Finite length for unramified GL_2

Christophe Breuil[a] Florian Herzig[b] Yongquan Hu[c] Stefano Morra[d]
Benjamin Schraen[e]

Abstract

Let p be a prime number and K a finite unramified extension of $\mathbb{Q}_p$. If p is large enough with respect to $[K : \mathbb{Q}_p]$ and under mild genericity assumptions, we prove that the admissible smooth representations of $\mathrm{GL}_2(K)$ that occur in Hecke eigenspaces of the mod p cohomology are of finite length. We also prove many new structural results about these representations of $\mathrm{GL}_2(K)$ and their subquotients.

Contents

[a]CNRS, Bâtiment 307, Faculté d'Orsay, Université Paris-Saclay, 91405 Orsay Cedex, France
[b]Dept. of Math., Univ. of Toronto, 40 St. George St., BA6290, Toronto, ON M5S 2E4, Canada
[c]Morningside Center of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; University of the Chinese Academy of Sciences, Beijing 100049, China
[d]Lab. d'Analyse, Géométrie, Algèbre, 99 Av. Jean Baptiste Clément, 93430 Villetaneuse, France
[e]Université Claude-Bernard-Lyon-I, Institut Camille Jordan, 69622 Villeurbanne, France

1 Introduction

1.1 The main results

Let p be a prime number, F a totally real number field and D a quaternion algebra of center F which is split at all p-adic places and at exactly one infinite place. In order to simplify this introduction we assume that p is inert in F (in the text we only need p unramified in F) and denote by v the unique p-adic place of F. To an absolutely irreducible continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ (here $\mathbb{F}$ is a sufficiently large finite extension of $\mathbb{F}_p$) and V^v a compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$ (here $\mathbb{A}_F^{\infty,v}$ is the ring of finite prime-to-v adèles of F), we associate the admissible smooth representation of $\mathrm{GL}_2(F_v)$ over $\mathbb{F}$:

$$\pi \stackrel{\text{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\overline{r}, H^1_{\text{ét}}(X_{V^v V_v} \times_F \overline{F}, \mathbb{F})\big), \tag{1}$$

where the inductive limit runs over compact open subgroups V_v of $(D \otimes_F F_v)^\times \cong \mathrm{GL}_2(F_v)$ and $X_{V^v V_v}$ is the smooth projective Shimura curve over F associated to D and $V^v V_v$. Throughout this introduction we fix π as in (1) such that $\pi \neq 0$. Recall that, when $F = \mathbb{Q}$ (and $X_{V^vV_v}$ is the compactified modular curve) and under very weak assumptions on $\overline{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$, the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π has been completely understood for quite some time (see [Eme], [CDP14]). Unfortunately, this is no longer the case when $F_v \neq \mathbb{Q}_p$ despite recent progress ([HW22], [BHH$^+$23], [BHH$^+$a], [BHH$^+$b], [Wan23], [Wan]). The main aim of the present work is to take a new step in the (long) journey towards the comprehension of the $\mathrm{GL}_2(F_v)$-representation π when $F_v \neq \mathbb{Q}_p$ by proving that, for $\overline{r}$ sufficiently generic and under a standard multiplicity one assumption (commonly referred to as "the minimal case"), π is of finite length.

Under similar assumptions, it was already known that π is absolutely irreducible if and only if $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is ([BHH$^+$a, Thm. 3.4.4.6(i)]), and that π is of length 3 when $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is reducible and $[F_v : \mathbb{Q}_p] = 2$ ([HW22] for $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ nonsplit, [BHH$^+$a, Thm. 3.4.4.6(ii)] for $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ split[1]). Hence the main contribution of this work is to prove that π is of finite length when $\overline{r}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ is reducible and $[F_v : \mathbb{Q}_p] \geq 3$. We also obtain many intermediate and aside results on (the irreducible constituents of) π.

Let us describe our most important results in more details.

