

# Commuting Hamiltonians and Hamilton-Jacobi multi-time equations

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November 29, 2012

Hamilton's variation principle can be shown to correspond to Fermat's *Principle* for a wave propagation in configuration space ( $q$ -space), and the Hamilton-Jacobi equation expresses Huygens' *Principle* for this wave propagation. Unfortunately this powerful and momentous conception of Hamilton, is deprived, in most modern reproductions, of its beautiful raiment as a superfluous accessory, in favour of a more colourless representation of the analytical correspondence.

*E. Schrödinger*, Quantisation as a Problem of eigenvalues  
(Part II), *Annalen der Physik*, 1926

# 1 Introduction

The problem of finding solutions of multi-time Hamilton-Jacobi equations, by which one usually means equations of the type

$$(MHJ) \quad \begin{cases} \frac{\partial}{\partial t_1} u(t_1, \dots, t_d, x) + H_1(t_1, \dots, t_d, x, \frac{\partial}{\partial x} u(t_1, \dots, t_d, x)) = 0 \\ \vdots \\ \frac{\partial}{\partial t_d} u(t_1, \dots, t_d, x) + H_d(t_1, \dots, t_d, x, \frac{\partial}{\partial x} u(t_1, \dots, t_d, x)) = 0 \end{cases}$$

with initial condition

$$u(0, \dots, 0, x) = f(x)$$

has been initiated by Rochet, Lions-Rochet and studied more recently by Barles-Tourin and Motta-Rampazzo ([Rochet, Lions-Rochet, Barles-Tourin, Motta-Rampazzo]).

Such a system of equations is well-known to be overdetermined, and in order to have a solution, we need that the Hamiltonians commute in a suitable sense. This is already obvious when applying the the method of characteristics. Besides a suitable commutation condition, we need to address the question of the type of solution one is looking for. It is well known, and was proved for more general equations in a recent paper by Dacorogna and Marcellini [Dac-Mar] that there are plenty of  $C^0$  solutions for such equations. We then need to select a particular “class” of solution, that is in some aspects, deemed to be the best suited to our problem. A classical choice is to look for viscosity solutions, which are the “right” solutions for optimal control and this is the type of solution considered in the above papers (except for [Dac-Mar]).

In Barles and Tourin’s paper, the existence of a viscosity solution is proved under the assumption that one of the Hamiltonians is coercive with controlled growth<sup>1</sup> (see [Barles-Tourin], page 1526, conditions (H1),(H2)), and more importantly that they both are

- (a). independent of  $(t_1, \dots, t_d)$
- (b). convex in  $p$
- (c). of class  $C^1$  and satisfying the commuting condition  $\{H_j, H_k\} = 0$

In fact the third condition can be weakened, as Barles and Tourin point out, to assume that the  $H_j$  are  $C^0$  and there are sequences of Hamiltonians  $H_j^\nu$  such that

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<sup>1</sup>Condition (H1) states that for each  $R$  there exists  $K_R$  such that  $|H_1(x, p)| \leq K_R$  and  $|\frac{\partial H_1}{\partial p}(x, p)| \leq K_R(1 + |x|)$  in  $\mathbb{R}^n \times B(0, R)$ . Moreover the authors assume  $du_0(x)$  to be bounded. These conditions are stronger than the assumptions we need here. See appendix A for more details.

(a).  $\lim_{\nu \rightarrow \infty} H_j^\nu = H_j$  in the  $C^0$  topology

(b).  $\{H_j^\nu, H_l^\nu\} = 0$

In other words, the  $H_j$  are limits of commuting Hamiltonians.

However this assumption is quite unpractical since it is already difficult to write commuting  $C^1$  Hamiltonians as nontrivial limits of commuting Hamiltonians.

The present paper has several goals.

First we solve the multi-time Hamilton-Jacobi equation in the framework of “variational solutions” defined by Sikorav, Chaperon and the second author in [Chaperon, Viterbo-Ottolenghi]. According to a result by Joukovskaia (cf [Joukovskaia]), if the Hamiltonian is convex in  $p$ , the variational solution must coincide with the viscosity solution, so that our results extend those of Barles and Tourin. For variational solutions, we prove that we only need the  $H_j$  to be  $C^0$ , and condition (b) is then replaced by a “commutation” condition best expressed in terms of symplectic invariants, refining the following

**Definition 1.1.** Let  $H, K$  be two  $C^0$  Hamiltonians. We shall say that  $H$  and  $K$   **$C^0$ -commute**, if and only if there are sequences  $H_\nu, K_\nu$  of  $C^1$  Hamiltonians such that, all limits being for the  $C^0$  topology, we have:

(a).  $\lim_{\nu \rightarrow \infty} H_\nu = H, \lim_{\nu \rightarrow \infty} K_\nu = K.$

(b).  $\lim_{\nu \rightarrow \infty} \{H_\nu, K_\nu\} = 0.$

A similar definition is given in section 3 for the time dependent case, that is for the case of  $H(t, z)$  and  $K(t, z)$ , and in appendix B for the case of equations depending on the function.

The two main theorems of this paper are

**Theorem 1.2.** *If two  $C^1$  Hamiltonians  $C^0$ -commute, then they commute in the usual sense (i.e. their Poisson bracket vanishes).*

**Theorem 1.3.** *Assume the Hamiltonians  $H_1(t_1, \dots, t_d, x, p), \dots, H_d(t_1, \dots, t_d, x, p)$  are Lipschitz, and  $C^0$ -commute. Then equation (MHJ) has a unique solution which induces on any path starting from the origin a variational solution the equation. If the  $H_j$  are convex in  $p$ , then  $u$  is a viscosity solution of each individual equation.*

*Remark 1.4.* This means that given any path  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^d$  such that  $\gamma(0) = 0$ , then  $v(s, x) = u(\gamma(s), x)$  is the variational solution of

$$(1.1) \quad \begin{cases} \frac{d}{ds}v(s, x) + \sum_{j=1}^d \gamma'_j(s) H_j(\gamma(s), x, \frac{\partial}{\partial x}v(s, x)) = 0 \\ v(0, x) = f(x) \end{cases}$$

Note that the important feature is that if we have several paths with the same endpoint  $(t_1, \dots, t_d)$  then the value of the variational solution above does not depend on the path from 0 to  $(t_1, \dots, t_d)$ .

The authors wish to thank Maxime Zavidovique for pointing out that the statement of the theorem in the published version of this paper was misleading.

*Remark 1.5.* The Lipschitz condition is only needed to guarantee the existence of the flow for the approximating Hamiltonians. Since if  $H$  is Lipschitz, we may approximate it by Hamiltonians  $H^\nu$  such that  $X_{H^\nu}$  will be complete, this condition is sufficient. In fact, it is enough that there should be constants  $A, B$  such that  $|X_H(q, p)| \leq A(|q| + |p|) + B$ , i.e. the Lipschitz norm of  $H$  on a ball grows at most linearly with the radius. Note also that in some cases, we may guarantee existence of the flow of the  $X_{H^\nu}$  for other reasons. For example when  $H$  is time-independent, and proper, since the  $H^\nu$  will satisfy the same assumption, and conservation of energy implies that the flow remains in a compact set, it is thus defined for all times.

*Remark 1.6.* The paper is not supposed to be written for specialists in symplectic topology, although a certain familiarity with the basic constructions of [Viterbo 1] and [Viterbo-Ottolenghi] is recommended. Appendices A and B, are of a more symplectic flavour, and really address the question of Hamilton-Jacobi equations from a symplectic topology viewpoint. In particular Appendix A addresses the question of the growth conditions one must impose on the Hamiltonian and initial condition from a purely geometric point of view, while Appendix B extends the main theorem for equations depending on the unknown function.

The authors warmly thank Franco Rampazzo for attracting their attention to this problem during a conference in Cortona, for communicating his lecture notes ([Rampazzo]) and for many interesting discussions. We also would like to thank F. Camilli, I. Capuzzo Dolcetta and A. Siconolfi for the superb organization of the Cortona conference. Even though it is probably not related, we mention the paper [Ramp-Suss] on commutation of Lipschitz vector fields.

## 2 Preliminary material

*We shall here assume that  $N$  is a connected manifold without boundary, and either compact or that all Hamiltonians are compact supported, as in [Viterbo 1]. However, we shall explain in Appendix A, how our results extend to non-compact situations, provided we have some estimate on the growth of Hamiltonians.*

Let  $T^*N$  be the cotangent bundle of the manifold  $N$  endowed with the canonical symplectic structure  $\sigma = \sum_{j=1}^n dp_j \wedge dx^j$ . To any function  $H(t, z)$  (where  $z = (x, p)$ ) on  $\mathbb{R} \times T^*N$ , usually called a Hamiltonian, we associate the time dependent vector field  $X_H$  defined by

$$\sigma(X_H, \xi) = -d_z H(t, z)\xi$$

and the corresponding Hamiltonian flow  $\varphi_{t_0}^t$  defined by

$$\begin{cases} \frac{d}{dt}\varphi_{t_0}^t = X_H(t, \varphi_{t_0}^t) \\ \varphi_{t_0}^{t_0} = Id \end{cases}$$

Let  $L$  be a Lagrangian submanifold of  $T^*N$  obtained from the zero section  $0_N = \{(x, 0) \in T^*N \mid x \in N\}$  by the Hamiltonian isotopy  $\varphi_{t_0}^t$ . We shall always assume that  $\varphi_{t_0}^t$  is well defined on  $0_N$  for all  $t$ . Then according to [Lau-Sik] there exists a generating function quadratic at infinity for  $L$  i.e. there exists a smooth function  $S : N \times \mathbb{R}^k \rightarrow \mathbb{R}$  such that

- (a).  $(x, \xi) \mapsto \frac{\partial S}{\partial \xi}(x, \xi)$  has 0 as a regular value (a Morse family in the terminology introduced by A. Weinstein in [Weinstein])
- (b).  $S(x, \xi) = Q(\xi)$  for  $\xi$  large enough, where  $Q$  is a non-degenerate quadratic form<sup>2</sup>.
- (c).

$$L = \{(x, \frac{\partial S}{\partial x}(x, \xi)) \mid \frac{\partial S}{\partial \xi}(x, \xi) = 0\}$$

In particular the critical points of  $S$  are in one to one correspondence with the points of  $L \cap 0_N$ .

In the rest of the paper, we shall shorten the expression “generating function quadratic at infinity” by “GFQI”.

Set  $S^\lambda = \{(x, \xi) \mid S(x, \xi) \leq \lambda\}$ ,  $E^\pm$  be the positive and negative eigenspaces of  $Q$ , and  $D^\pm$  be large discs in  $E^\pm$ . Since for  $c$  large enough  $S^{\pm c} = N \times Q^{\pm c}$  we have

$$H^*(S^c, S^{-c}) = H^*(N \times Q^c, N \times Q^{-c}) = H^*(N) \otimes H^*(D^-, \partial D^-)$$

so that, to each cohomology class  $\alpha \in H^*(N)$  we may associate a class, image of  $\alpha$  by the Thom isomorphism, and denoted by  $T\alpha$ . To such a class in  $H^*(S^c, S^{-c})$ , we may associate a minimax critical level

$$c(\alpha, S) = \inf\{\lambda \mid T\alpha \notin \text{Ker}(H^*(S^c, S^{-c}) \rightarrow H^*(S^\lambda, S^{-c}))\}$$

Now it is proved in [Viterbo 1] that given  $L$ ,  $S$  is essentially unique up to adding a constant, and more precisely, up to a global shift, the numbers  $c(\alpha, S)$  depend only on  $L$ , not on  $S$ , and they are thus denoted by  $c(\alpha, L)$ . Moreover, denoting

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<sup>2</sup>This is sometimes conveniently replaced by the condition  $|S(x, \cdot) - Q(\cdot)|_{C^1} < +\infty$ . We shall use these conditions interchangeably in the rest of the paper. It is easy to prove that existence of a generating function of one kind is equivalent to the existence of a generating function of the other kind.

by  $\gamma(L) = c(\mu, L) - c(1, L)$ , where  $1 \in H^0(N)$ ,  $\mu \in H^n(N)$  are generators,  $\gamma(L)$  vanishes if and only if  $L = 0_N$ .

Moreover let  $S_x(\xi) = S(x, \xi)$  be the restriction of  $S$  to the fibre over  $x$ . We can look for a minimax as above for the function  $S_x$ . Since the cohomology of the point is one dimensional, denoting its generator by  $1_x$ , we set

$$u_L(x) = c(1_x, S_x)$$

It has been proved by Chaperon, Sikorav (see also [Viterbo-Ottolenghi]), that such a function is a solution of the *Lagrangian Hamilton-Jacobi equation*, that is

$$(x, du_L(x)) \text{ is in } L \text{ for almost all } x \text{ in } N$$

In particular if  $L \subset H^{-1}(0)$ , we have a solution of the classical Hamilton-Jacobi equation

$$H(x, du_L(x)) = 0 \text{ a.e. on } N$$

We call  $u_L$  a **variational solution** of the equation.

For evolution equations, that is

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + H(t, x, \frac{\partial u}{\partial x}(t, x)) = 0 \\ u(0, x) = f(x) \end{cases}$$

the construction of variational solutions for a single equation can be rephrased as follows. Let  $\Lambda_0$  be a Lagrangian submanifold of  $T^*N$ , and  $H(t, z)$  a time dependent Hamiltonian. We consider  $\tilde{\Lambda}_{0,H} = \{(0, -H(0, z), z) \mid z \in \Lambda_0\}$ , and

$$L = \bigcup_{t \in \mathbb{R}} \Phi^t(\Lambda_{0,H}) \subset T^*(\mathbb{R} \times N)$$

In [Viterbo-Ottolenghi], it is also proved that for evolution equations the above class of solutions extends to  $C^0$  Hamiltonians: if  $H^\nu$  goes to  $H$ , then the solution  $u^\nu$  converges to  $u$ . Note that in the framework of viscosity solutions, this property is called **stability**. A priori, even though  $u$  is Lipschitz, hence according to Rademacher theorem is almost everywhere differentiable, we do not claim that  $u$  satisfies the above equation almost everywhere<sup>3</sup>.

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<sup>3</sup>This is the case for viscosity solutions, but is unknown for variational solutions.

## 2.1 Capacities for Hamiltonian flows

Let  $L_1, L_2$  be Lagrangian submanifolds generated by  $S_1, S_2$ . We may define  $c(\alpha, S_1 - S_2)$ , and as this does not depend on the choice of  $S_1, S_2$  but only on  $L_1, L_2$  we denote it by abuse of language  $c(\alpha, L_1 - L_2)$ , even though it is not really determined by the set

$$L_1 - L_2 = \{(x, p_1 - p_2) \mid (x, p_1) \in L_1, (x, p_2) \in L_2\}$$

but depends on both  $L_1$  and  $L_2$ .

