Géométrie pseudo-riemannienne, dynamique hyperbolique et représentations anosoviennes

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Introduction

Ce mémoire présente les travaux que j'ai effectués depuis la soutenance de ma thèse de doctorat. Si le point commun à ces travaux est la notion de sous-groupe anosovien, une classe de sous-groupes discrets des groupes de Lie réels semi-simples introduite par Labourie, c'est la géométrie lorentzienne (et plus généralement pseudo-riemannienne) qui m'a guidé vers cette thématique.

L'hyperboloïde à une nappe

Les objets géométriques que j'étudie sont tous liés à diverses généralisations (en diverses dimensions et signatures) de l'hyperboloïde à une nappe. On considérera donc

$$\mathcal{H} = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 - z^2 = 1\right\}.$$

Cette surface est souvent présentée par sa particularité d'être doublement réglée : elle possède deux feuilletages par droites. Si les droites de l'hyperboloïde peuvent s'interpréter par la géométrie euclidienne de l'espace ambiant (il s'agit des seules courbes de $\mathcal{H}$ qui sont des géodésiques de l'espace euclidien ambiant $\mathbb{R}^3$), elles n'ont pas de caractérisation intrinsèque : pour la métrique riemannienne induite, il s'agit de géodésiques comme les autres. La structure géométrique particulièrement adaptée à l'étude de $\mathcal{H}$ est une métrique lorentzienne : celle induite par la forme quadratique $q_{2,1}$ (dont $\mathcal{H}$ est le niveau $q_{2,1} = 1$) restreinte au fibré tangent $T\mathcal{H}$. Pour cette métrique, les droites de l'hyperboloïde sont exactement les géodésiques de type lumière (i.e. dont les vecteurs tangents sont isotropes pour $q_{2,1}$). Un autre intérêt de cette métrique lorentzienne est son homogénéité : l'action sur $\mathcal{H}$ du groupe $\mathrm{O}(2,1)$ des transformations linéaires qui préservent $q_{2,1}$ est isométrique et transitive. Ce point de vue sur l'hyperboloïde à une nappe $\mathcal{H}$ consiste à le considérer comme l'espace de *de Sitter* dS^2, qui est la variété lorentzienne modèle à courbure constante positive. Autrement dit, le rôle que tient $\mathcal{H}$ en géométrie lorentzienne est similaire à celui occupé par la sphère $\mathbb{S}^2$ en géométrie riemannienne.

L'action $\mathrm{O}(2,1) \curvearrowright \mathcal{H}$ est liée de près à la géométrie du plan hyperbolique $\mathbb{H}^2$. En effet, si l'on considère le modèle de l'hyperboloïde

$$\mathbb{H}^2 = \left\{(x,y,z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 - z^2 = -1,\ z > 0\right\},$$

on peut associer à un point $p \in \mathcal{H}$ une géodésique de $\mathbb{H}^2$, obtenue en intersectant l'espace tangent (vectoriel) $T_p\mathcal{H} \subset \mathbb{R}^3$ avec $\mathbb{H}^2$. D'un point de vue dynamique, ceci veut dire que l'on peut identifier $\mathcal{H}$ à l'espace des orbites (orientées) du *flot géodésique* du plan hyperbolique.

On peut aussi étudier l'hyperboloïde à une nappe comme une quadrique projective. Si l'on identifie $\mathbb{R}^3$ à une carte affine dans $\mathbb{RP}^3$, par exemple en envoyant $(x, y, z) \in \mathbb{R}^3$ sur $[x : y : z : 1] \in \mathbb{RP}^3$, l'hyperboloïde à une nappe $\mathcal{H}$ se compactifie en rajoutant un cercle à l'infini. On reconnaît le projectivisé $\overline{\mathcal{H}}$ du cône isotrope d'une forme quadratique $q_{2,2}$ de signature $(2, 2)$:

$$\overline{\mathcal{H}} = \left\{[x : y : z : t] \in \mathbb{RP}^3 \,\middle|\, x^2 + y^2 - z^2 - t^2 = 0\right\}.$$

Le groupe qui agit naturellement sur $\overline{\mathcal{H}}$ est donc $\mathrm{PO}(2, 2)$, et l'action de $\mathrm{O}(2, 1)$ considérée plus haut correspond au sous-groupe de $\mathrm{PO}(2, 2)$ qui préserve $\mathcal{H}$. Cette action ne préserve aucune métrique (riemannienne ou lorentzienne) sur $\overline{\mathcal{H}}$, mais seulement une structure conforme lorentzienne. La variété lorentzienne conforme que l'on obtient ainsi, appelée *univers d'Einstein* et notée $\mathrm{Ein}^{2,1}$, joue le rôle en géométrie lorentzienne conforme de la sphère conforme $\mathbb{S}^2$ en géométrie riemannienne conforme (c'est-à-dire de la sphère $\mathbb{S}^2$ munie de l'action du groupe conforme $\mathrm{PO}(3, 1)$, et pas seulement de son groupe d'isométries $\mathrm{O}(3)$).

Enfin, le point de vue adopté le plus souvent dans ce mémoire consiste à traiter $\overline{\mathcal{H}}$ comme le bord de l'espace *anti-de Sitter* :

$$\mathbb{A}\mathrm{d}\mathbb{S}^3 = \left\{[x : y : z : t] \in \mathbb{RP}^3 \,\middle|\, x^2 + y^2 - z^2 - t^2 < 0\right\}.$$

L'action de $\mathrm{PO}(2, 2)$ sur $\mathbb{A}\mathrm{d}\mathbb{S}^3$ est transitive et isométrique pour une métrique lorentzienne à courbure sectionnelle constante négative. Autrement dit, l'espace anti-de Sitter est l'analogue lorentzien de l'espace hyperbolique réel $\mathbb{H}^3$. Dans le cadre riemannien, toute 3-variété hyperbolique complète est isométrique à un quotient $\Gamma\backslash\mathbb{H}^3$ où Γ est un sous-groupe discret d'isométries de $\mathbb{H}^3$. Réciproquement, un tel sous-groupe produit une 3-variété hyperbolique quotient $\Gamma\backslash\mathbb{H}^3$, pourvu que Γ soit sans torsion. Dans le cadre lorentzien, les choses se compliquent : un sous-groupe discret sans torsion de $\mathrm{PO}(2, 2)$ n'agit en général pas de façon proprement discontinue sur $\mathbb{A}\mathrm{d}\mathbb{S}^3$, ce qui fait que l'on obtient pas de 3-variété quotient. Réciproquement, les 3-variétés anti-de Sitter ne s'obtiennent pas toutes comme le quotient de $\mathbb{A}\mathrm{d}\mathbb{S}^3$ par un groupe discret qui agit proprement discontinûment sur $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Par exemple, dans son article fondateur [Mes07], Mess a montré que les 3-variétés anti-de Sitter *globalement hyperbolique Cauchy-compactes* (essentiellement celles qui se rétractent sur une surface compacte de type espace) s'obtiennent comme des quotients $\Gamma\backslash\Omega$ d'un ouvert propre $\Omega \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$ par un sous-groupe discret $\Gamma < \mathrm{PO}(2, 2)$. Les variétés étudiées dans ce mémoire sont des généralisations en dimension et signature arbitraires des 3-variétés anti-de Sitter globalement hyperbolique Cauchy-compactes.

Géométrie hyperbolique pseudo-riemannienne

Le cadre principal de ce mémoire sera celui de l'espace hyperbolique pseudo-riemannien

$$\mathbb{H}^{p,q} := \left\{[x_1 : \cdots : x_{p+q+1}] \in \mathbb{RP}^{p+q} \,\middle|\, x_1^2 + \cdots + x_p^2 - x_{p+1} - \cdots - x_{p+q+1}^2 < 0\right\}$$

et de son bord

$$\partial\mathbb{H}^{p,q} := \left\{[x_1 : \cdots : x_{p+q+1}] \in \mathbb{RP}^{p+q} \,\middle|\, x_1^2 + \cdots + x_p^2 - x_{p+1} - \cdots - x_{p+q+1}^2 = 0\right\}.$$

Lorsque l'on sort du cadre riemannien (i.e. pour $q \neq 0$), un sous-groupe discret $\Gamma < \mathrm{PO}(p, q+1)$ n'agit pas toujours de façon proprement discontinue sur $\mathbb{H}^{p,q}$. Notre étude se portera sur les sous-groupes $\mathbb{H}^{p,q}$*-convexe-cocompacts* définis par Danciger, Guéritaud et Kassel dans [DGK18]. Comme le nom le suggère, un tel sous-groupe $\Gamma < \mathrm{PO}(p, q+1)$ a la particularité de préserver un convexe fermé $\mathcal{C} \subset \mathbb{H}^{p,q}$ sur lequel il agit de façon proprement discontinue et cocompacte (si cette définition suffit lorsque $q = 0$, la définition précise en signature quelconque est plus technique). On peut donc espérer, en restreignant l'étude de l'action $\Gamma \curvearrowright \mathbb{H}^{p,q}$ à l'action sur le convexe $\mathcal{C}$, arriver à une situation plus proche de la géométrie hyperbolique riemannienne. Une première similarité est la définition de l'*ensemble limite* $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$ d'un tel groupe, qui est égal à $\overline{\mathcal{C}} \cap \partial\mathbb{H}^{p,q}$. La dynamique $\Gamma \curvearrowright \Lambda_\Gamma$ est la même que dans le cadre riemannien : le groupe Γ est hyperbolique et cette action est topologiquement conjuguée à celle sur son bord de Gromov $\partial_\infty\Gamma$. Mais l'action de Γ sur $\partial\mathbb{H}^{p,q}$ n'est pas pour autant similaire au cas riemannien : lorsque $q \neq 0$, l'action de Γ sur le complémentaire $\partial\mathbb{H}^{p,q} \setminus \Lambda_\Gamma$ n'est pas proprement discontinue.

Je cherche depuis plusieurs années à comprendre la *nature fractale* de l'ensemble limite $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$ d'un sous-groupe $\mathbb{H}^{p,q}$-convexe-cocompact. En étudiant le lien entre cet ensemble limite et la structure pseudo-riemannienne conforme de $\mathbb{H}^{p,q}$, on se rend vite compte que Λ_Γ est homéomorphe à un sous-ensemble de la sphère $\mathbb{S}^{p-1}$, et que dans le cas où Λ_Γ est homéomorphe à $\mathbb{S}^{p-1}$, c'est automatiquement une sous-variété lipschitzienne. Ce comportement s'oppose à celui des sous-groupes quasi-fuchsiens en géométrie hyperbolique, où l'ensemble limite serait typiquement une sphère topologique de dimension de Hausdorff non entière (or la dimension de Hausdorff d'une sous-variété lipschitzienne est égale à sa dimension topologique).

Dans l'article [GM21], écrit en collaboration avec O. Glorieux, nous montrons qu'il est possible de définir une autre notion de dimension, que nous appelons *dimension de Hausdorff pseudo-riemannienne*, qui est plus adaptée pour décrire la géométrie fractale de l'ensemble limite. Tout comme en géométrie hyperbolique, nous interprétons cette dimension comme un *exposant critique*, c'est-à-dire comme un taux de croissance exponentielle

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) = \limsup_{R\to\infty} \frac{1}{R} \operatorname{Log} \operatorname{Card} \{\gamma \in \Gamma \,|\, d_{\mathbb{H}^{p,q}}(o, \gamma\cdot o) \leq R\}$$

qui demande une attention particulière afin d'être bien défini. En effet, la quantité $d_{\mathbb{H}^{p,q}}(x,y)$ représente la longueur d'une géodésique entre deux points $x, y \in \mathbb{H}^{p,q}$ mais n'est pas une distance. Pour que cet exposant ne dépende pas du point $o \in \mathbb{H}^{p,q}$ choisi, nous imposons à ce point d'appartenir à un convexe $\mathcal{C}$ sur lequel Γ agit de façon cocompacte, et nous montrons que la restriction de $d_{\mathbb{H}^{p,q}}$ à $\mathcal{C}$ vérifie une forme affaiblie d'inégalité triangulaire. Le lien entre cet exposant critique et la dimension de Hausdorff pseudo-riemannienne se fait alors par une adaptation de la *théorie de Patterson-Sullivan*.

Les deux autres articles présentés portant sur la géométrie hyperbolique pseudo-riemannienne ont pour point commun de se restreindre au cas de l'espace anti-de Sitter $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1} = \mathbb{H}^{d,1}$, et plus précisément aux groupes $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-*quasi-fuchsiens*, c'est-à-dire aux sous-groupes $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-convexe-cocompacts $\Gamma < \mathrm{PO}(d, 2)$ pour lesquels l'ensemble limite $\Lambda_\Gamma \subset \partial\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$ est homéomorphe à la sphère $\mathbb{S}^{d-1}$. Dans [MST23], co-écrit avec J-M. Schlenker et N. Tholozan, nous construisons des nouveaux exemples de tels sous-groupes lorsque $d \geq 4$, isomorphes à des groupes fondamentaux de *variétés de Gromov-Thurston*, une famille de variétés compactes possédant des métriques riemanniennes à courbure sectionnelle négative mais non localement symétriques. Dans [GM], co-écrit avec O. Glorieux, nous montrons que l'ensemble limite d'un groupe $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-fuchsien ne peut être une sous-variété de classe $\mathcal{C}^1$ que dans le cas très spécifique d'un réseau uniforme $\Gamma < \mathrm{O}(d,1) < \mathrm{PO}(d,2)$, renforçant ainsi l'idée que cet ensemble limite est un objet fractal.

Sous-groupes projectivement anosoviens

Un sous-groupe *projectivement anosovien* $\Gamma < \mathrm{SL}(d,\mathbb{R})$ est essentiellement un sous-groupe discret hyperbolique au sens de Gromov dont on peut réaliser le bord de Gromov comme un sous-ensemble fermé $\Lambda_\Gamma \subset \mathbb{RP}^{d-1}$, que l'on nommera à nouveau *ensemble limite*. Les sous-groupes $\mathbb{H}^{p,q}$-convexe-cocompacts en sont des exemples. C'est dans ce cadre plus général que se placent les deux autres articles présentés dans ce mémoire.

Dans l'article [GMT23], écrit en collaboration avec O. Glorieux et N. Tholozan, nous étudions la géométrie fractale de cet ensemble limite en lien avec un exposant critique qui cette fois-ci est de nature plus algébrique. Plus précisément, étant donné un sous-groupe projectivement anosovien $\Gamma < \mathrm{SL}(d,\mathbb{R})$, on peut considérer une application injective et équivariante $\xi : \partial_\infty\Gamma \to \mathbb{RP}^{d-1}$ dont l'image est l'ensemble limite Λ_Γ ainsi qu'une autre $\xi^* : \partial_\infty \to \mathbb{RP}^{d-1*}$ dont l'image est l'ensemble limite Λ_Γ^* de la représentation duale de Γ. Ces applications sont compatibles au sens où $\xi^{\mathrm{sym}} = (\xi,\xi^*) : \partial_\infty\Gamma \to \mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1*}$ est à valeurs dans la variété de drapeaux partiels

$$\mathcal{F}_{1,d-1} = \left\{([x],[\alpha]) \in \mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1*} \,\middle|\, \alpha(x) = 0\right\}.$$

Un sous-groupe projectivement anosovien $\Gamma < \mathrm{SL}(d,\mathbb{R})$ possède ainsi trois ensembles limites différents

$$\Lambda_\Gamma = \xi(\partial_\infty\Gamma) \subset \mathbb{RP}^{d-1}\ ; \quad \Lambda_\Gamma^* = \xi^*(\partial_\infty\Gamma) \subset \mathbb{RP}^{d-1*}\ ; \quad \Lambda_\Gamma^{\mathrm{sym}} = \xi^{\mathrm{sym}}(\partial_\infty\Gamma) \subset \mathcal{F}_{1,d-1}.$$

Notre travail relie ces ensembles limites aux *exposants critiques*

$$\delta_{1,2}(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R} \mathrm{Log}\,\mathrm{Card}\left\{\gamma \in \Gamma \,\middle|\, \mu_1(\gamma) - \mu_2(\gamma) \leq R\right\}$$

et

$$\delta_{1,d}(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R} \mathrm{Log}\,\mathrm{Card}\left\{\gamma \in \Gamma \,\middle|\, \mu_1(\gamma) - \mu_d(\gamma) \leq R\right\}\ .,$$

où $\mu_1(g) \geq \cdots \geq \mu_d(g)$ représentent les logarithmes des valeurs singulières de $g \in \mathrm{SL}(d,\mathbb{R})$ (i.e. la moitié des logarithmes des valeurs propres de $g^T g$). Si nous obtenons assez directement une majoration de la dimension de Hausdorff

$$\mathrm{Hdim}(\Lambda_\Gamma) \leq \delta_{1,2}(\Gamma)$$

valable pour les trois ensembles limites, il est en revanche plus délicat de minorer une dimension de Hausdorff. Nous parvenons à le faire pour l'ensemble limite symétrisé, la formule

$$\mathrm{Hdim}\left(\Lambda_\Gamma^{\mathrm{sym}}\right) \geq \delta_{1,d}(\Gamma)$$

se déduisant du résultat plus fort

$$\mathrm{Hdim}\left(\Lambda_\Gamma^{\mathrm{sym}}\right) \geq 2\delta_{1,d}(\Gamma)$$

valable pour les sous-groupe *fortement projectivement convexe-cocompacts* (dont les groupes $\mathbb{H}^{p,q}$-convexe-cocompacts sont des exemples), qui permettent une étude proche de celle effectuée dans [GM21], la géométrie hyperbolique pseudo-riemannienne étant remplacée par la géométrie de la *distance de Hilbert* d'un domaine proprement convexe de l'espace projectif. Un autre aspect de ce travail, utilisé dans les démonstrations mais qui est tout de même d'un intérêt indépendant, est l'égalité entre les exposants critiques décrits plus hauts (i.e. des taux de croissance de valeurs singulières) et des entropies (qui mesurent la croissance des valeurs propres).

Enfin, le dernier travail présenté [DMS24], issu d'une collaboration avec B. Delarue et A. Sanders, montre que s'il n'est pas possible de trouver un espace homogène remplissant le rôle de l'espace hyperbolique pour les sous-groupes discrets de $\mathrm{SO}(n,1)$, il est possible de trouver un remplaçant au flot géodésique. Plus précisément, nous considérons un espace homogène $\mathbb{L} \approx \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}^{\pm}(d-1,\mathbb{R})$ muni d'un flot ϕ^t qui commute avec l'action de $\mathrm{SL}(d,\mathbb{R})$ et nous montrons que, pour tout sous-groupe projectivement anosovien sans torsion $\Gamma < \mathrm{SL}(d,\mathbb{R})$, on peut trouver un ouvert $\widetilde{\mathcal{M}_\Gamma} \subset \mathbb{L}$ invariant à la fois par Γ et par le flot ϕ^t, sur lequel Γ agit librement et proprement discontinûment, et de façon à ce que le flot obtenu sur la variété quotient $\mathcal{M}_\Gamma = \Gamma\backslash\widetilde{\mathcal{M}_\Gamma}$ vérifie l'*axiome A de Smale*, une propriété forte d'hyperbolicité uniforme de ce flot.

Le fait d'associer un flot à un sous-groupe projectivement anosovien remonte à la définition originale de Labourie [Lab06]. Cependant, ce flot intervenant dans la définition n'est pas très pratique pour l'étude de tels sous-groupes, en particulier parce qu'il demande un choix à priori de paramétrage. Ce problème de choix de paramétrage a été réglé par Sambarino [Sam14, Sam24], introduisant ce qu'il appelle le *flot de réfraction*. Ce flot de réfraction est très efficace dans l'étude des sous-groupes anosoviens, par exemple des questions de comptage sur le groupe se résolvent par des propriétés ergodiques du flot. Sambarino arrive ainsi à l'estimée

$$\mathrm{Card}\left\{[\gamma] \in [\Gamma]_{\mathrm{prim}} \,\middle|\, \lambda_1(\gamma) \leq R\right\} = \frac{e^{h(\Gamma)R}}{h(\Gamma)R} + \mathrm{O}(1)$$

où le nombre $h(\Gamma) > 0$ représente l'entropie topologique du flot de réfraction, $[\Gamma]_{\mathrm{prim}}$ est l'ensemble des classes de conjugaison d'éléments primitifs de Γ et $\lambda_1(\gamma) \in \mathbb{R}$ est le logarithme de son rayon spectral. L'usage du flot de réfraction est tout de même limité par sa faible régularité : il s'agit d'un flot Hölder sur un espace métrique. Notre point de vue permet de réaliser ce flot de réfraction comme la restriction du flot lisse $\mathcal{M}_\Gamma \curvearrowleft \phi^t$ à un sous-ensemble compact invariant $\mathcal{K}_\Gamma \subset \mathcal{M}_\Gamma$. Ce nouveau point de vue permet d'étudier les sous-groupes projectivement anosoviens à l'aide d'outils de *dynamique différentiable*. Comme application, nous montrons le *mélange exponentiel* de ces flots, et en déduisons une asymptotique plus précise : l'existence d'un nombre $c \in (0, h(\Gamma))$ tel que

$$\mathrm{Card}\left\{[\gamma] \in [\Gamma]_{\mathrm{prim}} \,\middle|\, \lambda_1(\gamma) \leq R\right\} = \frac{e^{h(\Gamma)R}}{h(\Gamma)R}\left(1 + \mathrm{O}(e^{-cR})\right) .$$

Ce travail a aussi un intérêt géométrique. Par exemple, si l'on étudie un sous-groupe projectivement anosovien $\Gamma < \mathrm{SL}(d,\mathbb{R})$ qui est isomorphe au groupe fondamental d'une surface compacte sans bord Σ de genre au moins 2 (comme par exemple les images de *représentations de Hitchin*, l'exemple fondateur de la théorie), une question qui revient souvent est de savoir si l'on peut trouver une copie de la surface Σ, ou plutôt un plongement $\pi_1(\Sigma)$-équivariant de son revêtement universel $\widetilde{\Sigma}$, dans un espace homogène pour $\mathrm{SL}(d,\mathbb{R})$ (typiquement son espace symétrique riemannien $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$). Notre approche ne donne pas de telle surface, mais le compact invariant $\mathcal{K}_\Gamma \subset \mathcal{M}_\Gamma$ est une copie canonique du fibré unitaire tangent $T^1\Sigma$ dans une variété localement homogène $\mathcal{M}_\Gamma$ modelée sur $\mathbb{L}$.

Organisation du mémoire

Nous commencerons par une présentation des notions de géométrie hyperbolique pseudo-riemannienne et de sous-groupes anosoviens nécessaires à la lecture du mémoire.

Les trois parties suivantes portent sur les sous-groupes $\mathbb{H}^{p,q}$-convexe-cocompacts. Nous en verrons d'abord cinq familles d'exemples dans la partie 2. Les trois premières sont bien connues, mais la quatrième est jusqu'ici absente de la littérature sur le sujet. La dernière famille d'exemples est celle construite dans l'article [MST23] en partant de variétés de Gromov-Thurston. La partie 3 traite la régularité des ensembles limites. Si son but principal est de présenter les résultats de [GM], qui se restreint à la signature lorentzienne, une grande proportion est écrite en signature quelconque, à la fois pour obtenir un résultat plus général et pour clarifier l'importance de la signature lorentzienne dans [GM]. La partie 4 vient ensuite présenter la théorie de Patterson-Sullivan pseudo-riemannienne développée dans [GM21].

La suite du mémoire se place dans le cadre plus général des sous-groupes projectivement anosoviens de $\mathrm{SL}(d, \mathbb{R})$. La partie 5 résume les résultats obtenus dans [GMT23] sur les dimensions de Hausdorff des ensembles limites de ces sous-groupes. La partie 6 porte sur la construction d'un flot lisse associé à un sous-groupe projectivement anosovien issue de l'article [DMS24], et présente les propriétés différentiables et ergodiques de ce flot.

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1 Background on pseudo-Riemannian hyperbolic geometry and projective Anosov subgroups

1.1 Models for $\mathbb{H}^{p,q}$ and its boundary

Consider two integers p,q with $p \geq 1$ and $q \geq 0$, and equip the vector space $V = \mathbb{R}^{p+q+1}$ with the non-degenerate bilinear form $\langle \cdot,\cdot \rangle_{p,q+1}$ of signature $(p,q+1)$ defined by

$$\langle x,y \rangle_{p,q+1} := \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{d} x_j y_j \quad \forall x,y \in \mathbb{R}^{p+q+1}.$$

The pseudo-hyperbolic space $\mathbb{H}^{p,q}$ is defined as the projectivization of *timelike* vectors, i.e.

$$\mathbb{H}^{p,q} := \left\{ [x] \in \mathbb{P}(V) \,\middle|\, \langle x,x \rangle_{p,q+1} < 0 \right\}.$$

Two important cases are the (Riemannian) hyperbolic space $\mathbb{H}^p = \mathbb{H}^{p,0}$, and the (Lorentzian) anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^{p+1} = \mathbb{H}^{p,1}$.

The geometry of $\mathbb{H}^{p,q}$ is best described by considering its double cover

$$\widehat{\mathbb{H}}^{p,q} := \left\{ x \in V \,\middle|\, \langle x,x \rangle_{p,q+1} = -1 \right\}.$$

The restriction of $\langle \cdot,\cdot \rangle_{p,q+1}$ to tangent spaces

$$T_x \widehat{\mathbb{H}}^{p,q} = x^{\perp} := \{ v \in V \,|\, \langle x,v \rangle_{p,q+1} = 0 \}$$

endows $\widehat{\mathbb{H}}^{p,q}$ with a pseudo-Riemannian metric of signature (p,q) invariant under both the $\mathrm{O}(p,q+1)$-action and the antipodal map. It therefore descends to a $\mathrm{PO}(p,q+1)$-invariant pseudo-Riemannian metric on $\mathbb{H}^{p,q}$. Since the differential $d[\cdot] : T\widehat{\mathbb{H}}^{p,q} \to T\mathbb{H}^{p,q}$ is also a double cover, we may describe the tangent bundle of $\mathbb{H}^{p,q}$ as

$$T\mathbb{H}^{p,q} = \left\{ [x:v] \in \mathbb{P}(V \times V) \,\middle|\, \langle x,x \rangle_{p,q+1} < 0, \langle x,v \rangle_{p,q+1} = 0 \right\}.$$

With this description, the pseudo-Riemannian metric is expressed as

$$\Big([x:v_1], [x:v_2] \Big)_{[x]} = -\frac{\langle v_1,v_2 \rangle_{p,q+1}}{\langle x,x \rangle_{p,q+1}}.$$

If $q = 0$, we recover the (Klein model of the) real hyperbolic space $\mathbb{H}^p = \mathbb{H}^{p,0}$. If $q = 1$, we find the projective model of the anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^{p+1} = \mathbb{H}^{p,1}$.

The boundary of $\mathbb{H}^{p,q}$ is simply its boundary within the real projective space

$$\partial \mathbb{H}^{p,q} = \overline{\mathbb{H}^{p,q}} \setminus \mathbb{H}^{p,q} = \left\{ [x] \in \mathbb{P}(V) \,\middle|\, \langle x,x \rangle_{p,q+1} = 0 \right\}.$$

Notations and inner product for points of $\overline{\mathbb{H}^{p,q}}$

Given a point $x \in \mathbb{H}^{p,q} \subset \mathbb{RP}^{p+q}$, which is a line in $\mathbb{R}^{p,q+1}$, we will add a tilde to denote a lift $\widetilde{x} \in x \subset \mathbb{R}^{p+q+1}$ satisfying $\langle \widetilde{x}, \widetilde{x} \rangle_{p,q+1} = -1$, provided the expression in which it is used does not depend on such a lift (as there are two choices). With this convention, for $x, y \in \mathbb{H}^{p,q}$, we can define the number $|\langle \widetilde{x}, \widetilde{y} \rangle_{p,q+1}|$ (but not $\langle \widetilde{x}, \widetilde{y} \rangle_{p,q+1}$). Similarly, given $\xi \in \partial \mathbb{H}^{p,q}$ we will add a tilde to denote a choice of a lift $\widetilde{\xi} \in \xi \subset \mathbb{R}^{p+q+1}$, provided once again that the expression in which it is used does not depend on such a lift. For example, given $(x, \xi) \in \mathbb{H}^{p,q} \times \partial \mathbb{H}^{p,q}$, the expression $\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1} \neq 0$ is well defined, even though the number $\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1}$ is not.

1.2 Isometries of $\mathbb{H}^{p,q}$

The group of orientation preserving isometries of $\mathbb{H}^{p,q}$ is the group $\mathrm{PO}(p, q+1)$ of projective transformations of $\mathbb{RP}^{p+q}$ whose lifts to $\mathbb{R}^{p+q+1}$ preserve the bilinear form $\langle \cdot | \cdot \rangle_{p,q+1}$. It acts transitively on $\mathbb{H}^{p,q}$. The stabilizer of a point $x \in \mathbb{H}^{p,q}$ in $\mathrm{PO}(p, q+1)$ is isomorphic to $\mathrm{O}(p, q)$. For $x_0 = [0 : \cdots : 0 : 1]$, the associated embedding $\mathrm{O}(p, q) \subset \mathrm{PO}(p, q+1)$ corresponds to the standard inclusion by block-diagonal matrices

$$\begin{cases} \mathrm{O}(p,q) & \to & \mathrm{PO}(p,q+1) \\ A & \mapsto & \begin{bmatrix} A & \\ & 1 \end{bmatrix} \end{cases}$$

so $\mathbb{H}^{p,q}$ can be seen as the homogeneous space $\mathrm{PO}(p, q+1)/\mathrm{O}(p, q)$.

