SEMINAIRE DE M2

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INTRODUCTION

The purpose of the talk is to explain basic notions in complex geometry. In the first part, we define real and complex structures on topological spaces, which leads to the definitions of real and complex manifolds, almost complex structures and the Newlander-Nirenberg's theorem. In the second part, we study Kähler manifolds. They are complex manifolds in which the (1,1) form associated to the hermitian metric is closed. This special structure on the space gives rise to the Kähler identities and a nice form of the Hodge decomposition for de Rham cohomology. Finally, we end with an introduction to Chern-Weil theory, a basic construction in the theory of characteristic class. The theory associates to each vector bundle certain closed forms in the de Rham cohomology, which are topological invariants depending only on the vector bundles. We give an example in the case of Kähler manifolds, the Ricci form, which up to multiplication by a real constant, is the first Chern class of the anti-canonical bundle.

1. Complex manifold. Complex Structure. Newlander-Nirenberg's Theorem. (Reference: [5, 2])

In the following, let X be a topological space, Hausdorff, separable (having a countable basis for the topology). We introduce the structures of real and complex manifolds on X. A question one can ask is when does a real manifold admit the structure of a complex manifold. This question can be answered completely by studying a special kind of endomorphism on the tangent bundle of the manifold, which motivates the definition of an almost complex structure.

Definition 1.1. We say that X is a differentiable manifold if X can be covered by open sets U_i, together with a system of local charts $\phi_i : U_i \to \mathbb{R}^n$ such that the transition functions $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are differentiable. We say that X is smooth if all the transition functions are infinitely differentiable.

Definition 1.2. We say that X is a complex manifold if X can be covered by open sets U_i, together with a system of local charts $\phi_i : U_i \to \mathbb{C}^n$ such that the transition functions $\phi_j \circ \phi_i^{-1}$ are bi-holomorphic.

Definition 1.3. Let X be a smooth manifold. The tangent space at $x \in X$ is the real vector space of all derivations from the space of germs of $\mathscr{C}^\infty$ functions defined on a neighborhood of X to $\mathbb{R}$. We denote it by $T_{X,x}$.

Remarks 1.4. Suppose X is a complex manifold then X also has the structure of a real manifold. At every point $x \in X$, the real tangent space of X has the structure of a complex vector space. See Remarks 1.10.

Definition 1.5. The tangent bundle is defined as $T_X = \bigcup_{p\in X} T_{X,x}$, which can given the structure of a manifold, and a canonical projection $\pi : T_X \to X : (x, \delta) \mapsto x \in X$. It is an example of a differentiable real vector bundle.

Definition 1.6. Let X be a differentiable manifold. A real differentiable vector bundle of rank k on X consists of a differentiable manifold E and a differentiable map $\pi : E \to X$ such that
1) Each fibre $E_x = \pi^{-1}(x)$ has the structure of a real vector space
2) Each point $x \in X$ has an open neighborhood U and trivialization $\psi : E_{|U} \cong U \times \mathbb{R}^k$ such that $pr_U \circ \psi = \pi$ and for all $x \in U$, the map $\psi(x) : E_x \to \mathbb{R}^k$ is an isomorphism of real vector spaces.

Remarks 1.7. Similarly, one can define complex differentiable vector bundle, where the fibre are $\mathbb{C}$ vector spaces.

Definition 1.8. A differentiable morphism between vector bundles is a differentiable map such that the fibres are preserved.

Definition 1.9. An almost complex structure on a differentiable manifold X is an endomorphism J of the tangent bundle satisfying $J^2 = -\mathrm{Id}$.

Remarks 1.10. By the definition of morphism of vector bundles, each real tangent space $T_{X,x}$ is given an endomorphism $J_x : T_{X,x} \to T_{X,x}$ such that $J^2_{X,x} = -Id_x$. It is the same as giving $T_{X,x}$ the structure of a complex vector space, with $i{*}u = J_x(u)$ for $u \in T_{X,x}$, and T_X is given the structure of a differentiable complex vector bundle.

