeddings and harmonic maps

Theorem 7.1. *Every weakly coarse embedding $f : X \rightarrow Y$ between two pinched Hadamard manifolds is within bounded distance from a unique harmonic map $h : X \rightarrow Y$.*

Indeed we will prove a more general proposition using Definition 6.8.

Proposition 7.2. *Every rough Lipschitz map $f : X \rightarrow Y$ satisfying property \mathcal{C} between two pinched Hadamard manifolds is within bounded distance from a unique harmonic map $h : X \rightarrow Y$.*

The main new ingredients in the proof are the construction and the properties of a boundary map of f . These new ingredients which do not involve harmonic maps were explained in Chapter 6. We now explain how to adapt the proof of Theorem 1.1 using these new ingredients.

7.2 Rough Lipschitz harmonic maps

We first want to point out that Theorem 7.1 can not be extended to all rough Lipschitz maps.

Example 7.3. *There exists an injective Lipschitz map $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ from the hyperbolic plane to itself, that extends continuously to the visual boundary as the identity map, and which is not within bounded distance from any harmonic map.*

Proof. We will consider a map $f : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ that commutes to a parabolic subgroup of $\text{Isom}(\mathbb{H}^2)$. Let us work in the upper half-plane model. The map f is defined by

$$f(u, v) = (u, v + v^2) \quad u \in \mathbb{R}, v > 0,$$

so that $f \circ s_t = s_t \circ f$ where $s_t(u, v) = (t - u, v)$ for any $t \in \mathbb{R}$. Observe that f extends continuously to the visual compactification of \mathbb{H}^2 by the identity, and that f is 2-Lipschitz.

Assume by contradiction that there exists a harmonic map $h : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ within bounded distance from f .

First case: the map h is unique. In this case h also commutes to the isometries s_t , so that there exists a continuous function $g : [0, \infty] \rightarrow [0, \infty]$ such that

$$h(u, v) = (u, g(v)) \quad u \in \mathbb{R}, v > 0,$$

and with $g(0) = 0, g(\infty) = \infty$. Saying that h is harmonic is equivalent to requiring the function g to satisfy the differential equation

$$g g'' = (g')^2 - 1.$$

It follows that the harmonic map h coincides with one of the maps $h_a : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ defined by

$$h_a(u, v) = (u, \frac{1}{a} \sinh(av))$$

for some constant $a \geq 0$. Observe that none of the maps h_a is within bounded distance from f , hence the contradiction.

Second case: the map h is not unique. Let h_0, h_1 be two distinct harmonic maps within bounded distance from f . We want again to find a contradiction. We will use arguments similar to those in Chapter 5. Let $x_0 := (0, 1) \in \mathbb{H}^2$. We choose a sequence of points x_n in \mathbb{H}^2 for which the distances

$$d(h_0(x_n), h_1(x_n)) \text{ converge to } \delta := \sup_{x \in \mathbb{H}^2} d(h_0(x), h_1(x)) > 0$$

and we set $y_n := f(x_n)$. Let φ_n and ψ_n be the isometries of \mathbb{H}^2 fixing the point $\infty \in \partial\mathbb{H}^2$ and such that $\varphi_n(x_0) = x_n$ and $\psi_n(x_0) = y_n$. After

extraction, the sequence of maps $\psi_n^{-1} \circ f \circ \varphi_n$ converges to one of the maps $f_\beta : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ with $\beta \in [0, \infty]$ where

$$f_\beta : (u, v) \mapsto \left(\frac{u}{1+\beta}, \frac{v+\beta v^2}{1+\beta} \right) \quad \text{when } 0 \leq \beta < \infty$$

$$f_\infty : (u, v) \mapsto (0, v^2) \quad \text{when } \beta = \infty.$$

For $i = 0$ and 1 , the sequence of harmonic maps $h_{i,n} := \psi_n^{-1} \circ h_i \circ \varphi_n$ converges, after extraction, to a harmonic map $h_{i,\infty} : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ within bounded distance to f_β . The subharmonic function $x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ achieves its maximum value at $x = x_0$, hence is a constant function equal to δ . Therefore, by Corollary 5.19, the harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ take their values in the same geodesic Γ . This forces the equality $\beta = \infty$ and the geodesic Γ is the image of f_∞ . Now we write

$$f_\infty(u, v) = (0, e^{2F_\infty(u,v)}) \quad \text{and} \quad h_{0,\infty}(u, v) = (0, e^{2H_{0,\infty}(u,v)}),$$

where $F_\infty(u, v) = \log v$ and where $H_{0,\infty}$ is a harmonic function.

The function $G_\infty := F_\infty - H_{0,\infty}$ is then a bounded function on \mathbb{H}^2 such that $\Delta G_\infty = 1$. Such a function G_∞ does not exist. Indeed the function $G : x \mapsto 2 \log(\cosh(d(x_0, x)/2))$ also satisfies $\Delta G = 1$ and the function $G - G_\infty$ would be proper and harmonic, contradicting the maximum principle. \square

7.3 An overview of the proof of Proposition 7.2

Proof of Proposition 7.2. The strategy is the same as for Theorem 1.1:

Step 1 : smoothing f out. By Proposition 2.4 there exists a smooth map $\tilde{f} : X \rightarrow Y$ within bounded distance from f and whose first and second covariant derivatives are bounded on X . This function \tilde{f} is Lipschitz and still satisfies property \mathcal{C} . Hence we can assume that $f = \tilde{f}$.