We denote by  $\gamma(L_1 - L_2)$  the difference  $c(\mu, L_1 - L_2) - c(1, L_1 - L_2)$ . Now to  $\psi$  a compact supported symplectic isotopy, we may associate a symplectic invariant as follows:

**Definition 2.1.** Let  $\mathcal{L}$  be the set of Lagrangian submanifolds Hamiltonianly isotopic to the zero section,  $\mathcal{H}(T^*N)$  be the set of smooth time dependent Hamiltonians on  $\mathbb{R} \times T^*N$ ,  $\mathcal{DH}(T^*N)$  the set of time one maps of such Hamiltonians. We set for  $\psi \in \mathcal{DH}$ ,

$$\tilde{\gamma}(\psi) = \sup\{\gamma(\psi(L) - L) \mid L \in \mathcal{L}\}$$

We shall need the following

**Lemma 2.2.** *Let  $L_1, L_2, L_3 \in \mathcal{L}$ . Then*

$$\gamma(L_1 - L_3) \leq \gamma(L_1 - L_2) + \gamma(L_2 - L_3)$$

*Proof.* Assume first that  $L_2 = 0_N$ . Then we must prove

$$\gamma(L_1 - L_3) \leq \gamma(L_1) + \gamma(L_3)$$

But this follows from corollary 2.8 and Proposition 3.3 on page 693 of [Viterbo 1] according to which

$$c(1, -S) = -c(\mu, S)$$

and

$$c(uv, S_1 + S_2) \geq c(u, S_1) + c(v, S_2)$$

Indeed, we have  $\gamma(L) = \gamma(-L)$  and

$$c(1, S_1 - S_3) \geq c(1, S_1) + c(1, -S_3) = c(1, S_1) - c(\mu, S_3)$$

$$c(1, S_3 - S_1) \geq c(1, S_3) + c(1, -S_1) = c(1, S_3) - c(\mu, S_1)$$

using that  $c(1, S_3 - S_1) = -c(\mu, S_1 - S_3)$  this can be rewritten as

$$-c(1, S_1 - S_3) \leq c(\mu, S_3) - c(1, S_1)$$

$$c(\mu, S_1 - S_3) \leq c(\mu, S_1) - c(1, S_3)$$

and adding the inequalities we get

$$\gamma(S_1 - S_3) \leq \gamma(S_1) + \gamma(S_3).$$

Then we claim that

$$(2.1) \quad \gamma(L - L') = \gamma(\varphi(L) - \varphi(L'))$$

for any  $\varphi$  in  $\mathcal{DH}$ . Indeed, according to proposition 3.5 page 695 of [Viterbo 1], if  $L' = \rho^{-1}(0_N)$ , we have

$$\begin{aligned} \gamma(\varphi(L) - \varphi(L')) &= \gamma(\varphi(L) - \varphi\rho^{-1}(0_N)) = \\ \gamma((\varphi\rho^{-1})^{-1}(\varphi(L))) &= \gamma(\rho(L)) \end{aligned}$$

Since the last expression does not depend on the choice of  $\varphi$ , we get

$$\gamma(\varphi(L) - \varphi(L')) = \gamma(L - L')$$

Let now  $\varphi$  in  $\mathcal{DH}$  be such that  $\varphi(L_2) = 0_N$ , and let us apply the above equality to the  $L_j$ :

$$\begin{aligned} \gamma(L_1 - L_3) &= \gamma(\varphi(L_1) - \varphi(L_3)) \leq \gamma(\varphi(L_1)) + \gamma(\varphi(L_3)) = \\ \gamma(\varphi(L_1) - \varphi(L_2)) + \gamma(\varphi(L_2) - \varphi(L_3)) &= \gamma(L_1 - L_2) + \gamma(L_2 - L_3) \end{aligned}$$

□

We now have the following properties

**Proposition 2.3.**

(a).

$$\tilde{\gamma}(\psi) \geq 0$$

and  $\tilde{\gamma}(\psi) = 0$  if and only if  $\psi = \text{Id}$

(b).

$$\tilde{\gamma}(\psi^{-1}) = \tilde{\gamma}(\psi)$$

(c). (triangle inequality)

$$\tilde{\gamma}(\psi\varphi) \leq \tilde{\gamma}(\psi) + \tilde{\gamma}(\varphi)$$

(d). (invariance by conjugation)

$$\tilde{\gamma}(\varphi\psi\varphi^{-1}) = \tilde{\gamma}(\psi)$$



*Proof.* (a). Assume we have for all  $L$  in  $\mathcal{L}$ ,  $\gamma(\psi(L) - L) = 0$ . According to [Viterbo 1], corollary 2.3, we have  $\psi(L) = L$ . Now since this holds for any  $L$ , we must have  $\psi = \text{Id}$ .

(b). Indeed the change of variable in  $\mathcal{L}$  given by  $L = \psi(L')$  shows that

$$\tilde{\gamma}(\psi^{-1}) = \sup\{\gamma(\psi^{-1}(L) - L) \mid L \in \mathcal{L}\} = \sup\{\gamma(L' - \psi(L')) \mid L' \in \mathcal{L}\}$$

Since  $\gamma(-S) = \gamma(S)$  we see that the last expression equals  $\tilde{\gamma}(\psi)$

(c). Again by change of variable, we have

$$\begin{aligned} & \sup\{\gamma(\psi\varphi(L) - L) \mid L \in \mathcal{L}\} \leq \\ & \sup\{\gamma(\psi\varphi(L) - \varphi(L)) + \gamma(\varphi(L) - L) \mid L \in \mathcal{L}\} \leq \\ & \sup\{\gamma(\psi(L') - L') \mid L' \in \mathcal{L}\} + \sup\{\gamma(\varphi(L) - L) \mid L \in \mathcal{L}\} \end{aligned}$$

The second inequality follows from lemma 2.2

(d). Indeed let us make the change of variable  $L' = \varphi^{-1}(L)$ , we get

$$\tilde{\gamma}(\varphi\psi\varphi^{-1}) = \sup\{\gamma((\varphi\psi\varphi^{-1})(L) - L) \mid L \in \mathcal{L}\} = \sup\{\gamma(\varphi\psi(L') - \varphi(L')) \mid L' \in \mathcal{L}\}$$

Now using the equality (2.1), we get

$$\gamma(\varphi\psi(L') - \varphi(L')) = \gamma(\psi(L') - L')$$

and thus

$$\sup\{\gamma(\varphi\psi(L') - \varphi(L')) \mid L' \in \mathcal{L}\} = \sup\{\gamma(\psi(L') - L') \mid L' \in \mathcal{L}\} = \tilde{\gamma}(\psi)$$

□

**Definition 2.4.** We shall say that the sequence  $\varphi_n$  in  $\mathcal{DH}$  *c-converges* to  $\varphi$  if and only if

$$\lim_{n \rightarrow \infty} \tilde{\gamma}(\varphi_n^{-1}\varphi) = 0$$

*Remark 2.5.* Since our invariant is called  $\gamma$ , we should talk about  $\gamma$ -convergence. In fact our *c*-convergence is indeed related to  $\Gamma$ -convergence in the calculus of variations, but we want to avoid any confusion here.

We also need the following estimate:

**Proposition 2.6.** *Assume  $\psi$  is the time one map of the Hamiltonian  $H(t, z)$ . Then we have*

$$\tilde{\gamma}(\psi) \leq \|H\|_{C^0} = \sup_{(t,z) \in [0,1] \times \mathbb{R}^{2n}} H(t, z) - \inf_{(t,z) \in [0,1] \times \mathbb{R}^{2n}} H(t, z)$$

*As a consequence, if  $\varphi_n$  and  $\varphi$  are generated by  $H_n$  and  $H$ , and if  $H_n \rightarrow H$  in the  $C^0$  topology, then  $\varphi_n$   $c$ -converges to  $\varphi$ .*

*Proof.* We first prove that if  $H_0, H_1$  are Hamiltonians and  $\varphi_0, \varphi_1$  are their flows, we may normalize  $S_0, S_1$  generating functions of  $\varphi_0(L), \varphi_1(L)$  so that their critical values are those of

$$\mathcal{S}_j(x(1), \gamma, \xi) = S(x(0), \xi) + \int_0^1 p(t)\dot{x}(t) + H_j(t, x(t), p(t))dt$$

where  $S$  is generating function quadratic at infinity for  $L$ , and  $\gamma = (x, p) : [0, 1] \rightarrow T^*N$ . Indeed we have

$$\begin{aligned} D\mathcal{S}_j(x(1), \gamma, \xi)(\delta x(1), \delta \gamma, \delta \xi) = \\ \frac{\partial S}{\partial \xi}(x(0), \xi)\delta \xi + \left( p(1) - \frac{\partial S}{\partial x}(x(1), \xi) \right) \delta x(1) - \left( p(0) - \frac{\partial S}{\partial x}(x(0), \xi) \right) \delta x(0) + \\ \int_0^1 \left[ \left( \dot{x}(t) + \frac{\partial H}{\partial p}(x(t), p(t)) \right) \delta p(t) + \left( \dot{p}(t) - \frac{\partial H}{\partial x}(x(t), p(t)) \right) \delta x(t) \right] dt \end{aligned}$$

According to [Viterbo 1], [Théret] generating functions associated to a Lagrangian are “essentially unique”, up to a constant. Thus the critical values of two such functions differ by a global translation. Since  $\mathcal{S}_j(x(1), \gamma, \xi)$  is formally a generating function, and in particular has critical values coinciding up to translation with those of any other generating function, we may use its critical values to normalize the  $S_j$  (i.e. we replace  $S_j$  by  $S_j + c_j$  so that the critical values of  $S_j + c_j$  and  $S_j$  coincide).

In particular we claim that if  $H_0 \leq H_1$  we have  $c(\alpha, S_0) \leq c(\alpha, S_1)$ .

For this we argue as in the proof of proposition 4.6 from [Viterbo 1]. We consider the linear interpolation  $H_\lambda(t, x, p) = (1 - \lambda)H_0(t, x, p) + \lambda H_1(t, x, p)$ . Let  $\varphi_\lambda^t$  be the flow of  $H_\lambda$  and  $\varphi_\lambda$  be the time one map. The associated generating function  $S_\lambda(x, \xi)$  of  $\varphi_\lambda(L)$  is normalized as above. Now for a critical point of  $S_\lambda$ , we get an intersection point  $(x_\lambda, 0) = (x_\lambda(1), p_\lambda(1))$  of  $\varphi_\lambda(L) \cap 0_N$  and the critical values are those of

$$\mathcal{S}_\lambda(x(1), \gamma, \xi) = S(x(0), \xi) + \int_0^1 [p(t)\dot{x}(t) + H_\lambda(t, x(t), p(t))]dt$$

corresponding to critical points of the form  $(x_\lambda(1), \gamma_\lambda, \xi_\lambda)$  where

$$\gamma_\lambda(t) = (x_\lambda(t), p_\lambda(t)) = \varphi_\lambda^t(x_\lambda(0), p_\lambda(0))$$

and

$$\frac{\partial}{\partial \xi} S(x_\lambda(0), \xi_\lambda) = 0$$

Using the fact that generically  $x_\lambda = x_\lambda(1)$  is piecewise  $C^1$ , and  $S(x_\lambda, \xi_\lambda)$  is continuous, it is enough to know that for all but a finite set of values of  $\lambda$  we may write

$$\frac{d}{d\lambda} S_\lambda(x_\lambda, \xi_\lambda) = \int_0^1 \frac{d}{d\lambda} H_\lambda(t, x_\lambda(t), p_\lambda(t))$$

and this quantity is positive if  $H_0 \leq H_1$ .

Now  $\|H\|_{C^0} = C$  means that  $a \leq H \leq b$  with  $b - a \leq C$ . Then, since for the constant Hamiltonian  $h_a(x) = a$ ,

$$c(\mu, S_a) = c(1, S_a) = a$$

(again because of the above normalization) and we get  $a \leq c(1, S_H) \leq c(\mu, S_H) \leq b$  and we have  $\gamma(S_H) \leq b - a = \|H\|_{C^0}$ . □

**Corollary 2.7.** *Assume  $H_\nu \rightarrow H$  in the  $C^0$  topology, where  $H$  is in  $C^{1,1}$ , and  $\varphi_\nu^t, \varphi^t$  are the flows of  $H_\nu$  and  $H$ . Then for all  $t$ ,  $\varphi_\nu^t \xrightarrow{c} \varphi^t$ .*

*Proof.* Indeed  $\varphi^{-t} \varphi_\nu^t$  is the flow of  $H_\nu(t, \varphi^t(z)) - H(t, \varphi^t(z))$ , and clearly if  $H_\nu \rightarrow H$  in the  $C^0$  topology, this quantity goes to zero, hence  $\gamma(\varphi^{-1} \varphi_\nu^t)$  goes to zero, which is equivalent to  $\varphi_\nu^t \xrightarrow{c} \varphi^t$ . □

*Remark 2.8.* Given  $\varphi$  the time one map of a symplectomorphism, we may define its **Hofer norm** as follows ([Hofer]) Let  $\mathcal{H}(\varphi)$  be the set of (time-dependent) Hamiltonians on  $M$  such that the time one flow associated to  $H$  is  $\varphi$ .

Then

$$\|\varphi\| = \inf \left\{ \int_0^1 [\max_x H(t, x) - \min_x H(t, x)] dt \mid H \in \mathcal{H}(\varphi) \right\}$$

In fact, the proof of both proposition 2.6 and corollary 2.7 show more than stated: we prove that the identity map from  $(\mathcal{H}, \|\bullet\|)$  to  $(\mathcal{H}, \tilde{\gamma})$  is a contraction, or in less pedantic terms, that

$$\tilde{\gamma}(\varphi) \leq \|\varphi\|$$

### 3 Commuting Hamiltonians and the proof of theorem 1.2

Let  $f : M \rightarrow N$  be a diffeomorphism and  $X$  a vector field on  $M$ , then  $f_*X$  defined as the vector field:  $f_*X(y) = df(f^{-1}(y))X(f^{-1}(y))$  is the *push-forward* of  $X$ . Its main property is that if  $\varphi^t$  the flow of  $X$ , then

$$\frac{d}{dt}(f \circ \varphi^t)(x) = (f_*X)(f \circ \varphi^t(x))$$

since

$$\frac{d}{dt}(f \circ \varphi^t)(x) = df(\varphi^t(x))X(\varphi^t(x)) = df(f^{-1}f \circ \varphi^t(x))X(f^{-1}f \circ \varphi^t(x)) = f_*X(f \circ \varphi^t(x))$$

**Proposition 3.1.** *Let  $H, K$  be two time-independent Hamiltonians and assume  $\{H, K\}$  to be  $C^0$  small. Then denoting by  $\varphi^t, \psi^s$  the Hamiltonian flows of  $H$  and  $K$ ,  $t \mapsto \varphi^t \psi^s \varphi^{-t} \psi^{-s}$  is generated by a  $C^0$  small (time-dependent) Hamiltonian.*

*Proof.* Indeed we have, setting  $u = \varphi^t \psi^s \varphi^{-t} \psi^{-s}(x)$ :

$$\begin{aligned} \frac{d}{dt}(\varphi^t \psi^s \varphi^{-t} \psi^{-s})(x) &= \frac{d}{dt} \varphi^t(\psi^s \varphi^{-t} \psi^{-s}(x)) + d(\varphi^t \psi^s)(\varphi^{-t} \psi^{-s}(x)) \left( \frac{d}{dt} \varphi^{-t} \right)(\psi^{-s}(x)) = \\ &X_H(\varphi^t \psi^s \varphi^{-t} \psi^{-s}(x)) - d(\varphi^t \psi^s)(\varphi^{-t} \psi^{-s}(x)) X_H(\varphi^{-t} \psi^{-s}(x)) = \\ &X_H(u) - d(\varphi^t \psi^s)(\psi^{-s} \varphi^{-t}(u)) X_H(\psi^{-s} \varphi^{-t}(u)) = \\ &[X_H - (\varphi^t \psi^s)_* X_H](u), \end{aligned}$$

and since for each symplectic diffeomorphism,  $\rho$ , we have  $\rho_* X_L = X_{L \circ \rho^{-1}}$ , the vector field  $[X_H - (\varphi^t \psi^s)_* X_H]$  is Hamiltonian, with Hamiltonian function

$$L_s(t, x) = H(x) - H(\psi^{-s} \varphi^{-t}(x))$$

We may thus compute

$$\frac{\partial}{\partial s} L_s(t, x) = -\frac{d}{ds} H(\psi^{-s} \varphi^{-t}(x)) = dH(\psi^{-s} \varphi^{-t}(x)) \cdot X_K(\psi^{-s} \varphi^{-t}(x)) = \{H, K\}(\psi^{-s} \varphi^{-t}(x))$$

and since  $H(\varphi^{-t}(x)) = H(x)$ , we have  $L_0 = 0$  and we may then estimate

$$\begin{aligned} |L_s(t, x)| &= |L_s(t, x) - L_0(t, x)| \leq \int_0^s \left| \frac{\partial}{\partial \sigma} L_\sigma(t, x) \right| d\sigma \leq \\ &\int_0^s |\{H, K\}(\psi^{-\sigma} \varphi^{-t}(x))| d\sigma \leq s \|\{H, K\}\|_{C^0} \end{aligned}$$

Thus if  $\|\{H, K\}\|$  is small for the  $C^0$  topology, we have for each  $s$ , the flow  $t \mapsto \varphi^t \psi^s \varphi^{-t} \psi^{-s}$  is generated by a  $C^0$  small Hamiltonian.  $\square$

This allows us to introduce

**Definition 1.1.** Let  $H, K$  be two  $C^0$  Hamiltonians. We say that  $H$  and  $K$   **$C^0$ -commute** if and only if there exists sequences of smooth Hamiltonians  $H_n, K_n$  such that, in the  $C^0$  topology:

- (a).  $H_n$  goes to  $H$ ,  $K_n$  goes to  $K$ .
- (b).  $\{H_n, K_n\}$  goes to zero.

