Chaos and stochastic behaviour of deterministic evolutions

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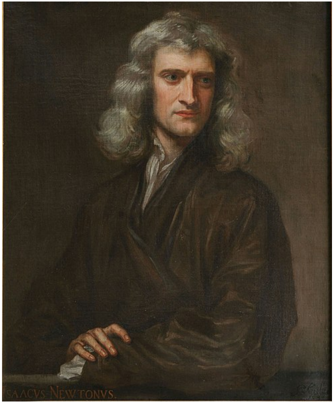
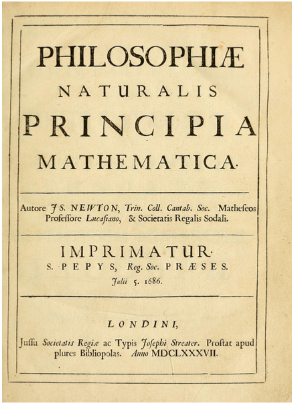
September 1st-2nd, 2021

The two-body problem (1687).





PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

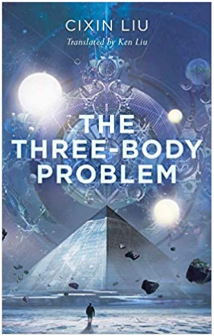
Autore J S. NEWTON, *Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtat apud plures Bibliopolas. *Anno* MDCLXXXVII.

Isaac Newton and the *Philosophiae Naturalis Principia Mathematica* (1687).

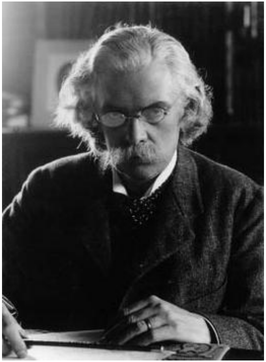
The N-body problem.



What about the 3-body problem ?

King Oscar's II prize (1889).





From left to right : King Oscar II of Sweden, Henri Poincaré, Gösta Mittag-Leffler.

Henri Poincaré's essay (1889).

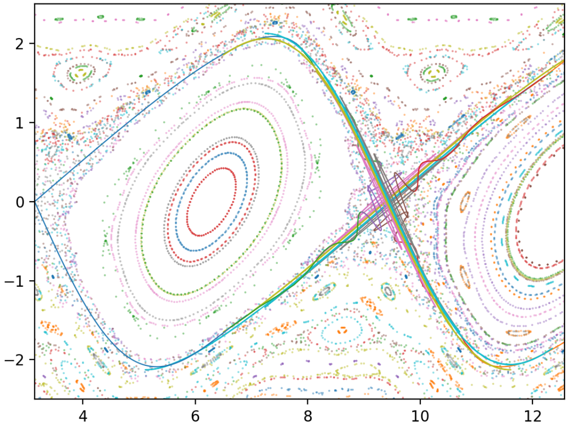
Que l'on cherche à se représenter la figure formée par ces deux courbes et leurs intersections en nombre infini dont chacune correspond à une solution doublement asymptotique, ces intersections forment une sorte de treillis, de tissu, de réseau à mailles infiniment serrées ; chacune de ces courbes ne doit jamais se recouper elle-même, mais elle doit se replier elle-même d'une manière très complexe pour venir couper une infinité de fois toutes les mailles du réseau.

On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer.

If one seeks to visualize the pattern formed by these two curves and their infinite number of intersections, each corresponding to a doubly asymptotic solution, these intersections form a kind of lattice work, a weave, a chain-link network of infinitely fine mesh ; each of the two curves can never cross itself, but it must fold back on itself in a very complicated way so as to recross all the chain-links an infinite number of times.

One will be struck by the complexity of this figure, which I am not even attempting to draw.

Poincaré's homoclinic tangle.



From *Henri Poincaré and his Homoclinic Tangle*, David NOLTE, 2020.