Step 2 : solving a bounded Dirichlet problem. We fix an origin $O \in X$. For any radius R we consider the unique harmonic map $h_R : B(O, R) \rightarrow Y$ satisfying the Dirichlet condition $h_R = f$ on the sphere $S(O, R)$.

Step 3 : estimating the distance $d(h_R, f)$. We will check in Section 7.4:

Proposition 7.4. *There exists a constant $\rho \geq 1$ such that, for any $R \geq 1$, one has $d(h_R, f) \leq \rho$.*

Step 4 : letting the sequence h_R converge to h . We prove this convergence as in Section 3.3. \square

The proofs of Steps 1, 2 and 4, as well as the proof of uniqueness, require only minor modifications from the ones for quasi-isometric maps. Thus, the remaining of this paper will be devoted to the proof of Step 3.

7.4 Interior estimate for rough Lipschitz

In this section we complete the proof of Proposition 7.4 whose structure is exactly the same as the proof of Proposition 3.5. We will just repeat quickly the arguments of Section 4 pointing out the changes in the choice of the many constants involved in the proof.

7.4.1 Strategy

Let X and Y be two Hadamard manifolds whose curvatures are pinched $-b^2 \leq K \leq -a^2 < 0$. Let $k = \dim X$ and $k' = \dim Y$. We fix two constants $M, N > 0$ as in Proposition 4.9. We set $\alpha = \frac{a}{2bk'N}$ so that, with the notation of Propositions 6.13 and 6.15, one has $\nu_\alpha = \frac{1}{2N}$. We set $\nu = 2\nu_\alpha = \frac{1}{N}$.

We start with a C^∞ Lipschitz map $f : X \rightarrow Y$ whose first and second covariant derivatives are bounded. We fix constants $c, C_1, C_2 \geq 1$ such that, f satisfies property \mathcal{C}_{C_1, C_2} as in Definition 6.8 and such that, for all x in X , one has

$$\|Df(x)\| \leq c, \quad \|D^2f(x)\| \leq bc^2. \quad (7.1)$$

We let $C_3 = C_{3, \alpha, \nu} \leq C_4 = C_{4, \alpha, \nu}$ be the two constants as in Proposition 6.13 and 6.15 :

$$C_3 = \frac{C_1 C_2^{1/N}}{1 - e^{-a/(2N)}}, \quad C_4 = \frac{C_1 C_2^{1/N}}{(1 - e^{-bk'\beta})(1 - e^{-a/(2N)})} \text{ where } \beta = \frac{\alpha^2}{2\alpha + c}.$$

Choosing ℓ_0 very large. We fix a point O in X . We introduce a fixed integer radius ℓ_0 depending only on a, b, k, k', c, C_1 and C_2 . This integer $\ell_0 \geq 1$ is only required to satisfy the three inequalities (7.2), (7.3) and (7.4) :

$$b\ell_0 > 1, \quad (7.2)$$

$$\ell_0 > 4n_0c/\alpha, \quad \text{where } n_0 \geq \frac{4e^{2ac}}{1 - e^{-a\beta}} \text{ is chosen with } MC_4e^{-an_0\alpha} \leq \frac{\alpha}{8c}, \quad (7.3)$$

$$16e^{-\frac{a\alpha\ell_0}{4}} < \theta_0 \text{ where } \theta_0 := e^{-2n_0bc}/2. \quad (7.4)$$

Choosing ρ very large. For $R > 0$, let $h_R : B(O, R) \rightarrow Y$ be the harmonic C^∞ map whose restriction to the sphere $\partial B(O, R)$ is equal to f . We let $\rho := \sup_{x \in B(O, R)} d(h_R(x), f(x))$. We argue by contradiction. If this supremum ρ is not uniformly bounded, we can fix a radius R such that ρ satisfies the three inequalities (4.6), (4.7) and (4.8) that we rewrite below:

$$a\rho > 8kbc^2\ell_0, \quad (7.5)$$

$$\frac{2^7(a\rho)^2}{\sinh(a\rho/2)} < \theta_0. \quad (7.6)$$

$$\rho > 4c\ell_0M(2^{10}e^{b\ell_0}k)^N. \quad (7.7)$$

We denote by x a point of $B(O, R)$ where the supremum is achieved: $d(h_R(x), f(x)) = \rho$. According to the boundary estimate (3.2), one has, using (7.5),

$$d(x, \partial B(O, R)) \geq \frac{a\rho}{3kbc^2} \geq 2\ell_0.$$

Getting a contradiction. We will focus on the restrictions of both maps f and h_R to this ball $B(x, \ell_0)$. We introduce the point $y := f(x)$. For ξ on the unit tangent sphere S_x , we will analyze the triangle inequality:

$$\theta_y(f(\xi_{\ell_0}), h_R(x)) \leq \theta_y(f(\xi_{\ell_0}), h_R(\xi_{\ell_0})) + \theta_y(h_R(\xi_{\ell_0}), h_R(x)), \quad (7.8)$$

and prove that on a subset $U_{\ell_0} \setminus A_{x, \alpha}(n_0)$ of the sphere, each term on the right-hand side is small (Lemmas 7.9 and 7.10) while the left-hand side is not always that small (Lemma 7.12), giving rise to the contradiction.