Remarks 1.11. Let M be a complex manifold. Then M has a natural almost complex structure. In particular, suppose the local coordinates of M are $z_1, z_2, ..., z_n$, $z_j = x_j + iy_j$. We can define $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}, J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$ locally, and glue them together. This definition does not depend on the charts, because the transition functions are holomorphic.

Definition 1.12. An almost complex structure J on a real manifold X is said to be integrable if there exists a complex manifold structure on X which induces J.

Question 1.13. *Given an almost complex structure on a real manifold M. Which ones are integrable?*

Given an almost complex structure J on a real manifold X, one can extend J to the complexified tangent bundle $T_X \otimes_{\mathbb{R}} \mathbb{C}$ by $J_x(u \otimes i + v \otimes 1) = J_x(u) \otimes i + J_x(v) \otimes 1$. We also have $J^2 = -\mathrm{Id}$ on $T_X \otimes \mathbb{C}$, hence in this action, J_x has eigenvalues i and $-i$.

Corollary 1.14. *The almost complex structure J on X induces a decomposition of sub-vector bundles:*

$$T_X \otimes \mathbb{C} = T^{1,0}_X \oplus T^{0,1}_X$$

where at each point $x \in X$, the space $T^{1,0}_{X,x}$ is the eigenspace with eigenvalue i and $T^{0,1}_{X,x}$ is the eigenspace with eigenvalue $-i$.

Theorem 1.15. *(Newlander-Nirenberg) An almost complex structure J is integrable if and only if*

$$[T^{0,1}_X, T^{0,1}_X] \subseteq T^{0,1}_X.$$

Remarks 1.16. Here, the Lie bracket is extended by $\mathbb{C}-linearity$ from the vector fields over X to complexified-vector fields.

If X is a complex manifold, one can prove the $(0,1)$ tangent bundle on X satisfies the above property by doing calculation on local coordinates. The other direction, however is a much more difficult theorem in analysis. For J smooth or differentiable of class $\mathscr{C}^k$, more difficult techniques in analysis (PDE) are required because the manifold X was only assumed to be differentiable, now it must also have the structure of a real analytic manifold. If M and J are assumed to be real analytic, then the theorem follows from the Frobenius theorem.

Theorem 1.17. *(Analytic version of Frobenius theorem) Let X be a complex manifold of dimension n, and let E be a holomorphic distribution of rank k over X, i.e. a holomorphic vector sub bundle of rank k of the holomorphic tangent bundle T_X. Then E is integrable in the holomorphic sense if and only if we have the integrability condition:*

$$[E, E] \subseteq E.$$

Here, the integrability in the holomorphic sense means that X is covered by open sets U such that there exists a holomorphic submersive map

$$\phi_U : U \to \mathbb{C}^{n-k}$$

satisfying

$$E_u = Ker(\phi_* : T_{U,u} \to T_{\mathbb{C}^{n-k},\phi(u)})$$

for every $u \in U$.

Proof. See [5], Theorem 2.26, page 51. □

Here, a differentiable complex vector bundle $\pi_E : E \to X$ over a complex manifold X is said to be holomorphic if we have trivializations $\tau_i : \pi_i^{-1}(U_i) \cong U_i \times \mathbb{C}^n$ such that the transition matrices $\tau_{i,j} = \tau_j \tau_i^{-1}$ have holomorphic coefficients.