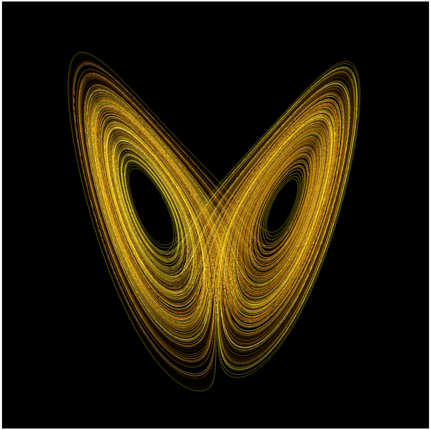
The 60's and chaos theory.





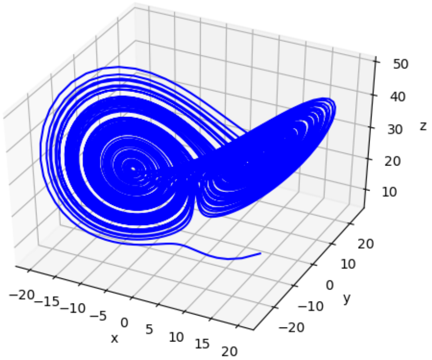
From left to right, top to bottom : Edward Lorenz, Stephen Smale, Yakov Sinai, Marina Ratner, Rufus Bowen. Photographies of M. Ratner and R. Bowen by George Bergman (Oberwolfach Institute).

A butterfly.



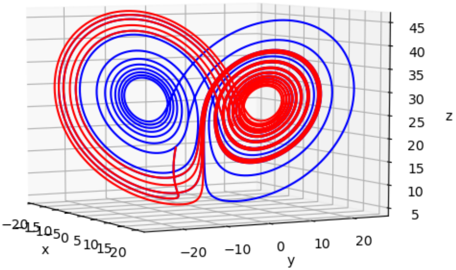
A Lorenz attractor.

An attracting butterfly.



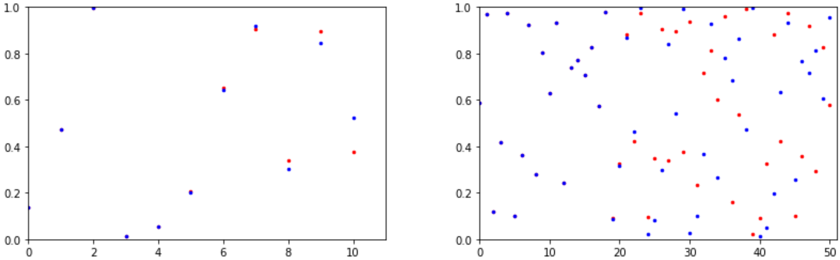
A trajectory converging to the Lorenz attractor with parameters $r = 28$, $\sigma = 28$, $b = 8/3$.

Sensitivity to initial conditions : Lorenz' system.



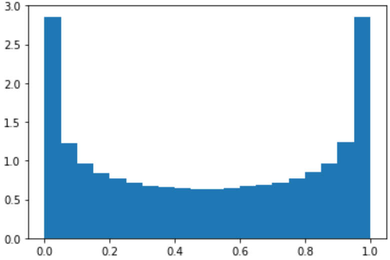
Two trajectories of Lorenz's system, with initial difference of order 10^{-4}.

Sensitivity to initial conditions : logistic map.



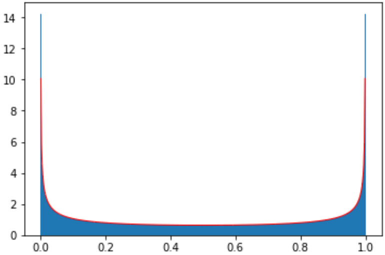
Two trajectories for the logistic map $x \mapsto 4x(1-x)$. On the left : initial difference of 10^{-3}. On the right : initial difference of 10^{-8}.

Histogram : logistic map, 1.



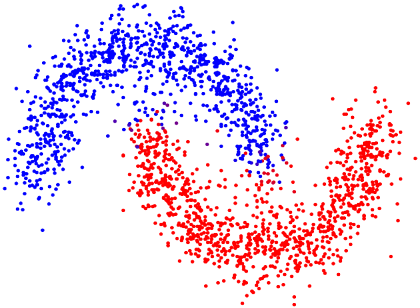
Histogram of a trajectory of length 10^6 for the logistic map $x \mapsto 4x(1-x)$, with 20 classes.