1.3 Geodesics

As in any pseudo-Riemannian manifold, geodesics of $\mathbb{H}^{p,q}$ are defined as geodesics of the Levi-Civita connection. Geodesic lines are intersections of $\mathbb{H}^{p,q}$ with projectivizations $\mathbb{P}(V) \subset \mathbb{RP}^{p+q}$ of 2-dimensional planes $V \subset \mathbb{R}^{p+q+1}$ such that $\mathbb{P}(V) \cap \mathbb{H}^{p,q} \neq \emptyset$.

The condition $\mathbb{P}(V) \cap \mathbb{H}^{p,q} \neq \emptyset$ means that V contains a negative direction, so the restriction of $\langle \cdot, \cdot \rangle_{p,q+1}$ to V has signature $(0, 2)$, $(1, 1)$ or $(0, 1)$. These 3 possible signatures correspond to the 3 types of geodesics in pseudo-Riemannian manifolds (respectively timelike, spacelike and lightlike).

Given two distinct points $x, y \in \mathbb{H}^{p,q}$, there is a unique geodesic line $(xy) \subset \mathbb{H}^{p,q}$ going through x and y, it has a simple description:

$$(xy) = \mathbb{P}(x \oplus y) \cap \mathbb{H}^{p,q}.$$

The type of the geodesic (xy) depends only on the inner product. It is (recall that $\widetilde{x}$ denotes a lift of x to $\mathbb{R}^{p+q+1}$ satisfying $\langle \widetilde{x}, \widetilde{x} \rangle_{p,q+1} = -1$, and $\widetilde{y}$ is defined in a similar way):

- Spacelike if and only if $|\langle \widetilde{x}, \widetilde{y} \rangle_{p,q+1}| > 1$,
- Lightlike if and only if $|\langle \widetilde{x}, \widetilde{y} \rangle_{p,q+1}| = 1$,
- Timelike if and only if $|\langle \widetilde{x}, \widetilde{y} \rangle_{p,q+1}| < 1$.

Not all geodesics of $\mathbb{H}^{p,q}$ have endpoints on $\partial\mathbb{H}^{p,q}$, which is a major difference with Riemannian hyperbolic geometry. However, the situation is nicer if we restrict ourselves to spacelike geodesics. Indeed, timelike geodesics are closed, so they never meet the boundary, and lightlike geodesics meet the boundary at exactly one point.

Lemma 1.1. *Given $x \in \mathbb{H}^{p,q}$ and $\xi \in \partial\mathbb{H}^{p,q}$, there is a unique geodesic $(x\xi)$ of $\mathbb{H}^{p,q}$ passing though x with endpoint ξ. It is spacelike if and only if $\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1} \neq 0$. In this case, it can be parametrized as $f(s)$ where:*

$$\begin{aligned}\widetilde{f(s)} &= \cosh(s)\widetilde{x} - \sinh(s)\left(\frac{\widetilde{\xi}}{\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1}} + \widetilde{x}\right) \\ &= e^{-s}\widetilde{x} - \frac{\sinh s}{\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1}}\widetilde{\xi}.\end{aligned}$$

Proof. Since $\mathbb{P}(x \oplus \xi)$ is the only projective line containing x and ξ, we have the existence and uniqueness of the geodesic. Since $\langle \widetilde{x}, \widetilde{x} \rangle_{p,q+1} = -1$ and $\left\langle \widetilde{\xi}, \widetilde{\xi} \right\rangle_{p,q+1} = 0$, the signature of the restriction of $\langle \cdot, \cdot \rangle_{p,q+1}$ to $x \oplus \xi$ is $(1,1)$ if and only if $\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1} \neq 0$. Since the formula for $f(s)$ is a unique speed parametrization of $\mathbb{P}(x \oplus \xi) \cap \mathrm{AdS}^{n+1}$, it is a parametrization of the geodesic $(x\xi)$. □

We will denote by $[x\xi) = f([0, +\infty))$ the half geodesic going from x to ξ. Pairs of points in $\partial\mathbb{H}^{p,q}$ do not always define a geodesic of $\mathbb{H}^{p,q}$. Given $\xi, \eta \in \partial\mathbb{H}^{p,q}$, there is a spacelike geodesic of $\mathbb{H}^{p,q}$ with endpoints ξ and η if and only if $\left\langle \widetilde{\xi}, \widetilde{\eta} \right\rangle_{p,q+1} \neq 0$, in which case we denote it by $(\xi\eta)$.

1.4 The geometry of $\partial\mathbb{H}^{p,q}$

The boundary $\partial\mathbb{H}^{p,q}$ carries a $\mathrm{PO}(p, q+1)$-invariant pseudo-Riemannian conformal class.

Definition 1.2. A lift of $\partial\mathbb{H}^{p,q}$ is a submanifold $\Sigma \subset V$ with the following properties:

- The restriction of the projection map $[\cdot] : V \setminus \{0\} \to \mathbb{P}(V)$ is a double cover $\Sigma \to \partial\mathbb{H}^{p,q}$,
- $-\xi \in \Sigma$ for any $\xi \in \Sigma$.

Lifts of $\partial\mathbb{H}^{p,q}$ exist, as proves the example

$$\Sigma = \left\{ x \in \mathbb{R}^{p,q+1} \,\middle|\, x_1^2 + \cdots + x_p^2 = x_{p+1}^2 + \cdots + x_{p+q+1}^2 = 1 \right\} = \mathbb{S}^{p-1} \times \mathbb{S}^q.$$

There is however no natural choice (whereas $\mathbb{H}^{p,q}$ has a natural lift $\widehat{\mathbb{H}}^{p,q}$). If $\Sigma \subset V$ is such a lift, then the flat pseudo-Riemannian metric of $\mathbb{R}^{p,q+1}$ restricts to a pseudo-Riemannian metric of signature $(p-1, q)$ on Σ. One can then consider its push-forward g_Σ on $\partial\mathbb{H}^{p,q}$. If $\Sigma, \Sigma' \subset V$ are two such lifts, there is a smooth function $\lambda : \Sigma \to \mathbb{R}_+$ such that the map $x \mapsto \lambda(x)x$ is a diffeomorphism from Σ to Σ'. The pull-back of the pseudo-Riemannian metric on Σ' by this map can be easily computed:

$$\begin{aligned}\langle d(\lambda \cdot x), d(\lambda \cdot x)\rangle_{p,q+1} &= \langle d\lambda \cdot x + \lambda \cdot dx, d\lambda \cdot x + \lambda \cdot dx\rangle_{p,q+1} \\ &= (d\lambda)^2 \underbrace{\langle x, x\rangle_{p,q+1}}_{=0} + 2\lambda d\lambda \underbrace{\langle x, dx\rangle_{p,q+1}}_{=0} + \lambda^2 \langle dx, dx\rangle_{p,q+1} \\ &= \lambda^2 \langle dx, dx\rangle_{p,q+1}.\end{aligned}$$

Pushed down to $\partial\mathbb{H}^{p,q}$, this translates to $g_{\Sigma'} = \lambda^2 g_\Sigma$, so there is a naturally defined conformal class of signature $(p-1, q)$ on $\partial\mathbb{H}^{p,q}$. Note that any metric g is this conformal class can be realised as $g = g_\Sigma$ for a unique submanifold $\Sigma \subset V$ with the above properties.

We will now see that certain open subsets of $\partial\mathbb{H}^{p,q}$ carry preferred metrics in the restricted conformal class once a base point in $\mathbb{H}^{p,q}$ has been chosen.

Definition 1.3. Let $x \in \mathbb{H}^{p,q}$. Its dual hyperplane is the set

$$x^* := \left\{ y \in \mathbb{H}^{p,q} \,\middle|\, \langle \widetilde{x}, \widetilde{y}\rangle_{p,q+1} = 0 \right\}.$$

The dual hyperplane x^* is a totally geodesic embedded copy of $\mathbb{H}^{p,q-1}$ in $\mathbb{H}^{p,q}$. Conversely, any totally geodesic embedded copy of $\mathbb{H}^{p,q-1}$ in $\mathbb{H}^{p,q}$ is equal to x^* for a unique point $x \in \mathbb{H}^{p,q}$. Note that if $q = 0$, then x^* is empty.

Definition 1.4. Let $x \in \mathbb{H}^{p,q}$. The affine domain associated to x is

$$U(x) = \left\{ y \in \mathbb{H}^{p,q} \,\middle|\, \langle \widetilde{x}, \widetilde{y}\rangle_{p,q+1} \neq 0 \right\} = \mathbb{H}^{p,q} \setminus x^*.$$

The pseudo-spherical domain associated to x is

$$\partial U(x) = \left\{ \xi \in \partial\mathbb{H}^{p,q} \,\middle|\, \left\langle \widetilde{x}, \widetilde{\xi}\right\rangle_{p,q+1} \neq 0 \right\} = \partial\mathbb{H}^{p,q} \setminus \partial x^*.$$

By definition, the affine domain $U(x)$ is the intersection of $\mathbb{H}^{p,q}$ with an affine patch. Consider the case of $x_0 = [0 : \cdots : 0 : 1] \in \mathbb{H}^{p,q}$. The affine domain $U(x_0)$ consists of points $[u_1 : \cdots : u_{p+q+1}] \in \mathbb{H}^{p,q}$ such that $u_{p+q+1} \neq 0$. The affine chart

$$[u_1 : \cdots : u_{p+q+1}] \mapsto \left(\frac{u_1}{u_{p+q+1}}, \ldots, \frac{u_{p+q}}{u_{p+q+1}} \right)$$

maps $U(x_0)$ to an open set W of $\mathbb{R}^{p+q}$, and sends geodesics to affine lines in $\mathbb{R}^{p+q}$.

More precisely, if we denote by $q_{p,q}$ the standard quadratic form of signature (p,q) on $\mathbb{R}^{p+q}$, then $W = q_{p,q}^{-1}((-\infty, 1))$ is the interior of the quadric $Q = q_{p,q}^{-1}(\{1\})$, which is the image of $\partial U(x_0)$ through the same map. If $q = 0$, then Q is a sphere, and we recover the Klein model of hyperbolic space. If $q = 1$, then Q is a one sheeted hyperboloid. This explains why we draw $\mathbb{A}\mathrm{d}\mathbb{S}^3 = \mathbb{H}^{2,1}$ as the interior of a one sheeted hyperboloid. As this is a projective model, geodesics are represented by straight lines (see Figure 1).

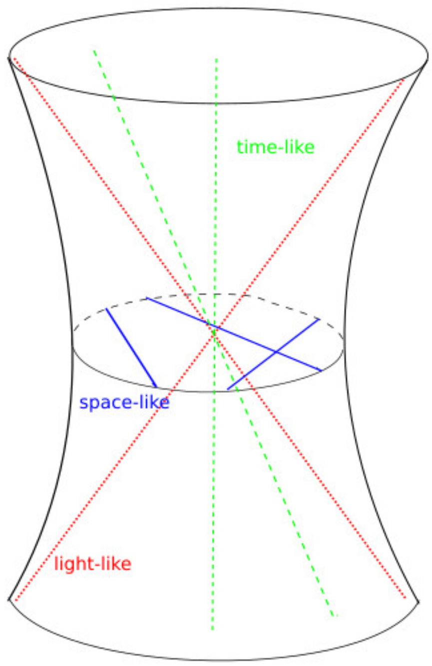

Figure 1: Geodesics of $\mathbb{A}\mathrm{d}\mathbb{S}^3 = \mathbb{H}^{2,1}$ in an affine domain.

Note that a similar description of $U(x)$ and $\partial U(x)$ is valid for any point $x \in \mathbb{H}^{p,q}$ (because the isometry group $\mathrm{PO}(p, q+1)$ acts transitively on $\mathbb{H}^{p,q}$).

Given $x \in \mathbb{H}^{p,q}$, the pseudo-spherical domain $\partial U(x)$ can be equipped with a pseudo-Riemannian metric g_x of signature $(p-1, q)$ such that $(\partial U(x), g_x)$ is isometric to the pseudo-Riemannian sphere $\mathbb{S}^{p-1,q}$. This is achieved by fixing a lift $\widetilde{x} \in \widehat{\mathbb{H}}^{p,q}$ of x and considering the submanifold

$$\begin{aligned}\Sigma_{\widetilde{x}} &= \left\{ y \in \mathbb{R}^{p+q+1} \,\middle|\, \langle y, y \rangle_{p,q+1} = 0,\, \langle \widetilde{x}, y \rangle_{p,q+1} = 1 \right\} \\ &= \left\{ \widetilde{x} + z \in \mathbb{R}^{p+q+1} \,\middle|\, z \in \widetilde{x}^{\perp},\, \langle z, z \rangle_{p,q+1} = 1 \right\}.\end{aligned}$$

The restriction of the flat pseudo-Riemannian metric to $\Sigma_{\widetilde{x}}$ has signature $(p-1, q)$ and constant sectional curvature $+1$. As it is invariant under both the stabiliser of $\widetilde{x}$ in $\mathrm{O}(p, q+1)$ and the antipodal map, it descends to a a pseudo-Riemannian metric g_x of signature $(p-1, q)$ on $\partial U(x)$, that has constant sectional curvature $+1$ and that is invariant under the stabiliser of x in $\mathrm{PO}(p, q+1)$.

1.5 Convex-cocompactness in $\mathbb{H}^{p,q}$

We follow the definitions in [DGK18]. Let us first give a precise definition of the appropriate notion of convexity in real projective geometry.

Definition 1.5. A subset $\Omega \subset \mathbb{RP}^{p+q}$ is properly convex if its closure is contained in some affine chart in which it is bounded and convex.

Definition 1.6. A group $\Gamma \subset \mathrm{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact if it is discrete and the action of Γ on $\mathbb{H}^{p,q}$ preserves a set Ω with the following properties:

1. Ω is closed in $\mathbb{H}^{p,q}$, is properly convex and has non empty interior.
2. The intersection of the closure $\overline{\Omega} \subset \mathbb{RP}^{p+q}$ with $\partial\mathbb{H}^{p,q}$ does not contain any non trivial line segment.
3. The action of Γ on Ω is properly discontinuous and cocompact.

Note that if $q = 0$, i.e. in usual (Riemannian) hyperbolic geometry, the hyperbolic space $\mathbb{H}^p = \mathbb{H}^{p,0}$ is properly convex and contains no trivial line segment in its boundary, so $\mathbb{H}^{p,0}$-convex-cocompactness is equivalent to the usual notion of convex-cocompactness.

We will focus most of our attention on the limit set of such a group.

Definition 1.7. If $\Gamma \subset \mathrm{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, and $\Omega \subset \mathbb{H}^{p,q}$ satisfies the conditions mentioned above, then the limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$ is the set of accumulation points of the Γ-orbit of a point in Ω.

This definition involves a choice of a properly convex set Ω satisfying the conditions described above, and a base point in Ω. However, the resulting limit set Λ_Γ only depends on the group Γ:

Proposition 1.8 ([DGK18])**.** *If $\Gamma \subset \mathrm{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, then Γ is Gromov-hyperbolic, and the action of Γ on Λ_Γ is topologically conjugate to the action on its Gromov boundary $\partial_\infty\Gamma$.*
Moreover, Λ_Γ does not depend on the choice of Ω or a point in Ω.

For $\mathbb{H}^{p,1}$-convex cocompact subgroups of $\mathrm{PO}(p, 2)$ for which the limit set is homeomorphic to $\mathbb{S}^{p-1}$, this was proved in [BM12]. In particular, the action of Γ on the set of triples of distinct points in Λ_Γ is properly discontinuous and cocompact, which will be of some use to us. Proposition 1.8 implies that infinite order elements of Γ have a north-south dynamic as for any hyperbolic group acting on its boundary.

Proposition 1.9. *If $\Gamma \subset \mathrm{PO}(p, q+1)$ is a convex cocompact group, then every infinite order element $\gamma \in \Gamma \setminus \{\mathrm{Id}\}$ acts on Λ_Γ with exactly two fixed points: $\gamma^\pm$. For every $\xi \in \Lambda_\Gamma \setminus \{\gamma^\pm\}$, we have $\lim_{n\to\pm\infty} \gamma^n \cdot \xi = \gamma^\pm$.*

1.6 Negative sets, convex hulls and black domains

One of the important properties of the limit set Λ_Γ is that it is negative.

Definition 1.10. A subset $\Lambda \subset \partial\mathbb{H}^{p,q}$ is negative if it lifts to a cone in $\mathbb{R}^{p,q+1} \setminus \{0\}$ on which all inner products for $\langle \cdot, \cdot \rangle_{p,q+1}$ of non collinear vectors are negative.

If Λ has at least three elements, this is equivalent to any triple $(\xi, \eta, \tau) \in \Lambda^3$ of pairwise distinct points satisfying $\left\langle \widetilde{\xi}, \widetilde{\eta} \right\rangle_{p,q+1} \left\langle \widetilde{\eta}, \widetilde{\tau} \right\rangle_{p,q+1} \left\langle \widetilde{\tau}, \widetilde{\xi} \right\rangle_{p,q+1} < 0$.

Note that the sign of $\left\langle \widetilde{\xi}, \widetilde{\eta} \right\rangle_{p,q+1} \left\langle \widetilde{\eta}, \widetilde{\tau} \right\rangle_{p,q+1} \left\langle \widetilde{\tau}, \widetilde{\xi} \right\rangle_{p,q+1}$ does not depend on a choice of lifts $\widetilde{\xi}, \widetilde{\eta}, \widetilde{\tau} \in \mathbb{R}^{p+q+1}$. This condition means that the intersection of the copy of $\mathbb{RP}^2$ spanned by ξ, η, τ with $\mathbb{H}^{p,q}$ is a totally geodesic copy of $\mathbb{H}^2$.

Proposition 1.11 ([DGK18]). *If $\Gamma < \mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, then its limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$ is negative.*

As a consequence, any two distinct points of Λ_Γ can be joined by a spacelike geodesic of $\mathbb{H}^{p,q}$.

Definition 1.12. Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. For every infinite order element $\gamma \in \Gamma$ we call the spacelike geodesic $(\gamma^-\gamma^+)$ the axis of γ.

Instead of considering the action on a properly convex set with non empty interior (as the ones involved in the definition of $\mathbb{H}^{p,q}$-convex cocompactness), we will work with the convex hull of the limit set (which can have empty interior).

Definition 1.13. If $\Gamma \subset \mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, we define $C(\Lambda_\Gamma)$ as the intersection of $\mathbb{H}^{p,q}$ with the convex hull of Λ_Γ defined in some affine chart containing a convex set Ω as defined above.

Another important subset of $\mathbb{H}^{p,q}$ associated to Λ is its black domain (or invisible domain).

Definition 1.14. Let $\Lambda \subset \partial\mathbb{H}^{p,q}$ be a negative set, and a lift $\widetilde{\Lambda} \subset \mathbb{R}^{p,q+1} \setminus \{0\}$ on which all inner products of non collinear vectors are negative. Its black domain is

$$E(\Lambda) = \mathbb{P}\left(\left\{u \in \mathbb{R}^{p,q+1} \,\middle|\, \langle u, v\rangle_{p,q+1} < 0 \ \forall v \in \widetilde{\Lambda}\right\}\right).$$

One can check that $\Omega(\Lambda)$ is convex, and that it contains $C(\Lambda)$.

Lemma 1.15. *If $x \in \Omega(\Lambda)$, then the dual hyperplane x^* is disjoint from $C(\Lambda)$.*

Proof. It comes from the definition of the black domain $\Omega(\Lambda)$ that x^* is disjoint from Λ. This can be translated as $\Lambda \subset \partial U(x)$. Since $U(x) \cup \partial U(x)$ is convex and it contains Λ, it must contain $\overline{C(\Lambda)}$. □

Negative sets are related to a local product structure of $\mathbb{H}^{p,q}$, itself coming from its conformal pseudo-Riemannian structure. Recall that the projection from $\partial\widehat{\mathbb{H}}^{p,q} = \mathbb{S}^{p-1} \times \mathbb{S}^q \subset \mathbb{R}^{p+q+1}$ to $\partial\mathbb{H}^{p,q} \subset \mathbb{RP}^{p+q}$ is a 2 to 1 covering. We will endow $\mathbb{S}^{p-1}$ and $\mathbb{S}^q$ with their round metrics and associated Riemannian distances, and say that a map $f : X \to Y$ between metric spaces is *distance-decreasing* if it satisfies $d_Y(f(x_1), f(x_2)) < d_X(x_1, x_2)$ whenever $x_1 \neq x_2$.

Lemma 1.16. *Let $\Lambda \subset \partial\mathbb{H}^{p,q}$ be a negative set. Then Λ is homeomorphic to a closed subset $F \subset \mathbb{S}^{p-1}$. More precisely, there is a distance-decreasing map $f : F \to \mathbb{S}^q$ such that Λ is the quotient by the antipodal map of the graph of f. Furthermore, if Λ is homeomorphic to $\mathbb{S}^{p-1}$ then $F = \mathbb{S}^{p-1}$.*

Proof. By definition of a negative set, we can consider a cone $\mathcal{C} \subset \mathbb{R}^{p+q+1}$ whose projection in $\mathbb{RP}^{p+q}$ is Λ and such that any non collinear vectors $v, w \in \mathcal{C}$ satisfy $\langle v, w\rangle_{p,q+1} < 0$. This condition means that two distinct points $(x_1, y_1), (x_2, y_2) \in \mathcal{C} \cap \mathbb{S}^{p-1} \times \mathbb{S}^q$ must satisfy $d_{\mathbb{S}^q}(y_1, y_2) < d_{\mathbb{S}^{p-1}}(x_1, x_2)$, i.e. $\mathcal{C} \cap \mathbb{S}^{p-1} \times \mathbb{S}^q$ is the graph of a non decreasing map $f : F \to \mathbb{S}^q$ for some subset $F \subset \mathbb{S}^{p-1}$.

Denote by $\pi : \mathbb{S}^{p-1} \times \mathbb{S}^q \to \partial\mathbb{H}^{p,q}$ the quotient by the antipodal map. The map

$$\psi : \begin{cases} F & \to & \Lambda \\ x & \mapsto & \pi((x, f(x)) \end{cases}$$

is a homeomorphism, so if Λ is assumed to be homeomorphic to $\mathbb{S}^{p-1}$ then F must be open by applying the invariance of domain to ψ^{-1}, so we must have $F = \mathbb{S}^{p-1}$. □

1.7 $\mathbb{H}^{p,q}$-convex-cocompact groups of maximal dimension

When $q = 0$, the notion of $\mathbb{H}^{p,0}$-convex cocompactness is equivalent to the usual notion in real hyperbolic geometry. This allows to construct examples in any signature: consider a convex cocompact group $\Gamma \subset \mathrm{O}(p,1)$, and its image through the standard embedding $\mathrm{O}(p,1) \hookrightarrow \mathrm{PO}(p,q+1)$. It is $\mathbb{H}^{p,q}$-convex cocompact. One can also consider any representation $\alpha : \Gamma \to \mathrm{O}(q)$, and the image of its graph $\{(\gamma,\alpha(\gamma)) | \gamma \in \Gamma\} \subset \mathrm{O}(p,1) \times \mathrm{O}(q)$ through the standard embedding $\mathrm{O}(p,1) \times \mathrm{O}(q) \hookrightarrow \mathrm{PO}(p,q+1)$. It is also $\mathbb{H}^{p,q}$-convex cocompact. The "largest" example one can construct in this way is obtained when Γ is a uniform lattice in $\mathrm{O}(p,1)$. In this case, the limit set Λ_Γ is homeomorphic to $\partial\mathbb{H}^p \approx \mathbb{S}^{p-1}$.

General $\mathbb{H}^{p,q}$-convex-cocompact subgroups cannot have a larger limit set.

Proposition 1.17. *If $\Gamma < \mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, then the limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$ is homeomorphic to a closed subset of $\mathbb{S}^{p-1}$. Furthermore, if Λ_Γ is homeomorphic to $\mathbb{S}^{p-1}$, then it is a Lipschitz submanifold of $\partial\mathbb{H}^{p,q}$.*

Proof. It follows from Lemma 1.16 and Proposition 1.11. □

Another way of comparing the sizes of discrete groups is the virtual cohomological dimension $\mathrm{vcd}(\Gamma)$. By the work of Bestvina-Mess [BM91, Corollary 1.4], a consequence of Proposition 1.17 is that any $\mathbb{H}^{p,q}$-convex cocompact subgroup $\Gamma < \mathrm{PO}(p,q+1)$ satisfies $\mathrm{vcd}(\Gamma) \le p-1$.

It would seem that when one has two possible definitions for the class of "largest" $\mathbb{H}^{p,q}$-convex cocompact subgroups: either those such that Λ_Γ is homeomorphic to $\mathbb{S}^{p-1}$, or those such that $\mathrm{vcd}(\Gamma) \le p-1$. Thankfully, we do not have to choose since they are equivalent.

Proposition 1.18 ([DGK23, Corollary 11.10])**.** *If $\Gamma < \mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact, then the following are equivalent:*

1. *The virtual cohomological dimension of Γ is equal to p,*
2. *The limit set Λ_Γ is homeomorphic to $\mathbb{S}^{p-1}$.*

Definition 1.19. We will say that a $\mathbb{H}^{p,q}$-convex cocompact subgroup $\Gamma < \mathrm{PO}(p,q+1)$ is *of maximal dimension* if Λ_Γ is homeomorphic to $\mathbb{S}^{p-1}$.

Although we will see results that apply to the general $\mathbb{H}^{p,q}$-convex-cocompact case, our point of focus, in particular all of our examples, will be of maximal dimension.

1.8 Projective Anosov subgroups

Definition 1.20. Let $g \in \mathrm{GL}(d,\mathbb{R})$. We denote by

$$\lambda_1(g) \ge \lambda_2(g) \ge \cdots \ge \lambda_d(g)$$

the logarithms of the moduli of the (complex) eigenvalues of g (with repetitions). We also denote by

$$\mu_1(g) \ge \mu_2(g) \ge \cdots \ge \mu_d(g)$$

the logarithms of its singular values, that is $\mu_i(g) = \frac{1}{2}\lambda_i(g^T g)$.

For $g \in \mathrm{PGL}(d,\mathbb{R})$, we define $\lambda_i(g)$ (resp. $\mu_i(g)$) by $\lambda_i(g) = \lambda_i(\widehat{g})$ (resp. $\mu_i(g) = \mu_i(\widehat{g})$) for some lift $\widehat{g} \in \mathrm{SL}^{\pm}(d,\mathbb{R})$ (i.e. $\det \widehat{g} = \pm 1$) of g.

Remark. In the case of $g \in \mathrm{PO}(p,q+1) \subset \mathrm{PGL}(p+q+1,\mathbb{R})$, we find that $\lambda_i(g) + \lambda_{p+q+2-i}(g) = 0$ when $1 \leq i \leq \min\{p,q+1\}$, and $\lambda_i(g) = 0$ when $\min\{p,q+1\} < i \leq \max\{p,q+1\}$.

Definition 1.21. A finitely generated subgroup $\Gamma < \mathrm{PGL}(d,\mathbb{R})$ (or $\Gamma < \mathrm{SL}^{\pm}(d,\mathbb{R})$) is called *projective Anosov* if there are constants $c, c' > 0$ such that

$$\mu_1(\gamma) - \mu_2(\gamma) \geq c|\gamma| - c'$$

for all $\gamma \in \Gamma$, where $|\gamma|$ denotes its word length with respect to some finite generating set.

Note that this is not the original definition of an Anosov subgroup [Lab06, GW12], but an equivalent characterisation taken from [BPS19, KLP17]. We will also use a characterisation in terms of eigenvalues.

Theorem 1.22 ([KP22])**.** *A finitely generated subgroup $\Gamma < \mathrm{SL}^{\pm}(d,\mathbb{R})$ (or $\Gamma < \mathrm{PGL}(d,\mathbb{R})$) is projective Anosov if and only if it is Gromov-hyperbolic and there are constants $c, c' > 0$ such that*

$$\lambda_1(\gamma) - \lambda_2(\gamma) \geq c|\gamma|_\infty - c'$$

for any $\gamma \in \Gamma$.

Let us list a few of the classic properties of projective Anosov subgroups. We will use the notation $\mathbb{RP}^{d-1*}$ for the projective space $\mathbb{P}(\mathbb{R}^{d*})$ of the dual space $\mathbb{R}^{d*}$, and identify it with the Grassmannian manifold $\mathcal{G}_{d-1}(\mathbb{R}^d)$ of hyperplanes in $\mathbb{R}^d$.

Proposition 1.23. *If $\Gamma < \mathrm{SL}^{\pm}(d,\mathbb{R})$ (or $\Gamma < \mathrm{PGL}(d,\mathbb{R})$) is projective Anosov, there exists a unique pair of Γ-equivariant maps $\xi : \partial_\infty\Gamma \to \mathbb{RP}^{d-1}$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{RP}^{d-1*}$ such that:*

(1) The maps ξ and ξ^* are bi-Hölder homeomorphisms onto their images.

(2) For $t,s \in \partial_\infty\Gamma$, one has $\xi(t) \subset \xi^*(s)$ if and only if $t = s$.

(3) For any $\gamma \in \Gamma$ of infinite order, $\lambda_1(\gamma) > 0$ and $e^{\lambda_1(\gamma)}$ is the modulus of a unique eigenvalue of (any matrix representing) γ. This eigenvalue is real and simple, and the corresponding eigenspace of γ is $\xi(\gamma_+) \in \mathbb{RP}^{d-1}$. Similarly, the eigenspace of the endomorphism $\alpha \mapsto \alpha \circ \gamma^{-1}$ of $\mathbb{R}^{d*}$ corresponding to the eigenvalue of modulus $e^{-\lambda_1(\gamma)}$ is $\xi^*(\gamma^-)$.

One of the consequences of the Anosov property is openness in moduli space.