Proof. (Sketch, see [5], Theorem 2.24, page 52) Assume X and J are analytic. The idea is that when complexifying both X, and $T_{X,\mathbb{R}}$, one will get a complex manifold and a holomorphic tangent bundle, and J can be extended to a holomorphic endomorphism of the holomorphic tangent bundle $T_{X_\mathbb{C}}$ of $X_\mathbb{C}$. We also have $J^2 = -\mathrm{Id}$. Let $E_\mathbb{C}$ be the eigenspace associated to the eigenvalue $-i$ of J. Then $E_\mathbb{C}$ is a holomorphic sub-bundle of $T_{X_\mathbb{C}}$. One has $E_{\mathbb{C},u} = T^{0,1}_{X,u}$ for all $u \in X$.
The bundle $E_\mathbb{C}$ satisfies the condition of the Frobenius theorem, and one can find an open cover for $X_\mathbb{C}$, such that for each $U \subseteq X_\mathbb{C}$, the map $\phi_U : U_\mathbb{C} \to \mathbb{C}^n$ is submersive and satisfies $E_{\mathbb{C},u} = \mathrm{Ker}(\phi_* : \mathrm{T_{U_\mathbb{C}}} \to \mathrm{\mathbb{C}^n})$. It implies $\phi_{*_{|T_U}}$ is an isomorphism and $\phi_{|U}$ is a diffeomorphism in a neighborhood of $u \in U$. One needs to check that the induced operator of the complex manifold structure from $\mathbb{C}^n$ on T_U is the same as the operator J. □

2. Hermitian metric, Kähler metric, Kähler identities, Hodge theory for Kähler manifolds. (Reference [[5, 2]])

We introduce the definition of Kähler metric, and gives a characterization in terms of connections on vector bundles. The existence of Kähler metric makes it easier to study the manifold. In particular, one has the Kähler identities, which are relationship between operators acting on differential forms. An important consequence of these identities is the Hodge-Decomposition of de Rham cohomology applying to complex manifolds.

2.1. **Hermitian metric and Kähler metrics.**

Let (X, J) be an almost complex manifold.

Definition 2.1. A Hermitian metric on X is a collection of Hermitian metrics h_x on tangent spaces $T_{X,x}$ (seen as a complex vector spaces via J_x). The metric h is said to be continuous (differentiable) if in local coordinates $x_1, x_2, ..., x_n$ for M, the map $x \to h_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is continuous (differentiable).

Remarks 2.2. By the definition one has $h(Ju, Jv) = h(iu, iv) = h(u, v)$.

Definition 2.3. In general, a Hermitian vector bundle is complex vector bundle E, together with Hermitian metrics h_x for each of its fibre such that, for all δ, β smooth sections of E, the map $x \to h_x(\delta, \beta)$ is infinitely differentiable.

Definition 2.4. Let $h = g - i\omega$, *i.e* $\omega = -\mathrm{Im(h)}, \mathrm{g} = \mathrm{Real(h)}$

Remarks 2.5. Any one of three forms h, g, ω uniquely determines the other two, for example, $\omega(u, v) = g(Ju, v)$ and $g(u, v) = \omega(u, Jv)$.

Remarks 2.6. Any almost complex manifold admits Hermitian metric. Choose an arbitrary Riemannian metric g', and define $g(X, Y) = g'(X, Y) + g'(JX, JY)$.

Proposition 2.7. *The metric g is a Riemannian metric, and ω is a real 2-form of type (1,1), i.e $\omega \in A^{1,1}(X) \cap A^2_{X,\mathbb{R}}$.*

Definition 2.8. We say that the Hermitian metric h is Kähler if J is integrable, and the 2-form ω is closed.

One has a characterization of Kähler metric in terms of connections on the tangent bundles.

Definition 2.9. Let $E \to X$ be a differentiable real vector bundle, and $\mathscr{C}^\infty_1(X, E)$ be the $\mathscr{C}^\infty$ sections of the bundle $\wedge T^*X \otimes E = Hom(T_X, E)$. A connection ∇ on E is an $\mathbb{R}-$ linear map

$$\nabla : \mathscr{C}^\infty(E) \to \mathscr{C}^\infty_1(X, E)$$

which satisfies the Leibniz rule: $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$.