Histogram : logistic map, 2.



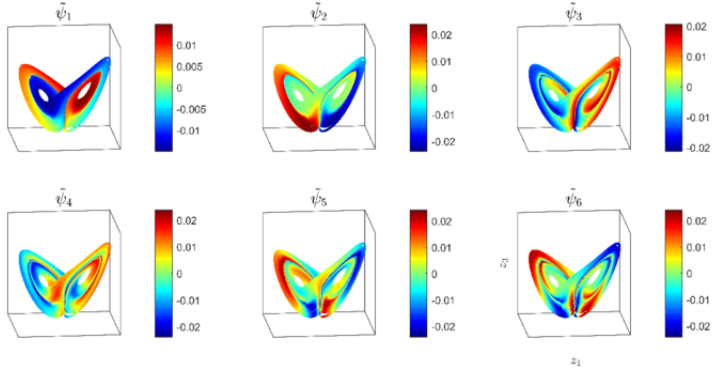
Histogram of a trajectory of length 10^8 for the logistic map $x \mapsto 4x(1-x)$, with 500 classes. In red : the graph of the invariant density.

Spectral clustering algorithms.



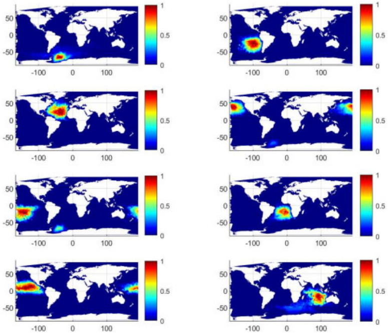
Partition of a set of points in two cluster using a spectral clustering algorithm.

Dynamical modes decomposition : Lorenz attractor.



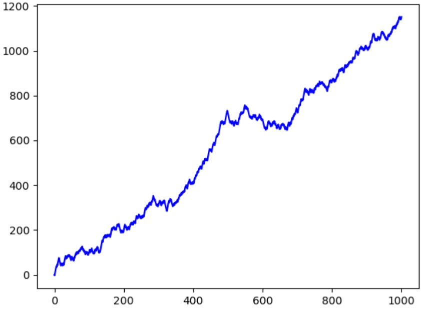
Dynamical mode decomposition of the Lorenz's flow. From *Ergodic Theory, Dynamic Mode Decomposition, and Computation of Spectral Properties of the Koopman Operator*, H. Arbabi and I. Mezic, SIAM Journal on Applied Dynamical Systems, 16(4), 2016.

Dynamical modes decomposition : oceanic currents.



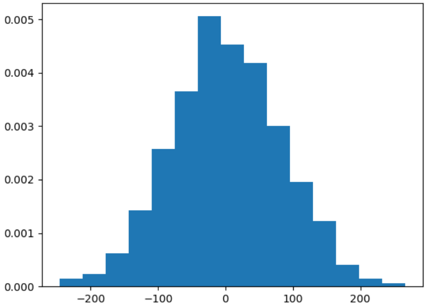
Dynamical mode decomposition of oceanic currents. From *Deep Lagrangian connectivity in the global ocean inferred from Argo floats*, R. Abernathey, C. Bladwell, G. Froyland and K. Sakellariou, preprint.

Birkhoff's ergodic theorem and the strong law of large numbers.



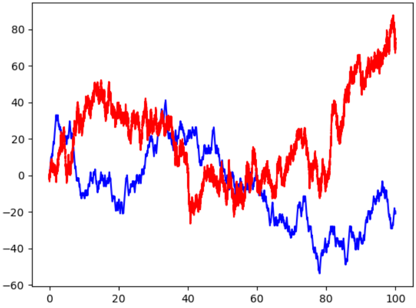
Integral of $1 + x(t)$ over a trajectory for Lorenz' flow up to time 1000.

The central limit theorem.



Histogram of the integral of $1 + x(t)$ over a trajectory for Lorenz' flow up to time 100. Histogram done with 10^3 random starting points and 15 classes.

The invariance principle(s).



In blue : integral of $x(t)$ over a trajectory for Lorenz' flow up to time 100, renormalized. In red : a simple random walk, renormalized.