Theorem 1.24 ([Lab06])**.** *Let $\Gamma < \mathrm{SL}^{\pm}(d,\mathbb{R})$ (or $\Gamma < \mathrm{PGL}(d,\mathbb{R})$) be projective Anosov, and let $\rho : \Gamma \to \mathrm{SL}^{\pm}(d,\mathbb{R})$ (or $\Gamma < \mathrm{PGL}(d,\mathbb{R})$) be a representation. If ρ is sufficiently close to the inclusion map, then it is faithful and $\rho(\Gamma)$ is projective Anosov.*

Projective Anosov subgroups of $\mathrm{PO}(p,q+1)$ are closely related to $\mathbb{H}^{p,q}$-convex cocompact groups, as was found out by Danciger, Guéritaud and Kassel. As a non degenerate bilinear form induces an isomorphism between a vector space and its dual space, projective Anosov subgroups of $\mathrm{PO}(p,q+1)$ can be understood with only one limit map.

Proposition 1.25. *If $\Gamma < \mathrm{PO}(p,q+1)$ is projective Anosov, there exists a unique Γ-equivariant map $\xi : \partial_\infty\Gamma \to \partial\mathbb{H}^{p,q} \subset \mathbb{RP}^{p+q}$ such that:*

(1) The map ξ is a bi-Hölder homeomorphism onto its image.

(2) For $t, s \in \partial_\infty \Gamma$, one has $\left\langle \widetilde{\xi}(t), \widetilde{\xi}(s) \right\rangle_{p,q+1} = 0$ if and only if $t = s$.

(3) For any $\gamma \in \Gamma$ of infinite order, $\lambda_1(\gamma) > 0$ and $e^{\lambda_1(\gamma)}$ is the modulus of a unique eigenvalue of any matrix representing γ. This eigenvalue is real and simple, and the corresponding eigenspace of γ is $\xi(\gamma^+) \in \partial \mathbb{H}^{p,q} \subset \mathbb{RP}^{p+q}$.

Theorem 1.26 ([DGK18, DGK23])**.** *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. Then Γ is projective Anosov and $\Lambda_\Gamma = \xi(\partial_\infty \Gamma)$.*

Conversely, if $\Gamma < \mathrm{PO}(p, q+1)$ is projective Anosov and $\xi(\partial_\infty \Gamma)$ is negative, then Γ is $\mathbb{H}^{p,q}$-convex cocompact.

In the case of $\mathbb{H}^{p,1}$-quasi-Fuchsian subgroups of $\mathrm{PO}(p, 2)$, this was proved by Barbot and Mérigot [BM12]. As negativity of the limit set is also an open condition, we get the following.

Theorem 1.27. *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, and let $\rho : \Gamma \to \mathrm{PO}(p, q+1)$ be a representation. If ρ is sufficiently close to the inclusion map, then it is faithful and $\rho(\Gamma)$ is projective Anosov.*

Another consequence is that the dynamics of elements of Γ acting on Λ_Γ (Proposition 1.9) extends to $\partial \mathbb{H}^{p,q}$. Indeed, any infinite order $\gamma \in \Gamma$ is *proximal*, i.e. satisfies $\lambda_1(\gamma) > \lambda_2(\gamma)$.

Proposition 1.28. *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, $\gamma \in \Gamma$ an infinite order element, and $\gamma^\pm \in \Lambda_\Gamma$ is attracting and repelling fixed points. For any $\xi \in \partial \mathbb{H}^{p,q}$ with $\left\langle \widetilde{\xi}, \widetilde{\gamma} \right\rangle_{p,q+1} \neq 0$, we have $\lim_{n \to +\infty} \gamma^n \cdot \xi = \gamma^+$.*

2 Examples of pseudo-Riemannian convex-cocompact groups

2.1 Globally hyperbolic anti-de Sitter spacetimes

Let us focus on the case of Lorentzian signature, that is the anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1} = \mathbb{H}^{d,1}$. In this setting, convex-cocompact subgroups of maximal dimension have codimension one, so it makes sense to refer to them as quasi-Fuchsian.

Definition 2.1. A subgroup $\Gamma < \mathrm{PO}(d, 2)$ is called $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-*quasi-Fuchsian* if it is $\mathbb{H}^{d,1}$-convex-cocompact of maximal dimension. If is called $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-*Fuchsian* if it preserves a totally geodesic copy of $\mathbb{H}^d$.

The original motivations for the study of $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian groups is that they are holonomies of globally hyperbolic Cauchy compact AdS manifolds (by AdS manifold, we mean a Lorentzian manifold that is locally isometric to $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$, or equivalently that has constant sectional curvature -1).

Recall that a $\mathcal{C}^1$ curve $c : I \to N$ in a Lorentzian manifold (N, h) is called *causal* if its tangent vector is everywhere causal, that is $h_{c(t)}(\dot{c}(t), \dot{c}(t)) \leq 0$ and $\dot{c}(t) \leq 0$ for all $t \in I$. Such a curve is called *inextensible* if none of its re-parametrisations can be extended to a causal curve defined on a larger interval.

Definition 2.2. A *Cauchy hypersurface* in a Lorentzian manifold is a topological hypersurface intersecting every inextensible causal curve at exactly one point. A Lorentzian manifold admitting a Cauchy hypersurface is called *globally hyperbolic*.

A globally hyperbolic Lorentzian manifold (N,h) always admits a *Cauchy temporal function*, i.e. a smooth function to $\mathbb{R}$ with no critical points whose level sets are spacelike Cauchy hypersurfaces, see e.g. [BS05]. Moreover, all smooth Cauchy hypersurfaces are diffeomorphic to each other, and if M is a smooth Cauchy hypersurface, then N is diffeomorphic to $M \times \mathbb{R}$.

Definition 2.3. A globally hyperbolic Lorentzian manifold is *Cauchy compact* if its Cauchy hypersurfaces are compact.

Definition 2.4. A globally hyperbolic Cauchy compact AdS manifold is *maximal* if it is not isometric to a proper open subset of another globally hyperbolic AdS manifold.

From now on, we abbreviate "globally hyperbolic Cauchy compact" into "GHC", and "globally hyperbolic maximal Cauchy compact" into "GHMC".

We recall here a description of GHMC AdS spacetimes, due to Mess [Mes07, ABB⁺07]. It is only stated in dimension $2+1$ in [Mes07], but the argument works in higher dimension as pointed in [Bar08]. For other proofs see [Bar08, Corollary 11.2] and [BM12, Proposition 4.8].

Theorem 2.5 (Mess)**.** *Let $\Gamma < \mathrm{PO}(d,2)$ be a torsion free $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian subgroup. The quotient $N = \Gamma\backslash\Omega(\Lambda_\Gamma)$ is a GHMC AdS manifold.*

Conversely, let N be a GHMC AdS manifold of dimension $d+1$ whose fundamental group is word-hyperbolic. Then there exists a torsion free $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian subgroup $\Gamma < \mathrm{PO}(d,2)$ such that N is isometric to $\Gamma\backslash\Omega(\Lambda_\Gamma)$.

The hypothesis of word-hyperbolicity of the fundamental group in the converse cannot be dropped, but it has many equivalent formulations in this context (see [MST23, Theorem 3.29] for a list). The case of $\mathbb{A}\mathrm{d}\mathbb{S}^3 = \mathbb{H}^{2,1}$ is quite specific, as we can use the exceptional isomorphism $\mathrm{PO}(2,2)_\circ \simeq \mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ to describe $\mathbb{H}^{2,1}$-quasi-Fuchsian groups.

Theorem 2.6 ([Mes07, Bar08])**.** *A torsion free subgroup $\Gamma < \mathrm{PO}(2,2)_\circ \simeq \mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ is $\mathbb{A}\mathrm{d}\mathbb{S}^3$-quasi-Fuchsian if and only if there is a closed oriented surface S, and two hyperbolic metrics h_1, h_2 on S such that*

$$\Gamma = \{(\rho_1(\gamma), \rho_2(\gamma)) \mid \gamma \in \pi_1(S)\}$$

where $\rho_1, \rho_2 : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)_\circ$ are the holonomy representations of h_1, h_2.

This is a Lorentzian analogue of the Bers simultaneous uniformization theorem [Ber72] for $\mathbb{H}^3$-quasi-Fuchsian groups. In this description of $\mathbb{A}\mathrm{d}\mathbb{S}^3$-quasi-Fuchsian groups (called the Mess parametrization), $\mathbb{A}\mathrm{d}\mathbb{S}^3$-Fuchsian groups correspond to pairs (ρ_1, ρ_2) where ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2,\mathbb{R})$ (i.e. they represent the same point in the Teichmüller space of S).

2.2 Representations of surface groups

Let us now turn to the $p = 2$ case. Note that a $\mathbb{H}^{2,q}$-convex-cocompact subgroup of maximal dimension is a hyperbolic group whose Gromov boundary is homeomorphic to the circle, so it must be virtually isomorphic to the fundamental group of a closed hyperbolic surface.

The Lie group $\mathrm{PO}(2,q+1)$ is of Hermitian type, and one can study maximal representations of surface groups. It is shown in [BILW05] that maximality is equivalent to the Anosov property with an additional property on the limit set which can be shown to be equivalent to negativity. Combining with the results of [DGK23], one gets the following:

Theorem 2.7 ([BILW05, DGK23])**.** *Let Γ be the fundamental group of a closed orientable hyperbolic surface, and let $q \geq 0$. A faithful representation $\rho : \Gamma \to \mathrm{PO}(2,q+1)$ is maximal if and only if its image is $\mathbb{H}^{2,q}$-quasi-Fuchsian.*

These representations are studied in depth in [CTT19].

2.3 Deformation of real hyperbolic lattices

Thanks to the stability result of Theorem 1.27, one way to produce $\mathbb{H}^{p,q}$-convex cocompact groups is to start with a convex-cocompact subgroup $\Gamma < \mathrm{O}(p,1)$ and construct a continuous path $(\rho_t)_{t\in\mathbb{R}}$ of representations of Γ into $\mathrm{PO}(p,q+1)$ such that ρ_0 is the standard block-diagonal inclusion. Theorem 1.27 then asserts that $\rho_t(\Gamma)$ is $\mathbb{H}^{p,q}$-convex-cocompact provided that t is small enough. It turns out that in the maximal dimension case (here when $\Gamma < \mathrm{O}(p,1)$ is a uniform lattice), this condition is also closed, as was proved for $q=1$ by Barbot [Bar15] and in the general case by Beyrer and Kassel [BK23].

Theorem 2.8 ([Bar15, BK23])**.** *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension, and let $(\rho_t)_{t\in\mathbb{R}}$ be a continuous path of representations of Γ into $\mathrm{PO}(p,q+1)$ such that ρ_0 is the inclusion. Then ρ_t is faithful and $\rho_t(\Gamma)$ is $\mathbb{H}^{p,q}$-convex-cocompact for all $t \in \mathbb{R}$.*

The standard construction of such a path of representations is given by bending deformations [JM87]. This is achieved by considering a uniform lattice $\Gamma < \mathrm{O}(p,1)$ that splits as an amalgamated product $\Gamma = \Gamma_1 *_\Delta \Gamma_2$ where Δ is a uniform lattice in $\mathrm{O}(p-1,1)$. Geometrically, this means that the hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^p$ (or orbifold if we do not assume Γ to be torsion free) contains a totally geodesic hypersurface $\Sigma = \Delta\backslash\mathbb{H}^{p-1}$ that separates M into two connected components $M \setminus \Sigma = M_1 \cup M_2$ (and $\Gamma_i = \pi_1(M_i)$ for $i = 1,2$). We can then consider any continuous path $(g_t)_{t\in\mathbb{R}}$ in the centraliser of O(p − 1,1) in $\mathrm{PO}(p,q+1)$ (which is commensurable to O(1,q)), and set $\rho_t : \Gamma \to \mathrm{PO}(p,q+1)$ to be the unique representation such that $\rho_t(\gamma) = \rho_0(\gamma)$ for $\gamma \in \Gamma_1$, and $\rho_t(\gamma) = g_t\rho_0(\gamma)g_t^{-1}$ for $\gamma \in \Gamma_2$.

In [MST23, Theorem 1.18], J-M. Schlenker, N. Tholozan and I produce examples of deformations of some specific lattices of $\mathrm{O}(3,1)$ in $\mathrm{PO}(3,2)$ that are not bending deformations.

Deformations of real hyperbolic lattices are not the only examples of $\mathbb{H}^{p,q}$-convex-cocompact subgroups of maximal dimension. If $q=2$, the space of $\mathbb{H}^{p,q}$-convex-cocompact representations of the fundamental group of an orientable surface of genus $g \geq 2$ has 2^{2g+1} connected components [CTT19, section 2.5]. One notable example is the Hitchin component in $\mathrm{PO}(2,3)$, i.e. the connected component containing representations obtained as the composition of the holonomy representation of a hyperbolic metric into $\mathrm{SO}(2,1)_\circ$ with the irreducible representation $\mathrm{SO}(2,1)_\circ \to \mathrm{PO}(2,3)$. In the next sections we will describe examples of $\mathbb{H}^{p,q}$-convex-cocompact subgroups of maximal dimension that are not isomorphic to lattices in $\mathrm{O}(p,1)$.

2.4 Complex hyperbolic lattices

Let us now describe an example that has not yet (to my knowledge) appeared in the literature. Given a semi-simple Lie group G without compact factors, the Killing form of G allows us to interpret the adjoint representation as $\mathrm{Ad}_G : G \to \mathrm{O}(p,q)$ where p is the dimension of the symmetric space of G and q is the dimension of a maximal compact subgroup. In the case of $G = \mathrm{SU}(d,1)$, we find $p = 2d$ and $q = d^2$.

Proposition 2.9. *Let $d \geq 1$, and let $\Gamma < \mathrm{SU}(d,1)$ be a uniform lattice. The subgroup*

$$\mathrm{Ad}_{\mathrm{SU}(d,1)}(\Gamma) < \mathrm{SO}(2d, d^2)$$

is $\mathbb{H}^{2d,d^2-1}$-convex-cocompact.

Remark. It also has maximal dimension, as $\mathrm{vcd}(\Gamma) = \dim \mathbb{H}^d_{\mathbb{C}} = 2d$.

There several ways of proving this, and we will present both a geometric proof, involving a recent geometric characterisation due to Beyrer-Kassel [BK23] using the existence of a $\mathrm{SU}(d,1)$-equivariant spacelike embedding $\mathbb{H}^d_{\mathbb{C}} \to \mathbb{H}^{2d,d^2-1}$, and an algebraic one based on the restricted root space decomposition of the Lie algebra $\mathfrak{su}(d,1)$.

2.4.1 A geometric approach

Let us start by defining the relevant geometric objects. For $z, w \in \mathbb{C}^{d+1}$, consider the standard hermitian form of signature $(d,1)$

$$\langle z, w \rangle = \overline{z_1} w_1 + \cdots + \overline{z_d} w_d - \overline{z_{d+1}} w_{d+1}.$$

The complex hyperbolic space is defined by

$$\mathbb{H}^d_{\mathbb{C}} = \left\{ [z] \in \mathbb{CP}^d \,\middle|\, \langle z, z \rangle < 0 \right\}.$$

The tangent space $T_{[z]}\mathbb{H}^d_{\mathbb{C}}$ can be identified with $\left\{ \dot{z} \in \mathbb{C}^{d+1} \,\middle|\, \langle z, \dot{z} \rangle = 0 \right\}$. The Riemannian metric on $\mathbb{H}^d_{\mathbb{C}}$ is defined as $\|\dot{z}\|^2 = 2\langle \dot{z}, \dot{z} \rangle$ (its sectional curvature ranges between -1 and $-\frac{1}{4}$).

Let $I_{d,1}$ be the diagonal matrix $I_{d,1} = \mathrm{Diag}(1, \ldots, 1, -1) \in \mathrm{GL}(d+1, \mathbb{C})$, so that by identifying $\mathbb{C}^{d+1}$ with column vectors, we have $\langle z, w \rangle = \overline{z}^T I_{d,1} w$. We can now consider the matrix group

$$\mathrm{SU}(d,1) = \left\{ g \in \mathrm{SL}(d+1, \mathbb{C}) \,\middle|\, \overline{g}^T I_{d,1} g = I_{d,1} \right\}$$

and its Lie algebra

$$\mathfrak{su}(d,1) = \left\{ X \in \mathfrak{sl}(d+1, \mathbb{C}) \,\middle|\, \overline{X}^T I_{d,1} + I_{d,1} X = 0 \right\}.$$

The Killing form B of $\mathfrak{su}(d,1)$ satisfies $B(X,Y) = 2(d+1)\mathrm{Tr}(XY)$ for $X, Y \in \mathfrak{su}(d,1)$, so we may consider the trace instead of the Killing form in what follows. Consider the $\mathrm{SU}(d,1)$-equivariant map

$$\varphi : \begin{cases} \left\{ z \in \mathbb{C}^{d+1} \,\middle|\, \langle z, z \rangle = -1 \right\} & \to & \mathfrak{su}(d,1) \\ z & \mapsto & \frac{i}{d+1} 1_{d+1} - \frac{i}{\langle z, z \rangle} z \overline{z}^T I_{d,1} \end{cases}.$$

It satisfies $\mathrm{Tr}(\varphi(z)\varphi(z)) = -\frac{1}{d+1}$, so it induces an embedding $\Phi : \mathbb{H}^d_{\mathbb{C}} \to \mathbb{H}^{2d,d^2-1}$. Its differential, for $\langle z,z\rangle = -1$, satisfies

$$\mathrm{Tr}(d_z\varphi(\dot z))^2) = 2\langle \dot z, \dot z\rangle.$$

In other terms Φ is a spacelike homothetic embedding, in particular $\Phi(\mathbb{H}^d_{\mathbb{C}})$ is a $2d$-dimensional spacelike submanifold of $\mathbb{H}^{2d,d^2-1}$ on which $\mathrm{Ad}_{\mathrm{SU}(d,1)}(\Gamma)$ acts properly discontinuously and cocompactly for any uniform lattice $\Gamma < \mathrm{SU}(d,1)$. According to [BK23, Corollary 1.14], this implies that $\mathrm{Ad}_{\mathrm{SU}(d,1)}(\Gamma)$ is $\mathbb{H}^{2d,d^2-1}$-convex-cocompact.

In order to describe the limit $\Lambda_{\mathrm{Ad}(\Gamma)}$, consider the boundary at infinity of $\mathbb{H}^d_{\mathbb{C}}$

$$\partial\mathbb{H}^d_{\mathbb{C}} = \left\{[z] \in \mathbb{CP}^d \,\middle|\, \langle z,z\rangle = 0\right\}.$$

The embedding $\Phi : \mathbb{H}^d_{\mathbb{C}} \to \mathbb{H}^{2d,d^2-1}$ extends to the boundary:

$$\partial\Phi : \left\{ \begin{array}{ccc} \partial\mathbb{H}^d_{\mathbb{C}} & \to & \mathbb{P}(\mathfrak{su}(d,1)) \\ {[z]} & \mapsto & \left[iz\bar z^T I_{d,1}\right] \end{array} \right. .$$

It satisfies $\mathrm{Tr}(\partial\Phi([z])\partial\Phi([z])) = 0$, so it can be seen as en embedding $\partial\Phi : \partial\mathbb{H}^d_{\mathbb{C}} \to \partial\mathbb{H}^{2d,d^2-1}$, and $\Lambda_{\mathrm{Ad}(\Gamma)} = \partial\Phi(\partial_\infty\mathbb{H}^d_{\mathbb{C}})$ for any uniform lattice $\Gamma < \mathrm{SU}(d,1)$. It is a smooth submanifold of $\partial\mathbb{H}^{2d,d^2-1}$. In order to understand its tangent spaces, it is easier to consider

$$\partial\varphi : \left\{ \begin{array}{ccc} \left\{z \in \mathbb{C}^{d+1}\setminus\{0\} \,\middle|\, \langle z,z\rangle = 0\right\} & \to & \mathfrak{su}(d,1) \\ z & \mapsto & iz\bar z^T I_{d,1} \end{array} \right.$$

and differentiate it to find

$$\begin{aligned} \mathrm{Tr}\left(d_z\partial\varphi(\dot z)^2\right) &= -\left(\dot z\bar z^T I_{d,1}\dot z\bar z^T I_{d,1} + 2\dot z\bar z^T I_{d,1} z\bar{\dot z}^T I_{d,1} + z\bar{\dot z}^T I_{d,1} z\bar{\dot z}^T I_{d,1}\right) \\ &= -\left(\langle z,\dot z\rangle^2 + 2\langle z,z\rangle\langle \dot z,\dot z\rangle + \langle \dot z,\dot z\rangle\right) \\ &= 0. \end{aligned}$$

It follows that $\partial\Phi(\partial\mathbb{H}^d_{\mathbb{C}})$ is a totally isotropic submanifold of the pseudo-Riemannian conformal manifold $\partial\mathbb{H}^{2d,d^2-1}$.

2.4.2 An algebraic approach

This paragraph assumes the reader to be familiar the structure theory of real semi-simple Lie algebras. It also assumes that $d \geq 2$, since the case of $\mathrm{SU}(1,1) \approx \mathrm{SL}(2,\mathbb{R})$ is well known. Let us start by considering a Cartan decomposition $\mathfrak{g} = \mathfrak{su}(d,1)$ as $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{k}^\perp$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of $G = \mathrm{SU}(d,1)$ (e.g. the Lie algebra of $\mathrm{S}(\mathrm{U}(d)\times\mathrm{U}(1))$), and $\mathfrak{k}^\perp$ is the orthogonal of $\mathfrak{k}$ for the Killing form B of $\mathfrak{g}$. We also fix a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{k}^\perp$ (i.e. any one-dimensional vector subspace since $\mathrm{rk}_{\mathbb{R}}(\mathfrak{g}) = 1$), and set $\mathfrak{m} \approx \mathfrak{u}(d-1)$ the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$. Consider a restricted root space decomposition of $\mathfrak{g} = \mathfrak{su}(d,1)$:

$$\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{2\alpha}\oplus\mathfrak{g}_{-\alpha}\oplus\mathfrak{g}_{-2\alpha}$$

where the restricted root spaces satisfy $\dim \mathfrak{g}_{\alpha} = \dim \mathfrak{g}_{-\alpha} = 2(d-1)$ and $\dim \mathfrak{g}_{2\alpha} = \dim \mathfrak{g}_{-2\alpha} = 1$. Given $X \in \mathfrak{a}$, we find $\lambda_1(\mathrm{Ad}(e^X)) = 2\alpha(X)$ and $\lambda_2(\mathrm{Ad}(e^X)) = \alpha(X)$. It follows that $\mu_1(\mathrm{Ad}(g)) - \mu_2(\mathrm{Ad}(g)) = \alpha(\mu(g))$ for any $g \in G$, where $\mu : G \to \mathfrak{a}^+$ is the Cartan projection. This directly implies that the restriction of Ad to any uniform lattice is projective Anosov.

Let $P < G$ be the parabolic subgroup with Lie algebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha}$, and $Q < \mathrm{SO}(B)$ the stabiliser of $\mathfrak{g}_{2\alpha}$. Since $\mathfrak{g}_{2\alpha}$ is an isotropic line in $\mathfrak{g}$, the homogeneous space $\mathrm{SO}(B)/Q$ can be interpreted as $\partial \mathbb{H}^{2d,d^2-1}$. As $\mathrm{Ad}(P) = \mathrm{Ad}(G) \cap Q$, we can consider the embedding

$$\Psi : \left\{ \begin{array}{ccc} G/P \approx \partial \mathbb{H}^d_{\mathbb{C}} & \to & \mathrm{SO}(B)/Q \approx \partial \mathbb{H}^{2d,d^2-1} \\ g \bmod P & \mapsto & \mathrm{Ad}(g) \bmod Q \end{array} \right. .$$

Note that by definition, for $X \in \mathfrak{a}^+$, the direction $\mathfrak{g}_{2\alpha}$ is the eigendirection of $\mathrm{Ad}(e^X) = e^{\mathrm{ad}(X)}$ for its largest eigenvalue $e^{2\alpha(X)}$. More generally, for a loxodromic element $g \in G$ with attracting fixed point $g^+ \in \partial \mathbb{H}^d_{\mathbb{C}}$, $\Psi(g^+)$ is the eigendirection of $\mathrm{Ad}(g)$ for its eigenvalue of largest modulus. Given a uniform lattice $\Gamma < G$, this means that $\Psi(\partial_\infty \mathbb{H}^d_{\mathbb{C}}) = \Lambda_{\mathrm{Ad}(\Gamma)} \subset \partial \mathbb{H}^{2d,d^2-1}$. In order to prove that $\Lambda_{\mathrm{Ad}(\Gamma)}$ is negative, consider the $\mathfrak{sl}_2$-triple (H,E,F) associated to the restricted root 2α, that is $H \in \mathfrak{a}^+$, $E \in \mathfrak{g}_{2\alpha}$ and $F \in \mathfrak{g}_{-2\alpha}$ satisfy $[H,E] = E$, $[H,F] = -F$ and $[E,F] = 2H$.

Consider the lift $\mathcal{L} = \{\mathrm{Ad}(g)E \mid g \in G\} \subset \mathfrak{g}$ of $\Lambda_{\mathrm{Ad}(\Gamma)}$. We wish to show that $B(X,Y) < 0$ for distinct $X, Y \in \mathcal{L}$. But the action of G on G/P is transitive on pairs of distinct points (because $\mathrm{rk}_{\mathbb{R}}(G) = 1$), so it is sufficient to prove $B(\mathrm{Ad}(e^F)E, E) < 0$. As in $\mathrm{SL}(2,\mathbb{R})$, we have $B(E,F) > 0$[1] and $\mathrm{Ad}(e^F)E = E - F - 2H$, therefore

$$\begin{aligned} B(\mathrm{Ad}(e^F)E, E) &= B(E - F - 2H, E) \\ &= -B(F,E) < 0. \end{aligned}$$

We have proved that $\mathrm{Ad}(\Gamma)$ is projective Anosov and that $\Lambda_{\mathrm{Ad}(\Gamma)}$ is negative, so by [DGK18] $\mathrm{Ad}(\Gamma)$ is $\mathbb{H}^{2d,d^2-1}$-convex-cocompact. We can also find from this construction that the limit set is a smooth manifold, and we can see that its tangent spaces are totally isotropic by differentiating

$$d_{e \bmod P}\Psi : \left\{ \begin{array}{ccc} \mathfrak{g}/\mathfrak{p} & \to & \mathfrak{so}(B)/\mathfrak{q} \\ X \bmod \mathfrak{p} & \mapsto & \mathrm{ad}(X) \bmod \mathfrak{q} \end{array} \right.$$

and composing with

$$\left\{ \begin{array}{ccc} \mathfrak{so}(B)/\mathfrak{q} & \to & T_{\mathfrak{g}_{2\alpha}} \partial \mathbb{H}^{2d,d^2-1} = \mathfrak{g}_{2\alpha}^{\perp}/\mathfrak{g}_{2\alpha} \\ \varphi \bmod \mathfrak{q} & \mapsto & \varphi(\mathfrak{g}_{2\alpha}) \bmod \mathfrak{g}_{2\alpha} \end{array} \right. .$$

Since $\mathfrak{g}_{2\alpha}^{\perp} = \mathfrak{p} \oplus \mathfrak{g}_{-\alpha}$, the range of $d_{e \bmod P}\Psi$ identifies with $\mathfrak{g}_{-\alpha}$, which is a totally isotropic subspace of $\mathfrak{g}$.

The embedding $\Phi : \mathbb{H}^d_{\mathbb{C}} \to \mathbb{H}^{2d,d^2-1}$ can also be recovered algebraically. Consider a maximal compact subgroup $K < G$ (recall that $K \approx \mathrm{SU}(d)$), and denote by $\mathfrak{z} \subset \mathfrak{g}$ the Lie algebra of its centre. Then $\mathfrak{z} \approx \mathfrak{u}(1)$ is a negative line in $\mathfrak{g}$ (the restriction of the Killing form to a compact subgroup is negative definite), so

[1] The restriction of B to $\mathfrak{a} \oplus \mathfrak{g}_{2_\alpha} \oplus \mathfrak{g}_{-2\alpha} \approx \mathfrak{sl}(2,\mathbb{R})$ is a multiple of the Killing form of $\mathfrak{sl}(2,\mathbb{R})$ because of its ad-invariance and the simplicity of $\mathfrak{sl}(2,\mathbb{R})$. This multiple is positive because both quadratic forms are positive on H.

by setting $H < \mathrm{SO}(B)$ the stabiliser of $\mathfrak{z}$, we can identify $\mathrm{SO}(B)/H$ with $\mathbb{H}^{2d,d^2-1}$. The embedding Φ now reads as

$$\begin{cases} G/K & \to & \mathrm{SO}(B)/H \approx \mathbb{H}^{2d,d^2-1} \\ g \,\mathrm{mod} K & \mapsto & \mathrm{Ad}(g)\,\mathrm{mod} H \end{cases}.$$

Remark. Given a semi-simple Lie group G without compact factors, if the restriction of the adjoint representation to a uniform lattice is projective Anosov, then G must be virtually isomorphic to $\mathrm{SU}(d,1)$, $d \geq 1$. Indeed, for a uniform lattice to have an Anosov representation, it must Gromov hyperbolic, so $\mathrm{rk}_{\mathbb{R}}(G) = 1$. The classification tells us that G is locally isomorphic to $\mathrm{SO}(d,1)$, $\mathrm{SU}(d,1)$, $\mathrm{Sp}(d,1)$ or the exceptional group F_4^{-20}. In the case of $\mathrm{SO}(d,1)$ ($d \geq 3$), the restricted root space decomposition has the form

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

where $\dim \mathfrak{g}_\alpha = d-1$, so for any element $g \in G$ we find $\mu_1(\mathrm{Ad}(g)) = \mu_2(\mathrm{Ad}(g)) = \alpha(\mu(g))$, where $\mu : G \to \mathfrak{a}^+$ is the Cartan projection. In the cases of $\mathrm{Sp}(d,1)$ ($d \geq 2$) and F_4^{-20}, the restricted root system is non reduced (as for $\mathrm{SU}(d,1)$) and the restricted root space decomposition has the form

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$$

where $\dim \mathfrak{g}_{2\alpha} = 3$ in the case of $\mathrm{Sp}(d,1)$, and $\dim \mathfrak{g}_{2\alpha} = 7$ for F_4^{-20}. In both cases we find $\mu_1(\mathrm{Ad}(g)) = \mu_2(\mathrm{Ad}(g)) = 2\alpha(\mu(g))$ for any $g \in G$.