Definition 7.5. Let $U_{\ell_0} = \{\xi \in S_x \mid d(y, h_R(\xi_{\ell_0})) \geq \rho - \ell_0\alpha/2\}$.

7.4.2 Measure estimate

Lemma 7.6. For ξ in S_x , one has $d(y, h_R(\xi_{\ell_0})) \leq \rho + c\ell_0$.

Proof. This is Lemma 4.2. □

Lemma 7.7. For ξ in S_x , and $r \leq \ell_0$, one has $\|Dh_R(\xi_r)\| \leq 2^8 kb\rho$.

Proof. This is Lemma 4.3. It uses (7.2) and (7.5). □

Lemma 7.8. Let $\sigma = \sigma_{x, \ell_0}$ be the harmonic measure on the sphere $S_x \simeq S(x, \ell_0)$ at the center point x . Then one has $\sigma(U_{\ell_0}) \geq \frac{\alpha}{3c}$.

Proof. Same as Lemma 4.4, using Lemma 7.6. □

7.4.3 Estimating the angles

Lemma 7.9. For ξ in $U_{\ell_0} \setminus A_{x, \alpha}(n_0)$, one has $\theta_y(f(\xi_{\ell_0}), h_R(\xi_{\ell_0})) \leq 4e^{-\frac{a\alpha\ell_0}{4}} < \frac{\theta_0}{4}$.

Proof. Same as Lemma 4.5, using (7.4). □

Lemma 7.10. For ξ in S_x , one has $\theta_y(h_R(\xi_{\ell_0}), h_R(x)) \leq \frac{2^5 (a\rho)^2}{\sinh(a\rho/2)} < \frac{\theta_0}{4}$.

Proof. Same as Lemma 4.6, relying on Lemma 7.11 and using both (7.5) and (7.6). □

Lemma 7.11. For all ξ in S_x and $r \leq \ell_0$, one has $d(y, h_R(\xi_r)) \geq \rho/2$.

Proof. Same as Lemma 4.7, using Lemma 7.7 and Condition (7.7). □

Lemma 7.12. There exist ξ, η in $U_{\ell_0} \setminus A_{x, \alpha}(n_0)$ with $\theta_y(f(\xi_{\ell_0}), f(\eta_{\ell_0})) \geq \theta_0$.

Proof of Lemma 7.12. Recall that $\sigma := \sigma_{x, \ell_0}$ denotes the harmonic measure at the point x for the sphere $S(x, \ell_0)$. Let $\sigma_0 := \frac{\alpha}{4c}$. According to Lemma 7.8, one has

$$\sigma(U_{\ell_0}) > \sigma_0 > 0.$$

Since the harmonic measure σ is $(M, 1/N)$ -Frostman (Proposition 4.9), one may apply (6.8) of Proposition 6.13 to σ and get, using (7.3), that

$$\sigma(A_{x, \alpha}(n_0)) \leq MC_3 e^{-\frac{an_0}{2N}} \leq \frac{\alpha}{8c} = \sigma_0/2.$$

Therefore, there exists an element $\xi \in U_{\ell_0} \setminus A_{x, \alpha}(\ell_0)$. One may now apply (6.10) of Proposition 6.15 to the harmonic measure $\sigma = \sigma_{x, \ell_0}$ and get, using (7.3) again, that

$$\sigma(B_{x, \alpha}^{\xi}(n_0)) \leq MC_4 e^{-\frac{an_0}{2N}} \leq \frac{\alpha}{8c} = \sigma_0/2.$$

Therefore, there exists an element $\eta \in U_{\ell_0} \setminus (A_{x, \alpha}(n_0) \cup B_{x, \alpha}^{\xi}(n_0))$. This element satisfies

$$\theta_y(f(\xi_{\ell_0}), f(\eta_{\ell_0})) \geq e^{-2n_0bc}/2 = \theta_0$$

because of (7.3), (7.4) and Proposition 6.15.e. □

End of the proof of Proposition 7.4. Let ξ, η be two vectors of $U_{\ell_0} \setminus A_{x, \alpha}(n_0)$ given by Lemma 7.12. Applying Lemmas 7.9 and 7.10 to ξ and η , one gets

$$\theta_y(f(\xi_{\ell_0}), f(\eta_{\ell_0})) \leq \theta_y(f(\xi_{\ell_0}), h_R(x)) + \theta_y(h_R(x), f(\eta_{\ell_0})) < \theta_0,$$

which contradicts Lemma 7.12. □

The first version of this paper containing Chapters 1 to 5 was released in February 2017. In this second version, Chapters 6 and 7 were added. In between, two related preprints were posted in the ArXiv: [30] and [39].

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