Fortunately this definition does not conflict with the standard one:

**Proposition 3.2.** *If  $H, K$  are  $C^1$ , and  **$C^0$ -commute**, then they commute in the usual sense (i.e.  $\{H, K\} = 0$ ).*

*Proof.* Remember that all Hamiltonians are compact supported. We shall explain in the appendix how to extend this to the general case.

We also assume temporarily that  $H, K$  are of class  $C^{1,1}$ . Let now  $H_n, K_n$  be a sequence of compact supported  $C^{1,1}$  Hamiltonians, such that

- (a).  $\lim_{n \rightarrow \infty} H_n = H, \lim_{n \rightarrow \infty} K_n = K$
- (b).  $\lim_{n \rightarrow \infty} \{H_n, K_n\} = 0$

We wish to prove that if  $H, K$  are  $C^{1,1}$  then  $\{H, K\} = 0$ . Let  $\varphi_n^t, \psi_n^s$  the flows of  $H_n, K_n$ , and set  $\rho_n(s, t, x) = \varphi_n^t \psi_n^s \varphi_n^{-t} \psi_n^{-s}(x)$ . The flow  $t \mapsto \rho_n(s, t, x)$  is the flow of the Hamiltonian  $L_s^n(t, x)$ .

Making use of the topology of  $c$ -convergence from in definition 2.4 above, and applying Corollary 2.7 we have that  $\varphi_n^t$   $c$ -converges to  $\varphi^t$ , and  $\psi_n^s$  to  $\psi^s$ . Applying the triangle inequality for  $\tilde{\gamma}$  we see that  $\varphi_n^t \psi_n^s \varphi_n^{-t} \psi_n^{-s}$   $c$ -converges to  $\varphi^t \psi^s \varphi^{-t} \psi^{-s}$ . On the other hand, according to proposition 3.1, since  $\{H_n, K_n\}$  is  $C^0$  small,  $\tilde{\gamma}(\varphi_n^t \psi_n^s \varphi_n^{-t} \psi_n^{-s})$  goes to zero and thus, using proposition 2.6, we get that  $\varphi^t \psi^s \varphi^{-t} \psi^{-s} = \text{Id}$ , that is  $H$  and  $K$  commute.

As we only needed that the Hamiltonians be  $C^{1,1}$  in order to define the flows of  $X_H, X_K$ , we now deal with the general case of  $C^1$  Hamiltonians.

Indeed, as this is a local property, we may assume that  $N = \mathbb{R}^n$  and by convolution with  $\rho^\varepsilon$ , where  $\rho^\varepsilon$  is close to the Dirac mass at the origin, we approximate simultaneously  $H, K$  by  $H^\varepsilon, K^\varepsilon$ , of class  $C^{1,1}$ , and  $H_n^\varepsilon = \rho^\varepsilon * H_n, K_n^\varepsilon = \rho^\varepsilon * K_n$ . Since we may assume  $\|H_n^\varepsilon - H_n\|_{C^1} \leq \varepsilon, \|K_n^\varepsilon - K_n\|_{C^1} \leq \varepsilon$

The sequences  $(H_n^\varepsilon)_n, (K_n^\varepsilon)_n$   $C^0$ -converge to  $H^\varepsilon, K^\varepsilon$ , but  $H^\varepsilon, K^\varepsilon$  being  $C^{1,1}$ , we may apply proposition 3.1 and we get that  $\{H^\varepsilon, K^\varepsilon\}$  is  $C^0$  small. In other words  $\{H^\varepsilon, K^\varepsilon\}$  goes to zero with  $\varepsilon$ . Now since  $H^\varepsilon \rightarrow H$  and  $K^\varepsilon \rightarrow K$  in the  $C^1$  norm, we have that  $\{H, K\} = \lim_{\varepsilon \rightarrow 0} \{H^\varepsilon, K^\varepsilon\} = 0$ . This concludes our proof.  $\square$

The same proof shows that the following definition of commutation is also compatible with the standard one

**Definition 3.3.** Let  $H, K$  be two  $C^0$  Hamiltonians on  $T^*N$ . We shall say that  $H$  and  $K$   **$c$ -commute** if and only if there exists sequences  $H_n, K_n$  such that if the flows  $H_n$  and  $H$  are denoted by  $\varphi_n^t, \varphi^t$ , those of  $K_n$  and  $K$  by  $\psi_n^t, \psi^t$ , we have

$$\begin{aligned}\varphi_n^t &\xrightarrow{c} \varphi^t \\ \psi_n^t &\xrightarrow{c} \psi^t \\ \varphi_n^t \psi_n^s \varphi_n^{-t} \psi_n^{-s} &\xrightarrow{c} Id\end{aligned}$$

Note that even though this is a very natural notion, it is not the one we will need to generalize our theorem.

### 3.1 The non-autonomous and multi-time case

#### 3.1.1 The non-autonomous case

Assume now  $H, K$  are time-dependent, that is of the form  $H(t, z)$  and  $K(t, z)$ . Commutation is then expressed by different formulas. If  $X_H, X_K$  are the Hamiltonian vector fields, we define

$$[H, K](t, z) = d_z H(t, z) X_K(t, z) + \frac{\partial H}{\partial t}(t, z) - \frac{\partial K}{\partial t}(t, z)$$

In the sequel, even for time dependent vector fields we denote by  $\{H, K\}(t, z)$  the Poisson bracket of  $H, K$  for fixed time, that is  $\{H, K\}(t, z) = d_x H(t, z) X_K(t, z)$ , so that

$$[H, K](t, z) = \{H, K\}(t, z) + \frac{\partial H}{\partial t}(t, z) - \frac{\partial K}{\partial t}(t, z)$$

The vanishing of  $[H, K]$  has the following interpretation. To  $H(t, z)$  we may associate the autonomous Hamiltonian

$$\tilde{H}(t, \tau, z) = \tau + H(t, z)$$

defined on  $(\mathbb{R}^2 \times M, d\tau \wedge dt + \omega)$ , and the associated flow is

$$\Phi^s : (t, \tau, x) \mapsto (t + s, \tau + H(t, z) - H(t + s, \varphi_t^{t+s}(z)), \varphi_t^{s+t}(z))$$

where  $\varphi_t^{t+s}$  is the flow on  $M$  of the non autonomous Hamiltonian  $H(t, x)$ .

Similarly to  $K$  we associate  $\Psi^s$ . Now it is easy to check that  $[H, K](t, z) = \{\tilde{H}, \tilde{K}\}(t, \tau, z)$ , the commutation of  $\Phi^t$  and  $\Psi^s$  implies obviously that

$$\psi_b^d \varphi_a^b(z) = \varphi_c^d \psi_a^c(z)$$

for any  $a, b, c, d$ .

With this definition, the results of the previous section can be extended to the non-autonomous case.

**Definition 3.4.** Let  $H(t, z), K(t, z)$  be two continuous Hamiltonians. We shall say that  $H$  and  $K$   $C^0$  commute if and only if there are sequences  $H_n, K_n$  such that  $[H_n, K_n]$  goes to zero in the  $C^0$  topology. If  $H, K$  are of class  $C^1$  this definition coincides with the condition  $[H, K] = 0$ .

### 3.1.2 The multi-time case

We shall limit ourselves to the case  $d = 2$ . We consider  $H(s, t, z), K(s, t, z)$  and we consider the Hamiltonian flows

$$\frac{d}{dt} \varphi_{t_0}^t(s; x) = X_H(s, t, \varphi_{t_0}^t(s; x))$$

$$\frac{d}{ds} \psi_{s_0}^s(t; x) = X_K(s, t, \psi_{s_0}^s(t; x))$$

Then, we consider the Hamiltonians on  $T^*(N \times \mathbb{R}^2)$  given by

$$\tilde{H}(s, t, \sigma, \tau, x) = \tau + H(s, t, z)$$

$$\tilde{K}(s, t, \sigma, \tau, x) = \sigma + K(s, t, z)$$

Denote their flow by  $\tilde{\Phi}^a, \tilde{\Psi}^b$  respectively. The last component of  $\tilde{\Phi}^a(s, t, \sigma, \tau)$  is given by  $\varphi_t^{t+a}(s; z)$  and the last component of  $\tilde{\Psi}^b(s, t, \sigma, \tau)$  by  $\psi_s^{s+b}(t; z)$ .

Now

$$\{\tilde{H}, \tilde{K}\} = \{H, K\}(s, t, z) + \frac{\partial H}{\partial t}(s, t, z) - \frac{\partial K}{\partial s}(s, t, z)$$

and the commutation of the flows  $\varphi_{s_0}^s(t; x)$  and  $\psi_{t_0}^t(s; x)$  is given by the vanishing of the above expression.

**Definition 3.5.** We set

$$\ll H, K \gg = \{H, K\}(s, t, z) + \frac{\partial H}{\partial t}(s, t, z) - \frac{\partial K}{\partial s}(s, t, z)$$

**Proposition 3.6.** *The “flows”  $\varphi_{s_0}^s(t; z)$  and  $\psi_{t_0}^t(s; z)$  commute, or more explicitly*

$$\varphi_{s_0}^s(t; \psi_{t_0}^t(s_0; z)) = \psi_{t_0}^t(s; \varphi_{s_0}^s(t_0; z))$$

*if and only if  $\ll H, K \gg = 0$ .*

*Remark 3.7.* By applying Proposition 3.2, we get that all our different notions of commutation (in the time-dependent or multi-time case) are well defined in the continuous case, and the definition is compatible with the usual definition in the  $C^{1,1}$  case.

## 4 Multi-time Hamilton-Jacobi equations and the proof of theorem 1.3

Consider the system of equations

$$(MHJ) \quad \begin{cases} \frac{\partial}{\partial t_1} u(t_1, \dots, t_d, x) + H_1(t_1, \dots, t_d, x, \frac{\partial}{\partial x} u(t_1, \dots, t_d, x)) = 0 \\ \vdots \\ \frac{\partial}{\partial t_d} u(t_1, \dots, t_d, x) + H_d(t_1, \dots, t_d, x, \frac{\partial}{\partial x} u(t_1, \dots, t_d, x)) = 0 \\ u(0, \dots, 0, x) = f(x) \end{cases}$$

For small  $t_1, \dots, t_d$ , and a  $C^1$  Hamiltonian, we can find a solution to the above equation provided the Hamiltonians commute in the sense of the previous section, that is the functions  $\tau_j + H_j(t_1, \dots, t_d, x)$  commute on  $T^*(\mathbb{R}^d \times N)$ , which we can rewrite as:

$$(H1) \quad \{H_j, H_k\}(t, z) + \frac{\partial H_j}{\partial t_k}(t, z) - \frac{\partial H_k}{\partial t_j}(t, z) = 0$$

### 4.1 The smooth case

Under assumption (H1) it is easy to check that the method of characteristics will yield a solution for  $t_j$  small.

We remind the reader of the geometric approach to the method of characteristics. Set

$$\Lambda_{f; H_1, \dots, H_d} = \{(0, \dots, 0, x, -H_1(0, \dots, 0, x, df(x)), \dots, -H_d(0, \dots, 0, x, df(x)), df(x)) \mid x \in N\}$$



This is an isotropic submanifold in  $T^*(N \times \mathbb{R}^d)$ , and the vector fields  $X_j$  associated to the  $\tilde{H}_j(t_1, \dots, t_d, \tau_1, \dots, \tau_d, x, p) = \tau_j + H_j(t_1, \dots, t_d, x, p)$  being in involution, are all tangents to

$$\Sigma = \bigcap_{j=1}^d \tilde{H}_j^{-1}(0)$$

and transverse to  $\Lambda_{f;H_1, \dots, H_d}$ . As a result the flows  $\tilde{\Phi}_j^t$  of  $X_j$  commute and defining

$$L_{f;H_1, \dots, H_d} = \bigcup_{t_1, \dots, t_d \in \mathbb{R}^d} \tilde{\Phi}_1^{t_1} \dots \tilde{\Phi}_j^{t_j} \dots \tilde{\Phi}_d^{t_d}(\Lambda_{f;H_1, \dots, H_d})$$

the commutation property insures that the order in which we take the  $\tilde{\Phi}_j^{t_j}$  is irrelevant, and that  $L_{f;H_1, \dots, H_d}$  is a Lagrangian submanifold contained in  $H_j^{-1}(0)$ , hence in  $\Sigma$ .

The following lemma is a straightforward extension of a result from [Viterbo-Ottolenghi]

**Lemma 4.1.** *The manifold  $L_{f;H_1, \dots, H_d}$  is Hamiltonianly isotopic to the zero section in  $T^*(N \times \mathbb{R}^d)$ .*

*Proof.* Consider the family  $\Phi_{j,\lambda}^\tau$  of flows of the Hamiltonian  $\tau_j + \lambda H(t_1, \dots, t_d, x, p)$ . We have  $\Phi_{j,1}^\tau = \Phi_j^\tau$  and  $\Phi_{j,0}^\tau$  is the translation by  $\tau$  in the  $t_j$  coordinate.

Now the family  $\Phi_{j,\lambda}^\tau$  is a continuous family of proper maps, and the family of Lagrangian submanifolds given by

$$L_{f;H_1, \dots, H_d}^\lambda = \bigcup_{t_1, \dots, t_d \in \mathbb{R}^d} \tilde{\Phi}_{1,\lambda}^{t_1} \dots \tilde{\Phi}_{j,\lambda}^{t_j} \dots \tilde{\Phi}_{d,\lambda}^{t_d}(\Lambda_{\lambda f;H_1, \dots, H_d})$$

is an exact proper symplectic isotopy of Lagrangian submanifolds.