In all three cases, we have found that $\mu_1(\mathrm{Ad}(g)) = \mu_2(\mathrm{Ad}(g))$ for any $g \in G$, so the restriction of the adjoint representation to a uniform lattice cannot be projective Anosov.

2.5 Gromov-Thurston spacetimes

So far the $\mathbb{H}^{p,q}$-convex-cocompact groups that we have considered have been obtained by the following processes:

- Consider a uniform lattice $\Gamma < \mathrm{O}(p,1)$ and consider deformations of the standard inclusion of Γ in $\mathrm{PO}(p,q+1)$.
- Consider representations of lattices $\Gamma < \mathrm{O}(p,1)$ that are not deformations of the standard inclusion of Γ in $\mathrm{PO}(p,q+1)$, e.g. the Hitchin component in $\mathrm{PO}(2,3)$.
- Consider representations of lattices in another Lie group $G \to \mathrm{PO}(p,q+1)$, e.g. the adjoint representation of $G = \mathrm{SU}(d,1)$.

Another approach consists in working with groups that are not isomorphic to a lattice in any Lie group.

Theorem 2.10. *For any $d \geq 4$, there are $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian subgroups $\Gamma < \mathrm{PO}(d,2)$ that are not isomorphic to a lattice in any Lie group.*

The first published construction works for $4 \leq d \leq 8$, and is due to Lee and Marquis [LM19]. They work with Coxeter groups, a very effective approach with the caveat of only working for small dimensions. Arbitrary dimensions $d \geq 4$ were treated in a collaboration with J-M. Schlenker and N. Tholozan [MST23]. In our approach, we construct representations of fundamental groups of Gromov-Thurston manifolds.

Definition 2.11. A *n-dihedral hyperbolic manifold* of dimension d is the data of a closed oriented hyperbolic manifold M of dimension d and two isometric involutions σ_1 and σ_2 of M with the following properties:

- The fixed loci of σ_1 and σ_2 are connected embedded totally geodesic hypersurfaces,
- The intersection $S = \mathrm{Fix}\,\sigma_1 \cap \mathrm{Fix}\,\sigma_2$ is connected,
- $\mathrm{Fix}\,\sigma_1$ and $\mathrm{Fix}\,\sigma_2$ intersect along S with an angle $\frac{\pi}{n}$.
- $\mathrm{Fix}\,\sigma_1$ and $\mathrm{Fix}\,\sigma_2$ are homologically trivial.

The existence of manifolds M of any dimension $d \geq 2$ with those properties is proved in [GT87]. Under these conditions, σ_1 and σ_2 generate a dihedral group of isometries of M of order $2n$, denoted D_n. We denote by R_n its cyclic subgroup of order n, spanned by $\rho = \sigma_1\sigma_2$, and by $\overline{M} = R_n \backslash M$ the quotient orbifold. We still denote by S its image in $\overline{M}$.

Definition 2.12. Let M be an n-dihedral hyperbolic manifold. For every $a \in \frac{1}{n}\mathbb{N}_{>0}$, we define the *Gromov-Thurston* manifold M^a of ramification a associated to M as the cyclically ramified cover of $\overline{M}$ along S of degree na.

To understand this construction visually, let $H_1 \subset \mathrm{Fix}\,\sigma_1$ be the closure of a connected component of $\mathrm{Fix}\,\sigma_1 \setminus S$, and $H_2 \subset \mathrm{Fix}\,\sigma_2$ the closure of a connected component of $\mathrm{Fix}\,\sigma_2 \setminus S$ chosen so that the oriented angle at S from H_1 to H_2 is $\frac{\pi}{n}$. We then consider the copies of H_1 and H_2 under the isometry $\rho = \sigma_1\sigma_2$ which we denote $H_{2i+1} = \rho^i(H_1)$ and $H_{2i} = \rho^{i-1}(H_2)$ for $i = 1, \ldots, n-1$.

When considering the action of the cyclic subgroup R_n, a fundamental domain is given by the union of two of the former small pieces, e.g. the domain bounded by H_1 and H_3 containing H_2 (see Figure 2).

Just as the hyperbolic manifold M is obtained by gluing n copies of this fundamental piece together, the Gromov-Thurston manifold M^a is obtained by gluing na copies of this same fundamental piece.

Theorem 2.13 ([MST23, Theorem 1.1])**.** *Let M be a n-dihedral hyperbolic manifold of dimension $d \geq 3$ and $a > 1$. There exist a faithful representation $\rho : \pi_1(M^a) \to \mathrm{PO}(d,2)$ whose image is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian.*

More precisely, there is a $na - 3$ parameter family of such representations [MST23, Theorem 1.2]. It was proved by Gromov and Thurston in [GT87] that M^a cannot carry a hyperbolic metric if $d \geq 4$ and $a \neq 1$, and more generally $\pi(M^a)$ cannot be isomorphic to a lattice in any Lie group [MST23, Remark 1.4], so Theorem 2.13 implies Theorem 2.10.

The proof of [MST23, Theorem 1.1] consists in treating M^a as a hyperbolic manifold with a cone singularity along S, and realizing this metric as the induced metric on a polyhedral hypersurface of $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$. Outside of S, this poses no problem as $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$ contains totally geodesic copies of $\mathbb{H}^d$. Locally around S, this is achieved by considering a particular class of polyhedral hypersurfaces that we call *hipped hypersurfaces* [MST23, Definition 5.1] (see Figure 3).

This leads to the construction of a *spacelike* AdS *structure* on M^a, that is an atlas $(U_i, \phi_i)_{i\in I}$ where $(U_i)_{i\in I}$ is an open cover of M^a and $\Phi_i : U_i \to \mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$ is a Lipschitz spacelike immersion. By proving that this spacelike AdS structure has the additional properties of being convex and ruled [MST23, Lemma 5.6], we can use a correspondence between convex ruled spacelike AdS structures on M^a and faithful representations $\pi_1(M^a) \to \mathrm{PO}(d,2)$ whose image is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian [MST23, Lemma 3.37].

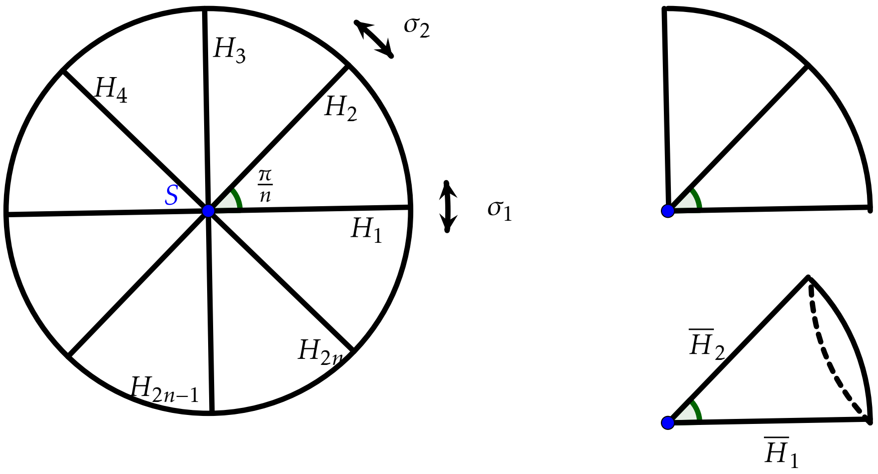

Figure 2: A n-dihedral manifold M, its fundamental piece and the quotient $\overline{M}$.

These representations are parametrised by the hipped hypersurfaces used to realise the cone singular metric of M^a locally in $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$. By parametrising these hipped surfaces by equilateral polygons in the de Sitter space dS^2 [MST23, Lemma 5.4], we show that the deformation space has dimension $na-1$ [MST23, Proposition 4.11].

Remark. This is not the first construction of geometric structures on Gromov-Thurston manifolds. In their paper [GT87], Gromov and Thurston show that they can be equipped with Riemannian metrics whose sectional curvature is pinched arbitrarily close to -1 (provided n is large enough). Kapovich [Kap07] showed that they carry convex projective structures, and by Fine and Premoselli [FP20] that they carry Einstein metrics of negative sectional curvature. In [MST23, Corollary 1.15], we also construct flat conformal structures on Gromov-Thurston manifolds M^a with $a < 1$ (through hyperbolic structures on $M^a \times \mathbb{R}$). The existence of these flat conformal structures was already mentioned in [GT87].

3 Non-differentiability of limit sets

This section discusses the regularity of the limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$ of an $\mathbb{H}^{p,q}$-convex-cocompact group $\Gamma < \mathrm{PO}(p,q+1)$ of maximal dimension (i.e. such that Λ_Γ is homeomorphic to $\mathbb{S}^{p-1}$), with the goal of presenting the results of the paper [GM] in which O. Glorieux and I studied the $q = 1$ case. We have seen in Proposition 1.17 that Λ_Γ is always a Lipschitz submanifold regardless of the signature. The main result of [GM] is that in the $q = 1$ case, it can only be $\mathcal{C}^1$ for $\mathbb{A}\mathrm{d}\mathbb{S}^{p+1}$-Fuchsian groups (Theorem 3.6). We will also discuss other signatures, which is why I included the proofs of several results from [GM] as the paper only covered Lorentzian signature.

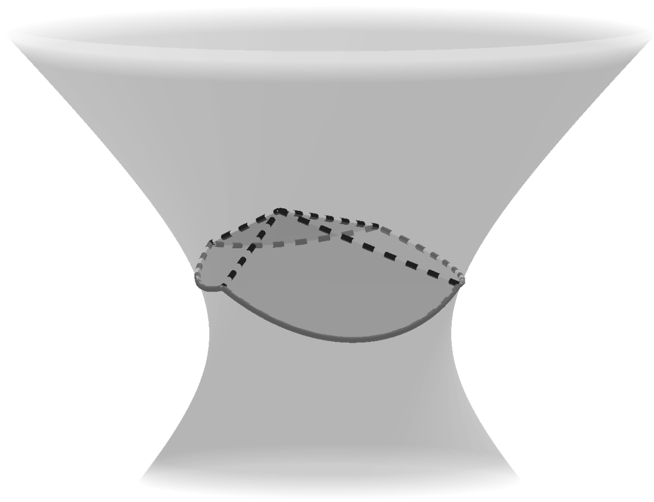

Figure 3: A hipped hypersurface in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

3.1 Zariski closures

One of the intermediate results of [GM] is a classification of Zariski closures of $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian groups: it is either a conjugate of $\mathrm{O}(d,1)$ (up to finite index), or $\mathrm{PO}(d,2)$ itself. The study of these Zariski closures is made easier by their semi-simplicity (which of course fails for general $\mathbb{H}^{p,q}$-convex-cocompact groups without any dimension assumption).

Proposition 3.1. *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension. The Lie algebra of the Zariski closure of Γ is the direct sum of a semi-simple Lie algebra and a Lie subalgebra of $\mathfrak{so}(q+1)$.*

We will call a vector subspace $V \subset \mathbb{R}^{p+q+1}$ non degenerate if the restriction of $\langle\cdot,\cdot\rangle_{p,q+1}$ to V is non degenerate, that is $V \cap V^{\perp} = \{0\}$.

Lemma 3.2. *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension, and let $V \subset \mathbb{R}^{p+q+1}$ be a vector subspace. If Γ preserves $\mathbb{P}(V)$, then V is non degenerate.*

Proof. By contradiction, we just have to eliminate the case of a totally isotropic subspace (by considering $V \cap V^{\perp}$), i.e. we may assume that $\mathbb{P}(V) \subset \partial\mathbb{H}^{p,q}$. Now $\Lambda_\Gamma \subset \mathbb{P}(V)$ is a closed Γ-invariant subset of Λ_Γ. Since the action of Γ on Λ_Γ is topologically conjugate to the action on its Gromov boundary $\partial_\infty\Gamma$, it is minimal (i.e. all orbits are dense). So we find a dichotomy: either $\Lambda_\Gamma \subset \mathbb{P}(V)$ or $\Lambda_\Gamma \cap \mathbb{P}(V) = \emptyset$. The first case is made impossible by the negativity of Λ_Γ, so we just have to rule out the second case.

We now assume that $\Lambda_\Gamma \cap \mathbb{P}(V) = \emptyset$. Let $\gamma \in \Gamma$ be of infinite order, and consider its attracting and repelling fixed points $\gamma^{\pm} \in \Lambda_\Gamma$ (see sections 1.5 and 1.8). Let $\xi \in \mathbb{P}(V)$. If $\left\langle \widetilde{\xi}, \widetilde{\gamma^-} \right\rangle_{p,q+1} \neq 0$, then $\gamma^n \cdot \xi \to \gamma^+$

as $n \to +\infty$ (Proposition 1.28), which would mean that $\gamma^+ \in \mathbb{P}(V)$. As this cannot be the case, we must have $\left\langle \widetilde{\xi}, \widetilde{\gamma^-} \right\rangle_{p,q+1} = 0$, and by density of repelling fixed points of elements of Γ in $\partial_\infty \Gamma$ we find that $\left\langle \widetilde{\xi}, \widetilde{\eta} \right\rangle_{p,q+1} = 0$ for all $\eta \in \Lambda_\Gamma$. Fix a lift $\widetilde{\xi} = (\xi_1, \xi_2) \in \mathbb{S}^{p-1} \times \mathbb{S}^q$ and consider a point $\widetilde{\eta} = (\eta_1, \eta_2) \in \pi^{-1}(\Lambda_\Gamma) \cap \mathbb{S}^{p-1} \times \mathbb{S}^q$ (using the notations of the proof of Lemma 1.16). The equality $\left\langle \widetilde{\xi}, \widetilde{\eta} \right\rangle_{p,q+1} = 0$ means that $d_{\mathbb{S}^{p-1}}(\xi_1, \eta_1) = d_{\mathbb{S}^q}(\xi_2, \eta_2)$. Now consider the distance decreasing function $f : \mathbb{S}^{p-1} \to \mathbb{S}^q$ from Lemma 1.16. Applying this to $\widetilde{\eta} = (\xi_1, f(\xi_1)) \in \pi^{-1}(\Lambda_\Gamma)$, we find that $f(\xi_1) = \xi_2$, hence $\xi \in \Lambda_\Gamma$, which is a contradiction. □

Lemma 3.3. *Let $\Lambda \subset \partial \mathbb{H}^{p,q}$ be a negative set that is homeomorphic to $\mathbb{S}^{p-1}$. The Lie algebra of*

$$G = \{g \in \mathrm{PO}(p, q+1) \,|\, g \cdot \xi = \xi, \forall \xi \in \Lambda\}$$

is isomorphic to a subalgebra of $\mathfrak{so}(q+1)$.

Proof. We use the notations of the proof of Lemma 1.16, and write $\pi^{-1}(\Lambda) = \Lambda^+ \sqcup \Lambda^- \subset \mathbb{S}^{p-1} \times \mathbb{S}^q$ where Λ^+ is the graph of the function $f : \mathbb{S}^{p-1} \to \mathbb{S}^q$ from Lemma 1.16. Let $\widehat{G} < \mathrm{O}(p, q+1)$ denote the lifts of G that act trivially on Λ^+. Its projection in G has index at most two, so the compactness of G and $\widehat{G}$ are equivalent.

Now $\widehat{G}$ also acts trivially on $V = \mathrm{Span}(\Lambda^+)$. Let us show that $V^\perp$ is negative definite. Let $v \in V^\perp \setminus \{0\}$, and decompose $v = (v_1, v_2) \in \mathbb{R}^p \times \mathbb{R}^{q+1}$. Note that if $v_1 = 0$, then $\langle v, v \rangle_{p,q+1} = -\|v_2\|^2 < 0$, so we my assume that $v_1 \neq 0$. Consider $x, y \in \mathbb{S}^{p-1}$ such that $x - y$ is collinear to v_1. The fact that v is orthogonal to both $(x, f(x))$ and $(y, f(y))$ for $\langle \cdot, \cdot \rangle_{p,q+1}$ means that

$$(x - y) \cdot v_1 = (f(x) - f(y)) \cdot v_2$$

where the dot product on each side represents the standard Euclidean inner products of $\mathbb{R}^p$ and $\mathbb{R}^{q+1}$. Since the left hand side is the inner product of collinear vectors, the Cauchy-Schwarz inequality yields

$$\left\| x - y \right\| \|v_1\| \leq \left\| f(x) - f(y) \right\| \|v_2\|.$$

As $v_1 \neq 0$, we also have $x \neq y$, so the left hand side cannot vanish, and neither can the right hand side, hence $v_2 \neq 0$. But f is distance decreasing[2], so $\left\| f(x) - f(y) \right\| < \left\| x - y \right\|$ and thus $\|v_1\| < \|v_2\|$, i.e. $\langle v, v \rangle_{p,q+1} < 0$.

It follows that $\widehat{G}$ is a closed subgroup of the group preserving a negative definite vector subspace $V^\perp$ and acting trivially on its orthogonal V, which is a compact group isomorphic to $\mathrm{O}(\dim V^\perp)$. □

Proof of Proposition 3.1. Consider a lift $\widehat{\Gamma} < \mathrm{O}(p, q+1)$. If the linear action of $\widehat{\Gamma}$ preserves a subspace $V \subset \mathbb{R}^{p+q+1}$, then it is non degenerate by Lemma 3.2, so $\mathbb{R}^{p+q+1} = V \oplus V^\perp$ and $\widehat{\Gamma}$ is totally reducible. It follows that the Zariski closure of Γ is reductive. Since the centraliser of Γ in $\mathrm{PO}(p, q+1)$ acts trivially on Λ_Γ, it is compact thanks to Lemma 3.3. □

Things are much simpler in the $q = 1$ case.

[2]By definition, f is distance decreasing for the Riemannian distances, but this is equivalent to being distance decreasing for the ambient Euclidean norms

Lemma 3.4 ([GM, Lemma 3.1]). *Let $\Gamma < \mathrm{PO}(d,2)$ be $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian. If there is a proper vector subspace $V \subset \mathbb{R}^{d+2}$ such that Γ preserves $\mathbb{P}(V)$, then Γ is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian.*

Sketch of proof. Let $V \subset \mathbb{R}^{d+2}$ be such a subspace. According to Lemma 3.2, V is non degenerate so its signature is $(k,2)$, $(k,1)$ or $(k,0)$ for some integer $k \geq 0$ (here k denotes the number of positive signs in the signature). In the first case, Γ acts on the totally geodesic copy $X = \mathbb{P}(V) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$ of $\mathbb{A}\mathrm{d}\mathbb{S}^{k+1}$. This is not possible because $\Lambda_\Gamma \approx \mathbb{S}^{d-1}$ would then be a negative subset of $\partial X \approx \partial\mathbb{A}\mathrm{d}\mathbb{S}^{k,1}$, thus homeomorphic to a subset of $\mathbb{S}^{k-1}$.

Signature $(k,0)$ can also be ruled out since $V^\perp$ would then have signature $(d-k,2)$. We are left with the Lorentzian signature $(k,1)$, in which case Γ preserves the totally geodesic copy $X = \mathbb{P}(V) \cap \mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$ of $\mathbb{H}^k$. As ∂X must contain Λ_Γ we find that $k = d$, and Γ is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian. ☐

Proposition 3.5 ([GM, Proposition 1.4]). *Let $\Gamma < \mathrm{PO}(d,2)$ be $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian. If Γ is not $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian, then it is Zariski dense in $\mathrm{PO}(d,2)$.*

Sketch of proof. If Γ is not $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian, let $G \leq \mathrm{SO}(d,2)_\circ$ be the identity component of the Zariski closure of lifts of elements of Γ to $\mathrm{O}(d,2)$. Not that finite index subgroups of Γ are not $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian (they have the same limit set), so by Lemma 3.4 G acts irreducibly on $\mathbb{R}^{d+2}$. According to [DSL11, Theorem 1], the only connected Lie subgroups of $\mathrm{SO}(d,2)_\circ$ with this property (other than $\mathrm{SO}(d,2)_\circ$ itself) are $\mathrm{U}(\frac{d}{2},1)$, $\mathrm{SU}(\frac{d}{2},1)$, $\mathbb{S}^1 \cdot \mathrm{SO}(\frac{d}{2},1)$ (when d is even) and the irreducible copy of $\mathrm{SO}(2,1)_\circ$ when $d = 3$. The first family of examples are not possible because they do not contain proximal elements, and the last one is ruled out by a cohomological dimension argument. ☐

3.2 Non differentiability in $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$

We can now turn to the main result of [GM].

Theorem 3.6 ([GM, Theorem 1.3]). *Let $\Gamma < \mathrm{PO}(d,2)$ be $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian. If the limit set Λ_Γ is a $\mathcal{C}^1$ submanifold of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$, then Γ is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian.*

The first step consists in understanding the signature of the restriction of the Lorentzian conformal class of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$ to tangent spaces of Λ_Γ. The Lorentzian signature is key here, as the following result fails in other signatures.

Lemma 3.7 ([GM, Lemma 4.1]). *If $\Gamma < \mathrm{PO}(d,2)$ is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian and Λ_Γ is a $\mathcal{C}^1$ submanifold of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$, then there is $\xi \in \Lambda_\Gamma$ such that $T_\xi\Lambda_\Gamma$ is spacelike.*

Proof. Let $f : \mathbb{S}^{d-1} \to \mathbb{S}^1$ be a distance-decreasing map such that the quotient by the antipodal map of its graph is Λ_Γ (Lemma 1.16).

Knowing that the graph of f is a $\mathcal{C}^1$-submanifold, we first want to show that f is $\mathcal{C}^1$. Using the Implicit Function Theorem, it is enough to know that the tangent space of the graph projects non trivially to the tangent space of $\mathbb{S}^{n-1}$. This is true because f is Lipschitz.

Since f is distance-decreasing, it cannot be onto, so it can be seen as a function $f : \mathbb{S}^{d-1} \to \mathbb{R}$. At a point $x \in \mathbb{S}^{d-1}$ where it reaches its maximum, it satisfies $d_x f = 0$, so the tangent space to Λ_Γ at $(x, f(x))$ is $T_x\mathbb{S}^{d-1} \times \{0\}$, which is spacelike. ☐

Remark. In $\mathbb{H}^{p,q}$, only the first part of the proof works: if $\Gamma < \mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension, then Λ_Γ is the antipodal projection of the graph of a $\mathcal{C}^1$ map $f : \mathbb{S}^{p-1} \to \mathbb{S}^q$. In particular, Λ_Γ is diffeomorphic to $\mathbb{S}^{p-1}$, i.e. it cannot be an exotic sphere.

One can then go from one spacelike tangent space to all tangent spaces being timelike using the topological dynamics of the action of Γ.

Corollary 3.8 ([GM, Corollary 4.2])**.** *If $\Gamma < \mathrm{PO}(d,2)$ is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian and Λ_Γ is a $\mathcal{C}^1$ submanifold of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$, then for all $\xi \in \Lambda_\Gamma$, the tangent space $T_\xi\Lambda_\Gamma$ is spacelike.*

By using the fact that the tangent space to Λ_Γ at the attracting fixed point γ^+ of an infinite order element $\gamma \in \Gamma$, and translating this fact into linear algebra, we get that Γ cannot contain regular elements.

Lemma 3.9 ([GM, Lemma 4.3])**.** *If $\Gamma < \mathrm{PO}(d,2)$ is $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-quasi-Fuchsian and Λ_Γ is a $\mathcal{C}^1$ submanifold of $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$, then any infinite order $\gamma \in \Gamma$ is conjugate in $\mathrm{PO}(d,2)$ to an element of $\mathrm{O}(d,1)$.*

The absence of regular elements is known to be an obstruction to Zariski density, which is the last step in the proof.

Proof of Theorem 3.6. If Γ is not $\mathbb{A}\mathrm{d}\mathbb{S}^{d+1}$-Fuchsian, then by Proposition 3.5 it must be Zariski dense in $\mathrm{PO}(d,2)$, in particular it must contain regular elements of $\mathrm{PO}(d,2)$ [AMS95, Ben97]. This is a contradiction with Lemma 3.9. □

3.3 Other signatures

Theorem 3.6 is not valid for arbitrary signature (p,q). One of the reasons is the failure of Proposition 3.5, and it is possible to find $\mathbb{H}^{p,q}$-convex-cocompact subgroups of maximal dimension groups that are not conjugates of lattices in $\mathrm{O}(p,1) < \mathrm{PO}(p,q+1)$ but that still are uniform lattices in a rank one Lie subgroup of $\mathrm{PO}(p,q+1)$. This can be achieved by considering a uniform lattice in $\mathrm{PSL}(2,\mathbb{R})$ then embedding $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PO}(2,3)$ via the irreducible representation, thus producing $\mathbb{H}^{2,2}$-convex-cocompact subgroups whose limit set is a smooth circle. Another example is the adjoint representation of $\mathrm{SU}(d,1)$ restricted to a uniform lattice that produces $\mathbb{H}^{2d,d^2-1}$-convex-cocompact subgroups whose limit set is smooth and diffeomorphic to $\mathbb{S}^{2d-1}$ (see Proposition 2.9 and the discussion that follows it).

In the first example, we can also consider deformations of these representations in $\mathrm{PO}(2,3)$ (i.e. Hitchin representations), and produce more $\mathbb{H}^{2,2}$-convex-cocompact subgroups whose limit set is a $\mathcal{C}^1$ curve. Is this case, the limit set is no longer smooth [PS17, Theorem D]. In the Zariski dense case, we have the following result of A. Zimmer.

Proposition 3.10 ([Zim21, Corollary 1.48])**.** *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension. If Λ_Γ is a $\mathcal{C}^2$ submanifold of $\partial\mathbb{H}^{p,q}$, then Γ is not Zariski dense in $\mathrm{PO}(p,q+1)$.*

A common trait of these non Fuchsian examples with a $\mathcal{C}^1$ limit set is that this limit set is isotropic for the pseudo-Riemannian conformal structure of $\mathbb{H}^{p,q}$, meaning that Corollary 3.8 fails in non Lorentzian signature. This is more than a coincidence.

Theorem 3.11. *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension. If Λ_Γ is a spacelike $\mathcal{C}^1$ submanifold of $\partial\mathbb{H}^{p,q}$, then Γ preserves a totally geodesic copy of $\mathbb{H}^p$.*

This can be proved using Theorem 4.19 presented in the next section and [MV24, Theorem 4], but let us present a sketch of proof in the spirit of [GM].

Sketch of proof. The proof of Lemma 3.9 works in arbitrary signature (p,q) under the assumption that Λ_Γ is spacelike. It follows that a maximal split torus of the Zariski closure G of Γ lies in a maximal split torus of $\mathrm{O}(p,1) < \mathrm{PO}(p,q+1)$, so G must itself be of the form $\mathrm{O}(p,1)\times K$ for some compact group K. □

More generally, one can prove that if the limit set Λ_Γ of a $\mathbb{H}^{p,q}$-convex-cocompact subgroup $\Gamma < \mathrm{PO}(p,q+1)$ of maximal dimension is a $\mathcal{C}^1$ submanifold, then the signature of the restriction of the pseudo-Riemannian conformal structure of $\partial\mathbb{H}^{p,q}$ to $T\Lambda_\Gamma$ is constant and of the form $(k,0)$ for some integer $0\leq k\leq p-1$. Adapting the proof of Lemma 3.9, we then find that the Zariski closure of Γ has rank at most $p-k$.

4 Pseudo-Riemannian Hausdorff dimension

Limit sets of discrete groups of isometries of real hyperbolic space (and more generally rank one symmetric spaces) are a central theme in hyperbolic geometry. They are especially nice for convex cocompact groups, as they have nice dynamical properties as well as a fractal nature that is well understood. Indeed, it is known that the Hausdorff dimension of the limit set is equal to the critical exponent, which is a dynamical invariant of the action on the hyperbolic space. With O. Glorieux, we extended this relation to $\mathbb{H}^{p,q}$-convex cocompact groups in [GM21]. It is important to understand that both sides of the equation have to be modified in the pseudo-Riemannian context. On one side, the critical exponent is a notion of metric geometry, but $\mathbb{H}^{p,q}$ is not a metric space. On the other side, the fact that limit sets of $\mathbb{H}^{p,q}$-convex-cocompact subgroups of maximal dimension are Lipschitz submanifolds implies that their Hausdorff dimension is always equal to $p-1$, making that notion uninteresting.

4.1 The critical exponent

Let us first recall the classic definition of the critical exponent in metric spaces. Let G be a countable group acting isometrically on a metric space (X,d), and $o\in X$. The **critical exponent** $\delta(G,X)$ is

$$\delta(G,X)=\limsup_{R\to\infty}\frac{1}{R}\operatorname{Log}\operatorname{Card}\{g\in G\,|\,d(g\cdot o,o)\leq R\}.$$

A simple computation based on the triangle inequality shows that this number does not depend on $o\in X$. It measures the exponential growth rate of the orbits of G in X. For example, by a simple argument of volume, we can see that the critical exponent of a uniform lattice of $\mathrm{PO}(p,1)$ acting on $\mathbb{H}^p$ is equal to $p-1$. More generally this applies to fundamental groups of closed Riemannian manifolds of negative curvature, where the critical exponent is equal to the exponential growth rate of the volume of balls.