Remarks 2.10. Similar to the exterior differential of forms, one can extend ∇ uniquely to $\mathscr{C}^\infty_k(E) \to \mathscr{C}^\infty_{k+1}(X, E)$.

Two popular connections in complex geometry are the Levi-Civita connection on the tangent bundle of a real manifold, and the Chern connection on holomorphic vector bundles of a complex manifold. They are characterized by the following propositions:

Proposition 2.11. *Let (X, g) be a Riemannian manifold. There exists a unique connection $\nabla : \mathscr{C}^\infty(T_X) \to A^1(T_X)$ on the tangent bundle T_X satisfying the properties:*
i) ∇ is compatible with g, i.e. $d(g(\chi, \psi)) = g(\chi, \nabla\psi) + g(\nabla\chi, \psi)$ for $\chi, \psi \in \mathscr{C}^\infty(T_X)$.
ii) ∇ is without torsion, i.e. it satisfies $\nabla_\chi\psi - \nabla_\psi\chi = [\chi, \psi]$
This connection is called the Levi-Civita connection on (X, g).

(See [3]: Chapter 2, section 8)

Proposition 2.12. *Let E be a holomorphic vector bundle, and h a hermitian metric on E. There exists a unique connection ∇ on E satisfying the following properties:*
i) ∇ is compatible with h, i.e. $d(h(\chi,\psi)) = h(\chi,\nabla\psi) + h(\nabla\chi,\psi)$
ii) $\nabla^{0,1} = \overline{\partial}_E$
This connection is called the Chern connection of (E,h).

Remarks 2.13. In a holomorphic vector bundle E of rank k, the operator $\overline{\partial}_E : \mathscr{C}^\infty(A^{0,q}\otimes E) \to \mathscr{C}^\infty(A^{0,q+1}\otimes E)$ is defined as follow. In a holomorphic trivialization of E, $\tau_U : E_{|U} \cong U \times \mathbb{C}^k$, any section α can be written as $(\alpha_1, ..., \alpha_k)$, where α_i are $\mathscr{C}^\infty$ forms of type $(0,q)$ on U. We then set $\overline{\partial}_E(\alpha)_{|U} = (\overline{\partial}\alpha_1, ..., \overline{\partial}\alpha_k)$. It is well-defined because E is a holomorphic vector bundle.

Theorem 2.14. *(Characterization of Kähler metric). The following properties are equivalent:*
i) The metric h is Kähler.
ii) The complex structure endomorphism J is flat for the Levi-Civita connection. This means that it satisfies:

$$\nabla(J\chi) = J\nabla\chi, \forall\chi \in A^0(T_{X,\mathbb{R}}).$$

iii) The Chern connection on the holomorphic tangent bundle $T_X = T_X^{1,0}$ coincides with the Levi-Civita connection on $T_{X,\mathbb{R}}$.

Proof. iii)$\to$ ii) and ii) $\to$ i): From the definition of the bundles, and connections. i)$\to$ iii), use the proposition that if h is a Kähler metric then locally, it is isomorphic to a constant metric up to the first order, i.e. in the neighborhood of every point $x \in X$, there exists holomorphic coordinates $z_1, z_2, ..., z_n$ such that in these coordinates $h = \mathrm{I_n} + \mathrm{O}(\sum_\mathrm{i} |\mathrm{z_i}|^2)$
(See [5] Theorem 3.13, page 72) □

2.2. Kähler identities. Hodge Theory for compact Kähler manifolds.

2.2.1. *Summary of Hodge Theory for Riemannian manifolds.*
(Reference: [5], chapter II.5)
Let X be a compact, oriented Riemannian manifold. The Riemannian metric $(-,-)$ on X induces the metric $(-,-)$ on $A^k(X)$, and hence a L^2 metric $(-.-)_{L^2}$ on the space $A^k(X)$ of $\mathscr{C}^\infty$ differential forms:

$$(\alpha,\beta)_{L^2} = \int_X (\alpha,\beta) Vol$$

where Vol is the volume form. In addition to the differential exterior d, by using the L^2 metric, one can define the following operators
1) The Hodge star operator