Note that we do not need here that the Hamiltonians commute, and we shall indeed use the lemma in the more general setting.  $\square$

Now assume the  $H_j$  are of class  $C^{1,1}$ . According to section 2,  $L_{f;H_1, \dots, H_d}$  has a GFQI. We thus have a variational solution  $u_{f;H_1, \dots, H_d}$  of the following equation

$$\begin{cases} \frac{\partial}{\partial t_1} u(t_1, t_2, \dots, t_d, x) + H_1(t_1, t_2, \dots, t_d, x, \frac{\partial}{\partial x} u(t_1, t_2, \dots, t_{q-1}, t_d, x)) = 0 \\ \vdots \\ \frac{\partial}{\partial t_2} u(0, t_2, \dots, t_d, x) + H_2(0, t_2, \dots, t_d, x, \frac{\partial}{\partial x} u(0, t_2, \dots, t_{q-1}, t_d, x)) = 0 \\ \vdots \\ \frac{\partial}{\partial t_d} u(0, 0, \dots, t_d, x) + H_d(0, 0, \dots, t_d, x, \frac{\partial}{\partial x} u(0, 0, \dots, 0, t_d, x)) = 0 \end{cases}$$

The commutation of the  $\tilde{\Phi}_j^{t_j}$  implies that for any permutation  $\sigma$  of  $\{1, \dots, n\}$ , we have

$$L_{f;H_1,\dots,H_d}^\sigma = \bigcup_{(t_1,\dots,t_q)\in\mathbb{R}^q} \tilde{\Phi}_{\sigma(1)}^{t_{\sigma(1)}} \dots \tilde{\Phi}_{\sigma(q)}^{t_{\sigma(q)}}(\Gamma(f)) = L_{f;H_1,\dots,H_d}$$

Thus the associated function  $u_{f;H_1,\dots,H_d}^\sigma = u_{f;H_{\sigma(1)},\dots,H_{\sigma(d)}}$  coincides with  $u_{f;H_1,\dots,H_d}$ . As a result,  $u$  is a solution of any equation obtained by replacing  $\frac{\partial}{\partial t_j}$  by  $\frac{\partial}{\partial t_{\sigma(j)}}$  and  $H_j$  by  $H_{\sigma(j)}$ . Looking at the first equation, we see that

$$\frac{\partial}{\partial t_{\sigma(1)}} u(t_1, t_2, \dots, t_d, x) + H_{\sigma(1)}(t_1, t_2, \dots, t_d, x, \frac{\partial}{\partial x} u(t_1, t_2, \dots, t_{q-1}, t_d, x)) = 0$$

and since  $\sigma(1)$  is arbitrary, we see that  $u$  solves  $(MHJ)$ .

Our problem is thus solved for initial data and equation of class  $C^{1,1}$ , and by a standard argument this extends to class  $C^1$ .

However we are really interested in the  $C^0$  case. This is the subject of the next section.

## 4.2 The continuous case

To deal with the  $C^0$  case, we must first extend the construction of the Lagrangian submanifold

$$L_{f;H_1,\dots,H_d}$$

for general Hamiltonians and initial conditions. Note that since the Hamiltonians do not commute anymore, order matters. We shall then consider for each permutation  $\sigma$  of  $\{1, \dots, d\}$ , the manifold

$$L_{f;H_1,\dots,H_d}^\sigma$$

As in Lemma 4.1, the manifold  $L_{f;H_1,\dots,H_d}^\sigma$  is Hamiltonianly isotopic to the zero section, thus each of the Lagrangians has a unique associated function  $u^\sigma$ .

If the  $H_j$  are only  $C^0$  and we consider a sequence  $H_j^\nu$  of  $C^2$  Hamiltonians converging to  $H_j$ , we shall then associate to these a sequence of Lagrangian submanifolds and a sequence of functions  $u_\nu^\sigma$ , which is, the solution of the set of equations given, for  $\sigma = id$ , by

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t_1} u(t_1, t_2, \dots, t_d, x) + \dots + H_1^\nu(t_1, t_2, \dots, t_d, \frac{\partial}{\partial x} u(t_1, t_2, \dots, t_{q-1}, t_d, x)) = 0 \\ \vdots \\ \frac{\partial}{\partial t_2} u(0, t_2, \dots, t_d, x) + \dots + H_2^\nu(0, t_2, \dots, t_d, \frac{\partial}{\partial x} u(0, t_2, \dots, t_{q-1}, t_d, x)) = 0 \\ \vdots \\ \frac{\partial}{\partial t_d} u(0, 0, \dots, t_d, x) + \dots + H_d^\nu(0, 0, \dots, t_d, \frac{\partial}{\partial x} u(0, 0, \dots, 0, t_d, x)) = 0 \\ \\ u(0, \dots, 0, x) = f^\nu(x) \end{array} \right.$$

We this want to prove that the conditions  $\lim_{\nu \rightarrow \infty} H_j^\nu = H_j$ , and  $\lim_{\nu \rightarrow \infty} \{H_j^\nu, H_k^\nu\} = 0$ , imply

$$\lim_{\nu \rightarrow \infty} |u_\sigma^\nu - u_\tau^\nu| = 0$$

for any permutations  $\sigma, \tau$ . One should notice that this does not say much about the  $L_{f; H_1^\nu, \dots, H_d^\nu}^\sigma$  and in particular we do not claim their convergence in the usual sense.

**Proposition 4.2.** *Assume  $\lim_{\nu \rightarrow \infty} f^\nu = f$ ,  $\lim_{m \rightarrow \infty} H_j^\nu = H_j$  and  $\lim_{m \rightarrow \infty} \{H_j^\nu, H_k^\nu\} = 0$ . Then  $u_{f; H_1, \dots, H_d} = \lim_{\nu \rightarrow \infty} u_\sigma^\nu$  is independent of  $\sigma$  and is a solution of the multi-time Hamilton-Jacobi equation (MHJ). Moreover  $u_{f; H_1, \dots, H_d}$  is a variational solution of each single equation. If the  $H_j$  are convex in  $p$  then variational and viscosity solutions do coincide (cf. [Joukovskaia]) and  $u_{f; H_1, \dots, H_d}$  is the viscosity solution of each single equation.*

The proof of this proposition will follow from a more general theorem, involving  $c$ -convergence. From now on, we shall only consider the case  $d = 2$ , since the general case is in all respects similar.

### 4.3 Capacity for manifolds with boundary

In the sequel we shall need to define  $\gamma$  on  $\mathcal{L}$  in the symplectic manifold  $T^*([0, 1] \times N)$  and  $\mathcal{D}_\omega(T^*([0, 1]^2 \times N))$  or more generally for the case of the cotangent bundle of a manifold  $M$  with boundary. Indeed let  $M$  be a manifold with boundary, and  $L$  be a Lagrangian submanifold in  $T^*N$  such that  $\partial L = L \cap T_{\partial N}^*N$ , assuming moreover the intersection to be transverse, we can consider the manifold  $\tilde{L} \subset T^*(\tilde{N})$  where  $\tilde{L}, \tilde{N}$  are the doubles of  $L, N$ . We shall write  $\tilde{N} = N_- \cup N_+$ ,  $\tilde{L} = L_- \cup L_+$  and we identify  $L, N$  with  $L_-, N_-$ . We also denote by  $\tau$  be the involution of  $\tilde{N}$  sending  $N_-$  to  $N_+$ .

Then denote by  $t$  the coordinate defining the collar of  $\partial M$ . By assumption, the reduction of  $L$  by  $\{t = t_0\}$ , denoted by  $L_{t_0}$  is a Lagrangian regular homotopy. We may modify this family near  $\{t = 0\}$ , so that it is constant for  $t$  close to 0. Then near  $\{t = 0\}$ ,  $L$  is uniquely defined by the real function  $\tau(t, z)$  defined on  $[0, \varepsilon] \times L_0$  such that  $L = \{(t, \tau(t, z), z) \mid z \in L_t\}$ . Then  $\tau(t, z) = \tau(z)$  near  $t = 0$ , and we may thus extend  $\tau$  and  $L_t$  to  $[-\varepsilon, \varepsilon] \times L_0$ , so that  $\tau(-t, z) = \tau(t, z)$  and  $L_{-t} = L_t$ . This way we may extend  $L$  to a Lagrangian manifold  $\tilde{L} \subset T^*\tilde{M}$ , uniquely defined by  $L$ .

Notice that if  $L$  is Hamiltonianly isotopic to the zero section, by a Hamiltonian preserving  $T_{\partial N}^*N$ , then clearly the same holds for  $\tilde{L}$ . Therefore  $\tilde{L}$  has a GFQI, and we can compute  $\gamma(\tilde{L})$ . We define  $\gamma(L) = \gamma(\tilde{L})$ . Note also that if  $S(x, \xi)$  is a GFQI for  $\tilde{L}$ , then  $(\tau^*S)(x, \xi) = S(\tau(x), \xi)$  also generates  $\tilde{L}$ . In fact any  $S$  generating  $L$  over  $T^*N$  has such an extension.

## 4.4 $S$ -commuting Hamiltonians

Remember that we denote by  $\mathcal{H}(M)$  be the set of smooth time dependent Hamiltonians on  $\mathbb{R} \times M$ , and by  $\mathcal{H}\mathcal{D}(M)$  the set of time one maps of such Hamiltonians.

We define  $\mathfrak{H}_\gamma(M)$  the completion of  $\mathcal{H}(M)$  for the metric  $\gamma$ , and  $\mathfrak{H}\mathcal{D}_\gamma(M)$  the completion of  $\mathcal{H}\mathcal{D}(M)$  for the metric  $\gamma$ .

Then the “time one flow” map  $\mathcal{H}(M) \rightarrow \mathcal{H}\mathcal{D}(M)$  extends to a continuous map  $\mathfrak{H}_\gamma(M) \rightarrow \mathfrak{H}\mathcal{D}_\gamma(M)$ .

Not only do we have an embedding of  $C^0(\mathbb{R} \times M)$  into  $\mathfrak{H}_\gamma(M)$ , but according to a theorem of V. Humilière ([Humilière]), Hamiltonians with some singularities also live in  $\mathfrak{H}_\gamma(M)$ .

Let  $\varphi^s, \psi^t$  be the flows of the autonomous Hamiltonians  $H$  and  $K$ . Then we introduce the following commutation condition. Consider  $\rho^{(s,t)} = \varphi^s \psi^t \varphi^{-s} \psi^{-t}$  for  $(s, t) \in [0, 1]^2$ , and the map

$$C(H, K) : (s, t, \sigma, \tau, z) \rightarrow (s, t, \tilde{\sigma}(t_1, t_2, z), \tilde{\tau}(t_1, t_2, z), \rho^{(s,t)}(z))$$

from  $T^*([0, 1]^2 \times N)$  into itself. We may choose  $\tilde{\sigma}, \tilde{\tau}$  such that  $C(H, K)$  is a Hamiltonian diffeomorphism, using the fact that  $\lambda \rightarrow \rho^{(s,\lambda t)}$  is an Hamiltonian isotopy starting from the identity, and these functions are well defined up to the addition of  $\frac{\partial c}{\partial s}(s, t)$  and  $\frac{\partial c}{\partial t}(s, t)$  respectively, where  $c$  is some  $C^1$  function.

If we are given Hamiltonians  $H(z), K(z)$  generating  $\varphi^s, \psi^t$ , we can assume  $s \rightarrow \rho^{(s,t)}$  is generated by

$$L_t(s, z) = H(z) - H(\psi^{-t} \varphi^{-s}(z))$$

We may also compute

$$\frac{\partial}{\partial t} L_t(s, z) = \{H, K\}(\psi^{-t} \varphi^{-s}(z))$$

and note that

$$\tilde{\tau}_1(t_1, t_2, z) = \tau_1 - L_{t_2}(t_1, \varphi_1^{t_1} \varphi_2^{t_2} \varphi_1^{-t_1} \varphi_2^{-t_2}(z)) = H_1(\varphi_1^{t_1} \varphi_2^{t_2} \varphi_1^{-t_1} \varphi_2^{-t_2}(z)) - H_1(\varphi_1^{-t_1} \varphi_2^{-t_2}(z))$$

$$\tilde{\tau}_2(t_1, t_2, z) = \tau_2 - t_1 \int_0^1 \{H_1, H_2\}(\varphi_1^{-\sigma t_1} \varphi_2^{-t_2}(z)) d\sigma$$

Note that if either  $M$  is compact and we impose the condition  $\int_M H_j(z) \omega^n$  or  $H_j$  are compact supported, then  $L_{t_2}(t_1, z)$  will satisfy the same condition, and this defines uniquely  $C(H, K)$  as the time one map of a Hamiltonian satisfying this normalization condition.

**Definition 4.3.** (a). We shall say that  $\varphi^s, \psi^t$   $S$ -commute up to  $\varepsilon$ , if and only if  $\gamma(C(H, K)) \leq \varepsilon$ .

- (b). Let  $H, K$  in  $\mathfrak{H}\mathcal{D}_\gamma(T^*N)$ , we say that they  $S$ -commute, if and only if there are sequences  $H_j, K_j$  such that  $H_j, K_j$  converge to  $H, K$  and  $\gamma(C(H_j, K_j))$  goes to zero.

We now have

**Lemma 4.4.** (a). If  $\varphi_1^t, \varphi_2^t$   $S$ -commute up to  $\varepsilon$  then  $\gamma(\rho^{(t_1, t_2)}) \leq \varepsilon$  for all pairs  $(t_1, t_2) \in [0, 1]^2$ . In particular  $S$ -commuting Hamiltonians  $c$ -commute.

(b). If the Hofer norm of  $\rho^{(t_1, t_2)}$ , is less than  $\varepsilon$ , then  $\varphi_1^t, \varphi_2^t$   $c$ -commute up to  $\varepsilon$ . In particular  $C^0$ -commuting Hamiltonians  $S$ -commute.

*Proof.* The first statement follows easily by symplectic reduction and the reduction inequality (see [Viterbo 1], page 705, prop. 5.1). The second statement follows from the proof of the inequality

$$d(U) \leq 2 \sup_x d(U_x)$$

in [Viterbo 4], where  $d$  is the displacement energy. □

## 4.5 Different suspensions of Hamiltonian isotopies

Let  $H(t, z)$  be a time-dependent Hamiltonian on  $T^*N$ . We may associate to  $H$  the Hamiltonians on  $T^*(\mathbb{R} \times N)$  given by  $\tilde{H}(t, \tau, z) = \tau + H(t, z)$  and  $\hat{H}(s; t, \tau, z) = tH(st, z)$ , whith respective flows

$$\Phi^s : (t, \tau, z) \rightarrow (t + s, \tau - H(t, \varphi_t^{t+s}(z)), \varphi_t^{t+s}(z))$$

$$\hat{\Phi}^s : (t, \tau, z) \rightarrow (t, \tau + H(0, z) - H(st, \varphi_0^{st}(z)) - \int_0^s H(\sigma t, \varphi_H^{\sigma t}(z)) d\sigma, \varphi_0^{st}(z))$$

Note that if  $H$  is  $C^0$  small, we have, denoting by  $T_s$  the translation on the  $t$  variable,  $\gamma(\Phi^s T_{-s})$  and  $\gamma(\hat{\Phi}^s)$  are small, in other words,  $\Phi^s$  is  $c$ -close to  $T_s$  and  $\hat{\Phi}^s$  is  $c$ -close to  $Id$ .

## 4.6 Proof of the main theorem

The construction of variational solutions for a single equation can be rephrased as follows. Let  $\Lambda_0$  be a Lagrangian submanifold of  $T^*N$ , and  $H(t, z)$  a time dependent Hamiltonian. We consider  $\tilde{\Lambda}_{0,H} = \{(0, -H(0, z), z) \mid z \in \Lambda_0\}$ , and  $L = \bigcup_{t \in \mathbb{R}} \Phi^t(\Lambda_{0,H})$ .

We can obtain  $L$  as follows. Consider the map

$$\begin{aligned} F : T^*(\mathbb{R} \times \mathbb{R} \times N) &\rightarrow T^*(\mathbb{R} \times \mathbb{R} \times N) \\ (s, \sigma, \zeta) &\mapsto (s, \tilde{\sigma}, \Phi^s(\zeta)) \end{aligned}$$

and  $L_{0,H}$  any Lagrangian submanifold in  $T^*(\mathbb{R} \times N)$  such that  $\tilde{\Lambda}_{0,H} \subset L_{0,H}$ . Let  $t : T^*(\mathbb{R} \times N) \rightarrow \mathbb{R}$  be the projection on the first coordinate of  $\mathbb{R} \times N$ .