The main problem when it comes to defining this invariant in pseudo-Riemannian hyperbolic geometry is that $\mathbb{H}^{p,q}$ is not a metric space: if $q>0$, there are no $\mathrm{PO}(p,q+1)$ invariant distances on $\mathbb{H}^{p,q}$. The starting point of our work is the search for a good replacement for the distance on the convex hull $C(\Lambda_\Gamma)$ of a $\mathbb{H}^{p,q}$-convex cocompact group $\Gamma\subset\mathrm{PO}(p,q+1)$, which will lead to an $\mathbb{H}^{p,q}$ critical exponent. Instead of defining the distance as an infimum of lengths of all curves joining two points (which would ultimately lead to 0), we just focus on geodesics.

Definition 4.1. Let $x, y \in \mathbb{H}^{p,q}$. If x and y are joined by a spacelike geodesic, we define $d_{\mathbb{H}^{p,q}}(x,y)$ as the length of this spacelike geodesic. In other cases, we set $d_{\mathbb{H}^{p,q}}(x,y) = 0$.

A simple computation (see [GM21, Proposition 3.2]) shows the following formula[3]

$$d_{\mathbb{H}^{p,q}}(x,y) = \cosh^{-1}\left(\max\left\{1, \left|\langle \widetilde{x}, \widetilde{y}\rangle_{p,q+1}\right|\right\}\right). \tag{1}$$

This function is not a distance, and the first part of our work consists in finding a weak form of the triangle inequality when looking at the convex hull $C(\Lambda_\Gamma)$.

Theorem 4.2. *If $\Gamma \subset \mathrm{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, there is a constant $k_\Gamma > 0$ such that $d_{\mathbb{H}^{p,q}}(x,y) \leq d_{\mathbb{H}^{p,q}}(x,z) + d_{\mathbb{H}^{p,q}}(z,y) + k_\Gamma$ for all $x, y, z \in C(\Lambda_\Gamma)$.*

Sketch of proof. Thanks to the formula (1) and the inequality $\mathrm{Log}\, t \leq \cosh^{-1}(t) \leq \mathrm{Log}\, t + 2$ for any $t \geq 1$, it is enough to show that the function $F : C(\Lambda_\Gamma)^3 \to \mathbb{R}$ defined by

$$F(x,y,z) = \frac{\langle \widetilde{x}, \widetilde{y}\rangle_{p,q+1}}{\langle \widetilde{x}, \widetilde{z}\rangle_{p,q+1} \langle \widetilde{z}, \widetilde{y}\rangle_{p,q+1}}$$

is bounded. For this, we first notice that it extends continuously to $\overline{C(\Lambda_\Gamma)}^2 \times C(\Lambda_\Gamma)$ (essentially because the formula defining F also makes sense for x or y in Λ_Γ), then use the cocompactness of the action $\Gamma \curvearrowright C(\Lambda_\Gamma)$ in order to confine z to a compact fundamental domain. ☐

Remark. This idea of using a weakened triangle inequality in order to relate Hausdorff dimensions and critical exponents has been recently used in [DKO24].

Definition 4.3. Let $\Gamma \subset \mathrm{PO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact. The *critical exponent* of Γ is

$$\delta_{\mathbb{H}^{p,q}}(\Gamma) = \limsup_{R\to\infty} \frac{1}{R} \mathrm{Log}\,\mathrm{Card}\left\{\gamma \in \Gamma \,\middle|\, d_{\mathbb{H}^{p,q}}(\gamma o, o) \leq R\right\}$$

where $o \in C(\Lambda_\Gamma)$.

Thanks to Theorem 4.2, it does not depend on the choice of a point $o \in C(\Lambda_\Gamma)$. Another way of defining $\delta_{\mathbb{H}^{p,q}}(\Gamma)$ is as the convergence exponent of the *Poincaré series*

$$P(s) = \sum_{\gamma\in\Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma\cdot o, o)}.$$

It satisfies

$$\begin{cases} P(s) < +\infty & \text{if} \quad s > \delta_{\mathbb{H}^{p,q}}(\Gamma) \\ P(s) = +\infty & \text{if} \quad s < \delta_{\mathbb{H}^{p,q}}(\Gamma) \end{cases}.$$

[3]Recall that by convention, a lift $\widetilde{x} \in \mathbb{R}^{p+q+1}$ of a point $x \in \mathbb{H}^{p,q}$ is assumed to be in $\widehat{\mathbb{H}}^{p,q}$, i.e. to satisfy $\langle \widetilde{x}, \widetilde{x}\rangle_{p,q+1} = -1$, unless otherwise specified.

4.2 Pseudo-Riemannian Hausdorff dimension

Hausdorff dimension and measures rely on the choice of a distance, so in order to define a replacement for the Hausdorff dimension of the limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^{p,q}$, we must find a replacement for the distance on $\partial\mathbb{H}^{p,q}$. This is achieved by using standard constructions in negatively curved metric geometry.

Definition 4.4. Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. Consider $x \in C(\Lambda_\Gamma)$ and $\xi, \eta \in \Lambda_\Gamma$. The Gromov product of ξ and η centred at x is

$$(\xi|\eta)_x = \frac{1}{2}\mathrm{Log}\left|\frac{2\left\langle\widetilde{\xi},\widetilde{x}\right\rangle_{p,q+1}\langle\widetilde{x},\widetilde{\eta}\rangle_{p,q+1}}{\left\langle\widetilde{\xi},\widetilde{\eta}\right\rangle_{p,q+1}}\right|.$$

The Gromov distance between ξ and η seen from x is

$$d_x(\xi,\eta) = e^{-(\xi|\eta)_x} = \sqrt{\left|\frac{\left\langle\widetilde{\xi},\widetilde{\eta}\right\rangle_{p,q+1}}{2\left\langle\widetilde{\xi},\widetilde{x}\right\rangle_{p,q+1}\langle\widetilde{x},\widetilde{\eta}\rangle_{p,q+1}}\right|}.$$

It has the same interpretation as in metric spaces: for $x,y,z \in C(\Lambda_\Gamma)$, the Gromov product is defined by

$$(x|y)_z = \frac{1}{2}\left(d_{\mathbb{H}^{p,q}}(x,z) + d_{\mathbb{H}^{p,q}}(y,z) - d_{\mathbb{H}^{p,q}}(x,y)\right).$$

We then have

$$(\xi|\eta)_x = \lim_{\substack{y\to\xi \\ z\to\eta}} (y|z)_x.$$

Note that the Gromov distance d_x has a nice pseudo-Riemannian interpretation. As seen in section 1.4, one x is chosen there is a natural pseudo-Riemannian metric on the open $\partial U(x) \subset \partial\mathbb{H}^{p,q}$, and $\Lambda_\Gamma \subset \partial U(x)$ when $x \in C(\Lambda_\Gamma)$. Then two distinct points $\xi, \eta \in \Lambda_\Gamma$ are related by a spacelike geodesic whose length is $\cos^{-1}(1 - 2d_x(\xi,\eta)^2)$.

Even though the Gromov distance d_x is symmetric and positive on distinct points, it does not satisfy the triangle inequality. But here again, there is a weaker version of the triangle inequality that will be sufficient for our purpose.

Proposition 4.5 ([GM21, Lemma 3.17])**.** *If $\Gamma \subset \mathrm{PO}(p,q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, there is a constant $\lambda_\Gamma \geq 1$ such that*

$$\forall x \in C(\Lambda_\Gamma)\, \forall \xi,\eta,\tau \in \Lambda_\Gamma \quad d_x(\xi,\eta) \leq \lambda_\Gamma\left(d_x(\xi,\tau) + d_x(\tau,\eta)\right).$$

The proof is based on the cocompactness of the action of Γ on the space $\Lambda_\Gamma^{(3)}$ of pairwise distinct triples of points in Λ_Γ, a general fact for hyperbolic groups. From there, we define a ball in Λ_Γ as

$$B_x(\xi,r) = \left\{\eta \in \Lambda_\Gamma \,\middle|\, d_x(\xi,\eta) \leq r\right\}$$

for $x \in C(\Lambda_\Gamma), \xi \in \Lambda_\Gamma$ and $r \geq 0$. We can then adapt the definitions of Hausdorff measure and dimension to our setting. For $E \subset \Lambda_\Gamma$, $s > 0$ and $\varepsilon > 0$ we set:

$$\mathrm{H}^{s,\varepsilon}_{d_x}(E) = \inf\Big\{\sum r_i^s \,\Big|\, E \subset \bigcup B_x(\xi_i, r_i), \xi_i \in E, r_i \leq \varepsilon\Big\}.$$

Since $\mathrm{H}^{s,\varepsilon}_{d_x}(E)$ increases as ε decreases, we can consider:

$$\mathrm{H}^{s}_{d_x}(E) = \lim_{\varepsilon \to 0} \mathrm{H}^{s,\varepsilon}_{d_x}(E) \in [0, +\infty].$$

Finally, the pseudo-Riemannian Hausdorff dimension of E is:

$$\mathrm{Hdim}_{d_x}(E) = \inf\Big\{s > 0 \,\Big|\, \mathrm{H}^{s}_{d_x}(E) = 0\Big\}.$$

Given two points $x, y \in C(\Lambda_\Gamma)$, the Gromov distances d_x and d_y satisfy a bi-Lipschitz type inequality

$$\forall \xi, \eta \in \Lambda_\Gamma \quad e^{-2d_{\mathbb{H}^{p,q}}(x,y) - 2k_\Gamma} \leq \frac{d_x(\xi,\eta)}{d_y(\xi,\eta)} \leq e^{2d_{\mathbb{H}^{p,q}}(x,y) + 2k_\Gamma}$$

from which one deduces that the pseudo-Riemannian Hausdorff dimension $\mathrm{Hdim}_{d_x}(\Lambda_\Gamma)$ does not depend on x.

Theorem 4.6. *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. For any $x \in C(\Lambda_\Gamma)$, we have*

$$\mathrm{Hdim}_{d_x}(\Lambda_\Gamma) = \delta_{\mathbb{H}^{p,q}}(\Gamma).$$

Furthermore, given $x \in C(\Lambda_\Gamma)$, it is possible to find a Riemannian metric on $\partial U(x)$ whose induced distance is larger than d_x (see the proof of [GM21, Proposition 5.1]). Intuitively, the fact that $d_x(\xi, \eta)$ is small means that the pseudo-Riemannian geodesic segment in $\partial U(x)$ joining ξ and η is close to being isotropic, but this does not imply that it is short for the Riemannian metric. It follows that the pseudo-Riemannian Hausdorff dimension is smaller than the usual (Riemannian) Hausdorff dimension. Note that since the limit set is the locally the graph of a Lipschitz map defined on a closed subset of $\mathbb{S}^{p-1}$, the usual Hausdorff dimension is at most $p-1$.

Proposition 4.7 ([GM21, Proposition 5.1])**.** *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. For any $x \in C(\Lambda_\Gamma)$, we have*

$$\mathrm{Hdim}_{d_x}(\Lambda_\Gamma) \leq \mathrm{Hdim}(\Lambda_\Gamma) \leq p-1$$

where $\mathrm{Hdim}(\Lambda_\Gamma)$ denotes its Hausdorff distance for any Riemannian distance on an open neighbourhood of Λ_Γ.

4.3 Pseudo-Riemannian Patterson-Sullivan theory

The proof of Theorem 4.6 relies on the existence of a measure for which a pseudo-Riemannian ball of radius r has volume roughly $r^{\delta_{\mathbb{H}^{p,q}}(\Gamma)}$ (this condition is known as Ahlfors-David regularity for metric space).

Proposition 4.8. *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, and $x \in C(\Lambda_\Gamma)$. If there exist a measure μ on Λ_Γ and constants $s, C > 0$ such that*

$$\frac{1}{C} \leq \frac{\mu(B_x(\xi,r))}{r^s} \leq C$$

for all $r \in (0,1)$, then

$$\mathrm{Hdim}_{d_x}(\Lambda_\Gamma) = s.$$

This is a classic result for metric spaces whose proof relies on the Vitali Lemma, a way of modifying the radii of a covering by balls in order to get disjoint balls that still cover the space if the radius is replaced by a larger one. This intermediate result uses the triangle inequality, so it must be adapted to our situation.

Lemma 4.9 (Vitali Lemma for d_x, [GM21, Lemma 5.3])**.** *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, and $x \in C(\Lambda_\Gamma)$. Given a subset $J \subset \Lambda_\Gamma$ and a bounded function $r : J \to (0,+\infty)$, there is a subset $I \subset J$ such that:*

- *The balls $B_x(\xi, r(\xi))$ are disjoint for distinct points $\xi \in I$.*
- $\bigcup_{\xi\in J} B_x(\xi, r(\xi)) \subset \bigcup_{\eta\in I} B_x(\eta, 5\lambda_\Gamma^2 r(\eta))$.

Proof of Proposition 4.8. Let $\varepsilon > 0$, and consider the open cover $\Lambda_\Gamma \subset \bigcup_{\xi\in\Lambda_\Gamma} B_x(\xi, \frac{\varepsilon}{5\lambda_\Gamma^2})$. By Lemma 4.9, we can find a (necessarily countable) subset $J \subset \Lambda_\Gamma$ such that $\Lambda_\Gamma \subset \bigcup_{\xi\in\Lambda_\Gamma} B_x(\xi, \varepsilon)$ and the balls $B_x(\xi, \frac{\varepsilon}{5\lambda_\Gamma^2})$ for $\xi \in \Lambda_\Gamma$ are pairwise disjoint. Since $\mathrm{H}^{s,\varepsilon}_{d_x}(\Lambda_\Gamma) \leq \sum_{\xi\in J} \varepsilon^s$, we find:

$$\begin{aligned}
\mathrm{H}^{s,\varepsilon}_{d_x}(\Lambda_\Gamma) &\leq \sum_{\xi\in J} \varepsilon^s \\
&\leq (5\lambda_\Gamma^2)^s C \sum_{\xi\in J} \mu\left(B_x(\xi, \frac{\varepsilon}{5\lambda_\Gamma^2})\right) \\
&\leq \left(5\lambda_\Gamma^2\right)^s C\mu(\Lambda_\Gamma). \qquad (2)
\end{aligned}$$

We now consider an arbitrary countable collection (ξ_i, r_i) of points of Λ_Γ and radii such that $\Lambda_\Gamma \subset \bigcup B_x(\xi_i, r_i)$. We have $\mu(\Lambda_\Gamma) \leq \sum \mu(B_x(\xi_i, r_i)) \leq C \sum r_i^s$, so by considering the infimum amongs all covers we find

$$\mu(\Lambda_\Gamma) \leq C\mathrm{H}^{s,\varepsilon}_{d_x}(\Lambda_\Gamma) \qquad (3)$$

Combining (2) and (3) and letting $\varepsilon \to 0$, we get:

$$\frac{\mu(\Lambda_\Gamma)}{C} \leq \mathrm{H}^s_{d_x}(\Lambda_\Gamma) \leq \left(5\lambda_\Gamma^2\right)^s C\mu(\Lambda_\Gamma).$$

The fact that $\mathrm{H}^s_{d_x}(\Lambda_\Gamma)$ is both positive and finite implies that $\mathrm{Hdim}_{d_x}(\Lambda_\Gamma) = s$. □

The quest for a measure μ satisfying the Ahlfors-David regularity condition of Proposition 4.8 is achieved through Patterson-Sullivan theory. The idea, first developed by Patterson in $\mathbb{H}^2$ then Sullivan in $\mathbb{H}^d$ [Pat76, Sul79], is that this condition can be achieved by requiring that $\gamma_*\mu$ is absolutely continuous with respect to μ for any $\gamma \in \Gamma$, with an imposed Radon-Nikodym derivative. This condition is easier to handle (and obtain) if instead we work with families of measures on Λ_Γ indexed by points of $C(\Lambda_\Gamma)$ that we call conformal densities. First, we will need to recall the definition of Busemann functions.

Definition 4.10. Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. The Busemann function centred at $\xi \in \Lambda_\Gamma$, is the function β_ξ on $C(\Lambda_\Gamma)^2$ defined by:

$$\forall x,y \in C(\Lambda) \ \ \beta_\xi(x,y) = \ln\left(\left|\frac{\left\langle \widetilde{\xi}, \widetilde{x}\right\rangle_{p,q+1}}{\left\langle \widetilde{\xi}, \widetilde{y}\right\rangle_{p,q+1}}\right|\right).$$

Just as for the Gromov product, there is an interpretation in terms of differences of distances:

$$\lim_{z\to\xi} d_{\mathbb{H}^{p,q}}(z,x) - d_{\mathbb{H}^{p,q}}(z,y) = \beta_\xi(x,y).$$

Both notions are related by $(\xi|\eta)_x = \frac{1}{2}(\beta_\xi(x,y)+\beta_\eta(x,y))$ for any $y \in (\xi\eta)$. We also have the cocycle relation

$$\forall \xi \in \Lambda_\Gamma \ \forall x,y,z \in C(\Lambda_\Gamma) \quad \beta_\xi(x,y)+\beta_\xi(y,z) = \beta_\xi(x,z).$$

Definition 4.11. Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. A conformal density of dimension s on Λ_Γ is a family of measures $\nu = (\nu_x)_{x\in C(\Lambda_\Gamma)}$ on Λ_Γ satisfying the following conditions:

1. $\forall \gamma \in \Gamma$, $\gamma_*\nu_x = \nu_{\gamma x}$ (where $\gamma_*\nu(E) = \nu(\gamma^{-1}E)$)
2. The measures are all absolutely continuous with respect to each other, and $\frac{d\nu_x}{d\nu_y}(\xi) = e^{-s\beta_\xi(x,y)}$ forall $x,y \in C(\Lambda_\Gamma)$ and $\xi \in \Lambda_\Gamma$.
3. $\mathrm{supp}\,(\nu_x) = \Lambda_\Gamma$

Remark. Fixing a base point $o \in C(\Lambda_\Gamma)$, we can recover ν_x for any $x \in C(\Lambda_\Gamma)$ from ν_o because of condition 2. So an alternative approach is to start with a single measure ν whose support is Λ_Γ and such that, for any $\gamma \in \Gamma$, the measure $\gamma_*\nu$ is absolutely continuous with respect to ν with density $\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{-s\beta_\xi(\gamma\cdot o,o)}$ at $\xi \in \Lambda_\Gamma$.

In order to prove that conformal densities are Ahlfors-David regular, we replace pseudo-Riemannian balls in Λ_Γ with shadows.

Definition 4.12. Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, $x,y \in C(\Lambda)$ and $r > 0$. The shadow $\mathcal{S}_r(x,y)$ is

$$\mathcal{S}_r(x,y) = \left\{\xi \in \Lambda_\Gamma \,\middle|\, [x\xi) \cap B_{C(\Lambda_\Gamma)}(y,r) \neq \emptyset\right\}$$

where $B_{C(\Lambda_\Gamma)}(y,r) = \{z \in C(\Lambda_\Gamma) \,|\, d_{\mathbb{H}^{p,q}}(y,z) \leq r\}$ is the pseudo-Riemannian ball.

Remark. This is slightly different from the usual definition of shadows as we require that points in shadows lie on the limit set.

There are explicit relations between shadows and balls in Λ_Γ.

Lemma 4.13 ([GM21, Corollaries 3.20 and 3.21]). *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, $\xi \in \Lambda_\Gamma$ and $r \in (0,1)$.*

- *If $y \in [x\xi)$ is such that $d_{\mathbb{H}^{p,q}}(x,y) = -\mathrm{Log} r$, then $B_x(\xi, r) \subset \mathcal{S}_{\ln 6}(x,y)$.*
- *Let $t > 0$. If $y \in [x\xi)$ is such that $d_{\mathbb{H}^{p,q}}(x,y) = t + k_\Gamma - \mathrm{Log} r - \frac{\mathrm{Log} 8}{2}$, then $\mathcal{S}_t(x,y) \subset B_x(\xi, r)$.*

The main interest in shadows is that they are subsets of the limit set Λ_Γ parametrised by points in the convex hull $C(\Lambda_\Gamma)$, where we can use the cocompactness of the action of Γ. One of the main technical results needed to work with shadows is a control on Busemann functions.

Lemma 4.14 ([GM21, Lemma 3.11]). *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, $x, y \in C(\Lambda_\Gamma)$ and $r > 0$. For all $\xi \in \mathcal{S}_r(x,y)$, one has:*

$$d_{\mathbb{H}^{p,q}}(x,y) - 2r - 2k_\Gamma \leq \beta_\xi(x,y) \leq d_{\mathbb{H}^{p,q}}(x,y) + k_\Gamma.$$

We then get a pseudo-Riemannian version of Sullivan's Shadow Lemma.

Theorem 4.15 ([GM21, Theorem 4.7]). *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, and let ν be a conformal density of dimension s. For any $x \in C_\Gamma(\Lambda)$ and $r > 0$ large enough, there is a constant $C(r) > 0$ satisfying:*

$$\frac{1}{C(r)} e^{-s d_{\mathbb{H}^{p,q}}(x,y)} \leq \nu_x(\mathcal{S}_r(x,y)) \leq C(r) e^{-s d_{\mathbb{H}^{p,q}}(x,y)}$$

for all $y \in C(\Lambda_\Gamma)$.

Sketch of proof. The first step is to move from an arbitrary pair (x,y) to a pair of the form $(x, \gamma \cdot x)$ for some $\gamma \in \Gamma$. This is achieved by using the cocompactness of the action on $C(\Lambda_\Gamma)$, and the relation

$$\mathcal{S}_r(x,y) \subset \mathcal{S}_{r + d_{\mathbb{H}^{p,q}}(y,z) + k_\Gamma}(x,z)$$

for $x, y, z \in C(\Lambda_\Gamma)$ (see the proof of [GM21, Theorem 4.7]). We now want to estimate $\nu_x(\mathcal{S}_r(x, \gamma \cdot x))$ for some $\gamma \in \Gamma$, and we start with a change of variables.

$$\begin{aligned} \nu_x(\mathcal{S}_r(x, \gamma \cdot x)) &= \nu_x\left(\gamma \cdot \mathcal{S}_r(\gamma^{-1} \cdot x, x)\right) \\ &= \nu_{\gamma^{-1} \cdot x}\left(\mathcal{S}_r(\gamma^{-1} \cdot x, x)\right) \\ &= \int_{\mathcal{S}_r(\gamma^{-1} \cdot x, x)} e^{-s\beta_\xi(\gamma^{-1} \cdot x, x)} d\nu_x(\xi). \end{aligned}$$

From there, assuming[4] $s \geq 0$ we easily get an upper bound thanks to the lower bound of the Buneman function from Lemma 4.14:

$$\nu_x(\mathcal{S}_r(x, \gamma \cdot x)) \leq \nu_x(\Lambda_\Gamma) e^{2s(r + k_\Gamma)} e^{-s d_{\mathbb{H}^{p,q}}(x, \gamma \cdot x)}.$$

For the lower bound, still use Lemma 4.14:

$$\nu_x(\mathcal{S}_r(x, \gamma \cdot x)) \geq \nu_x\left(\mathcal{S}_r(\gamma^{-1} \cdot x, x)\right) e^{-s k_\Gamma} e^{-s d_{\mathbb{H}^{p,q}}(x, \gamma \cdot x)}.$$

One concludes by finding a lower bound on $\nu_x(\mathcal{S}_r(y,x))$ uniform in $y \in C(\Lambda_\Gamma)$ provided that r is large enough. This is possible because $\mathcal{S}_r(y,x)$ is close to being the whole limit set Λ_Γ when r is large. For a precise statement and proof, see [GM21, Corollary 4.5]. □

[4]We omit the $s < 0$ case which is also treated in the proof of [GM21, Theorem 4.6] as it is *a posteriori* unnecessary: there are no conformal densities of negative dimension. However the proof of this fact is by contradiction and uses the Shadow Lemma in this case, thus its presence in the paper.

Theorem 4.15 also estimates the volume of balls in the limit set thanks to Lemma 4.13.

Theorem 4.16 ([GM21, Theorem 4.8])**.** *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact, ν be a conformal density of dimension s and $x \in C_\Gamma(\Lambda)$. There is $c > 0$ such that for all $\xi \in \Lambda_\Gamma$ and $r \in (0,1)$, we have*

$$\frac{1}{c} \le \frac{\nu_x\left(B_x(\xi,r)\right)}{r^s} \le c.$$

Combining Theorem 4.16 and Proposition 4.8, we see that the existence of a conformal density of dimension s implies that $s = \mathrm{Hdim}_{d_x}(\Lambda_\Gamma)$, so Theorem 4.6 will be proved if we can show the existence of a conformal density of dimension $\delta_{\mathbb{H}^{p,q}}(\Gamma)$. The classic Patterson-Sullivan construction works very well here.

Theorem 4.17 ([GM21, Theorem 4.2])**.** *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. There exists a conformal density of dimension $\delta_{\mathbb{H}^{p,q}}(\Gamma)$.*

We will see the proof of Theorem 4.17 under the technical assumption that the Poincaré series

$$P(s) = \sum_{\gamma\in\Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma\cdot o, o)}$$

diverges at $s = \delta_{\mathbb{H}^{p,q}}(\Gamma)$. This happen to be true for all $\mathbb{H}^{p,q}$-convex cocompact [GM21, Corollary 4.12], but the proof uses the existence of the Patterson-Sullivan density. This is why the construction in [GM21, Section 7.1] uses a modification of the Poincaré series involving the Patterson function introduced in [Pat76]. In conclusion, the construction that we give below does give the Patterson-Sullivan density, but is not sufficient to prove its existence.

Proof of Theorem 4.17 assuming the Poincaré series diverges at $s = \delta_{\mathbb{H}^{p,q}}(\Gamma)$. Recall from the remark following Definition 4.11 that we can fix a base point $o \in \Lambda_\Gamma$ and focus on finding a measure μ whose support is Λ_Γ and such that, for any $\gamma \in \Gamma$, the measure $\gamma^*\mu$ is absolutely continuous with respect to μ with density $\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{-\delta_{\mathbb{H}^{p,q}}(\Gamma)\beta_\xi(\gamma\cdot o, o)}$ at $\xi \in \Lambda_\Gamma$.

The idea is to construct μ as a limit of probability measures μ_s defined for $s > \delta_{\mathbb{H}^{p,q}}(\Gamma)$ and supported on the orbit $\Gamma\cdot o$, with weights chosen so that they concentrate towards Λ_Γ as $s \to \delta_{\mathbb{H}^{p,q}}(\Gamma)$. This is achieved by setting

$$\mu_s = \frac{1}{P(s)} \sum_{\gamma\in\Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma\cdot o, o)} \Delta_{\gamma\cdot o}$$

where Δ_x is the Dirac mass at $x \in C(\Lambda_\Gamma)$. Considering μ_s as a measure on the compact space

$$\overline{C(\Lambda_\Gamma)} = C(\Lambda_\Gamma) \cup \Lambda_\Gamma \subset \overline{\mathbb{H}^{p,q}}.$$

Using the compactness of the space of probability measures over a compact topological space we can consider a sequence $(s_k)_{k\ge0}$ of numbers $s_k < \delta_{\mathbb{H}^{p,q}}(\Gamma)$ with $s_k < \delta_{\mathbb{H}^{p,q}}(\Gamma)$ such that μ_{s_k} converges weakly to a probability measure μ on $\overline{C(\Lambda_\Gamma)}$. As any compact subset $K \subset C(\Lambda_\Gamma)$ only contains finitely many elements of the orbit $\Gamma\cdot o$ (a consequence of proper discontinuity), we find that $\mu_s(K) \to 0$ as $s \to \delta_{\mathbb{H}^{p,q}}(\Gamma)$, thus $\mathrm{Supp}(\mu) \subset \Lambda_\Gamma$.

For any $\gamma \in \Gamma$ and $s > \delta_{\mathbb{H}^{p,q}}(\Gamma)$, we have

$$\begin{aligned}
\gamma_* \mu_s &= \frac{1}{P(s)} \sum_{\gamma' \in \Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma' \cdot o, o)} \gamma_* \Delta_{\gamma' \cdot o} \\
&= \frac{1}{P(s)} \sum_{\gamma' \in \Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma' \cdot o, o)} \Delta_{\gamma\gamma' \cdot o} \\
&= \frac{1}{P(s)} \sum_{\gamma' \in \Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma^{-1}\gamma' \cdot o, o)} \Delta_{\gamma' \cdot o} \\
&= \frac{1}{P(s)} \sum_{\gamma' \in \Gamma} e^{-s d_{\mathbb{H}^{p,q}}(\gamma' \cdot o, \gamma \cdot o)} \Delta_{\gamma' \cdot o}.
\end{aligned}$$

It follows that $\gamma_* \mu_s$ is absolutely continuous with respect to μ_s, with density

$$\frac{d\gamma_* \mu_s}{d\mu_s}(z) = e^{-s(d_{\mathbb{H}^{p,q}}(z, \gamma \cdot o) - d_{\mathbb{H}^{p,q}}(z, o))}$$

at $z \in \Gamma \cdot o$. As $d_{\mathbb{H}^{p,q}}(z, \gamma \cdot o) - d_{\mathbb{H}^{p,q}}(z, o) \to \beta_\xi(\gamma \cdot o, o)$ when $z \to \xi \in \Lambda_\Gamma$, we find that $\gamma_* \mu$ is absolutely continuous with respect to μ, with density

$$\frac{d\gamma_* \mu}{d\mu}(\xi) = e^{-\delta_{\mathbb{H}^{p,q}}(\Gamma) \beta_\xi(\gamma \cdot o, o)}$$

at $\xi \in \Lambda_\Gamma$. It also follows from this formula that $\gamma_* \mu$ and μ have the same support, so $\mathrm{Supp}(\mu)$ is Γ-invariant and closed, as well as non empty (μ is a probability measure). It follows from the minimality of the action on Λ_Γ that $\mathrm{Supp}(\mu) = \Lambda_\Gamma$. $\square$

Remark. When we focus on a single measure μ associated to a fixed base point like we just did, it is natural to ask of the finite measure μ to be a probability measure. However, if we work with a conformal density $\nu = (\nu_x)_{x \in C(\Lambda_\Gamma)}$, the fact that ν_o is a probability measure does not imply the same property at points outside the orbit $\Gamma \cdot o$. This is actually important in (Riemannian) hyperbolic geometry, as the function $x \mapsto \mu_x(\Lambda_\Gamma)$ has its importance in spectral theory: it is an eigenfunction of the Laplacian realising the bottom of the spectrum. The relation between Patterson-Sullivan measures and spectral theory in the pseudo-Riemannian setting is the subject of an ongoing project in collaboration with B. Delarue and C. Guillarmou.