$$* : A^k(X) \cong A^{n-k}(X),$$

characterized by $(\alpha,\beta)_{L^2} = \int_X \alpha \wedge *\beta$.
2) The formal adjoint operator d^* of d for the L^2 metric:

$$d^* : A^k(X) \to A^{k-1}(X)$$

It is characterized by $(\alpha, d^*\beta)_{L^2} = (d\alpha,\beta)_{L^2}$, and can also be defined by $d^* = (-1)^k *^{-1} d*$ on $A^k(X)$.

3) The Laplace-Beltrami operator $\Delta_d = dd^* + d^*d : A^k(X) \to A^k(X)$.

The Laplace-Beltrami operator can be shown to be elliptic, hence there is a decomposition: $A^k(X) = \mathscr{H}^k \oplus Imd \oplus Imd^*$, where $\mathscr{H}^k$ is the vector space of Δ_d harmonic differential forms of degree k: $\alpha \in \mathscr{H}^k$ iff $\Delta_d\alpha = 0$ iff $d\alpha = 0$ and $d^*\alpha = 0$.

Corollary 2.15. *Let $\mathscr{H}^k$ be the vector space of Δ_d- harmonic differential forms of degree k. Then the natural map*

$$\mathscr{H}^k \to H^k(X, \mathbb{R})$$

which to α associates the class of the closed form α in $H^k_{DR}(X, \mathbb{R}) = H^k(X, \mathbb{R})$ is an isomorphism. Similar when replacing $\mathbb{R}$ by $\mathbb{C}$.

2.2.2. *Hodge Theory for compact Kähler manifolds.*
Let X be a compact Kähler manifold, and $A^{p,q}$ be the space of differential forms of bi-degree (p, q). In the case of complex manifolds, we have the following operators:
1) The exterior differential d can be written $d = \partial + \overline{\partial}$ where $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$ and $\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$.
2)The operators ∂ and $\bar{\partial}$ have formal adjoints ∂^* and $\overline{\partial}^*$, which can also be defined by $\partial^* = - * \overline{\partial} *$ and $\overline{\partial}^* = - * \partial *$.
3) The Lefschetz operator defined by taking exterior product with ω :

$$L : A^k(X) \to A^{k+2}(X) : \alpha \mapsto \omega \wedge \alpha$$

4) The Λ operator is a formal adjoint of L, which can also be defined by

$$\Lambda = (-1)^k * L*$$

5) The Laplacian operators Δ_d, Δ_∂, $\Delta_{\bar{\partial}}$.

By the structure of Kähler manifolds, we have relationships between those operators, called the Kähler identities:

Theorem 2.16. *The operators $L, \Lambda, \partial, \bar{\partial}^*$ satisfy*

$$[\Lambda, \overline{\partial}] = -i\partial^*, \quad [\Lambda, \partial] = i\overline{\partial}^*, \quad [\overline{\partial}^*, L] = i\partial, \ [\partial^*, L] = -i\overline{\partial}.$$

Proof. The idea of the proof is to do calculation on local coordinates. See [5] Prop 6.5, page 139. □

The Kähler identities imply a strong relationship among the three Laplacian operators $\Delta_d, \Delta_\partial$ and $\Delta_{\bar{\partial}}$:

Corollary 2.17. *Let (X, ω) be a Kähler manifold, and let $\Delta_X, \Delta_\partial, \Delta_{\overline{\partial}}$ be the Laplacians associated respectively to the operators $d, \partial, \overline{\partial}$. Then:*

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$$

Remarks 2.18. The corollary implies that Δ_d is bi-homogeneous: $\Delta_d(A^{p,q}(X) \subseteq A^{p,q}(X))$. Therefore, if $\alpha \in A^k(X)$ is harmonic then its components are also harmonic. This implies the decomposition

$$\mathscr{H}^k(X) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}$$

where $\mathscr{H}^{p,q}$ is the set of forms of type (p, q) which are harmonic for Δ_d or Δ_∂.