**Proposition 4.5.**  *$L$  is the reduction of  $F(L_{0,H})$  by the coisotropic submanifold  $\{s = t\}$ .*

*Proof.* Indeed, we look for the set of  $(s, \tilde{\sigma}, \Phi^s(0, -H(0, z), z))$  such that  $t = s$ . But  $\Phi^s(t, \tau, z) = (t + s, \tau + H(t, z) - H(t + s, \varphi_t^{t+s}(z)), \varphi_t^{t+s}(z))$ . Intersecting with  $t = s$  means that we consider the points  $\Phi^s(0, \tau, z)$  where  $(0, \tau, z) \in L_{0,H}$  and by assumption this means that  $(0, \tau, z) \in \tilde{\Lambda}_{0,H}$ .  $\square$

Now we will first estimate  $\gamma(L_{0,H})$  and then deduce the inequality

$$\gamma(L) \leq \tilde{\gamma}(F) + \gamma(L_{0,H})$$

**Lemma 4.6.** *Any isotopy  $\Xi^s$  of  $\{t = 0\}$  in  $T^*(\mathbb{R} \times N)$  can be generated by an arbitrarily small Hamiltonian preserving the hypersurfaces  $\{t = t_0\}$ .*

*Proof.* Indeed if we have an isotopy of the type  $(0, \tau, x, p) \mapsto (0, \tau + \tau^s(x, p), x, p)$ , it is induced by a Hamiltonian  $H(t, x, p)$  and only depends on  $\frac{\partial H}{\partial t}(0, x, p) = \frac{\partial}{\partial s} \tau^s(x, p)$ . The extension of  $H$  to  $T^*(\mathbb{R} \times N)$  can be arbitrarily small, for example we may take  $H(s; t, \tau, x, p) = \chi(t) \frac{\partial}{\partial s} \tau^s(x, p)$ , where  $\dot{\chi}(0) = 1$  and  $\chi$  has arbitrarily small  $C^0$  norm.  $\square$

**Proposition 4.7.** *For any positive  $\varepsilon$ , we may choose  $L_{0,H}$  so that we have*

$$\gamma(L_{0,H}) \leq \gamma(\Lambda_0) + \varepsilon$$

*Remark 4.8.* Note that necessarily we have  $\gamma(L_{0,H}) \geq \gamma(\Lambda_0)$ . This follows from the reduction inequality.

*Proof.* We notice that  $L_{0,0} = \mathbb{R} \times \{0\} \times \Lambda_0$  satisfies obviously  $\gamma(L_{0,0}) = \gamma(\Lambda_0)$ . Using the notation  $L_{0,0} \cap \{t = 0\} = \tilde{\Lambda}_{0,0}$ , the previous lemma allows us to find a Hamiltonian isotopy,  $\Xi^s$  on  $T^*(\mathbb{R} \times N)$  such that

- $\Xi^s$  preserves  $t = 0$
- on  $t = 0$ , we have  $\Xi^1(\tilde{\Lambda}_{0,0}) = \tilde{\Lambda}_{0,H}$
- $\Xi^s$  is generated by an arbitrarily  $C^0$  small Hamiltonian.

Therefore

$$\gamma(L_{0,H}) = \gamma(\Xi^1(\Lambda_{0,0})) \leq \tilde{\gamma}(\Xi^1) + \gamma(\Lambda_{0,0}) \leq \varepsilon + \gamma(\Lambda_{0,0})$$

$\square$

**Corollary 4.9.** *We have  $\gamma(L) \leq \tilde{\gamma}(F) + \gamma(\Lambda_0)$ . Moreover if  $L_1$  is the Lagrangian submanifold associated to  $H_1, \Lambda_1$  and  $L_2$  to  $H_2, \Lambda_2$ , we have*

$$\gamma(L_1 - L_2) \leq \tilde{\gamma}(F_1 \circ F_2^{-1}) + \gamma(\Lambda_1 - \Lambda_2)$$

*Proof.* The first inequality follows from the second by setting

$$L_1 = L, F_1 = F, \Lambda_1 = \Lambda, L_2 = 0_{\mathbb{R} \times N}, F_2 = \text{Id}, \Lambda_2 = O_N$$

Since  $L_1$  and  $L_2$  are the reductions of  $F_1(L_{1,H_1})$  and  $F_2(L_{1,H_1})$ , using the reduction inequality it will be enough to prove that

$$\gamma(F_1(L_{1,H_1}) - F_2(L_{2,H_2})) \leq \tilde{\gamma}(F_1 \circ F_2^{-1}) + \gamma(\Lambda_1 - \Lambda_2)$$

Since the map  $F_2 \circ F_1^{-1}$  sends  $F_1(L_{1,H_1})$  to  $F_2(L_{1,H_1})$ , we have by the triangle inequality and the symplectic invariance of  $\gamma$ ,

$$\begin{aligned} \gamma(F_1(L_{1,H_1}) - F_2(L_{2,H_2})) &\leq \gamma(F_1(L_{1,H_1}) - F_2(L_{1,H_1})) + \gamma(F_2(L_{1,H_1}) - F_2(L_{2,H_2})) \leq \\ &\tilde{\gamma}(F_2 \circ F_1^{-1}) + \gamma(L_{1,H_1} - L_{2,H_2}) \end{aligned}$$

so that using invariance by reduction, we have

$$\gamma(L_1 - L_2) \leq \tilde{\gamma}(F_2 \circ F_1^{-1}) + \gamma(L_{1,H_1} - L_{2,H_2})$$

Finally we claim that for any positive  $\varepsilon$  we may find  $L_{1,H_1}, L_{2,H_2}$  such that  $\gamma(L_{1,H_1} - L_{2,H_2}) \leq \gamma(\Lambda_1 - \Lambda_2) + \varepsilon$ . Indeed remember that  $L_{1,H_1}, L_{2,H_2}$  are not uniquely defined, but just required to be Lagrangian submanifolds contained in  $\tau + H_1(t, z) = 0$  and  $\tau + H_2(t, z) = 0$  respectively, with reductions  $L_1, L_2$ . Since  $\gamma(L_{1,0} - L_{2,0}) = \gamma(\Lambda_1 - \Lambda_2)$  and we saw in Lemma 4.6 that we may go from  $L_{1,0}$  to  $L_{1,H_1}$  by applying a Hamiltonian with  $C^0$  norm less than  $\varepsilon$ , and similarly from  $L_{2,0}$  to  $L_{2,H_1}$  we get that  $\gamma(L_{1,H_1} - L_{1,0})$  and  $\gamma(L_{2,H_2} - L_{2,0})$  are arbitrarily small. We may thus conclude

$$\gamma(L_{1,H_1} - L_{2,H_2}) \leq \gamma(L_{1,H_1} - L_{1,0}) + \gamma(L_{1,0} - L_{2,0}) + \gamma(L_{2,0} - L_{2,H_2}) \leq 2\varepsilon + \gamma(\Lambda_1 - \Lambda_2)$$

Since  $\varepsilon$  is arbitrarily small, we get the announced inequality.  $\square$

Our goal here is to prove the following theorem

**Theorem 4.10.** *Assume the Hamiltonians  $H_1(t_1, \dots, t_d, x, p), \dots, H_d(t_1, \dots, t_d, x, p)$  are Lipschitz, and  $S$ -commute. Then equation (MHJ) has a unique variational solution. If the  $H_j$  are convex in  $p$ , then  $u$  is a viscosity solution of each individual equation.*

We limit ourselves to the case  $d = 2$ .

We consider the flows  $\Phi_1^s, \Phi_2^s$  associated to  $\tau_1 + H_1(t_1, t_2, z), \tau_2 + H_2(t_1, t_2, z)$ , and associated isotopies depending on 2 parameters. More precisely  $\varphi_1^{(t_1, t_1+s)}(t_2; z)$  is the flow of  $(t_1, z) \rightarrow H(t_1, t_2, z)$  between times  $t_1$  and  $t_1 + s$  ( $t_2$  being considered as fixed). Similarly  $\varphi_2^{(t_2, t_2+s)}(t_1; z)$  is the flow associated to the Hamiltonian  $(t_2, z) \rightarrow H_2(t_1, t_2, z)$  between times  $t_2$  and  $t_2 + s$ .

Then  $\Phi_1^s, \Phi_2^s$  are symplectomorphisms from  $T^*(\mathbb{R}^2 \times N)$  given by

$$\Phi_1^s(t_1, t_2, \tau_1, \tau_2, z) = (t_1 + s, t_2, \tau_1 + \hat{\tau}_1^s(t_1, t_2, z), \tau_2 + \hat{\tau}_2^s(t_1, t_2, z), \varphi_1^{(t_1, t_1+s)}(t_2; z))$$

$$\Phi_2^s(t_1, t_2, \tau_1, \tau_2, z) = (t_1 + s, t_2, \tau_1 + \check{\tau}_1^s(t_1, t_2, z), \tau_2 + \check{\tau}_2^s(t_1, t_2, z), \varphi_2^{(t_2, t_2+s)}(t_1; z))$$

Note that since  $L_1 = \bigcup_{t_1 \in \mathbb{R}} \Phi_1^{t_1}(\tilde{\Lambda}_0)$  is not contained in the hypersurface  $\tau_2 + H_2(t_1, t_2, x, p) = 0$ , in order to get a Lagrangian submanifold after applying the flow  $\Phi_2^s$  we must move  $L_1$  in the  $\frac{\partial}{\partial \tau_2}$  direction, so that its image is included in this hypersurface. We have

$$\tilde{L}_1 = \{(t_1, 0, -H_1(t_1, 0, \varphi^{(0, t_1)}(z)), -H_2(0, 0, z) + \tau_2^{t_1}(z), \varphi_1^{(0, t_1)}(0; z)) \mid z = (x, df(x)) \in T^*N\}$$

and indeed we may check that on  $\tilde{L}_1$ ,  $\tau_2 + H_2(t_1, t_2, z)$  vanishes (note that  $\tilde{\Lambda}_1$  is contained in  $\{t_2 = 0\}$ ), where

$$\tau_2^s(z) = - \int_0^s \frac{\partial H_1}{\partial t_2}(\sigma, 0, \varphi_1^{(0, \sigma)}(0; z)) d\sigma$$

We thus define a “normalization map”, defined over  $\{t_2 = 0\}$  :

$$V_{1,2} : (t_1, 0, \tau_1, \tau_2, z) \mapsto (t_1, 0, \tau_1, -H_2(t_1, 0, z), z)$$

and similarly

$$V_{2,1} : (0, t_2, \tau_1, \tau_2, z) \mapsto (0, t_2, -H_1(t_1, 0, z), \tau_2, z)$$

These maps are not diffeomorphisms, but their restriction to  $\tilde{L}_1$  (resp.  $\tilde{L}_2$ ) is one, and since it preserves the Liouville form  $(\tau_2 dt_2 + \tau_1 dt_1 + \lambda)$ , its restriction to  $\tilde{L}_1$  (resp.  $\tilde{L}_2$ ) extends to a Hamiltonian isotopy  $\tilde{V}_{1,2}$  (resp.  $\tilde{V}_{2,1}$ ) which according to lemma 4.6 are generated by arbitrarily  $C^0$ -small Hamiltonians <sup>4</sup>. We will use the maps

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<sup>4</sup>Note that  $\tilde{V}_{1,2}$  and  $\tilde{V}_{2,1}$  depend on  $\tilde{L}_1$  and  $\tilde{L}_2$  respectively.



$$W_{1,2} : (s_1, s_2, \sigma_1, \sigma_2, \zeta) \mapsto (s_1, s_2, \hat{\sigma}_1, \hat{\sigma}_2, \Phi_2^{s_2} \tilde{V}_{1,2} \Phi^{-s_2})$$

and

$$W_{2,1} : (s_1, s_2, \sigma_1, \sigma_2, \zeta) \mapsto (s_1, s_2, \check{\sigma}_1, \check{\sigma}_2, \Phi_1^{s_1} \tilde{V}_{2,1} \Phi_1^{-s_1})$$

also generated by  $C^0$ -small Hamiltonians.

Now we claim that the following submanifolds are lagrangian

$$L_{1,2} = \bigcup_{(s_1, s_2) \in \mathbb{R}^2} \Phi_1^{s_1} \tilde{V}_{2,1} \Phi_2^{s_2} (\Lambda_0)$$

$$L_{2,1} = \bigcup_{(s_1, s_2) \in \mathbb{R}^2} \Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} (\Lambda_0)$$

It will later be useful to notice that

$$\Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} \Phi_2^{-s_2} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1} = \left( \Phi_2^{s_2} \tilde{V}_{1,2} \Phi_2^{-s_2} \right) \circ \left( \Phi_2^{s_2} \Phi_1^{s_1} \Phi_2^{-s_2} \Phi_1^{-s_1} \right) \circ \left( \Phi_1^{s_1} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1} \right)$$

We then consider the map  $C$  as above:

$$\begin{aligned} T^*(\mathbb{R}^2 \times \mathbb{R}^2 \times N) &\longrightarrow T^*(\mathbb{R}^2 \times \mathbb{R}^2 \times N) \\ (s_1, s_2, \sigma_1, \sigma_2, \zeta) &\longrightarrow (s_1, s_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \Phi_1^{s_1} \Phi_2^{s_2} \Phi_1^{-s_1} \Phi_2^{-s_2} (\zeta)) \end{aligned}$$

where  $\zeta = (t_1, t_2, \tau_1, \tau_2, z)$ , and the map  $\tilde{C} = W_{1,2} \circ C \circ W_{2,1}^{-1}$ .

We do not claim that  $\tilde{C}$  sends  $L_{1,2}$  to  $L_{2,1}$ , since these manifolds are in  $T^*(\mathbb{R}^2 \times N)$  and not  $T^*(\mathbb{R}^2 \times \mathbb{R}^2 \times N)$ .