By *Patterson-Sullivan* density, we mean the conformal density constructed in the proof of Theorem 4.17. Let us finish this section by mentioning the uniqueness and ergodicity of conformal measures.

Theorem 4.18 ([GM21, Theorem 4.13]). *Let $\Gamma < \mathrm{PO}(p, q+1)$ be $\mathbb{H}^{p,q}$-convex cocompact. The Patterson-Sullivan conformal density of to a multiplicative constant, and it is ergodic: any Γ-invariant measurable subset of Λ_Γ has zero of full measure for Patterson-Sullivan measures.*

4.4 Isotropic tangent spaces

Let us focus on the maximal dimension case in this section. At this point of the discussion, all that we know about the $\mathbb{H}^{p,q}$-critical exponent and the pseudo-Riemannian Hausdorff dimension is summarised in the following line:

$$0 < \delta_{\mathbb{H}^{p,q}}(\Gamma) = \mathrm{Hdim}_{pR}(\Lambda_\Gamma) \leq p-1.$$

If Γ is conjugate to a uniform lattice in $\mathrm{O}(p,1) < \mathrm{PO}(p,q+1)$, then $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1$. But we have not encountered any sign pointing to different values of the $\mathbb{H}^{p,q}$-critical exponent. Although we will see in the next section that the value $p-1$ is a characterisation of subgroups preserving a totally geodesic copy of $\mathbb{H}^p$, let us look at a (failed) naive attempt to prove that any $\mathbb{H}^{p,q}$-convex-cocompact subgroup of maximal dimension must satisfy $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1.$, and see what we can learn from it (Theorem 4.19).

We know (Proposition 1.17) that the limit set of a $\mathbb{H}^{p,q}$-convex-cocompact subgroup of maximal dimension is a Lipschitz submanifold of $\partial\mathbb{H}^{p,q}$. This has two interesting consequences. One is that it naturally carries a class of measures: those that are absolutely continuous with respect to the Lebesgue class in Lipschitz charts. The other is that it possesses tangent spaces at almost every point (for the aforementioned measure class).

A naive approach to conformal densities would be to look for them in this class. There is a simple solution: every $x \in C(\Lambda_\Gamma)$ defines a pseudo-Riemannian metric h_x on the open subset $\partial U(x) \subset \partial\mathbb{H}^{p,q}$, and $\Lambda_\Gamma \subset \partial U(x)$. We can therefore consider the restriction of h_x to tangent spaces of Λ_Γ, and use the volume element of this metric to define a measure Vol_x on Λ_Γ.

To be more precise, consider the antipodal quotient map $\pi : \mathbb{S}^{p-1} \times \mathbb{S}^q \to \partial\mathbb{H}^{p,q}$ and a distance-decreasing function $f : \mathbb{S}^{p-1} \to \mathbb{S}^q$ such that $\pi(\mathrm{Gr}(f)) = \Lambda_\Gamma$, so that we have a global Lipschitz chart

$$\varphi : \begin{cases} \mathbb{S}^{p-1} & \to & \Lambda_\Gamma \\ u & \mapsto & \pi(u, f(u)) \end{cases}.$$

For $x \in C(\Lambda_\Gamma)$, and a measurable subset $A \subset \Lambda_\Gamma$, the formula is

$$\mathrm{Vol}_x(A) = \int_{\varphi^{-1}(A)} \sqrt{\det \varphi^* h_x}\, \mathrm{dvol}_{\mathbb{S}^{p-1}}. \tag{4}$$

The facts that the measure is independent of charts and that $g_* h_x = h_{g\cdot x}$ for any $g \in \mathrm{PO}(p,q+1)$ imply that $\gamma_* \mathrm{Vol}_x = \mathrm{Vol}_{\gamma\cdot x}$ for any $\gamma \in \Gamma$. For $x, y \in C(\Lambda_\Gamma)$, the metrics h_x and h_y are in the same conformal class of $\partial U(x) \cap \partial U(y)$, and the conformal factor is

$$(h_y)_\xi = \left(\frac{\left\langle \widetilde{x}, \widetilde{\xi} \right\rangle_{p,q+1}}{\left\langle \widetilde{y}, \widetilde{\xi} \right\rangle_{p,q+1}} \right)^2 (h_x)_\xi = e^{2\beta_\xi(x,y)} (h_x)_\xi.$$

So the measures Vol_x and Vol_y are in the same class with

$$\frac{d\,\mathrm{Vol}_x}{d\,\mathrm{Vol}_y}(\xi) = e^{-(p-1)\beta_\xi(x,y)}.$$

At this point, the family $(\mathrm{Vol}_x)_{x\in C(\Lambda_\Gamma)}$ seems to be a conformal density of dimension $p-1$, with only one missing item in the definition: the support of Vol_x. Here again, the Γ-invariance means that either Vol_x is the zero measure or it has full support. Looking at the coordinate formula (4) of Vol_x, the integrand $\sqrt{\det \varphi^* h_x}$ (defined at points where Λ_Γ has a tangent space) vanishes if and only if the tangent space is degenerate for the pseudo-Riemannian conformal structure of $\partial\mathbb{H}^{p,q}$. As the only possible dimension of a conformal density is $\delta_{\mathbb{H}^{p,q}}(\Gamma)$, we get the following statement.

Theorem 4.19. *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension. If $\delta_{\mathbb{H}^{p,q}}(\Gamma) < p-1$, then almost all tangent spaces of Λ_Γ are degenerate.*

4.5 Rigidity results

Let us turn to the question of rigidity, i.e. describing the equality case $\delta_{\mathbb{H}^{p,q}}(\Gamma) = p-1$. A famous theorem of Bowen [Bow79] in dimension 3 and Yue [Yue96] in higher dimension shows that the critical exponent of a convex cocompact representation of a uniform lattice of $\mathrm{O}(p,1)$ into $\mathrm{PO}(p+1,1)$ is greater than $p-1$ with equality if and only if the group is Fuchsian, that is conjugate to a subgroup of $\mathrm{O}(p,1)$. In [GM21], we prove an analogous statement for $\mathbb{H}^{2,1} = \mathbb{A}\mathrm{d}\mathbb{S}^3$.

Theorem 4.20 ([GM21, Theorem 1.5])**.** *Let $\Gamma < \mathrm{PO}(2,2)$ be $\mathbb{A}\mathrm{d}\mathbb{S}^3$-quasi-Fuchsian. Then*

$$\delta_{\mathbb{A}\mathrm{d}\mathbb{S}^3}(\Gamma) \leq 1,$$

with equality if and only if Γ is $\mathbb{A}\mathrm{d}\mathbb{S}^3$-Fuchsian.

The proof we propose in [GM21] mixes geometric and ergodic tools. Although specificities of the signature of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, such as the fact that $\mathbb{A}\mathrm{d}\mathbb{S}^3$-quasi-Fuchsian groups are surface groups, occur several times in the proof (e.g. [GM21, Lemma 6.27] relies on some Teichmüller theory via Bonahon's geodesic currents), several arguments could probably be modified in more general settings. There is however an argument which is central to the proof and only applies to $\mathbb{A}\mathrm{d}\mathbb{S}^3$: the existence of a Lipschitz spacelike hypersurface $\Sigma \subset \Gamma\backslash\Omega(\Lambda_\Gamma)$ on which induced length metric is isometric to a hyperbolic surface. Indeed, the fact that a $\mathbb{H}^{p,q}$-quasi-Fuchsian subgroup $\Gamma < \mathrm{PO}(p,q+1)$ needs not be isomorphic to a lattice in $\mathrm{O}(p,1)$ when $p \geq 4$ means that there is not Lipschitz spacelike manifold of dimension $p-1$ on which the induced length metric is isometric to a hyperbolic manifold.

The general signature has since been solved by Mazzoli and Viaggi.

Theorem 4.21 ([MV24, Theorem 4])**.** *Let $\Gamma < \mathrm{PO}(p,q+1)$ be $\mathbb{H}^{p,q}$-convex-cocompact of maximal dimension. If $\delta_{\mathbb{H}^{p,q}} = p-1$, then Γ preserves a totally geodesic copy of $\mathbb{H}^p$.*

Their proof relies on the existence of a maximal (i.e. with vanishing mean curvature) spacelike p-dimensional submanifold of $\Gamma\backslash\Omega(\Lambda_\Gamma)$, the existence of which is due to Seppi, Smith and Toulisse [SST23], and a rigidity result of Ledrappier and Wang [LW10] on the volume entropy of Riemannian manifolds with Ricci curvature bounded from below.

Remark. Theorem 4.20 can also be recovered from a result of Bishop and Steger [BS91]. In order to understand this, we need to consider the relationship between critical exponent and entropy. There is a bijective correspondence between the set $[\Gamma]_{\rm hyp}$ of conjugacy classes of infinite order elements in Γ and closed spacelike geodesics in $\Gamma\backslash\Omega(\Lambda_\Gamma)$. Denote by $\ell_{\mathbb{H}^{p,q}}([\gamma])$ the length of this geodesic for $[\gamma]\in[\Gamma]_{\rm hyp}$. The $\mathbb{H}^{p,q}$-entropy of Γ is

$$h_{\mathbb{H}^{p,q}}(\Gamma)=\limsup_{R\to+\infty}\frac{1}{R}\operatorname{Log}\operatorname{Card}\left\{[\gamma]\in[\Gamma]_{\rm hyp}\,\middle|\,\ell_{\mathbb{H}^{p,q}}([\gamma])\leq R\right\}.$$

Adapting some classic arguments of hyperbolic geometry, we can prove

$$h_{\mathbb{H}^{p,q}}(\Gamma)=\delta_{\mathbb{H}^{p,q}}(\Gamma)$$

for any $\mathbb{H}^{p,q}$-quasi-Fuchsian group. In the $\mathbb{H}^{2,1}$ case, assuming that $\Gamma<\mathrm{PO}(2,2)$ is torsion free and $\mathbb{A}\mathrm{d}\mathbb{S}^3$-quasi-Fuchsian, we have seen that the exceptional isomorphism $\mathrm{PO}(2,2)_\circ\approx\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$ attributes to Γ a pair of Fuchsian representations $\rho_1,\rho_2:\Gamma\to\mathrm{PSL}(2,\mathbb{R})$, i.e. faithful representations whose images are uniform lattices, such that Γ corresponds to $\{(\rho_1(\gamma),\rho_2(\gamma))\,|\,\gamma\in\Gamma\}<\mathrm{PSL}(2,\mathbb{R})\times\mathrm{PSL}(2,\mathbb{R})$. Now every conjugacy class $[\gamma]\in[\Gamma]$ corresponds to closed geodesics in the hyperbolic surfaces $\rho_1(\Gamma)\backslash\mathbb{H}^2$ and $\rho_2(\Gamma)\backslash\mathbb{H}^2$, whose lengths will be denoted by $\ell_1([\gamma])$ and $\ell_2([\gamma])$. It is proved in [Glo17, Proposition 2.3] that

$$\ell_{\mathbb{A}\mathrm{d}\mathbb{S}^3}([\gamma])=\frac{\ell_1([\gamma])+\ell_2([\gamma])}{2}.$$

Now given two Fuchsian representations $\rho_1,\rho_2:\pi_1(\Sigma)\to\mathrm{PSL}(2,\mathbb{R})$ of the fundamental group of a closed surface Σ, it follows from [BS91] that

$$\limsup_{R\to+\infty}\frac{1}{R}\operatorname{Log}\operatorname{Card}\{[\gamma]\in[\pi_1(\Sigma)]\,|\,\ell_1([\gamma])+\ell_2([\gamma])\leq R\}\leq\frac{1}{2}$$

with equality if and only if ρ_1 is conjugate to ρ_2.

4.6 Geometric interpretation of the upper bound on the critical exponent

The fact that the entropy has a rigid upper bound, rather than a lower bound as in the case of quasi-Fuchsian groups acting on $\mathbb{H}^2$, is commonly understood to be a higher rank feature (see the discussion following Theorem A in [PS17]). The lower bound in rank one is best understood when replacing the entropy or critical exponent with the Hausdorff dimension of the limit set: non Fuchsian examples have a fractal limit set, thus a larger Hausdorff dimension. A basic understanding of the pseudo-Riemannian Hausdorff dimension also explains why we should expect an upper bound in the pseudo-Riemannian setting: non Fuchsian examples have a limit set with degenerate tangent spaces (Theorem 4.19), so they can be covered by "fewer" pseudo-Riemannian balls (recall that a pseudo-Riemannian ball of radius 0 is a light cone), so their pseudo-Riemannian Hausdorff dimension should be smaller.

5 Critical exponents, entropies and Hausdorff dimension of limit sets for projective Anosov subgroups

The paper [GMT23], written in collaboration with O. Glorieux and N. Tholozan, is in many ways an extension of the results of [GM21] to the more general setting of projective Anosov subgroups of $\mathrm{SL}(d,\mathbb{R})$, with the addition of proving equalities between critical exponents (growth rates of singular values) and entropies (growth rates of eigenvalues) in a broad setting. When geometric arguments are needed, the pseudo-Riemannian hyperbolic geometry used in [GM21] is replaced with convex projective geometry.

5.1 Results

Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. Then Γ is Gromov hyperbolic and comes with two injective equivariant maps $\xi : \partial_\infty\Gamma \to \mathbb{RP}^{d-1}$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{RP}^{d-1*}$. We denote by ξ^{sym} the map $(\xi,\xi^*) : \partial_\infty\Gamma \to \mathcal{F}_{1,d-1}$ where

$$\mathcal{F}_{1,d-1} = \left\{([x],[\alpha]) \in \mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1*} \,\middle|\, \alpha(x) = 0\right\}$$

is the partial flag manifold. We will refer to the images of these maps as the *limit sets* of Γ, denoted

$$\Lambda_\Gamma = \xi(\partial_\infty\Gamma) \subset \mathbb{RP}^{d-1}\ ;\quad \Lambda^*_\Gamma = \xi^*(\partial_\infty\Gamma) \subset \mathbb{RP}^{d-1*}\ ;\quad \Lambda^{\mathrm{sym}}_\Gamma = \xi^{\mathrm{sym}}(\partial_\infty\Gamma) \subset \mathcal{F}_{1,d-1}.$$

We define the *simple root critical exponent* of Γ by

$$\delta_{1,2}(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R} \operatorname{Log}\operatorname{Card}\left\{\gamma \in \Gamma \,\middle|\, \mu_1(\gamma) - \mu_2(\gamma) \leq R\right\}$$

and the *highest weight critical exponent* of Γ by

$$\delta_{1,d}(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R} \operatorname{Log}\operatorname{Card}\left\{\gamma \in \Gamma \,\middle|\, \mu_1(\gamma) - \mu_d(\gamma) \leq R\right\}\ .$$

These critical exponents are relevant for different reasons: the projective Anosov property means that $\mu_1(\gamma) - \mu_2(\gamma)$ grows linearly with the word length of γ, so $\delta_{1,2}(\Gamma)$ can be seen as a "measure" of the Anosov property. The critical exponent $\delta_{1,d}(\Gamma)$ is the critical exponent associated to the Hilbert metric on $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$ seen as the projectivization of the cone of positive definite quadratic forms on $\mathbb{R}^d$. Our main result compares these two critical exponents with the Hausdorff dimension of $\Lambda^{\mathrm{sym}}_\Gamma$ with respect to a Riemannian metric on $\mathcal{F}_{1,d-1}$.

Our first comparison result between Hausdorff dimensions concerns strongly projectively convex-cocompact subgroups of $\mathrm{SL}(d,\mathbb{R})$, introduced by Crampon and Marquis [CM14]. It is shown in [DGK23] that these groups are projective Anosov.

Theorem 5.1 ([GMT23, Theorem 1.1]). *Assume $d \geq 3$, and let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a strongly projectively convex-cocompact subgroup. Then*

$$2\delta_{1,d}(\Gamma) \leq \mathrm{Hdim}(\Lambda^{\mathrm{sym}}_\Gamma) \leq \delta_{1,2}(\Gamma)\ .$$

For projective Anosov subgroups that are not convex-cocompact, composing with the representation of $\mathrm{SL}(d,\mathbb{R})$ into $\mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^d))$ gives the following weaker result:

Corollary 5.2 ([GMT23, Corollary 1.2]). *Assume $d \geq 2$, and let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. Then*

$$\delta_{1,d}(\Gamma) \leq \mathrm{Hdim}(\Lambda^{\mathrm{sym}}_\Gamma) \leq \delta_{1,2}(\Gamma)\ .$$

Note that Theorem 5.1 is "sharp" in the sense that both inequalities become equalities when Γ is a convex cocompact subgroup in $\mathrm{SO}(d-1,1) \subset \mathrm{SL}(d,\mathbb{R})$. Corollary 5.2 is weaker since we always have $\delta_{1,d}(\Gamma) \leq \frac{1}{2}\delta_{1,2}(\Gamma)$ when $d \geq 3$, so at most one of the inequalities can be an equality. However, it cannot be sharpened in full generality. For instance, let $\Gamma < \mathrm{SL}(2,\mathbb{R})$ be a uniform lattice and let $\rho_{irr}, \rho_{red} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ denote respectively the irreducible and reducible representations. Then $\rho_{irr}(\Gamma)$ and $\rho_{red}(\Gamma)$ are projective Anosov with limit set a smooth curve (of Hausdorff dimension 1). However, their critical exponents differ:

- $\rho_{irr}(\Gamma) < \mathrm{SO}(2,1)$ is convex cocompact and both equalities in Theorem 5.1 are reached

$$2\delta_{1,3}(\rho_{irr}(\Gamma)) = \delta_{1,2}(\rho_{irr}(\Gamma)) = 1 = \mathrm{Hdim}(\Lambda_\Gamma^{\mathrm{sym}}) \,.$$

- $\rho_{red}(\Gamma)$ is not convex cocompact and the lower bound in Corollary 5.2 is reached

$$\delta_{1,3}(\rho_{red}(\Gamma)) = \frac{1}{2}\delta_{1,2}(\rho_{red}(\Gamma)) = 1 = \mathrm{Hdim}(\Lambda_\Gamma^{\mathrm{sym}}) \,.$$

The common right-hand side inequality in Theorem 5.1 and Corollary 5.2 is also valid for the other limit sets.

Theorem 5.3 ([GMT23, Theorem 4.1])**.** *Assume $d \geq 2$, and let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. Then*

$$\mathrm{Hdim}(\Lambda_\Gamma) \leq \delta_{1,2}(\Gamma) \,.$$

This inequality was proven independently by Pozzetti, Sambarino and Wienhard in [PSW21]. There are many situations where this upper bound on the Hausdorff dimension is reached, including some Zariski dense examples: the equality $\mathrm{Hdim}(\Lambda_\Gamma) = \delta_{1,2}(\Gamma)$ holds for images of Hitchin representations [PS17, Theorem B].

The lower bounds on the Hausdorff dimension seem however to be rarely reached, leading us in [GMT23] to conjecture the following rigidity statement:

Conjecture 5.4. *Assume $d \geq 3$, and let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a strongly projectively convex cocompact subgroup. If $2\delta_{1,d} = \mathrm{DimH}(\Lambda_\Gamma^{\mathrm{sym}})$, then Γ is conjugated to a subgroup of $\mathrm{SO}(d-1,1)$.*

For $\mathbb{H}^{p,q}$-convex-cocompact subgroups $\Gamma < \mathrm{PO}(p,q+1)$, we have $2\delta_{1,d}(\Gamma) = \delta_{\mathbb{H}^{p,q}}(\Gamma)$, so [MV24, Theorem 4] proves this conjecture for $\mathbb{H}^{p,q}$-quasi-Fuchsian groups.

5.2 Critical exponents and entropies

Let us introduce entropies as growth rates of eigenvalue gaps for discrete linear groups.

Definition 5.5. Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a discrete subgroup. We define the *simple root entropy* of Γ as

$$h_{1,2}(\Gamma) = \limsup_{R\to\infty} \frac{1}{R} \mathrm{Log}\,\mathrm{Card}\,\{[\gamma] \in [\Gamma] \,\big|\, \lambda_1(\gamma) - \lambda_2(\gamma) \leq R\} \,,$$

and the *highest weight entropy* of Γ as

$$h_{1,d}(\Gamma) = \limsup_{R\to\infty} \frac{1}{R} \operatorname{Log} \operatorname{Card}\{[\gamma] \in [\Gamma] \,\big|\, \lambda_1(\gamma) - \lambda_d(\gamma) \leq R\} ,$$

where $[\Gamma]$ denotes the set of conjugacy classes in Γ.

The term "entropy" comes from the analogy with the geodesic flow of a closed negatively curved manifold, whose closed orbits are in bijection with conjugacy classes in the fundamental group, and whose topological entropy equals the exponential growth rate of lengths of closed orbits. For projective Anosov subgroups, the highest weight entropy is equal to the topological entropy of a flow associated to Γ (see [Sam14]), and for some specific subgroups as images of Hitchin representations it is also the case for the simple root entropy (these flows are studied in [BCLS18]).

For sufficiently nice discrete groups of isometries of a negatively curved manifold, the critical exponent equals the entropy. For a Zariski dense Θ-Anosov group, Sambarino obtained in [Sam14] precise counting estimates for

$$\operatorname{Card}\{\gamma \in \Gamma \,\big|\, \mu_1(\gamma) - \mu_d(\gamma) \leq R\} ,$$

for projective Anosov subgroups, implying in particular that $h_{1,d}(\Gamma) = \delta_{1,d}(\Gamma)$. The tools he uses, however, do not seem to apply to simple root critical exponents in general, so the novelty in the following result concerns the simple root exponent/entropy.

Theorem 5.6 ([GMT23, Theorem 2.30]). *Assume $d \geq 2$, and let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. Then*

- $h_{1,2}(\Gamma) \leq \delta_{1,2}(\Gamma)$,
- $h_{1,d}(\Gamma) = \delta_{1,d}(\Gamma)$.

If Γ is moreover Zariski dense in $\mathrm{SL}(d,\mathbb{R})$, then

- $h_{1,2}(\Gamma) = \delta_{1,2}(\Gamma)$.

The reason for Zariski density in the second part is that we need to find some regular elements $\gamma \in \Gamma$ (more precisely, we need a controlled gap between $\mu_2(\gamma)$ and $\mu_3(\gamma)$), this is achieved thanks to a famous result of Abels, Margulis and Soifer [AMS95] stating that up to multiplying by elements in a finite subset of Γ, elements become loxodromic provided that Γ is Zariski-dense. For the simple root entropy, we only need control on the gap between $\mu_1(\gamma)$ and $\mu_2(\gamma)$, which is provided by the projective Anosov condition.

The equality $h_{1,2}(\Gamma) = \delta_{1,2}(\Gamma)$ actually holds as soon as the Zariski closure of Γ is semi-simple (by applying the more general [GMT23, Theorem 2.31] to the Zariski closure). A typical example where we don't know whether the equality $\delta_{1,2} = h_{1,2}$ holds is for Barbot representations: deformations of uniform lattices of $\mathrm{SL}(2,\mathbb{R})$ inside $\mathrm{Aff}(\mathbb{R}^2) \subset \mathrm{SL}(3,\mathbb{R})$ [Bar01, Bar10].

5.3 Convex projective geometry and Hausdorff dimensions

We will not discuss the proof of the upper bound (Theorem 5.3) on the Hausdorff dimension in depth, as it is a rather classic argument. In a few words, let us say that a Hausdorff dimension is bounded from above by finding an appropriate cover of the set. In self-similar settings, this is achieved by considering translates of a fixed ball by elements of the group. This works well in our case because the distortion of such an element $\gamma \in \Gamma$ is comparable to $e^{\mu_2(\gamma)-\mu_1(\gamma)}$.

The proof of the lower bound in Theorem 5.1 follows a strategy similar to that of [GM21], with the Hilbert geometry of a proper convex domain in $\mathbb{RP}^{d-1}$ replacing the pseudo-Riemannian geometry of $\mathbb{H}^{p,q}$. Recall that an open subset $\Omega \subset \mathbb{RP}^{d-1}$ is called *properly convex* if its closure is contained in an affine chart, in which it is convex and bounded.

Definition 5.7 ([DGK23, Definition 1.1]). Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be an infinite discrete subgroup.

- Let $\Omega \subset \mathbb{RP}^{d-1}$ be a Γ-invariant properly convex open subset. The action $\Gamma \curvearrowright \Omega$ is called *strongly convex cocompact* if Ω is strictly convex with $\mathcal{C}^1$ boundary, and for some $x \in \Omega$, the convex hull in Ω of $\overline{\Gamma \cdot x} \cap \partial\Omega$ is non-empty and has compact quotient by Γ.
- The group $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is *strongly projectively convex-cocompact* if it admits a strongly convex cocompact action on a properly convex open subset $\Omega \subset \mathbb{RP}^{d-1}$.

Theorem 5.8 ([DGK23, Theorem 1.15]). *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be strongly projectively convex-cocompact. Then Γ is projective Anosov, and if $\Omega \subset \mathbb{RP}^{d-1}$ is a properly convex open subset on which Γ acts strongly convex-cocompactly, then $\overline{\Gamma \cdot x} \cap \partial\Omega = \Lambda_\Gamma$ for any $x \in \Omega$.*

Although the converse is false (e.g. uniform lattices in $\mathrm{SL}(2,\mathbb{R})$), one can go from projective Anosov subgroups to strongly projectively convex-cocompact subgroups by considering an appropriate representation of $\mathrm{SL}(d,\mathbb{R})$.

Proposition 5.9. *Let $I : \mathrm{SL}(d,\mathbb{R}) \to \mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^{d*}))$ be the representation given by the action on the space of quadratic forms. If $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is projective Anosov, then $I(\Gamma) < \mathrm{SL}(\mathrm{Sym}^2(\mathbb{R}^{d*}))$ is strongly projectively convex-cocompact.*

This is a consequence of [DGK23, Theorem 1.15 (vi)] and the fact that $I(\mathrm{SL}(d,\mathbb{R}))$ preserves the cone of positive definite quadratic forms.

Sketch of proof of Corollary 5.2 assuming Theorem 5.1. Let $n = \dim \mathrm{Sym}^2(\mathbb{R}^{d*}) = \frac{d(d+1)}{2}$. For $g \in \mathrm{SL}(d,\mathbb{R})$, one finds

$$\mu_1(I(g)) = 2\mu_1(g)\,, \quad \mu_2(I(g)) = \mu_1(g) + \mu_2(g)\,, \quad \mu_n(I(g)) = 2\mu_d(g)\,,$$

so for any discrete subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$ we have

$$\delta_{1,2}(I(\Gamma)) = \delta_{1,2}(\Gamma)\,, \quad 2\delta_{1,n}(I(\Gamma)) = \delta_{1,d}(\Gamma).$$

If Γ is projective Anosov, then the smooth $\mathrm{SL}(d,\mathbb{R})$-equivariant embedding

$$\begin{cases} \mathcal{F}_{1,d-1} & \to & \mathbb{P}\left(\mathrm{Sym}^2(\mathbb{R}^{d*})\right) \times \mathbb{P}\left(\mathrm{Sym}^2(\mathbb{R}^{d*})^*\right) \\ ([x],[\alpha]) & \mapsto & ([\alpha \otimes \alpha], [\iota_x \otimes \iota_x]) \end{cases},$$

where $\iota_x \in (\mathbb{R}^{d*})^*)$ is the evaluation at $x \in \mathbb{R}^d$, sends $\Lambda_\Gamma^{\mathrm{sym}}$ to $\Lambda_{I(\Gamma)}^{\mathrm{sym}}$, therefore

$$\mathrm{Hdim}\left(\Lambda_\Gamma^{\mathrm{sym}}\right) = \mathrm{Hdim}\left(\Lambda_{I(\Gamma)}^{\mathrm{sym}}\right).$$

□

The geometry of a properly convex open subset $\Omega \subset \mathbb{RP}^{d-1}$ can be studied through its *Hilbert metric* of d_Ω.

Definition 5.10. Let $\Omega \subset \mathbb{RP}^{d-1}$ be a properly convex open subset, and $x, y \in \Omega$. Let $a, b \in \partial\Omega$ denote the intersections of the projective line (xy) with $\partial\Omega$, ordered so that x lies between a and y. Then the *Hilbert distance* between x and y is

$$d_\Omega(x,y) = \frac{1}{2}\mathrm{Log}[a;x;y;b].$$

If a group $\Gamma < \mathrm{SL}(d,\mathbb{R})$ acts strongly convex cocompactly on Ω, we can define the *Hilbert critical exponent*

$$\delta_\Omega(\Gamma) = \limsup_{R\to+\infty} \mathrm{Log}\,\mathrm{Card}\left\{\gamma \in \Gamma \,\middle|\, d_\Omega(x, \gamma\cdot x) \le R\right\}$$

where $x \in \Omega$ is any point. Now any conjugacy class $[\gamma] \in [\Gamma]$ determines a closed geodesic in $\Gamma\backslash\Omega$, whose length will be denoted by $\ell_\Omega([\gamma])$. We can define the *Hilbert entropy* as

$$h_\Omega(\Gamma) = \limsup_{R\to+\infty} \frac{1}{R}\mathrm{Log}\,\mathrm{Card}\left\{[\gamma] \in [\Gamma] \,\middle|\, \ell_\Omega([\gamma]) \le R\right\}.$$

Just as we saw in $\mathbb{H}^{p,q}$ (Definition 4.4), we can define the Gromov product and Gromov quasi-distance on $\partial\Omega$.