Corollary 2.19. *Let X be a Kähler manifold. We have the following decomposition for the de Rham cohomology groups with complex coefficients:*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the set of classes representable by a harmonic form of type (p,q)

Remarks 2.20. In fact, one can show that the decomposition does not depend on the choice of the Kähler metric by showing that $H^{p,q} = K^{p,q}$ where $K^{p,q}$ is the subspace consisting of de Rham cohomology classes which are representable by a closed form of type (p,q).

3. Chern Weil Theory (References: [2, 6, 1, 4])

First, we give a construction of the Chern-Weil homomorphism. Given a vector bundle E on a manifold X, one can construct a connection ∇ on E, and calculate its curvature. The curvature of a connection is a 2-form with coefficients in $End(E)$. One can use invariant polynomials to evaluate the curvature, and get certain closed forms in the de Rham cohomology. These forms can be shown to be independent of the connection chosen, hence become topological invariants of the vector bundles. The Chern classes are defined by using certain kinds of invariant polynomials.

In the last part of the talk, we define the Ricci curvature tensor of a Riemannian manifold. If X is a Kähler manifold, one can define the Ricci form which up to a real constant is equal to the first Chern form of the anti-canonical line bundle K_X^*. It implies that the Ricci form in this case is a closed (1,1)- form, independent of the Riemannian metric. It is thus a topological invariant of the complex manifold, and depends only on the topology of X and the class of the complex structure J.

The exposition for Chern-Weil theory follows from [2, 6] using the language of connections and vector bundles. The original (and a little more general) exposition was to use principal bundles, and connections for principal bundles (See the appendix of [1]). The exposition for the Ricci form follows from [4] .

3.1. **Chern-Weil theory.**

Denote $\mathscr{C}^\infty(E)$ the $\mathscr{C}^\infty$ sections of E, and $\mathscr{C}^\infty_k(X,E)$ the $\mathscr{C}^\infty$ sections of the bundle $\wedge^k T^*X \otimes E$. We have

$$\mathscr{C}^\infty(E) \otimes_{\mathbb{C}} \mathscr{C}^\infty(\wedge^k T^*X) \cong \mathscr{C}^\infty_k(X,E)$$

Definition 3.1. A (linear) connection ∇ on E is a $\mathbb{C}$ linear differential operator $\nabla : \mathscr{C}^\infty(E) \to \mathscr{C}^\infty_1(X,E)$ such that for any $f \in \mathscr{C}^\infty(X), s \in \mathscr{C}^\infty(X,E)$, the following Leibniz rule holds:

$$\nabla(fs) = (df)s + f\nabla(s)$$

Remarks 3.2. One can always construct a connection on a complex vector bundle by defining locally and glueing them using partition of unity.

Remarks 3.3. Moreover, one can canonically extend ∇ to a map:

$$\nabla : \mathscr{C}^\infty_k \to \mathscr{C}^\infty_{k+1}(E)$$

such that for any $f \in \wedge^k(T^*X), s \in \mathscr{C}^\infty(E)$, we have:

$$\nabla(fs) = (df)s + (-1)^{deg(f)} f \wedge \nabla(s)$$

Remarks 3.4. The map ∇ and $\{\mathscr{C}_k^\infty(E)\}_k$ does not form not a complex, and the obstruction is given by the notion of curvature.

Definition 3.5. The curvature R^∇ of a connection ∇ is defined by

$$R^\nabla = \nabla \circ \nabla : \mathscr{C}^\infty(E) \to \mathscr{C}_2^\infty(M, E)$$

Lemma 3.6. *The curvature satisties:*

$$R^\nabla_{X,Y} s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \quad X, Y \in TM, s \in \mathscr{C}^\infty(E)$$

Lemma 3.7. *Let* $\alpha \in \mathscr{C}_k^\infty(X, E)$ *then* $\nabla^2(\alpha) = R^\nabla \wedge \alpha$.