However, let  $\Delta$  be the coistropic submanifold of  $T^*(\mathbb{R}^2 \times \mathbb{R}^2 \times N)$  given by

$$\Delta = \{s_1 = t_1, s_2 = t_2\}$$

**Proposition 4.11.**

$$[\tilde{C}(0_{\mathbb{R}^2} \times L_{2,1})]_{\Delta} = L_{1,2}$$

*Proof.*

$$(0_{\mathbb{R}^2} \times L_{1,2}) = \{(s_1, s_2, 0, 0, \Phi_1^{t_1} \tilde{V}_{2,1} \Phi_2^{t_2} (\zeta)) \mid \zeta \in \tilde{\Lambda}_0\}$$

then

$$\begin{aligned} \tilde{C}(0_{\mathbb{R}^2} \times L_{2,1}) &= \{(s_1, s_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} \Phi_2^{-s_2} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1} (\zeta)) \mid \zeta \in L_{2,1}\} = \\ &\{(s_1, s_2, \tilde{\sigma}_1, \tilde{\sigma}_2, (\Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} \Phi_2^{-s_2} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1}) \circ (\Phi_1^{t_1} \tilde{V}_{2,1} \Phi_2^{t_2}) (\zeta)) \mid \zeta \in \tilde{\Lambda}_0, (t_1, t_2) \in \mathbb{R}^2\} \end{aligned}$$

Now if  $t_1(\zeta) = t_2(\zeta) = 0$  we have

$$t_1((\Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} \Phi_2^{-s_2} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1}) \circ (\Phi_1^{t_1} \tilde{V}_{2,1} \Phi_2^{t_2})(\zeta)) = t_1$$

$$t_2((\Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} \Phi_2^{-s_2} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1}) \circ (\Phi_1^{t_1} \tilde{V}_{2,1} \Phi_2^{t_2})(\zeta)) = t_2$$

Thus intersection of this with  $\{s_1 = t_1, s_2 = t_2\}$  will be given by

$$\begin{aligned} \{(s_1, s_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1} \Phi_2^{-s_2} \tilde{V}_{2,1}^{-1} \Phi_1^{-s_1} \circ \Phi_1^{s_1} \tilde{V}_{2,1} \Phi_2^{s_2}(\zeta) \mid \zeta \in \tilde{\Lambda}_0\} = \\ \{(s_1, s_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \Phi_2^{s_2} \tilde{V}_{1,2} \Phi_1^{s_1}(\zeta) \mid \zeta \in \tilde{\Lambda}_0\} \end{aligned}$$

and we must figure out its projection according to the equivalence relation given by

$$(s_1, s_2, \sigma_1, \sigma_2, t_1, t_2, \tau_1, \tau_2, z) \sim (s'_1, s'_2, \sigma'_1, \sigma'_2, t'_1, t'_2, \tau'_1, \tau'_2, z')$$

if and only if

$$s'_1 = s_1 = t_1 = t'_1, s'_2 = s_2 = t_2 = t'_2, \sigma_1 + \tau_1 = \sigma'_1 + \tau'_1, \sigma_2 + \tau_2 = \sigma'_2 + \tau'_2$$

Since the map

$$(s_1, s_2, \sigma_1, \sigma_2, t_1, t_2, \tau_1, \tau_2, z) \longrightarrow (s_1 - t_1, s_2 - t_2, \sigma_1, \sigma_2, t_1, t_2, \sigma_1 + \tau_1, \sigma_2 + \tau_2, z)$$

is a symplectomorphism, the reduction of  $X = \tilde{C}(0_{\mathbb{R}^2} \times L_{2,1})$  is given by

$$X_\Delta = \{(s_1, s_2, \sigma_1 + \tau_1, \sigma_2 + \tau_2, z) \mid (s_1, s_2, \sigma_1, \sigma_2, s_1, s_2, \tau_1, \tau_2, z) \in X\}$$

and this yields  $L_{1,2}$  as announced. □

**Proposition 4.12.** *Let  $\varphi_1^t, \varphi_2^t$   $S$ -commute up to  $\varepsilon$ , then we have*

$$\gamma(L_{1,2} - L_{2,1}) \leq \varepsilon$$

*Proof.* Indeed it is enough to prove that  $\tilde{\gamma}(\tilde{C})$  is small by assumption ( $\Phi^t, \Psi^s$   $S$ -commute up to  $\varepsilon$ ). Since  $\gamma(C)$  is small, and  $W_{1,2}, W_{2,1}^{-1}$  are generated by  $C^0$  small Hamiltonians,  $\gamma(W_{1,2}), \gamma(W_{2,1})$  are also small, and using the triangle inequality, we get that  $\tilde{\gamma}(\tilde{C})$  is small.

Now if  $\tilde{\gamma}(\tilde{C}) \leq \varepsilon$

$$\gamma(\tilde{C}(0_{\mathbb{R}^2} \times L_{2,1}) - 0_{\mathbb{R}^2} \times L_{1,2}) \leq \varepsilon$$

Using the reduction inequality we obtain

$$\gamma(\tilde{C}(0_{\mathbb{R}^2} \times L_{2,1}) - 0_{\mathbb{R}^2} \times L_{1,2})_\Delta \leq \varepsilon$$

and according to the above proposition 4.11, we get

$$\gamma(L_{2,1} - L_{1,2}) \leq \varepsilon$$

□

*Proof of theorem 1.3.* Indeed we just proved that if  $\varphi_1^t, \varphi_2^t$   $S$ -commute up to  $\varepsilon$  on  $[0, T]$ , we have  $\gamma(L_{1,2} - L_{2,1}) \leq \varepsilon$  and thus, by the reduction inequality we have

$$|u_{1,2}(t, x) - u_{2,1}(t, x)| \leq \varepsilon$$

on  $[0, T] \times N$ .

Now assume we have sequences  $H_1^\nu, H_2^\nu$  such that  $H_1^\nu$   $c$ -converges to  $H_1$ ,  $H_2^\nu$   $c$ -converges  $H_2$ , and  $H_1^\nu, H_2^\nu$   $S$ -commute up to  $\epsilon_\nu$  where  $\lim_{\nu \rightarrow 0} \epsilon_\nu = 0$ . Then we have

$$|u_{1,2}^\nu(t, x) - u_{2,1}^\nu(t, x)| \leq \epsilon_\nu$$

on  $[0, T] \times N$ .

Now we know that  $u_{1,2}^\nu$  converges to a variational solution of

$$\begin{cases} \frac{\partial}{\partial t_1} \hat{v}(t_1, t_2, x) + H_1(t_1, t_2, x, \frac{\partial}{\partial x} \hat{v}(t_1, t_2, x)) = 0 \\ \hat{v}(0, t_2, x) = \hat{w}(t_2, x) \end{cases}$$

where  $\hat{w}$  is the variational solution of the equation

$$\begin{cases} \frac{\partial}{\partial t_2} \hat{w}(t_2, x) + H_2(0, t_2, x, \frac{\partial}{\partial x} \hat{w}(t_2, x)) = 0 \\ \hat{w}(0, x) = f(x) \end{cases}$$

on the other hand,  $u_{2,1}^\nu$  converges to a variational solution of

$$\begin{cases} \frac{\partial}{\partial t_2} \check{v}(t_1, t_2, x) + H_2(t_1, t_2, x, \frac{\partial}{\partial x} \check{v}(t_1, t_2, x)) = 0 \\ \check{v}(t_1, 0, x) = \check{w}(t_1, x) \end{cases}$$

where  $\check{w}$  is the variational solution of the equation

$$\begin{cases} \frac{\partial}{\partial t_1} \check{w}(t_1, x) + H_1(t_1, 0, x, \frac{\partial}{\partial x} \check{w}(t_1, x)) = 0 \\ \check{w}(0, x) = f(x) \end{cases}$$

Since we proved that  $u_{1,2}^\nu(t, x) - u_{2,1}^\nu(t, x)$  goes to zero, the sequences have a common limit,  $v$ , and this will be a variational solution of both

$$\frac{\partial}{\partial t_2} v(t_1, t_2, x) + H_2(t_1, t_2, x, \frac{\partial}{\partial x} v(t_1, t_2, x)) = 0$$

and

$$\frac{\partial}{\partial t_1} v(t_1, t_2, x) + H_1(t_1, t_2, x, \frac{\partial}{\partial x} v(t_1, t_2, x)) = 0$$

and such that  $v(0, 0, x) = f(x)$ .

In other words,  $v$  is a variational solution of (MHJ).  $\square$

*Remark 4.13.* Note that the condition on the function need not be given by “functions”. It is enough to have a “ $C^0$ - coisotropic submanifold”  $C \subset T^*(\mathbb{R}^d \times N)$ , that is the Hausdorff limit of submanifolds  $C_\nu$  such that  $\sigma_{TC_\nu}$  goes to zero for the  $C^0$  topology, and  $T^*(\mathbb{R}^d \times N) \rightarrow \mathbb{R}^d$  given by  $(t_1, \dots, t_d, \tau_1, \dots, \tau_d, x, p) \rightarrow (t_1, \dots, t_d)$  when restricted to the leaves of the foliation is uniformly proper. We let the reader adapt such a generalisation into our framework. We also mention that the notion of  $C^0$ -Lagrangian submanifold has already been used by several authors, related to the fact that if a  $C^0$ -limit of Lagrangian submanifolds is a smooth manifold, it has to be Lagrangian.

## A Appendix : Symplectic invariants and the main theorem in the non-compact case

The aim of this appendix is to extend our construction and results to the case where  $N$  is a non-compact manifold. Of course, if all our Hamiltonians are compact supported, the statements and proofs are straightforward adaptations of the compact case. We refer to [Barles] and [F-M] for the study, from a different point of view, of Hamilton-Jacobi equations in the non-compact setting.

Let us first consider the problem of solving a single equation in our variational framework. For this we start from the isotropic manifold  $\Lambda_f$ , and consider  $L = \bigcup_{t \in \mathbb{R}} \Phi^t(\Lambda_{f,H})$ . For this to be defined, we need that the Hamiltonian flow is complete, or at least that its restriction to  $\Lambda_{0,H}$  is complete <sup>5</sup>

For a general Hamiltonian, we cannot hope to define variational solutions unless the associated flow is complete, but this condition will not suffice, as the following example shows.

Consider the equation

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<sup>5</sup>This means that  $\Phi^s(z)$  is defined for all  $z$  in  $\Lambda_{f,H}$ .

$$\begin{cases} \frac{\partial u}{\partial t} - x^2 - \frac{1}{4} \left| \frac{\partial u}{\partial x} \right|^2 = 0 \\ u(0, x) = 0 . \end{cases}$$

The variational solution will be  $u(t, x) = \tan(t)x^2$  which goes to infinity as  $t$  goes to  $\frac{\pi}{2}$ .

This happens because the Lagrangian submanifold associated to the equation is given by

$$L = \{(t, \tau, x, p) \mid \exists a \in \mathbb{R}, x = a \cos t, p = 2a \sin t, \tau = a^2\}$$

and this being vertical for  $t = \frac{\pi}{2}$ , cannot contain the graph of the differential of some function.

We see that the trouble comes from the fact that the flow at time  $t = \frac{\pi}{2}$  brings over  $\{x = 0\}$  points which are arbitrarily far. To avoid this phenomenon we introduce the notion of “finite propagation speed Hamiltonian”. We shall see that such a flow can be suitably approximated by compact supported flows, and such approximation can be used to define the variational solutions of the associated Hamilton-Jacobi equation.

We denote by  $\pi : T^*N \rightarrow N$  the projection, by  $\mathcal{W}$  some exhaustion  $(W_\mu)_{\mu \in \mathbb{N}}$  of  $T^*N$  by neighbourhoods of the zero section, and by  $(\Omega_\mu)_{\mu \in \mathbb{N}}$  some exhaustion<sup>6</sup> of  $N$ .

We shall assume that for all  $\mu$ , and all compact subsets  $K$  in  $N$ ,  $W_\mu \cap \pi^{-1}(K)$ , is compact.

Our first task will be to determine when a sequence  $(\varphi_\mu^t)$  of **compact supported** Hamiltonian isotopies **approximates** a given Hamiltonian isotopy,  $\psi^t$ , from the variational solution point of view. This will be based on

**Proposition A.1.** *Let  $\sigma^t$  be a compact supported Hamiltonian isotopy,  $L = \sigma^1(0_N)$ ,  $\rho^t$  another compact supported Hamiltonian isotopy generated by  $H(t, x, p)$ , and set  $\rho = \rho^1$ . Assume we have  $|H(t, x, p)| \leq \varepsilon$  on*

$$\mathcal{U}((\rho^t)_{t \in [0,1]}, L, \Omega) = \{(t, \rho^t(z)) \mid t \in [0, 1], z \in L, \rho(z) \in \pi^{-1}(\Omega)\}$$

for some open set  $\Omega$  in  $N$ .

Then for all  $x$  in  $\Omega$  we have

$$|u_{\rho(L)}(x) - u_L(x)| \leq \varepsilon$$

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<sup>6</sup>By exhaustion of a set  $X$ , we mean a non-decreasing sequence  $(Y_\mu)_{\mu \in \mathbb{N}}$  of subsets of  $X$  such that  $\bigcup_{\mu \in \mathbb{N}} Y_\mu = X$ .

In particular for  $\varepsilon = 0$ , this means that if  $H$  vanishes on

$$\bigcup_{t \in [0,1]} \{t\} \times \{\rho^t(L \cap \rho^{-1}(\pi^{-1}(\Omega)))\}$$

we have

$$u_{\rho(L)}(x) = u_L(x) \text{ for } x \text{ in } \Omega.$$

*Proof.* We first assume  $|H| \leq \varepsilon$  everywhere. Then according to Proposition 2.6 we have  $\gamma(\rho(L) - L) \leq \varepsilon$  and the inequality  $|c(1_x, L') - c(1_x, L)| \leq \gamma(L' - L)$  implies

$$|u_{\rho(L)}(x) - u_L(x)| \leq \gamma(\rho(L) - L) \leq \varepsilon.$$

Now to deal with the general case, assume  $(H_\tau)_{\tau \in [0,1]}$  is a smooth one-parameter family of compact supported Hamiltonians with  $H_0 = H$ , and for all  $\tau$  in  $[0, 1]$

$$H_\tau = H_0 \text{ on } \mathcal{U}((\rho^t), L, \Omega).$$

We claim that  $c(1_x, \rho_\tau(L)) = c(1_x, L)$ .

Now notice that  $c(1_x, \rho_\tau(L))$  belongs to the set  $C_\tau(x, L)$  of critical values of

$$(x, p, \xi) \longrightarrow \int_0^1 [p(s)\dot{x}(s) - H_\tau(s, x(s), p(s))] ds + S(x(0), \xi)$$

where  $S$  is a generating function quadratic at infinity for  $L$ .

The set  $C_\tau(x, L)$  is equal to the set of values of

$$\int_0^1 [p_\tau(s)\dot{x}_\tau(s) - H_\tau(s, x_\tau(s), p_\tau(s))] ds + F(x_\tau(0), p(0))$$

where

$$(a) \quad (x_\tau(s), p_\tau(s)) = \rho_\tau^s(x_\tau(0), p_\tau(0))$$

$$(b) \quad (x_\tau(0), p_\tau(0)) \in L$$

$$(c) \quad x_\tau(1) = x$$

$$(d) \quad dF = \lambda \text{ on } L$$

Since by assumption,  $z \in L$  and  $\rho(z) \in \pi^{-1}(\Omega)$  imply  $H_\tau = H_0$  on  $\{(s, \rho^s(z)), s \in [0, 1]\}$ , we conclude  $\rho_\tau^s(z) = \rho^s(z)$ .

Thus for  $x$  in  $\Omega$ , the set  $C_\tau(x, L)$  does not depend on  $\tau$ . Since  $C_\tau(x, L)$  is the set of critical values of the action, it has measure zero, and using the fact that  $u_{\rho_\tau(L)}(x) = c(1_x, \rho_\tau(L))$  depends continuously on  $\tau$ , we conclude that it must be independent of  $\tau$  for  $x$  in  $\Omega$ .

From the above argument follows that

$$u_{\rho_\tau(L)}(x) = u_{\rho(L)}(x) \text{ for all } x \text{ in } \Omega .$$

Now if  $|H_0| \leq \varepsilon$  on  $\mathcal{U}((\rho^t)_{t \in [0,1]}, L, \Omega)$  we may find a linear homotopy

$$H_\tau(s, z) = (1 - \tau\chi(s, z))H_0(s, z)$$

where  $\chi(s, z) = 0$  for  $(s, z) \in \mathcal{U}((\rho^t), L, \Omega)$  and  $\chi = 1$  outside a neighbourhood of  $\mathcal{U}((\rho^t), L, \Omega)$ .

By suitably choosing  $\chi$ , we may assume the following inequality holds

$$|H_1| \leq \varepsilon + \delta$$

with  $\delta > 0$  arbitrarily small. Since  $H_\tau = H_0$  on  $\mathcal{U}((\rho^t), L, \Omega)$ , we have

$$u_{\rho_1(L)}(x) = u_{\rho(L)}(x)$$

for  $x$  in  $\Omega$ . Since  $|H_1| \leq \varepsilon + \delta$  everywhere, we may apply the first part of the proof and obtain

$$|u_{\rho_1(L)}(x) - u_L(x)| \leq \varepsilon + \delta$$

and thus

$$|u_{\rho(L)}(x) - u_L(x)| \leq \varepsilon + \delta .$$

Finally,  $\delta$  being arbitrarily small, we conclude that

$$|u_{\rho(L)}(x) - u_L(x)| \leq \varepsilon$$

□

Our next task is to associate to the non compact supported Hamiltonian isotopy  $\psi^t$  a sequence  $(\varphi_\mu^t)_{\mu \in \mathbb{N}}$  of compact supported ones, such that  $c(1_x, \varphi_\mu^t(L))$  converges -or rather its restriction to any compact set stabilizes- for fixed  $x, t$ , as  $\mu$  goes to infinity.