Definition 5.11. Let $\Omega \subset \mathbb{RP}^{d-1}$ be a properly convex open subset, $x \in \Omega$ and $\xi, \eta \in \partial\Omega$. The Gromov product $(\xi|\eta)_x$ is defined by

$$(\xi|\eta)_x = \lim_{k\to+\infty} \frac{1}{2}(d_\Omega(x_k,x) + d_\Omega(y_k,y) - d_\Omega(x_k,y_k)$$

where (x_k) and (y_k) are sequences in Ω such that $x_k \to \xi$ and $y_k \to \eta$. The Gromov quasi-distance is

$$d_x(\xi,\eta) = e^{-(\xi|\eta)_x}.$$

Even though the Gromov quasi-distance is not a distance, it can be shown to satisfy the same type of generalised triangle inequality as in the $\mathbb{H}^{p,q}$-convex cocompact case (see [GM21, Lemma 3.17] or Proposition 4.5 in this memoir), which is enough to make sense of the Hausdorff dimension $\mathrm{Hdim}_{d_x}(\Lambda_\Gamma)$ for a strongly projectively convex-cocompact subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$.

Proposition 5.12 ([GMT23, Proposition 3.5])**.** *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be acting strongly convex-cocompactly on a proper convex domain $\Omega \subset \mathbb{RP}^d$. Then, for any $x \in \Omega$,*

$$\mathrm{Hdim}_{d_x}(\Lambda_\Gamma) = 2\delta_{1,d}(\Gamma).$$

Sketch of proof of Proposition 5.12. Translating a theorem of Coornaert [Coo93] relating Hausdorff dimension and critical exponent for actions of Gromov hyperbolic groups on Gromov hyperbolic spaces, we find the equality

$$\delta_\Omega(\Gamma) = \mathrm{Hdim}_{d_x}(\Lambda_\Gamma)\,.$$

By a result of Coornaert and Knieper in this same general setting [CK02], we find

$$\delta_\Omega(\Gamma) = h_\Omega(\Gamma).$$

Now since the closed geodesic in $\Gamma\backslash\Omega$ corresponding to $[\gamma] \in [\Gamma]$ lifts to $\Omega \subset \mathbb{RP}^d$ to the line whose endpoints in $\partial\Omega$ are the eigendirection of γ for its eigenvalues of largest and smallest moduli, a simple computation shows that

$$\ell_\Omega([\gamma]) = \frac{\lambda_1(\gamma) - \lambda_d(\gamma)}{2}$$

therefore $h_\Omega(\Gamma) = 2h_{1,d}(\Gamma)$. Applying Theorem 5.6, we also have that $h_{1,d}(\Gamma) = \delta_{1,d}(\Gamma)$, hence the conclusion. □

The last missing piece in order to prove Theorem 5.1 is a comparison between the Riemannian Hausdorff dimension $\mathrm{Hdim}(\Lambda_\Gamma^{\mathrm{sym}})$ and the dimension $\mathrm{Hdim}_{d_x}(\Lambda)$ computed with respect to the Gromov quasi-distance d_x.

Recall from the definition of strong projective cocompactness that we assume the boundary $\partial\Omega$ to be $\mathcal{C}^1$. For $p \in \partial\Omega$, the tangent space $T_p\partial\Omega$ corresponds to a hyperplane in $\mathbb{RP}^d$, thus to a point $p^* \in \mathbb{RP}^{d*}$. In the specific case of $p = \xi(t) \in \Lambda_\Gamma$ for some $t \in \partial_\infty\Gamma$, we find $p^* = \xi^*(t)$.

Lemma 5.13 ([GMT23, Lemma 3.6])**.** *Given Riemannian distances $d_{\mathbf{P}}$ and $d^*_{\mathbf{P}}$ on $\mathbb{RP}^d$ and $\mathbb{RP}^{d*}$, there is a constant $C > 0$ such that:*

$$\forall p, q \in \partial\Omega \quad d_x(p,q) \le C\sqrt{d_{\mathbf{P}}(p,q)\,d^*_{\mathbf{P}}(p^*,q^*)}$$

The inequality

$$\mathrm{Hdim}_{d_x}(\Lambda_\Gamma) \le \mathrm{Hdim}(\Lambda_\Gamma^{\mathrm{sym}})$$

then follows from elementary comparison results between Hausdorff dimensions. The proof of Lemma 5.13 is essentially some elementary planar Euclidean geometry.

6 Locally homogeneous axiom A flows

In [DMS24], written in collaboration with B. Delarue and A. Sanders, we construct a natural real analytic flow associated to any projective Anosov subgroup, then prove that this flow satisfies Smale's Axiom A and is exponentially mixing. Smale's Axiom A implies that the interesting dynamics of this flow occur on an invariant compact subset. An important feature of our construction is that the restriction to this invariant compact subset is equivalent to Sambarino's *refraction flow* [Sam14, Sam24]. The novelty is that we embed Sambarino's refraction flow into a smooth setting, thus allowing us to import tools from smooth dynamics. In particular we manage to prove a certain set of conditions, proved to imply exponential mixing for sufficiently smooth flows by Stoyanov [Sto11], are met by the flow we construct.

Theorem 6.1 ([DMS24]). *There is a* $\mathrm{SL}(d,\mathbb{R})$*-homogeneous space* $\mathbb{L}$*, equipped with a flow* ϕ^t *that commutes with the action of* $\mathrm{SL}(d,\mathbb{R})$*, such that for any torsion free projective Anosov subgroup* $\Gamma < \mathrm{SL}(d,\mathbb{R})$*, there is a* $\Gamma \times \{\phi^t\}$*-invariant open subset* $\widetilde{\mathcal{M}_\Gamma} \subset \mathbb{L}$ *on which* Γ *acts properly discontinuously and freely, and such that the quotient flow* $\mathcal{M}_\Gamma = \Gamma \backslash \widetilde{\mathcal{M}_\Gamma} \curvearrowleft \phi^t$ *satisfies Smale's Axiom A.*

Furthermore, if Γ *is irreducible, then the quotient flow mixes exponentially fast with respect to any Gibbs equilibrium measure associated to a Hölder potential.*

6.1 The flow space

We will work with the open set $\mathbb{L} \subset \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^{d*})$ defined by

$$\mathbb{L} := \left\{ [v:\alpha] \in \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^{d*}) \,\middle|\, \alpha(v) \neq 0 \right\}.$$

This real analytic manifold comes with a real analytic action of $\mathbb{R}$ defined by

$$\phi^t([v:\alpha]) = [e^t v : e^{-t}\alpha], \quad \forall\, t \in \mathbb{R},\ \forall\, [v:\alpha] \in \mathbb{L}.$$

The flow ϕ^t commutes with the action $\mathrm{SL}(d,\mathbb{R}) \curvearrowright \mathbb{L}$ given by $g \cdot [v:\alpha] = [g\cdot v : \alpha \circ g^{-1}]$.

In order to describe the geometry of $\mathbb{L}$, it is practical to work with the affine quadric hypersurface

$$\mathbb{L}_1 := \left\{ (v,\alpha) \in \mathbb{R}^d \times \mathbb{R}^{d*} \,\middle|\, \alpha(v) = 1 \right\}$$

which is an $\mathrm{SL}(d,\mathbb{R})$-equivariant double cover of $\mathbb{L}$ through the restriction of the projection $\pi : \left(\mathbb{R}^d \times \mathbb{R}^{d*}\right) \setminus \{(0,0)\} \to \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^{d*})$. The description of the tangent space

$$T_{(v,\alpha)}\mathbb{L}_1 = \left\{ (w,\beta) \in \mathbb{R}^d \times \mathbb{R}^{d*} \,\middle|\, \alpha(w) + \beta(v) = 0 \right\}$$

shows that it carries a real analytic pseudo-Riemannian metric of signature $(d, d-1)$ given by

$$((w,\beta),(w',\beta'))_{(v,\alpha)} = \beta(w') + \beta'(w)\,, \quad (w,\beta),(w',\beta') \in T_{(v,\alpha)}\mathbb{L}_1\ .$$

It also carries a real analytic contact form given by the restriction of the tautological 1-form of the cotangent bundle $T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^{d*}$:

$$\tau_{(v,\alpha)}(w,\beta) = \alpha(w) = -\beta(v)\,, \quad (w,\beta) \in T_{(v,\alpha)}\mathbb{L}_1\ .$$

Both structures descend to $\mathrm{SL}(d,\mathbb{R})$-invariant structures on $\mathbb{L}$. The Reeb vector field of the contact structure integrates to the flow ϕ^t.

The space $\mathbb{L}$ fibres over the open set

$$\mathbb{RP}^{d-1} \overset{\pitchfork}{\times} \mathbb{RP}^{d-1*} = \left\{ ([v],[\alpha]) \in \mathbb{RP}^{d-1} \times \mathbb{RP}^{d-1*} \,\middle|\, \alpha(v) \neq 0 \right\}$$

through the projection

$$p : \left\{ \begin{array}{ccc} \mathbb{L} & \to & \mathbb{RP}^{d-1} \overset{\pitchfork}{\times} \mathbb{RP}^{d-1*} \\ [v:\alpha] & \mapsto & ([v],[\alpha]) \end{array} \right. . \tag{5}$$

The fibres of p are equal to the orbits of the flow ϕ^t, giving $\mathbb{L}$ the structure of a real analytic principal $\mathbb{R}$-bundle (i.e. a real affine line bundle).

Remark. In Lie theoretic language, we can interpret $\mathbb{L}$ as the homogeneous space $\mathrm{SL}(d,\mathbb{R})/S$ where

$$S = \left\{ \begin{pmatrix} \frac{1}{\det A} & \\ & A \end{pmatrix} \middle| \det A = \pm 1 \right\} \approx \mathrm{SL}^{\pm}(d-1,\mathbb{R}) ,$$

and the flow ϕ^t is the diagonal flow given by right multiplication by the one parameter subgroup

$$\left\{ e^{tX} \,\middle|\, t \in \mathbb{R} \right\} , \quad X = \begin{pmatrix} e^t & \\ & e^{-\frac{t}{d-1}} 1_{d-1} \end{pmatrix} .$$

The projection $p : \mathbb{L} \to \mathbb{RP}^{d-1} \overset{\pitchfork}{\times} \mathbb{RP}^{d-1*}$ can also be seen as the projection $\mathrm{SL}(d,\mathbb{R})/S \to \mathrm{SL}(d,\mathbb{R})/L$ where

$$L = \left\{ \begin{pmatrix} \frac{1}{\det A} & \\ & A \end{pmatrix} \middle| A \in \mathrm{GL}(d-1,\mathbb{R}) \right\} \approx \mathrm{GL}(d-1,\mathbb{R}) .$$

Definition 6.2. Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be projective Anosov, and consider its limit maps $\xi : \partial_\infty\Gamma \to \mathbb{RP}^{d-1}$ and $\xi^* : \partial_\infty\Gamma \to \mathbb{RP}^{d-1*}$. We introduce the following subset of $\mathbb{RP}^{d-1} \overset{\pitchfork}{\times} \mathbb{RP}^{d-1*}$:

$$\Omega_\Gamma := \left\{ (\ell, H) \in \mathbb{RP}^{d-1} \overset{\pitchfork}{\times} \mathbb{RP}^{d-1*} \middle| \forall x \in \partial_\infty\Gamma \ \ell \pitchfork \xi^*(x) \text{ or } \xi(x) \pitchfork H \right\}.$$

Then we define

$$\widetilde{\mathcal{M}_\Gamma} := p^{-1}(\Omega_\Gamma) \subset \mathbb{L}.$$

Theorem 6.3 ([DMS24, Lemma 1.4, Theorem 1]). *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be projective Anosov. The set $\widetilde{\mathcal{M}_\Gamma} \subset \mathbb{L}$ is open and $\Gamma \times \{\phi^t\}$-invariant, and the action $\Gamma \curvearrowright \widetilde{\mathcal{M}_\Gamma}$ is properly discontinuous. If moreover Γ is torsion free, this action is free.*

The openness and invariance of $\widetilde{\mathcal{M}_\Gamma}$ are quite straightforward. The proper discontinuity of the action $\Gamma \curvearrowright \widetilde{\mathcal{M}_\Gamma}$ requires some understanding of the linear action $\Gamma \curvearrowright \mathbb{R}^d$ of a projective Anosov subgroup, which is possible by seeing $\mathbb{R}^d \setminus \{0\}$ as the total space of the tautological bundle over $\mathbb{RP}^{d-1}$, since the projective action $\Gamma \curvearrowright \mathbb{RP}^{d-1}$ is well understood thanks to the convergence property (see Theorem 1.1 and Definitions 4.2, 4.25 in [KLP17]):

Proposition 6.4. *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be projective Anosov. For any unbounded sequence $\gamma_N \in \Gamma$ with boundary limit points $\gamma_+ = \lim_{N\to+\infty} \gamma_N \in \partial_\infty\Gamma$ and $\gamma_- = \lim_{N\to+\infty} \gamma_N^{-1} \in \partial_\infty\Gamma$, there is a subsequence γ_{N_k} for which the actions on $\mathbb{RP}^{d-1}$ and $\mathbb{RP}^{d-1*}$ obey the following dynamics as $k \to +\infty$:*

1. *$\gamma_{N_k} \cdot \ell \to \xi(\gamma_+)$ for all $\ell \in \mathbb{RP}^{d-1}$ with $\ell \pitchfork \xi^*(\gamma_-)$;*
2. *$\gamma_{N_k}^{-1} \cdot \ell \to \xi(\gamma_-)$ for all $\ell \in \mathbb{RP}^{d-1}$ with $\ell \pitchfork \xi^*(\gamma_+)$;*
3. *$\gamma_{N_k} \cdot H \to \xi^*(\gamma_+)$ for all $H \in \mathbb{RP}^{d-1*}$ with $\xi(\gamma_-) \pitchfork H$;*
4. *$\gamma_{N_k}^{-1} \cdot H \to \xi^*(\gamma_-)$ for all $H \in \mathbb{RP}^{d-1*}$ with $\xi(\gamma_+) \pitchfork H$;*

and all these convergences are uniform on compact subsets.

The lift to $\mathbb{R}^d \setminus \{0\}$ obeys the following rule, which is the central argument in the proof of proper discontinuity in Theorem 6.3:

Lemma 6.5 ([DMS24, Lemma 3.2])**.** *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be projective Anosov, and $\gamma_k \in \Gamma$ a sequence admitting distinct boundary limits $\gamma_+ = \lim \gamma_k \in \partial_\infty \Gamma$ and $\gamma_- = \lim \gamma_k^{-1} \in \partial_\infty \Gamma$. For any sequence $v_k \to v \in \mathbb{R}^d \setminus \{0\}$ such that $[v] \pitchfork \xi^*(\gamma_-)$, one has $\gamma_k \cdot v_k \to \infty$ as $k \to \infty$.*

6.2 Smale's Axiom A

Let us start this section with a few classical definitions from dynamical systems.

Definition 6.6. Let $\mathcal{M}$ be a metrizable topological space and $\phi^t : \mathcal{M} \to \mathcal{M}$ a continuous flow defined for all $t \in \mathbb{R}$ which has no fixed points.

- The *non-wandering set* $\mathcal{NW}(\phi^t)$ of the flow ϕ^t is the set of all points $x \in \mathcal{M}$ for which there are sequences $x_N \to x$ in $\mathcal{M}$ and $t_N \to +\infty$ in $\mathbb{R}$ such that $\phi^{t_N}(x_N) \to x$.
- The set $\mathcal{P}(\phi^t)$ of *periodic points* of the flow ϕ^t consists of all points $x \in \mathcal{M}$ for which there exists $T > 0$ with $\phi^T(x) = x$.
- Let $\mathcal{K} \subset \mathcal{M}$ be a compact ϕ^t-invariant set and E a continuous vector bundle over $\mathcal{K}$ equipped with a continuous flow $\phi^t_E : E \to E$ lifting ϕ^t over $\mathcal{K}$. Then ϕ^t_E is *uniformly contracting* (resp. *expanding*) on E if for some (hence any) continuous bundle norm $\|\cdot\|$ on E there are constants $C, c > 0$ such that for all $p \in \mathcal{K}$ and all $v \in E_p$ one has
$$\left\|\phi^t_E(v)\right\|_{\phi^t(p)} \leq C e^{-c|t|} \|v\|_p$$
for all $t \geq 0$ (resp. $t \leq 0$).

Definition 6.7. Let $\mathcal{M}$ be a smooth manifold and $\phi^t : \mathcal{M} \to \mathcal{M}$ a smooth flow generated by a complete nowhere vanishing vector field $X : \mathcal{M} \to T\mathcal{M}$. A compact ϕ^t-invariant set $\mathcal{K} \subset \mathcal{M}$ is called *hyperbolic* for the flow ϕ^t if the restriction of the tangent bundle $T\mathcal{M}$ to $\mathcal{K}$ admits a decomposition
$$T\mathcal{M}|_{\mathcal{K}} = E^0 \oplus E^{\mathrm{s}} \oplus E^{\mathrm{u}}$$
where $E^0_p = \mathbb{R} \cdot X(p)$ for all $p \in \mathcal{K}$ and E^{s}, E^{u} are $d\phi^t$-invariant continuous sub-bundles such that $d\phi^t$ is uniformly contracting (resp. expanding) on E^{s} (resp. E^{u}).

Definition 6.8 (c.f. [Sma67, §II.5 (5.1)])**.** The flow ϕ^t is an *Axiom A* flow if the non-wandering set $\mathcal{NW}(\phi^t)$ is compact and hyperbolic and coincides with the closure in $\mathcal{M}$ of the set of periodic points $\mathcal{P}(\phi^t)$.

We now turn back to projective Anosov subgroups.

Definition 6.9. Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. The *transverse limit set* is
$$\Lambda^{\pitchfork}_\Gamma = \{(\xi(s), \xi^*(t)) \,|\, s, t \in \partial_\infty \Gamma,\ s \neq t\} \subset \mathbb{RP}^{d-1} \overset{\pitchfork}{\times} \mathbb{RP}^{d-1*}.$$
The *lifted basic set* is
$$\widetilde{\mathcal{K}_\Gamma} = p^{-1}\left(\Lambda^{\pitchfork}_\Gamma\right) \subset \mathbb{L}.$$

Lemma 6.10 ([DMS24, Lemma 3.7 and 3.8]). *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. The lifted basic set $\widetilde{\mathcal{K}_\Gamma} \subset \mathbb{L}$ is a closed $\Gamma \times \{\phi^t\}$-invariant subset contained in $\widetilde{\mathcal{M}_\Gamma}$, and the action $\Gamma \curvearrowright \widetilde{\mathcal{K}_\Gamma}$ is cocompact.*

The cocompactness of the action $\Gamma \curvearrowright \widetilde{\mathcal{K}_\Gamma}$ is proved by using some appropriate Hopf coordinates on $\mathbb{L}$, showing that the action $\Gamma \curvearrowright \widetilde{\mathcal{K}_\Gamma}$ is conjugate to an action $\Gamma \curvearrowright \partial_\infty\Gamma^{(2)} \times \mathbb{R}$ studied by Sambarino in [Sam14] and therefore proving the conjugacy between the quotient flow on $\mathcal{K}_\Gamma = \Gamma\backslash\widetilde{\mathcal{K}_\Gamma}$ and Sambarino's refraction flow.

Theorem 6.11 ([DMS24, Theorem A]). *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a projective Anosov subgroup. The quotient flow $\mathcal{M}_\Gamma = \Gamma\backslash\widetilde{\mathcal{M}_\Gamma} \curvearrowleft \phi^t$ is an axiom A flow, and its non wandering set is $\mathcal{K}_\Gamma = \Gamma\backslash\widetilde{\mathcal{K}_\Gamma}$.*

Remark. A flow $\mathcal{M} \curvearrowleft \phi^t$ on a smooth manifold is called *Anosov* if $\mathcal{M}$ is compact and is a hyperbolic set. Despite the terminology, the flow $\mathcal{M}_\Gamma \curvearrowleft \phi^t$ associated to a projective Anosov subgroup is never Anosov when $d \geq 3$, and is only Anosov for uniform lattices in $\mathrm{SL}(2,\mathbb{R})$.

A simple study of the differential of the flow $\mathbb{L} \curvearrowleft \phi^t$ shows some predisposition towards hyperbolicity. Indeed, recall the double cover

$$\mathbb{L}_1 := \left\{(v,\alpha) \in \mathbb{R}^d \times \mathbb{R}^{d*} \,\middle|\, \alpha(v) = 1\right\}$$

and the description of the tangent space

$$T_{(v,\alpha)}\mathbb{L}_1 = \left\{(w,\beta) \in \mathbb{R}^d \times \mathbb{R}^{d*} \,\middle|\, \alpha(w) + \beta(v) = 0\right\}.$$

There is a natural splitting

$$T\mathbb{L}_1 = E^{\mathrm{s}} \oplus E^0 \oplus E^{\mathrm{u}}$$

where

$$\begin{aligned} E^{\mathrm{u}}_{(v,\alpha)} &= \ker\alpha \times \{0\}, \\ E^{\mathrm{s}}_{(v,\alpha)} &= \{0\} \times \ker\iota_v, \qquad \forall (v,\alpha) \in \mathbb{L}_1. \\ E^0_{(v,\alpha)} &= \mathbb{R}\cdot(v,-\alpha), \end{aligned}$$

These distributions project to an $\mathrm{SL}(d,\mathbb{R}) \times \{d\phi^t\}$-equivariant splitting of the tangent bundle $T\mathbb{L}$:

$$T\mathbb{L} = E^{\mathrm{s}} \oplus E^0 \oplus E^{\mathrm{u}}$$

This decomposition is related to the flow ϕ^t by the formula $E^0_{[v:\alpha]} = \mathbb{R}\cdot \frac{d}{dt}\big|_{t=0}\phi^t([v:\alpha])$. The action of the differential $d\phi^t$ of the flow ϕ^t on E^{u} and E^{s} has a very simple expression. For $[v:\alpha] \in \mathbb{L}$, $(w,0) \in \ker\alpha \times \{0\} = E^{\mathrm{u}}_{[v:\alpha]}$, $(0,\beta) \in \{0\} \times \ker\iota_v = E^{\mathrm{s}}_{[v:\alpha]}$ and $t \in \mathbb{R}$ we find:

$$d_{[v:\alpha]}\phi^t(w,0) = (e^t w, 0)\,, \quad d_{[v:\alpha]}\phi^t(0;\beta) = (0, e^{-t}\beta)\,.$$

One can be tempted to believe that this formula automatically implies dilation on E^{u} and contraction on E^{s}. However, these notions involve a Riemannian metric on $\mathcal{M}_\Gamma$. Considering a lift of such a Riemannian metric to $\widetilde{\mathcal{M}_\Gamma}$, the ratio that should grow exponentially fast is

$$\frac{\left\|d_{[v:\alpha]}\phi^t(w,0)\right\|_{\phi^t([v:\alpha])}}{\|(w,0)\|_{[v:\alpha]}} = e^t\frac{\|(w,0)\|_{[e^t v:e^{-t}\alpha]}}{\|(w,0)\|_{[v:\alpha]}}.$$

The comparison would be made trivial if it were possible to choose this Riemannian metric to be constant. This is not possible because the Γ-action on $\mathbb{R}^d \times \mathbb{R}^{d*}$ does not preserve any norm. Instead, the proof of the hyperbolicity in Theorem 6.11 consists in showing that the actions of the differential $E^{\mathrm{u}}|_{\mathcal{K}_\Gamma} \curvearrowleft d\phi^t$ and $E^{\mathrm{s}}|_{\mathcal{K}_\Gamma} \curvearrowleft d\phi^t$ are conjugate to the flows appearing in the original definition of an Anosov subgroup given in [Lab06, GW12].

6.3 Exponential mixing

Axiom A flows are known to admit many invariant measures. A large class of interesting invariant measures is given by Gibbs states. They are ergodic measures μ_U supported on the non wandering set $\mathcal{K}$ of an Axiom A flow $\mathcal{M} \curvearrowleft \phi^t$ associated to Hölder functions $U : \mathcal{K} \to \mathbb{R}$. Their existence was established by Bowen and Ruelle [BR75, Theorem 3.3] via a variational principle. Another possible approach is to see them as limits of measures supported on periodic orbits. Denote by $\mathcal{P}$ the set of periodic orbits. For $c \in \mathcal{P}$, denote by $\ell(c)$ its period and by λ_c the Lebesgue measure of length $\ell(c)$ supported on c. Given a Hölder potential $U \in \mathcal{C}^\alpha(\mathcal{K})$, let

$$\ell_u(c) = \int_c u \, d\lambda_c.$$

Then the Gibbs equilibrium state is the weak limit []

$$\mu_u = \lim_{T \to +\infty} \frac{1}{\mathrm{Card}\,\{c \in \mathcal{P} \,|\, \ell_u(c) \le T\}} \sum_{\{c \in \mathcal{P} \,|\, \ell_u(c) \le T\}} e^{\ell_u(c)} \frac{\lambda_c}{\ell(c)}.$$

Let $U \in C^\alpha(\mathcal{K}, \mathbb{R})$ with unique Gibbs measure μ_U. Given $F, G \in C^\alpha(\mathcal{K}, \mathbb{R})$, the correlation function is defined by

$$\mathrm{c}^t(F, G; U) = \left| \int_{z \in \mathcal{K}} F(z) \cdot G(\phi^t(z)) \, d\mu_U(z) - \int_{z \in \mathcal{K}} F(z) \, d\mu_U(z) \int_{z \in \mathcal{K}} G(z) \, d\mu_U(z) \right|.$$

The flow ϕ^t is mixing with respect to μ_U for all Hölder observables if for all $F, G \in C^\alpha(\mathcal{K}, \mathbb{R})$ one has $\mathrm{c}^t(F, G; U) \to 0$ as $t \to \infty$ and exponentially mixing if there exists $c_\alpha(U), C_\alpha(U) > 0$ such that

$$\forall\, t \in \mathbb{R} : \ \mathrm{c}^t(F, G; U) \le C_\alpha(U) e^{-c_\alpha(U)|t|} \|F\|_\alpha \|G\|_\alpha.$$

Exponential mixing is also called exponential decay of correlations. It is usually considered as a difficult condition to prove, and the most famous contribution towards exponential mixing in the context of hyperbolicity is the work of Dologopyat [Dol98], relating rates of mixing of Anosov flows with quantitative measurements of the non integrability of the joint distribution $E^{\mathrm{u}} \oplus E^{\mathrm{s}}$. Stoyanov [Sto11] adapted this method to the Axiom A setting, in which some additional constraints on the (possibly fractal) geometry of the non wandering set must be controlled. In [DMS24], we manage to apply the work of Stoyanov and prove:

Theorem 6.12 ([DMS24, Theorem C]). *Let $\Gamma < \mathrm{SL}(d, \mathbb{R})$ be a torsion free irreducible projective Anosov subgroup. The flow $\mathcal{M}_\Gamma \curvearrowleft$ is exponentially mixing for all Hölder observables with respect to Gibbs equilibrium states associated to Hölder potentials.*

The work of Stoyanov actually proves more than exponential mixing, but some rather precise estimates on transfer operators. This stronger result can also be understood in terms of the Ruelle zeta function

$$\zeta_\Gamma(s) = \prod_{[\gamma]\in[\Gamma]_{\mathrm{prim}}} \left(1 - e^{-s\lambda_1([\gamma])}\right)^{-1}$$

where $[\Gamma]_{\mathrm{prim}}$ denotes the set of conjugacy classes of primitive elements in Γ (i.e. that are not positive powers of other elements). This Euler product converges for $\mathrm{Re}(s) < h_{\mathrm{top}}(\Gamma)$, where $h_{\mathrm{top}}(\Gamma)$ is the topological entropy of the flow $\mathcal{K}_\Gamma \curvearrowleft \phi^t$.

The resolution of Smale's conjecture for Axiom A flows due to Dyatlov-Guillarmou [DG16, DG18] and Borns-Weil-Shen [BWS21] immediately implies the global meromorphic continuation of ζ_Γ. It is known [PS98] that exponential mixing is equivalent to the existence of an open neighbourhood of the vertical line Re(s) $= h_{\mathrm{top}}(\Gamma)$ on which $\zeta_\Gamma(s)$ is holomorphic except at $s = h_{\mathrm{top}}(\Gamma)$ where it has a simple pole. The spectral estimates on Ruelle transfer operators achieved by Stoyanov [Sto11], when combined with the work Pollicott-Sharp [PS98] (see also Dolgopyat-Pollicott [DP98]) imply a zero-free strip to the left of the simple pole $h_{\mathrm{top}}(\Gamma) \in \mathbb{C}$.

Theorem 6.13 ([DMS24, Theorem D]). *Suppose $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is a torsion-free projective Anosov subgroup. Then the associated Ruelle zeta function $\zeta_\Gamma(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = h_{\mathrm{top}}(\Gamma)$.*

If $\Gamma < \mathrm{SL}(V)$ is irreducible, then ζ_Γ has a zero-free spectral gap: there exists $\varepsilon > 0$ such that ζ_Γ is holomorphic and nowhere vanishing in the strip $h_{\mathrm{top}}(\Gamma) - \varepsilon < \mathrm{Re}(s) < h_{\mathrm{top}}(\Gamma)$.