Remarks 3.8. The curvature R^∇ is $\mathscr{C}^\infty(X)$ linear, hence it might be thought of as an element of $\mathscr{C}^\infty(End(E))$ with coefficients in $\mathscr{C}^\infty(\wedge^2 T^* X)$, where $End(E)$ is the vector bundle with fibre $End(E_x, E_x)$. To get a form on the cohomology, we want a map of bundle $End(E) \to \underline{\mathbb{C}}$, and it can be done using the idea of invariant polynomials.

Definition 3.9. Let V be a vector space, and $\mathfrak{g} = End(V)$. An (ad)-invariant symmetric k-linear form on $\mathfrak{g}$ is a symmetric multilinear form $S : \mathfrak{g} \otimes \mathfrak{g} \otimes .... \otimes \mathfrak{g} \to \mathbb{C}$ satisfying $S(A_1, A_2, ..., A_k) = S(gA_1g^{-1}, ...gA_kg^{-1})$ for $A_1, ..., A_k \in \mathfrak{g}$ and $g \in GL_n(\mathbb{C})$.

Remarks 3.10. To any such form, we can associate an invariant polynomial $P(A) = S(A, A, ..., A)$ satisfying $P(gAg^{-1}) = P(A)$. Then P is a polynomial with complex coefficients, homogeneous of degree k, variables $a_{i,j}$ are the entries of the matrix A. For example, the trace and the determinant of A are invariant polynomials. More generally, let $\det(\mathrm{Id} + \mathrm{A}) = \sum \sigma_\mathrm{k}(\mathrm{A})$ where $\sigma_k(A)$ are homogeneous parts of degree k then $\sigma_k(A)$ then $\sigma_k(A)$ are examples of invariant polynomials.

Lemma 3.11. *Let* $E \xrightarrow{p} X$ *be a vector bundle with fiber V, and let S be an invariant symmetric multilinear form on End(V). Then S induces a multilinear bundle map:*

$$S^E : End(E) \otimes End(E) \otimes .... \otimes End(E) \to \underline{\mathbb{C}}$$

given in a frame f, by $S^f(A_1, ..., A_k) = S(A_1^f, ..., A_k^f)$, *where* A_j^f *is the element in* $End(V)$ *defined by* $A_j \in End(E)$ *and the frame* f. *The invariance of* S *ensures that* S^f *is independent of the frame and hence* S^E *is well-defined.*

Theorem 3.12. *(Chern-Weil) Let* $E \xrightarrow{p} X$ *be a vector bundle with fiber V, and* P *an invariant polynomial on* $End(V)$ *of degree k, and* ∇ *a connection on E, with curvature* R^∇. *Then the 2k-form* $P(R^\nabla)$ *is closed, and moreover, the cohomology class* $P(R^\nabla) \in H^{2k}(X, \mathbb{C})$ *is independent of the choice of connection* ∇.

Proof. To show $P(R^\nabla)$ is closed, use Bianchi's identity: $[\nabla, R^\nabla] = 0$ and the following lemma:

Lemma 3.13. *Let* $\alpha_j \in \mathscr{C}_{p_j}^\infty(End(E))$ *and S an invariant symmetric multilinear form of degree k. Denote by* $e_j = p_1 + ... + p_{j-1}$. *Then*

$$d(S(\alpha_1, ..., \alpha_k)) = \sum_{j=1}^k S(\alpha_1, ..., (-1)^{e_j} \nabla \alpha_j, ..., \alpha_k)$$

To show independence, take two connections ∇ and ∇'. Consider the curve connecting the two $\nabla^t = \nabla + t(\nabla' - \nabla)$ then $\nabla^0 = \nabla$ and $\nabla^1 = \nabla'$. In order to show

$$P(R^{\nabla}) = P(R^{\nabla'}) + d\beta$$

we will first show that

$$\frac{d}{dt}P(R^{\nabla^t}) = d\beta^t$$

then take $\beta = \int_0^1 \beta^t dt$ for the above.
(See [2, 1] for the calculation.)