We denote by  $H_\mu(t, z)$  and  $H(t, z)$  the Hamiltonians generating  $\varphi_\mu^t$  and  $\psi^t$ , and by  $K_{\mu, \nu}(t, z)$  the Hamiltonian generating  $\rho_{\mu, \nu}^t = \varphi_\nu^t \varphi_\mu^{-t}$ .

The equality  $\varphi_\mu^t = \varphi_\nu^t = \psi^t$  on  $\mathcal{U}$  is equivalent to  $H_\mu(t, z) = H_\nu(t, z) = H(t, z)$  on  $\psi^s(\mathcal{U})$  for all  $s \in [0, T]$ .

Thus if  $\varphi_\mu^s = \varphi_\nu^s = \psi^s$  on

$$W_\mu \cap (\psi^t)^{-1}(\pi^{-1}(\Omega_\mu))$$

we have.

$$\varphi_\nu^s \varphi_\mu^{-s} = id \text{ on } \psi^s(W_\mu \cap (\psi^t)^{-1}\pi^{-1}(\Omega_\mu))$$

hence

$$K_{\mu,\nu}(s, z) = 0 \text{ on } \psi^s(W_\mu \cap (\psi^t)^{-1}\pi^{-1}(\Omega_\mu))$$

so that, according to proposition A.1,

$$u_{\varphi_\mu^t(L)}(x) = u_{\varphi_\nu^t(L)}(x) \text{ for } x \text{ in } \pi^{-1}(\Omega_\mu) \text{ } t \in [0, T]$$

This motivates the

**Definition A.2.** Let  $(\psi^t)_{t \in [0, T]}$  be a Hamiltonian isotopy. A sequence of **compact supported** Hamiltonian isotopies,  $(\varphi_\mu^t)_{\mu \in \mathbb{N}}$  is an **exhausting sequence** associated to  $\mathcal{W}$  and  $(\Omega_\mu)_{\mu \in \mathbb{N}}$  if for all  $\mu$ , we have

$$\varphi_\mu^t = \psi^t \text{ on } \bigcup_{s \in [0, T]} W_\mu \cap (\psi^s)^{-1}(\pi^{-1}(\Omega_\mu))$$

The above argument implies

**Proposition A.3.** Let  $(\varphi_\mu^t)$  be an exhausting sequence for  $(\psi^t)_{t \in [0, T]}$  associated to  $\mathcal{W}$  and  $(\Omega_\mu)_{\mu \in \mathbb{N}}$ . Then for any  $L$ , image of the zero section by a compact supported Hamiltonian isotopy and contained in some  $W_\mu$ , and, for all  $\alpha, \beta \geq \mu$  and  $t \in [0, T]$

$$u_{\varphi_\alpha^t(L)}(x) = u_{\varphi_\beta^t(L)}(x) \quad \forall x \in \Omega_\mu$$

We may therefore set

**Definition A.4.** Given  $(\psi^t)_{t \in [0, T]}$  and an exhausting sequence  $(\varphi_\mu^t)$  associated to  $\mathcal{W}$ ,  $(\Omega_\mu)_{\mu \in \mathbb{N}}$  we define  $u_{\psi^t(L)}(x)$  to be the common value of the  $u_{\varphi_\mu^t(L)}(x)$  for  $\mu$  large enough. Since  $u_{\psi^t(L)}$  is a solution of the Hamilton-Jacobi equation, we call it the **variational solution associated to the exhausting sequence**.

We now need to solve two difficulties.

First, it is not clear, given  $(\psi^t)_{t \in [0, T]}$ ,  $\mathcal{W}$ ,  $(\Omega_\mu)_{\mu \in \mathbb{N}}$  whether we may find an associated exhausting sequence. Second, one could wonder whether  $u_{\psi^t(L)}(x)$  will depend on the choice of the exhausting sequence. But this will not be the case, since if  $\check{\varphi}_\mu^t, \hat{\varphi}_\mu^t$  are exhausting sequences, we get a new exhausting sequence by intertwining them. Remember that we assume  $W_\mu \cap \Omega_\mu$  is compact for all  $\mu$ .



**Definition A.5.** (a). We shall say that  $(\psi^t)_{t \in [0, T]}$  has **finite propagation speed** with respect to  $\mathcal{W}$  if and only if for each  $\mu$ , there exists  $\nu$  such that

$$\bigcup_{t \in [0, T]} \psi^t(W_\mu) \subset W_\nu$$

(b).  $\mathcal{W}$  is a **convex exhaustion** if the fibers of  $W_\mu$  are convex, that is for all  $x \in N$   $W_\mu \cap \pi^{-1}(x)$  is convex in  $T_x^*N$ .

Note that the composition of finite propagation speed Hamiltonians has finite propagation speed (with respect to the same family  $\mathcal{W}$ ). However this is not necessarily true for its inverse  $((\psi^t)^{-1})_{t \in [0, T]}$ .

The set of Hamiltonians isotopies having, together with their inverse, finite propagation speed, is a group for the composition law

$$(\varphi^t) \circ (\psi^t) = (\varphi^t \psi^t)$$

Our last result for this section is

**Proposition A.6.** *Let  $\mathcal{W}$  be an exhaustion. If  $(\psi^t)_{t \in [0, T]}$  has finite propagation speed with respect to  $\mathcal{W}$ , then, for any exhaustion  $(\Omega_\mu)_{\mu \in \mathbb{N}}$  of  $N$ , there exists an exhausting sequence  $(\varphi_\mu^t)$  associated to  $\mathcal{W}$ .*

The proof makes use of the obvious lemma:

**Lemma A.7.** *Let  $K \subset U$  be a compact subset and  $(\psi^t)_{t \in [0, 1]}$  a Hamiltonian isotopy such that*

$$V = \bigcup_{t \in [0, 1]} \psi^t(K) \subset U$$

*Then there is a Hamiltonian isotopy,  $\varphi^t$ , supported near  $V$  such that*

$$\varphi^t = \psi^t \text{ on } K \text{ for all } t \text{ in } [0, 1]$$

*Proof.* Indeed, if  $\chi$  is a function equal to 1 on  $V$ , vanishing outside a neighbourhood of  $U$ , and  $H(t, z)$  generates  $\psi^t$ , then  $\chi(z)H(t, z)$  generates  $\varphi^t$ .  $\square$

*Proof.* We may now prove the proposition.

Since for all  $\mu$  there exists  $\nu$  such that

$$\psi^s(W_\mu) \subset W_\nu$$

for all  $s$  in  $[0, T]$ , we have

$$\psi^s(W_\mu) \cap \pi^{-1}(\Omega_\mu) \subset W_\nu \cap \pi^{-1}(\Omega_\mu) \subset W_\nu \cap \pi^{-1}(\Omega_\nu)$$

Thus

$$\bigcup_{s \in [0, T]} W_\mu \cap (\psi^s)^{-1}(\pi^{-1}(\Omega_\mu)) = \bigcup_{s \in [0, T]} (\psi^s)^{-1}(\psi^s(W_\mu) \cap \pi^{-1}(\Omega_\mu)) \subset \bigcup_{s \in [0, T]} (\psi^s)^{-1}(K_\nu)$$

We denote by  $\tilde{K}_{\nu, T}$  this last set, and note that it is compact. Therefore

$$\bigcup_{s \in [0, T]} \psi^s(\tilde{K}_{\nu, T})$$

and we may find an isotopy  $\varphi_\mu^t$ , compact supported, such that  $\varphi_\mu^t = \psi^t$  on  $\tilde{K}_{\nu, T}$ .  $\square$

Now this yields solutions of the evolution Hamilton-Jacobi equations in case the initial condition  $f$  has compact support. Of course formally the general case can be reduced to this one, since solving

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + H(t, x, \frac{\partial u}{\partial x}(t, x)) = 0 \\ u(0, x) = f(x) \end{cases}$$

is equivalent to solving

$$\begin{cases} \frac{\partial v}{\partial t}(t, x) + H(t, x, \frac{\partial v}{\partial x}(t, x) + df(x)) = 0 \\ v(0, x) = 0 \end{cases}$$

by the change of function  $v(t, x) = u(t, x) - f(x)$  (and this change of function preserves variational solutions).

However we would like to have a criterion for  $H$  and an independent one for  $f$  in order to isolate the difficulties.

**Proposition A.8.** *Let  $\mathcal{W}$  be a convex exhaustion. Let  $C_{\mathcal{W}}(N)$  be the union over all sets  $W_\mu$  of the exhaustion, of the  $C^0$  closure of the set of  $C^1$  functions with graph in  $W_\mu$ . Let  $\mathcal{H}_{\mathcal{W}}^1(\mathbb{R} \times T^*N)$  be the set of  $C^{1,1}$  Hamiltonians having finite propagation speed with respect to  $\mathcal{W}$ . Let  $f$  such that the graph of  $df$  is in  $W_\mu$ , the function  $u_{\psi^t(L)}$  does not depend on the choice of the exhausting sequence. Moreover it is a solution of the Hamilton-Jacobi equation*

$$(HJ) \quad \begin{cases} \frac{\partial u}{\partial t}(t, x) + H(t, x, \frac{\partial u}{\partial x}(t, x)) = 0 \\ u(0, x) = f(x) \end{cases}$$

*Remark A.9.* Let  $\psi^t$  be the flow of  $H$  and  $\Lambda_0 \subset T^*N$  be a Lagrangian submanifold.

We first notice that the manifold we constructed,  $L$  in  $T^*(\mathbb{R} \times N)$  giving the solution of (HJ) and the family  $L_t = \psi^t(L)$  give essentially the same information. Indeed, if  $u(t, x) = c(1_{(t,x)}, L)$  and  $v(t, x) = c(1_x, L_t)$  we have  $u(t, x) - v(t, x) = d(t)$  for some continuous function  $d$ . Given  $v$ , we can easily determine  $d$  or  $u$ , since  $\frac{\partial}{\partial x}v(t, x) = \frac{\partial}{\partial x}u(t, x)$ , hence

$$\frac{\partial}{\partial x}u(t, x) = -H(t, x, \frac{\partial}{\partial x}v(t, x))$$

hence

$$u(t, x) = f(x) + \int_0^t H(s, x, \frac{\partial}{\partial x}v(s, x))ds$$

*Proof.* Clearly if  $\Lambda_0$  is the graph of  $df$ , we wish to replace it by a Lagrangian manifold  $\Lambda_\mu$ , such that there exists an isotopy  $\Lambda_\mu^s$  from  $\Lambda_0$  to  $\Lambda_\mu$ , such that

$$\varphi_\mu^t(\Lambda_\mu^s) = \psi^t(\Lambda_\mu^s) = \psi^t(\Lambda_0)$$

over  $\pi^{-1}(\Omega_\mu)$  for all  $t \in [0, T]$ .

But  $\varphi_\mu^t = \psi^t$  on  $W_\mu \cap (\psi^s)^{-1}\pi^{-1}(\Omega_\mu)$  so if  $\Lambda_\mu^s = \Lambda_0$  over  $\pi^{-1}(A_\mu)$ , where  $A_\mu$  will be determined later, we have

$$\varphi_\mu^t(\Lambda_\mu^s) = \psi^t(\Lambda_0)$$

over

$$\psi^t(W_\mu \cap (\psi^s)^{-1}\pi^{-1}(\Omega_\mu) \cap \pi^{-1}(A_\mu))$$

By compactness of

$$W_\mu \cap (\psi^s)^{-1}\pi^{-1}(\Omega_\mu)$$

we may find  $A_\mu$  such that  $W_\mu \cap (\psi^s)^{-1}\pi^{-1}(\Omega_\mu) \subset \pi^{-1}(A_\mu)$ , and therefore according to proposition A.3

$$u_{\varphi_\mu^t(\Lambda_\mu^s)} = u_{\psi^t(\Lambda_0)}$$

on  $\Omega_\mu$ .

Now if  $\Lambda_0$  is the graph of  $df$ , we may take for  $\Lambda_\mu^s$  the graph of  $df_\mu^s$  where

$$f_\mu^s(x) = (1 - s\chi_\mu(x))f(x)$$

and

$$\begin{cases} \chi_\mu(x) = 0 \text{ on } A_\mu \\ \chi_\mu(x) = 1 \text{ near infinity} \end{cases}$$

Then  $\Lambda_\mu^s$  will coincide with  $\Lambda_0$  over  $A_\mu$ . We still need to prove that  $\Lambda_\mu^s$  is in  $W_\mu$  for all  $s$  in  $[0, 1]$ . Now this coincides with the graph of  $(1 - s\chi_\mu(x))df(x)$  outside a compact set. Since the graph of  $df$  is in  $W_\mu$ , this will be also in some  $W_\nu$  for some  $\nu$ , provided  $W_\nu$  is fiber convex. □

*Remark A.10.* In practice most  $W_\mu$  are fiber convex, and if this is not the case, we can always convexify them. However, it is not clear that a  $\psi^t$  having finite propagation speed with respect to  $\mathcal{W}$  will have also finite propagation speed with respect to the convexified exhaustion  $\mathcal{W}'$ , unless the convexification of  $W_\mu$ , is contained in some  $W_\nu$ .

**Example A.11.** (a). The Barles-Tourin condition defined for an autonomous Hamiltonian  $H(x, p)$  on  $T^*(\mathbb{R}^n)$  can be rewritten as

- $$|D_p H(x, p)| \leq f(|p|)(1 + |x|)$$

- $$g_-(|p|) \leq |H(x, p)| \leq g_+(|p|)$$

where  $\lim_{\xi \rightarrow \infty} g_\pm(\xi) = +\infty$

This implies finite propagation speed for the exhaustion by the  $U_r = \{(x, p) \mid |p| \leq r\}$ , since the assumptions and conservation of energy imply that the image of  $U_r = \{(x, p) \mid |p| \leq r\}$  by the flow remains in some  $U_{r'}$ . In such a region, the first inequality implies, by Gronwall's lemma, finite propagation speed.

(b). Consider the  $U_{(a,b)} = \{(x, p) \mid |p| \leq ad(x, x_0) + b\}$ , and assume

- $$|D_x H(t, x, p)| \leq C(|p|)$$

- $$|D_p H(t, x, p)| \leq C(1 + d(x, x_0))$$

Then the first assumption, together with Gronwall's lemma implies that the flow sends  $U_{a,b}$  to some  $U_{a',b'}$ .

The proposition implies that if  $\mathcal{W}$  is an exhaustion, and  $f$  a function on  $N$  such that the graph of  $df$  is contained in  $W_\mu$  for some  $\mu$ , and if  $H(t, x, p)$  generates a flow having finite propagation speed with respect to the  $\mathcal{W}$ , then we may find a variational solution of

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) + H(t, x, \frac{\partial u}{\partial x}(t, x)) &= 0 \\ u(0, x) &= f(x) \end{aligned}$$

by considering solutions of

$$\begin{cases} \frac{\partial u_\mu}{\partial t}(t, x) + H_\mu(t, x, \frac{\partial}{\partial x} u_\mu(t, x)) = 0 \\ u_\mu(0, x) = f(x) \end{cases}$$

where  $H_\mu$  is a compact supported Hamiltonian generating an exhausting sequence  $(\varphi_\mu)_{\mu \in \mathbb{N}}$  for the flow  $\psi^t$  of  $H$ .