This has applications to counting problems. Consider the orbit counting function

$$N_\Gamma(t) = \mathrm{Card}\left\{[\gamma] \in [\Gamma]_{\mathrm{prim}} \,\middle|\, \lambda_1([\gamma]) \le t\right\}.$$

Theorem 6.14 ([DMS24, Theorem D]). *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a torsion free irreducible projective Anosov subgroup. Then there exists $c > 0$ such that*

$$N_\Gamma(t) = \frac{e^{h_{\mathrm{top}}(\Gamma)t}}{h_{\mathrm{top}}(\Gamma)t}\left(1 + \mathrm{O}(e^{-ct})\right).$$

This refines the estimate $N_\Gamma(t) \sim \frac{e^{h_{\mathrm{top}}(\Gamma)t}}{h_{\mathrm{top}}(\Gamma)t}$ obtained by Sambarino [Sam14].

6.4 Examples

Let us now describe two settings in which the Axiom A flow ϕ^t on $\mathcal{M}_\Gamma$ is directly related to a geodesic flow: $\mathbb{H}^{p,q}$-convex cocompact groups and Benoist groups.

6.4.1 $\mathbb{H}^{p,q}$-convex-cocompact groups

If a subgroup $\Gamma < \mathrm{SO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact, it is more natural to work with the space like geodesic flow of $\mathbb{H}^{p,q}$ than with the flow space $\mathbb{L}$.

Definition 6.15. The *spacelike unit tangent bundle* of $\mathbb{H}^{p,q}$ is

$$\begin{aligned} T^1\mathbb{H}^{p,q} &= \Big\{[x:v] \in T\mathbb{H}^{p,q} \,\Big|\, ([x:v],[x:v])_{[x]} = 1\Big\} \\ &= \Big\{[x:v] \in \mathbb{P}(\mathbb{R}^{p+q+1} \times \mathbb{R}^{p+q+1}) \,\Big|\, \langle x,x\rangle_{p,q+1} < 0, \langle x,v\rangle_{p,q+1} = \langle v,v\rangle_{p,q+1} + \langle x,x\rangle_{p,q+1} = 0\Big\}. \end{aligned}$$

The geodesic flow $\varphi^t : T\mathbb{H}^{p,q} \to T\mathbb{H}^{p,q}$ leaves $T^1\mathbb{H}^{p,q}$ invariant, and we find

$$\varphi^t([x:v]) = [\cosh t\, x + \sinh t\, v : \sinh t\, x + \cosh t\, v] \quad \forall [x:v] \in T^1\mathbb{H}^{p,q}.$$

Flow lines of this *space-like geodesic flow* have endpoints in the boundary

$$\partial\mathbb{H}^{p,q} = \Big\{[x] \in \mathbb{RP}^{p+1} \,\Big|\, \langle x,x\rangle_{p,q+1} = 0\Big\}.$$

They are given, for $[x:v] \in T^1\mathbb{H}^{p,q}$, by

$$[x:v]_\pm := \lim_{t\to\pm\infty} \pi(\varphi^t([x:v])) = [x \pm v] \in \partial\mathbb{H}^{p,q}.$$

Note that we always have $[x:v]_+ \neq [x:v]_-$. The isomorphism

$$\Phi^{p,q+1} : \begin{cases} \mathbb{R}^{p+q+1} & \to & \mathbb{R}^{p+q+1} \\ [v] & \mapsto & \big[\langle v, \cdot\rangle_{p,q+1}\big] \end{cases}$$

allows us to replace the flow space

$$\mathbb{L} = \Big\{[v:\alpha] \in \mathbb{P}(\mathbb{R}^{p+q+1} \times \mathbb{R}^{p+q+1}) \,\Big|\, \alpha(v) > 0\Big\}$$

and the flow

$$\phi^t([v:\alpha]) = [e^t v : e^{-t}\alpha]$$

with the flow space

$$\mathbb{L}^{p,q+1} = \Big\{[v_1:v_2] \in \mathbb{P}(\mathbb{R}^{p+q+1} \times \mathbb{R}^{p+q+1} \,\Big|\, \langle v_1,v_2\rangle_{p,q+1} > 0\Big\}$$

equipped with the flow

$$\phi^t([v_1:v_2]) = ([e^t v_1 : e^{-t} v_2]), \qquad t \in \mathbb{R}.$$

The map

$$\Phi_\partial^{p,q+1} : \begin{cases} T^1\mathbb{H}^{p,q} & \to & \mathbb{L}^{p,q+1} \\ [x:v] & \mapsto & [x+v : x-v] \end{cases}$$

is an $\mathrm{SO}(p,q+1)$-equivariant embedding that intertwines the flows $\Phi_\partial^{p,q+1} \circ \varphi^t = \phi^t \circ \Phi_\partial^{p,q+1}$. Its image is the subspace $\mathbb{L}_\partial^{p,q+1} \subset \mathbb{L}$ defined by

$$\mathbb{L}_\partial^{p,q+1} = \Big\{[v_1:v_2] \in \mathbb{L}^{p,q+1} \,\Big|\, [v_1],[v_2] \in \partial\mathbb{H}^{p,q}\Big\}$$

and the inverse $\left(\Phi_\partial^{p,q+1}\right)^{-1} : \mathbb{L}_\partial^{p,q+1} \to T^1\mathbb{H}^{p,q}$ is given by

$$\left(\Phi_\partial^{p,q+1}\right)^{-1}([v_1:v_2]) = [\langle v_1,v_2\rangle_{p,q+1}\, v_1 - v_2 : \langle v_1,v_2\rangle_{p,q+1}\, v_1 + v_2].$$

Definition 6.16. If $\Gamma < \mathrm{SO}(p, q+1)$ is projective Anosov, we consider

$$\begin{aligned}\widetilde{\mathcal{M}}_{\Gamma,\partial} &= \widetilde{\mathcal{M}}_\Gamma \cap \mathbb{L}_\partial^{p,q+1} \\ &= \left\{[v_1 : v_2] \in \mathbb{L}_\partial^{p,q+1} \,\middle|\, \langle v_1, v\rangle_{p,q+1} \neq 0 \text{ or } \langle v_2, v\rangle_{p,q+1} \neq 0 \ \forall\,[v] \in \Lambda_\Gamma\right\}.\end{aligned}$$

If $\Gamma < \mathrm{SO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex-cocompact, we already know that it acts properly discontinuously on the open set $\Omega(\Lambda_\Gamma) \subset \mathbb{H}^{p,q}$, and therefore also on $T^1\Omega(\Lambda_\Gamma) \subset T^1\mathbb{H}^{p,q}$. Our approach produces a larger discontinuity domain for the Γ-action on $T^1\mathbb{H}^{p,q}$:

Lemma 6.17 ([DMS24, Lemma 6.9])**.** *If $\Gamma < \mathrm{SO}(p, q+1)$ is $\mathbb{H}^{p,q}$-convex cocompact, then*

$$\widetilde{\mathcal{K}}_\Gamma \subset \Phi_\partial^{p,q+1}\left(T^1\Omega(\Lambda_\Gamma)\right) \subset \widetilde{\mathcal{M}}_{\Gamma,\partial}.$$

Note that the quotient flow on $\Gamma\backslash\widetilde{\mathcal{M}_{\Gamma,\partial}}$ is complete, whereas the spacelike geodesic flow of $\Gamma\backslash\Omega(\Lambda_\Gamma)$ is not. In the $\mathbb{H}^{p,q}$-convex-cocompact setting, our results become:

Theorem 6.18 ([DMS24, Theorem 12])**.** *Suppose $\Gamma < \mathrm{SO}(p, q+1)$ is a non-trivial torsion-free $\mathbb{H}^{p,q}$-convex cocompact subgroup. The (possibly incomplete) space-like geodesic flow*

$$\phi^t : T^1(\Gamma\backslash\Omega(\Lambda_\Gamma)) \to T^1(\Gamma\backslash\Omega(\Lambda_\Gamma))$$

is the restriction of a complete Axiom A flow

$$\phi^t : \mathcal{M}_\Gamma \to \mathcal{M}_\Gamma$$

with a unique basic hyperbolic set $\mathcal{K}_\Gamma$ such that $\mathcal{K}_\Gamma \subset T^1(\Gamma\backslash\Omega(\Lambda_\Gamma)) \subset \mathcal{M}_\Gamma$. Moreover,

1. *If Γ is irreducible, the restriction of the space-like geodesic flow ϕ^t to the basic hyperbolic set $\mathcal{K}_\Gamma$) mixes exponentially for all Hölder observables with respect to every Gibbs equilibrium state with Hölder potential.*

2. *The Ruelle zeta function ζ_Γ constructed using the periods of the space-like geodesic flow admits global meromorphic continuation to $\mathbb{C}$ with a simple pole at $h_{\mathrm{top}}(\phi^t)$, and assuming Γ is irreducible, ζ_Γ is nowhere vanishing and analytic in a strip $h_{\mathrm{top}}(\phi^t) - \varepsilon < \mathrm{Re}(z) < h_{\mathrm{top}}(\phi^t)$ for some $\varepsilon > 0$.*

3. *If Γ is irreducible, the spacelike geodesic flow satisfies the prime orbit theorem with exponentially decaying error term:*

   $$N_\Gamma(\tau) = \frac{e^{h_{\mathrm{top}}(\phi^t)\tau}}{h_{\mathrm{top}}(\phi^t)\tau}\left(1 + O\left(e^{-(c - h_{\mathrm{top}}(\phi^t))\tau}\right)\right)$$

   for some $0 < c < h_{\mathrm{top}}(\phi^t)$.

6.4.2 Strictly convex divisible domains

We will denote by $\pi : \mathbb{R}^d \setminus \{0\} \to \mathbb{RP}^{d-1}$ the canonical projection.

Definition 6.19. A discrete subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$ *divides* a properly convex open subset $\mathcal{C} \subset \mathbb{RP}^{d-1}$ if the action of Γ on $\mathbb{RP}^{d-1}$ preserves $\mathcal{C}$ and $\Gamma \curvearrowright \mathcal{C}$ is properly discontinuous and cocompact.

A *strictly convex domain* $\mathcal{C} \subset \mathbb{RP}^{d-1}$ is a properly convex open subset whose boundary $\partial\mathcal{C}$ does not contain any non-trivial projective segment.

A discrete subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is called a *Benoist subgroup* if it divides a non-empty strictly convex domain.

Proposition 6.20 ([Ben04, Théorème 1.1])**.** *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a Benoist subgroup. Then Γ is projective Anosov, it divides a unique strictly convex domain $\mathcal{C}_\Gamma \subset \mathbb{RP}^{d-1}$ and one has*

$$\Lambda_\Gamma = \partial\mathcal{C}_\Gamma.$$

Furthermore, the boundary $\partial\mathcal{C}_\Gamma \subset \mathbb{RP}^{d-1}$ is a C^1 submanifold, and the limit maps ξ, ξ^* are related by

$$T_{\xi(t)}\partial\mathcal{C}_\Gamma = d_x\pi(\xi^*(t)), \quad \forall t \in \partial_\infty\Gamma, \forall x \in \xi(t).$$

Benoist also proved in [Ben04] that the boundary $\partial\mathcal{C}_\Gamma$ is never C^2, unless Γ is conjugate to a uniform lattice in $\mathrm{SO}(d-1,1)$ (in which case $\mathcal{C}_\Gamma$ is an ellipsoid).

Recall that the Hilbert distance (Definition 5.10) makes any properly convex open set $\mathcal{C} \subset \mathbb{RP}^{d-1}$ a metric space $(\mathcal{C}, d_\mathcal{C})$. If $\mathcal{C}$ is strictly convex, then geodesics are intersections of projective lines with $\mathcal{C}$. In order to define a geodesic flow, we will work with *sphere bundles* of manifolds, i.e. the fibre bundle $\mathbb{S}M$ over a manifold M with fibre $\mathbb{S}_xM = (T_xM \setminus \{0\})/\mathbb{R}_+^*$ over $x \in M$, where $\mathbb{R}_+^*$ acts by multiplication. The ray spanned by a non-zero tangent vector $v \in T_xM$ will be denoted by $[v] \in \mathbb{S}_xM$.

If $\mathcal{C} \subset \mathbb{RP}^{d-1}$ is a properly convex open subset, a pair $(\ell,[v]) \in \mathbb{S}\mathcal{C}$ defines a parametrization $c_{\ell,[v]} : \mathbb{R} \to \mathcal{C}$ of the intersection of $\mathcal{C}$ with the projective line going through ℓ and tangent to v, satisfying $d_\mathcal{C}(\ell, c_{\ell,[v]}(t)) = |t|$ for any $t \in \mathbb{R}$, and $\dot{c}_{\ell,[v]}(0) \in \mathbb{R}_+^* \cdot v$.

Definition 6.21. The *Benoist-Hilbert flow* is the flow $\phi^t_{\mathrm{BH}} : \mathbb{S}\mathcal{C} \to \mathbb{S}\mathcal{C}$ defined by

$$\phi^t_{\mathrm{BH}}(\ell,[v]) := (c_{\ell,[v]}(t), [\dot{c}_{\ell,[v]}(t)]), \quad \forall (\ell,[v]) \in \mathbb{S}\mathcal{C}.$$

Let us now focus on the case of a torsion-free Benoist subgroup $\Gamma < \mathrm{SL}(d,\mathbb{R})$, and denote by $\mathcal{N}_\Gamma = \Gamma\backslash\mathcal{C}_\Gamma$ the quotient manifold. As the action $\Gamma \curvearrowright \mathbb{S}\mathcal{C}_\Gamma$ commutes with the Benoist-Hilbert flow, we can also define a flow ϕ^t_{BH} on the quotient manifold $\mathbb{S}\mathcal{N}_\Gamma = \Gamma\backslash\mathbb{S}\mathcal{C}_\Gamma$.

Note that $\mathbb{S}\mathcal{N}_\Gamma$ is a smooth manifold, but the regularity of ϕ^t_{BH} is exactly that of $\partial\mathcal{C}_\Gamma$.

Proposition 6.22 ([Ben04, Théorème 1.1, Théorème 1.2])**.** *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a torsion-free Benoist subgroup. There exists $0 < \alpha < 1$ such that the Benoist-Hilbert flow $\phi^t_{\mathrm{BH}} : \mathbb{S}\mathcal{N}_\Gamma \to \mathbb{S}\mathcal{N}_\Gamma$ is a topologically transitive $C^{1+\alpha}$ Anosov flow.*

The Benoist-Hilbert flow $\phi^t_{\mathrm{BH}} : \mathbb{S}\mathcal{N}_\Gamma \to \mathbb{S}\mathcal{N}_\Gamma$ cannot be directly related to our flow space $(\mathbb{L}, \phi^t)$. Indeed, each non-trivial element $\gamma \in \Gamma$ corresponds to a periodic orbit of both flows, but they have different periods: $\lambda_1(\gamma)$ for the flow ϕ^t on $\mathcal{M}_\Gamma$ and $\frac{1}{2}(\lambda_1(\gamma) - \lambda_d(\gamma))$ for the Benoist-Hilbert flow.

In order to make the two coincide, we will work with the adjoint representation $\mathrm{Ad} : \mathrm{SL}(d,\mathbb{R}) \to \mathrm{SL}(\mathfrak{sl}(d,\mathbb{R}))$, as it satisfies $\lambda_1(\mathrm{Ad}(g)) = \lambda_1(g) - \lambda_d(g)$ for any $g \in \mathrm{SL}(d,\mathbb{R})$ (meaning that $\lambda_1 - \lambda_d$ is the highest weight of the adjoint representation).

Lemma 6.23 ([GW12, Prop. 4.3]). *If $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is projective Anosov, then so is $\mathrm{Ad}(\Gamma) < \mathrm{SL}(\mathfrak{sl}(d,\mathbb{R}))$, and its limit map is given by $\xi_{\mathrm{Ad}}(t) = [v \otimes \alpha] \in \mathbb{P}(\mathfrak{sl}(d,\mathbb{R}))$ at $t \in \partial_\infty\Gamma$, where $[v] = \xi(t)$ and $[\alpha] = \xi^*(t)$.*

Just as in the study of $\mathbb{H}^{p,q}$-convex cocompact subgroups, we can use the non-degenerate symmetric bilinear form $(X,Y) \mapsto \mathrm{Tr}(XY)$ (which is a multiple of the Killing form of $\mathfrak{sl}(d,\mathbb{R})$) and work with the flow space

$$\mathbb{L}_{\mathrm{Ad}} = \{[X:Y] \in \mathbb{P}(\mathfrak{sl}(d,\mathbb{R}) \times \mathfrak{sl}(d,\mathbb{R})) \,|\, \mathrm{Tr}(XY) > 0\}$$

equipped with the flow

$$\phi^t([X:Y]) = ([e^t X : e^{-t} Y]), \qquad t \in \mathbb{R}.$$

There is a difference in regularities between the Benoist-Hilbert flow ϕ^t_{BH} on $\mathbb{S}\mathcal{N}_\Gamma$ and the flow ϕ^t on $\mathcal{M}_{\mathrm{Ad}(\Gamma)}$, as the basic set $\mathcal{K}_{\mathrm{Ad}(\Gamma)}$ is not a C^1 submanifold of $\mathcal{M}_{\mathrm{Ad}(\Gamma)}$. It is however better than the expected Hölder regularity.

Lemma 6.24 ([DMS24, Lemma 6.16 and 6.17]). *Let $\Gamma < \mathrm{SL}(d,\mathbb{R})$ be a torsion free Benoist subgroup. The basic set $\mathcal{K}_{\mathrm{Ad}(\Gamma)}$ is a Lipschitz submanifold of $\mathcal{M}_{\mathrm{Ad}(\Gamma)}$, and there is a Hölder homeomorphism*

$$\Psi : \mathbb{S}\mathcal{N}_\Gamma \to \mathcal{K}_{\mathrm{Ad}(\Gamma)}$$

with Lipschitz inverse such that

$$\Psi \circ \phi^t_{\mathrm{BH}} = \phi^{2t} \circ \Psi\,, \quad \forall t \in \mathbb{R}.$$

The loss of regularity from $\mathcal{C}^{1,\alpha}$ to Lipschitz when going from ϕ^t_{BH} to $\phi^t|_{\mathcal{K}_{\mathrm{Ad}(\Gamma)}}$ should be thought of as being compensated by the gain of regularity in the stable/unstable foliations.

Theorem 6.25 ([DMS24, Theorem G]). *If $\Gamma < \mathrm{SL}(d,\mathbb{R})$ is a torsion-free Benoist subgroup and $W = \mathfrak{sl}(V)$, then the Benoist-Hilbert flow ϕ^t_{BH} mixes exponentially for all Hölder observables with respect to every Gibbs equilibrium state with Hölder potential.*

References

[ABB+07] Lars Andersson, Thierry Barbot, Riccardo Benedetti, Francesco Bonsante, William M. Goldman, François Labourie, Kevin P. Scannell, and Jean-Marc Schlenker, *Notes on: "Lorentz spacetimes of constant curvature" [Geom. Dedicata* **126** *(2007), 3–45; mr2328921] by G. Mess*, Geom. Dedicata **126** (2007), 47–70. MR MR2328922

[AMS95] H. Abels, G. A. Margulis, and G. A. Soifer, *Semigroups containing proximal linear maps*, Israel J. Math. **91** (1995), no. 1-3, 1–30. MR 1348303

[Bar01] T. Barbot, *Flag Structures on Seifert Manifolds*, Geom. Topol. **5** (2001), 227–266.

[Bar08] Thierry Barbot, *Causal properties of AdS-isometry groups. I. Causal actions and limit sets*, Adv. Theor. Math. Phys. **12** (2008), no. 1, 1–66. MR 2369412

[Bar10] T. Barbot, *Three-dimensional Anosov flag manifolds*, Geom. Topol. **14** (2010), no. 1, 153–191.

[Bar15] Thierry Barbot, *Deformations of Fuchsian AdS representations are quasi-Fuchsian*, J. Differential Geom. **101** (2015), no. 1, 1–46. MR 3356068

[BCLS18] Martin Bridgeman, Richard Canary, Fran¸cois Labourie, and Andres Sambarino, *Simple root flows for Hitchin representations*, Geom. Dedicata **192** (2018), 57–86. MR 3749423

[Ben97] Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. **7** (1997), no. 1, 1–47.

[Ben04] ———, *Convexes divisibles. I*, Algebraic groups and arithmetic, Tata Inst. Fund. Res., Mumbai, 2004, pp. 339–374.

[Ber72] Lipman Bers, *Uniformization, moduli, and Kleinian groups*, Bull. London Math. Soc. **4** (1972), 257–300. MR 348097

[BILW05] Marc Burger, Alessandra Iozzi, Fran¸cois Labourie, and Anna Wienhard, *Maximal representations of surface groups: symplectic Anosov structures*, Pure Appl. Math. Q. **1** (2005), no. 3, 543–590. MR 2201327

[BK23] Jonas Beyrer and Fanny Kassel, *$\mathbb{H}^{p,q}$-convex cocompactness and higher higher teichmüller spaces*, 2023.

[BM91] Mladen Bestvina and Geoffrey Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. **4** (1991), no. 3, 469–481. MR 1096169

[BM12] T. Barbot and Q. Mérigot, *Anosov AdS representations are quasi-Fuchsian*, Groups Geom. Dyn. **6** (2012), no. 3, 441–483.

[Bow79] Rufus Bowen, *Hausdorff dimension of quasicircles*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 11–25. MR 556580

[BPS19] Jairo Bochi, Rafael Potrie, and Andrés Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 11, 3343–3414. MR 4012341

[BR75] R. Bowen and D. Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), no. 3, 181–202.

[BS91] Christopher Bishop and Tim Steger, *Three rigidity criteria for* $\mathrm{PSL}(2,\mathbf{R})$, Bull. Amer. Math. Soc. (N.S.) **24** (1991), no. 1, 117–123. MR 1065010

[BS05] Antonio N. Bernal and Miguel Sánchez, *Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes*, Comm. Math. Phys. **257** (2005), no. 1, 43–50. MR 2163568

[BWS21] Y. Borns-Weil and S. Shen, *Dynamical zeta functions in the nonorientable case*, Nonlinearity **34** (2021), no. 10, 7322–7334.

[CK02] M. Coornaert and G. Knieper, *Growth of conjugacy classes in Gromov hyperbolic groups*, Geom. Funct. Anal. **12** (2002), no. 3, 464–478. MR 1924369

[CM14] M. Crampon and L. Marquis, *Geometrical finiteness in Hilbert geometry*, Ann. Inst. Fourier **64** (2014), no. 6, 2299–2377 (French).

[Coo93] Michel Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. **159** (1993), no. 2, 241–270. MR 1214072

[CTT19] Brian Collier, Nicolas Tholozan, and Jérémy Toulisse, *The geometry of maximal representations of surface groups into* $SO_0(2,n)$, Duke Math. J. **168** (2019), no. 15, 2873–2949. MR 4017517

[DG16] S. Dyatlov and C. Guillarmou, *Pollicott–Ruelle resonances for open systems*, Ann. Henri Poincaré **17** (2016), no. 11, 3089–3146.

[DG18] ———, *Afterword: Dynamical zeta functions for Axiom A flows*, Bull. Amer. Math. Soc. (2018), no. 55, 337–342.

[DGK18] J. Danciger, F. Guéritaud, and F. Kassel, *Convex cocompactness in pseudo-Riemannian hyperbolic spaces.*, Geom. Dedicata **192** (2018), 87–126.

[DGK23] ———, *Convex cocompact actions in real projective geometry*, 2023, arXiv:1704.08711, to appear in Ann. Sci. Éc. Norm. Supér.

[DKO24] Subhadip Dey, Dongryul M. Kim, and Hee Oh, *Ahlfors regularity of patterson-sullivan measures of anosov groups and applications*, 2024.

[DMS24] Benjamin Delarue, Daniel Monclair, and Andrew Sanders, *Locally homogeneous axiom a flows i: projective anosov subgroups and exponential mixing*.

[Dol98] D. Dolgopyat, *On decay of correlations in Anosov flows*, Annals of Mathematics **147** (1998), no. 2, 357–390.

[DP98] D. Dolgopyat and M. Pollicott, *Addendum to: "Periodic orbits and dynamical spectra" [Ergodic Theory Dynam. Systems* **18** *(1998), no. 2, 255–292] by V. Baladi*, Ergodic Theory Dynam. Systems **18** (1998), no. 2, 293–301.

[DSL11] Antonio J. Di Scala and Thomas Leistner, *Connected subgroups of* $SO(2,n)$ *acting irreducibly on* $\mathbb{R}^{2,n}$, Israel J. Math. **182** (2011), 103–121. MR 2783967

[FP20] Joel Fine and Bruno Premoselli, *Examples of compact Einstein four-manifolds with negative curvature*, J. Amer. Math. Soc. **33** (2020), no. 4, 991–1038. MR 4155218

[Glo17] Olivier Glorieux, *Counting closed geodesics in globally hyperbolic maximal compact AdS 3-manifolds*, Geom. Dedicata **188** (2017), 63–101. MR 3639625

[GM] Olivier Glorieux and Daniel Monclair, *Regularity of limit sets of ads quasi-fuchsian groups*, to appear in Mathematics Research Letters.

[GM21] O. Glorieux and D. Monclair, *Critical exponent and Hausdorff dimension in pseudo-Riemannian hyperbolic geometry*, Int. Math. Res. Not. IMRN (2021), no. 18, 13661–13729.

[GMT23] O. Glorieux, D. Monclair, and N. Tholozan, *Hausdorff dimension of limit sets for projective Anosov representations*, J. Éc. Polytech., Math. **10** (2023), 1157–1193.

[GT87] M. Gromov and W. Thurston, *Pinching constants for hyperbolic manifolds,* Invent. Math. **89** (1987), no. 1, 1–12. MR 892185

[GW12] O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and applications.*, Invent. Math. **190** (2012), no. 2, 357–438.

[JM87] Dennis Johnson and John J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math., vol. 67, Birkhäuser Boston, Boston, MA, 1987, pp. 48–106. MR 900823 (88j:22010)

[Kap07] Michael Kapovich, *Convex projective structures on Gromov-Thurston manifolds*, Geom. Topol. **11** (2007), 1777–1830. MR 2350468

[KLP17] M. Kapovich, B. Leeb, and J. Porti, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math. **3** (2017), no. 4, 808–898.

[KP22] F. Kassel and R. Potrie, *Eigenvalue gaps for hyperbolic groups and semigroups*, J. Mod. Dyn. **18** (2022), 161–208.

[Lab06] F. Labourie, *Anosov flows, surface groups and curves in projective space.*, Invent. Math. **165** (2006), no. 1, 51–114.

[LM19] Gye-Seon Lee and Ludovic Marquis, *Anti–de Sitter strictly GHC-regular groups which are not lattices*, Trans. Amer. Math. Soc. **372** (2019), no. 1, 153–186. MR 3968766

[LW10] Fran¸cois Ledrappier and Xiaodong Wang, *An integral formula for the volume entropy with applications to rigidity*, J. Differential Geom. **85** (2010), no. 3, 461–477. MR 2739810

[Mes07] G. Mess, *Lorentz spacetimes of constant curvature.*, Geom. Dedicata **126** (2007), 3–45.

[MST23] Daniel Monclair, Jean-Marc Schlenker, and Nicolas Tholozan, *Gromov-thurston manifolds and anti-de sitter geometry*, 2023.

[MV24] Filippo Mazzoli and Gabriele Viaggi, *Volume, entropy, and diameter in* $\mathrm{so}(p,q+1)$*-higher teichmüller spaces*.

[Pat76] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Mathematica **136** (1976), no. none, 241 – 273.

[PS98] M. Pollicott and R. Sharp, *Exponential error terms for growth functions on negatively curved surfaces*, Amer. J. Math. **120** (1998), no. 5, 1019–1042.

[PS17] Rafael Potrie and Andrés Sambarino, *Eigenvalues and entropy of a Hitchin representation*, Invent. Math. **209** (2017), no. 3, 885–925. MR 3681396

[PSW21] M.B. Pozzetti, A. Sambarino, and A. Wienhard, *Conformality for a robust class of non-conformal attractors*, J. Reine Angew. Math. **774** (2021), 1–51.

[Sam14] A. Sambarino, *Quantitative properties of convex representations*, Comment. Math. Helv. **89** (2014), no. 2, 443–488.

[Sam24] ______, *A report on an ergodic dichotomy*, Ergodic Theory Dynam. Systems **44** (2024), no. 1, 236–289.

[Sma67] S. Smale, *Differentiable dynamical systems. With an appendix to the first part of the paper: "Anosov diffeomorphisms" by John Mather*, Bull. Am. Math. Soc. **73** (1967), 747–817.

[SST23] Andrea Seppi, Graham Smith, and Jérémy Toulisse, *On complete maximal submanifolds in pseudo-hyperbolic space*.

[Sto11] L. Stoyanov, *Spectra of Ruelle transfer operators for Axiom A flows*, Nonlinearity **24** (2011), no. 4, 1089–1120.

[Sul79] Dennis Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Inst. Hautes Études Sci. Publ. Math. (1979), no. 50, 171–202. MR 556586

[Yue96] Chengbo Yue, *Dimension and rigidity of quasi-Fuchsian representations*, Ann. of Math. (2) **143** (1996), no. 2, 331–355. MR 1381989

[Zim21] Andrew Zimmer, *Projective Anosov representations, convex cocompact actions, and rigidity*, J. Differential Geom. **119** (2021), no. 3, 513–586. MR 4333029