□

Proposition 3.14. *Let $\mathbb{C}[\mathfrak{g}]$ be the graded algebra of invariant polynomials. The homomorphism:*

$$\mathbb{C}[\mathfrak{g}] \to (\oplus H^k(X, \mathbb{C}), \wedge), P \to [P(R^{\nabla})]$$

is a ring homomorphism. It is called the Chern-Weil homomorphism.

In case E is a holomorphic vector bundle over a complex manifold M, the curvature form of E, with respect to some Hermitian metric is not just a 2-form, but is a (1,1) form. The Chern-Weil homomorphism takes the form :

$$\mathbb{C}[\mathfrak{g}]_k \to H^{k,k}(X, \mathbb{C}), P \to [P(R^{\nabla})]$$

3.2. **Chern classes.** Let $E \xrightarrow{\pi} X$ be a complex vector bundle of rank r, and let $\sigma_k(A)$ be invariant polynomials in the Remark 3.10. The Chern forms of a vector bundle of rank r endowed with a connection are defined by

$$c_k(E, \nabla) = \sigma_k(\frac{i}{2\pi}R^{\nabla}) \in A^{2k}_{\mathbb{C}}(X)$$

and the k-th Chern class is the induced cohomology class $c_k(E) = [c_k(E, \nabla)] \in H^{2k}(M, \mathbb{C})$.

3.2.1. *The Ricci form as curvature form on the canonical bundle.* (Reference: [4]) Let (X, J, h) be a Kähler manifold of complex dimension n. Let $g = Real(h)$. The manifold (X, g, J) is then a Riemannian manifold and has the Levi-Civita connection ∇. Let R be its curvature tensor then R can be given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathscr{C}^{\infty}(T_{X,\mathbb{R}})$$

This defines a tensor of type $(3, 1)$. The Ricci tensor of (X, g) is a 2-form defined by

$$Ric(X, Y) = Tr(V \to R(V, X)Y)$$

On a Kähler manifold, one can show that $Ric(X, Y) = Ric(JX, JY)$ hence as in the case of a Hermitian metric (See Prop 2.7), $Ric(JX, Y)$ is skew symmetric in X and Y.

Definition 3.15. The Ricci form ρ of a Kähler manifold is defined by

$$\rho(X, Y) = Ric(JX, Y), \quad \forall X, Y \in T_X$$

On the manifold X, we can form the n-th exterior power of the holomorphic cotangent bundle. This is a holomorphic vector bundle of rank 1, denoted by K_X, and is called the canonical bundle of X:

$$K_X = \wedge^n A^{1,0}(X) = A^{n,0}(X)$$

In local coordinates, a trivialisation of K_X is given by $dz^1 \wedge dz^2 \wedge .... \wedge dz^n$, and the transition functions are given by Jacobian determinants. A Hermitian metric on the anti-canonical line bundle K_X^* is given by a nowhere non vanishing section of $K_X \otimes \overline{K_X} = A^{n,n}(X)$. On a Kähler manifold, the volume form $\frac{\omega^n}{n!}$ is a non-vanishing section of $A^{n,n}(X)$, and it can serve as a Hermitian metric on K_X^*. By Prop 2.12, there is the Chern connection on K_X, and one can define its curvature r^* and the first Chern class $c_1(X) = c_1(K_X^*)$.

Proposition 3.16. *The curvature of the anti-canonical bundle (with its induced metric) is given by $r^* = -i\rho$. In particular, the Ricci form is closed, and its cohomology class is independent of the Kähler metric: $[\rho] = 2\pi c_1(X)$.*
(See [4] *Prop 17.4, page 120)*

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