It follows from the above that  $u(t, x) = \lim_{\mu \rightarrow \infty} u_\mu(t, x)$  defines a function independent of the choice of the exhausting sequence. In the regular case (i.e. for smooth data), this will of course be a solution of the equation almost everywhere.

We may of course combine theorem 4.10 with the above proposition to get the following:

**Theorem A.12.** *Let  $\mathcal{W}$  denote an exhausting sequence of neighbourhoods of  $0_N$ . Let  $\mathfrak{C}_\mathcal{W}(N)$  be the union over all sets  $W_\mu$  of the exhaustion, of the  $\gamma$  closures of the set of  $C^1$  functions with graph in  $W_\mu$ . Let  $\mathcal{H}_\mathcal{W}^1(\mathbb{R} \times T^*N)$  be the set of  $C^{1,1}$  Hamiltonians having finite propagation speed with respect to  $\mathcal{W}$ . Consider the closure  $\mathfrak{H}_{\gamma, \mathcal{W}}(\mathbb{R} \times T^*N)$  of  $\mathcal{H}_\mathcal{W}^1(\mathbb{R} \times T^*N)$  in  $\mathfrak{H}_\gamma(\mathbb{R} \times T^*N)$ . Finally we assume that*

$$(f, H_1, \dots, H_d) \in \mathfrak{C}(N) \times \mathfrak{H}_{\gamma, \mathcal{W}}(\mathbb{R} \times T^*N)^d$$

then there is a solution of (MHJ) with data  $(f, H_1, \dots, H_d)$ .

## B Appendix : equations involving the unknown function

For equations of the type

$$\frac{\partial u}{\partial t_j} + H_j(t_1, \dots, t_d, x, \frac{\partial}{\partial x} u(x), u(x)) = 0$$

the same proofs hold, but we have to replace the cotangent space  $T^*M$  which is symplectic by the jet space  $J^1(M)$ , a contact manifold, on which we use the contact form  $\alpha = dz - p dx$ . We also replace Lagrangian submanifolds, by Legendrian ones, which are manifolds of dimension  $\dim(M)$  on which  $\alpha$  vanishes.

Since the notion of generating function applies “**even better**” to a Legendrian submanifold of  $J^1(M)$ , and according to [Théret 2], we also have uniqueness of generating functions, all the constructions of [Viterbo 1] apply. Similarly, since the Hamiltonian flow of  $\tau + H(t, x, u, p)$  is also well-defined, the constructions from [Viterbo-Ottolenghi] also carry through.

The details of proofs are left to the reader. However, we will make explicit the commutativity condition for  $H_1(t_1, t_2, x, p, u)$  and  $H_2(t_1, t_2, x, p, u)$ .

This is easy even in a general contact manifold. Indeed, define the Reeb vector field  $R$ , by the equalities

$$\begin{aligned}i_R\alpha &= 1 \\i_Rd\alpha &= 0\end{aligned}$$

the associated Hamiltonian vector field must satisfy

$$\begin{aligned}i_{\tilde{X}_H}\alpha &= -H \\i_{\tilde{X}_H}d\alpha &= dH + i_R(dH)\alpha\end{aligned}$$

(note that for  $H = -1$  we recover  $R$ ) which in our case, since  $\alpha = du - pdx$ ,  $R = \frac{\partial}{\partial u}$  translates to

$$\tilde{X}_H(x, p, u) = (pH_p - H)\frac{\partial}{\partial u} + H_p\frac{\partial}{\partial x} - (H_x + pH_u)\frac{\partial}{\partial p}$$

We then have  $L_{\tilde{X}_H}\alpha = H_u\alpha$  and this shows that  $\tilde{X}_H$  preserves the contact distribution  $\alpha = 0$ . Since the commutator of such vector fields must also preserve the contact distribution, we may define

**Definition B.1.** Let  $H_1(x, p, u)$  and  $H_2(x, p, u)$  be two functions on  $J^1(M)$ . We set

$$\begin{aligned}[H_1, H_2](x, p, u) &= \\ \{H_1, H_2\}_s(x, p, u) &+ \\ \frac{\partial H_2}{\partial u}(x, p, u) \left( p\frac{\partial H_1}{\partial p}(x, p, u) - H_1(x, p, u) \right) &- \frac{\partial H_1}{\partial u}(x, p, u) \left( p\frac{\partial H_2}{\partial p}(x, p, u) - H_2(x, p, u) \right)\end{aligned}$$

where

$$\{H_1, H_2\}_s = \frac{\partial H_1}{\partial p}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial p}$$

**Proposition B.2.**

$$\tilde{X}_{[H_1, H_2]} = [\tilde{X}_{H_1}, \tilde{X}_{H_2}]$$

Now in the multi-time dependent case, for  $H_1(t_1, t_2, x, p, u)$ ,  $H_2(t_1, t_2, x, p, u)$ , we can consider

$$K_1(t_1, t_2, \tau_1, \tau_2, x, p, u) = \tau_1 + H_1(t_1, t_2, x, p, u), K_2(t_1, t_2, \tau_1, \tau_2, x, p, u) = \tau_2 + H_2(t_1, t_2, x, p, u)$$

as functions on  $J^1(\mathbb{R}^2 \times M)$  and set

**Definition B.3.** Let  $H_1, H_2$  as above. We define

$$[[H_1, H_2]] = [K_1, K_2]$$

The following is the natural extension of Theorem 1.2 :

**Proposition B.4.** Assume we have sequences  $H_\nu, K_\nu$  such that  $H_\nu \xrightarrow{C^0} H$ ,  $K_\nu \xrightarrow{C^0} K$ ,  $[H_\nu, K_\nu] \xrightarrow{C^0} 0$ . Then  $[H, K] = 0$ .

## C Appendix : Coisotropic submanifolds of symplectic manifolds

Let  $(P, \sigma)$  be a symplectic manifold, and  $C$  a coisotropic submanifold, that is

$$(T_z C)^\perp \subset T_z C \quad \forall z \in C$$

where

$$(T_z C)^\perp := \{v \in T_z P : \sigma(v, u) = 0 \quad \forall u \in T_z C, \}$$

here  $\sigma$  is a closed and non degenerated 2-form, e.g. if  $P = T^*M$ , then  $\sigma = d\theta$ ,  $\theta = \sum_i p_i dx^i$  is the Liouville 1-form. Let  $H : T^*M \rightarrow \mathbb{R}$ , then the Hamiltonian vector field  $X_H$  related to  $H$  is so defined:

$$\sigma(X_H, \cdot) = -dH,$$

and the Poisson brackets of two Hamiltonian functions  $H$  e  $K$  are

$$\{K, H\}(z) = dK(z) \cdot X_H(z) \quad (z = (x, p)).$$

Let suppose that  $C$  is defined by zero's of functions<sup>7</sup>,  $H^a : P \rightarrow \mathbb{R}$ ,  $a = 1, \dots, \text{codim } C$ :

$$C : H^a = 0 \quad \text{rk } dH^a|_{H^a=0} = \max = \text{codim } C$$

Thus:  $u \in T_z C \iff dH^a u = 0$ . Standard homomorphism theorem between linear spaces gives existence (and uniqueness, by the max rank) of the Lagrange multipliers  $\lambda_a = \lambda_a(z, v)$ :

$$\forall v \in (T_z C)^\perp : [dH^a u = 0 \Rightarrow \sigma(v, u) = 0] \Rightarrow \omega(v, \cdot) = \sum_a \lambda_a dH^a$$

The linear independence of the forms  $dH^a$ ,  $a = 1, \dots, \text{codim } C$ , implies that the following Hamiltonian vector fields  $(X_a)_{a=1, \dots, \text{codim } C}$ ,

$$\sigma(X_a, \cdot) = -dH^a,$$

span  $(T_z C)^\perp$ . Since  $(T_z C)^\perp \subset T_z C$ , the vectors  $X_a(z)$  are in  $T_z C$ , and their integral paths, the *characteristics* of  $X_a$ , are in  $C$ , hence

$$\sigma(X_a, X_b) = 0.$$

---

<sup>7</sup>it is equivalent to ask that the normal bundle of  $C$ , with respect to some Riemannian metric, is trivial ([Guillemin-Pollack] p. 77.)

From the definition of Poisson brackets  $\{\cdot, \cdot\}$ ,

$$\sigma(X_a, X_b) = \{H_a, H_b\},$$

and from the well known Lie algebra morphism  $([\cdot, \cdot]: \text{Lie brackets})$

$$[X_a, X_b] = X_{\{H_a, H_b\}},$$

one obtains that the fields  $(X_a)_{a=1, \dots, \text{codim} C}$  are in *involution*:

$$[X_a, X_b] = 0.$$

In other words, by Frobenius' theorem,  $(TC)^\perp$  is a *integrable distribution*, the *characteristic* distribution. Notice that the involution condition of the Hamiltonians

$$\{H_a, H_b\} = 0$$

is intrinsic, constitutive, of the coisotropic submanifold  $C$ .

## D Appendix: Geometric Theory of Hamilton-Jacobi equations on coisotropic submanifolds

Let us consider  $d$  functionally independent and in involution Hamiltonians  $\tilde{H}_a$ ,  $a = 1, \dots, d$ , of the following structure:

$$\tilde{H}_a : T^*(N \times \mathbb{R}^d) \longrightarrow \mathbb{R}$$

$$(x_1, \dots, x_n, t_1, \dots, t_d, p_1, \dots, p_n, \tau_1, \dots, \tau_d) \longmapsto \tau_a + H_a(t_1, \dots, t_d, x, p)$$

The submanifold

$$C = \bigcap_{a=1}^d \tilde{H}_a^{-1}(0)$$

is coisotropic (because  $\{\tilde{H}_a, \tilde{H}_b\} = 0$ ),  $\dim C = 2n + d$ ,  $\text{co-dim} C = d$ .

We call *geometric solution* of the Hamilton-Jacobi multi-equation related to  $C$  every *Lagrangian submanifold*<sup>8</sup>  $\Lambda \subset T^*(N \times \mathbb{R}^d)$  belonging to  $C$ :

$$\Lambda \subset C.$$

---

<sup>8</sup>that is, *i*)  $\sigma|_\Lambda = 0$  and *ii*)  $\dim \Lambda = \dim N \times \mathbb{R}^d = n + d$ .



The Lagrangian submanifolds which are graphs of a section of the cotangent bundle  $T^*(N \times \mathbb{R}^d)$  are image of the differential of functions  $S$ , defined on  $N \times \mathbb{R}^d$ ,  $S : N \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,

$$\Lambda = \text{im } dS = \left\{ (x, t, p, \tau) : p = \frac{\partial S}{\partial x}(x, t), \tau = \frac{\partial S}{\partial t}(x, t) \right\}$$

Thus, every globally transverse Lagrangian submanifold  $\Lambda$ , solving the H-J multi-equation related to  $C$ , represents, by its generating function  $S$ , also a *classic* solution of it:

$$(A1) \quad \frac{\partial S}{\partial t_a}(x, t) + H_a(t_1, \dots, t_d, x, \frac{\partial S}{\partial x}(x, t)) = 0, \quad a = 1, \dots, d.$$

• The characteristics method. The following proposition suggests us as to built geometric solutions of the H-J multi-equation:

**Proposition** *Let  $\Lambda \subset C$ ,  $\Lambda$  be Lagrangian and  $C = \cap_{a=1}^d \tilde{H}_a^{-1}(0)$  coisotropic. Then the Hamiltonian vector fields  $X_{\tilde{H}_a}$ ,  $a = 1, \dots, d$ , are tangent to  $\Lambda$ .*

*Proof.* If  $v \in T_x \Lambda \Rightarrow v \in T_x C \Rightarrow v \in \cap_{a=1}^d \ker d\tilde{H}_a(x)$ , that is  $d\tilde{H}_a(x)v = 0$ ,  $\forall a = 1, \dots, d$ . It holds that  $\sigma(X_{\tilde{H}_a}, v) = -d\tilde{H}_a v = 0$ , that is, the fields  $X_{\tilde{H}_a}$  are skew-orthogonal to  $\Lambda$ ,  $X_{\tilde{H}_a}(x) \in (T_x \Lambda)^\perp = T_x \Lambda$ , where the last equality holds since  $\Lambda$  is Lagrangian.  $\square$

• Multi-valued Variational Solutions. The Liouville 1-form of  $T^*(N \times \mathbb{R}^d)$  is

$$\theta = \sum_{j=1}^n p_j dx^j + \sum_{a=1}^d \tau_a dt^a$$

The restriction (pull-back) of  $\theta$  on the Lagrangian submanifolds gives closed 1-forms, indeed, by denoting  $j$  the inclusion map  $j : \Lambda \hookrightarrow T^*(N \times \mathbb{R}^d)$ ,

$$0 = \sigma|_\Lambda = j^* \sigma = j^* d\theta = dj^* \theta = d(\theta|_\Lambda).$$

Let  $f : N \rightarrow \mathbb{R}$  be the initial data for the multi-equation (A1), then a primitive function of  $\theta|_\Lambda$  (on  $\Lambda$ , parametrized by  $\chi, t^1, \dots, t^d$ ) is given by

$$\bar{S}(\chi, t^1, \dots, t^d) = f(\chi) + \int_{\tilde{\Phi}^{t^1} \dots \tilde{\Phi}^{t^a} \dots \tilde{\Phi}^{t^d}(z)} \sum_{j=1}^n p_j dx^j - \sum_{a=1}^d H_a dt^a,$$

$$z = (\chi, 0; df(\chi), -H_a(0, \chi, df(\chi))) \in T^*(N \times \mathbb{R}^d)$$

For small  $t^1, \dots, t^d$ , the  $x$ -components of  $\tilde{\Phi}^{t^1} \dots \tilde{\Phi}^{t^a} \dots \tilde{\Phi}^{t^d}(z)$ , i.e.

$$x(\chi, t) = \text{pr}_N \pi_{N \times \mathbb{R}^d} \tilde{\Phi}^{t^1} \dots \tilde{\Phi}^{t^a} \dots \tilde{\Phi}^{t^d}(z),$$

admits inverse:  $\chi = \chi(x, t)$ . In such a (scarcely meaningful) case, the function

$$S(x, t) := \bar{S}(\chi(x, t), t)$$

represents the classic solution to the Cauchy problem.

## E Appendix: A short account on the developments of the theory of the multi-equations of Hamilton-Jacobi

After a pioneering paper by Tulczyjew [Tulczyjew], the study Hamilton-Jacobi problems in general co-isotropic submanifolds of co-dimension greater than one inside  $T^*Q$ , that is, extending the standard case  $\frac{\partial S}{\partial t} + H(t, x, \frac{\partial S}{\partial x}) = 0$ , was further developed by Benenti and Tulczyjew in [Benenti-Tulczyjew] and then in [Benenti], where, among other things, in this last paper first it was considered generating functions with finite auxiliary parameters for Lagrangian submanifolds of  $T^*Q \times T^*Q$  representing symplectic relations (canonical transformations). Examples of this new framework are the integrable Dirac systems reconsidered by [Lichnerowicz] and [Menzio-Tulczyjew]. More recently, new examples of H-J multi-equations arose from economics [Rochet], one can see the bibliography in [Barles-Tourin]. The use of symplectic topology and the notion of variational solution for the H-J equation have been introduced by Chaperon and Sikorav, and then developed in [Viterbo 2], where, in particular, the uniqueness of such a solution is deduced by the essential global uniqueness of the generating functions proved in [Viterbo 1]. We recall that these variational solutions coincide with the viscosity solutions in the case of  $p$ -convex Hamiltonians [Joukovskaia], but it is not true in general case, lacking a (expected) Markov condition (cf. [Viterbo 2], [Viterbo-Ottolenghi]).

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