We set $K \stackrel{\mathrm{def}}{=} F_v$, $f \stackrel{\mathrm{def}}{=} [K : \mathbb{Q}_p]$ and $q \stackrel{\mathrm{def}}{=} p^f$. We denote by ω the mod p cyclotomic character of $\mathrm{Gal}(\overline{K}/K)$ (that we consider as a character of $K^\times$ via local class field theory, where uniformizers correspond to geometric Frobenius elements), and by ω_f, ω_{2f} Serre's fundamental characters of the inertia subgroup I_K of $\mathrm{Gal}(\overline{K}/K)$ of level f, $2f$ respectively. In this introduction, we say that $\overline{r}$ is *generic* if the following conditions are satisfied:

(i) $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible;

(ii) for $w \nmid p$ such that either D or $\overline{r}$ ramifies at w, the framed deformation ring of $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ over the Witt vectors $W(\mathbb{F})$ is formally smooth;

(iii) $\overline{r}|_{I_K}$ is up to twist of form

$$\begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ 0 & 1 \end{pmatrix} \text{ with } \max\{12, 2f+1\} \leq r_j \leq p - \max\{15, 2f+4\}$$

or

$$\begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & \\ & \omega_{2f}^{q(\mathrm{same})} \end{pmatrix} \text{ with } \begin{cases} \max\{12, 2f+1\} \leq r_j \leq p - \max\{15, 2f+4\} & j > 0 \\ \max\{13, 2f+2\} \leq r_0 \leq p - \max\{14, 2f+3\}. \end{cases}$$

Note that (iii) implies $p \geq \max\{27, 4f+5\}$ and that (ii) can be made explicit ([Sho16], [BHH$^+$23, Rk. 8.1.1]). The bounds on r_j in (iii) are such that all the results mentioned in this introduction except one hold (in the paper many results actually require weaker bounds, and a few results require stronger bounds). By [BHH$^+$23, Thm. 1.9] (for $\overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$ semisimple) and [Wan23, Thm. 6.3(ii)] (for $\overline{r}|_{\mathrm{Gal}(\overline{K}/K)}$ non-semisimple) for $\overline{r}$ generic there is a unique integer $r \geq 1$ (the

[1] [BHH$^+$a, Thm. 3.4.4.6] is stated in the global setting of compact unitary groups but the proof is the same.

"multiplicity") such that, for any (absolutely) irreducible representation σ of $\mathrm{GL}_2(\mathcal{O}_K)$ over $\mathbb{F}$, we have $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\sigma, \pi) \in \{0, r\}$ (the notation $\overline{r}$ and r is somewhat unfortunate but is consistent with [BHH+23, § 8]).

In the sequel we let $\overline{\rho} \stackrel{\mathrm{def}}{=} \overline{r}^\vee|_{\mathrm{Gal}(\overline{K}/K)}$, where $\overline{r}^\vee$ is the dual of $\overline{r}$.

If π_1 and π_2 are representations of a group, we denote by $\pi_1 — \pi_2$ an arbitrary *nonsplit* extension of π_2 by π_1 (so π_1 is a subrepresentation and π_2 is a quotient). We say a finite length representation is *uniserial* if it has a unique composition series, in which case we write $\pi_1 — \pi_2 — \pi_3 — \cdots$, where π_i are the (irreducible) graded pieces. Finally we let $B(K)$ be the subgroup of upper triangular matrices in $\mathrm{GL}_2(K)$.

Theorem 1.1.1. *Assume that $\overline{r}$ is generic and that $r = 1$.*

(i) *If $\overline{\rho}$ is irreducible then π is irreducible supersingular.*

(ii) *If $\overline{\rho}$ is split, i.e. $\overline{\rho} \cong \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$, then*

$$\pi \cong \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_2 \otimes \chi_1\omega^{-1}) \oplus \pi' \oplus \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_1 \otimes \chi_2\omega^{-1}),$$

where $\pi' = 0$ if $K = \mathbb{Q}_p$ and π' has length $\in \{1, \ldots, f-1\}$ with distinct supersingular constituents if $K \neq \mathbb{Q}_p$.

(iii) *If $\overline{\rho}$ is nonsplit, i.e. $\overline{\rho} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $* \neq 0$, then*

$$\pi \cong \Big(\mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_2 \otimes \chi_1\omega^{-1}) —— \pi' —— \mathrm{Ind}_{B(K)}^{\mathrm{GL}_2(K)}(\chi_1 \otimes \chi_2\omega^{-1}) \Big),$$

where $\pi' = 0$ if $K = \mathbb{Q}_p$ and π' is uniserial of length $\in \{1, \ldots, f-1\}$ with distinct supersingular constituents if $K \neq \mathbb{Q}_p$.

Part (i) was known ([BHH+a, Thm. 3.4.4.6(i)], as already mentioned), (ii) easily follows from Theorem 3.2.3(i) with the first statement of [BHH+a, Thm. 1.3.11] and from Corollary 3.2.7(iv), and (iii) follows from Theorem 4.4.8(ii) and Corollary 4.4.10.

Theorem 1.1.1 implies that π is of finite length and multiplicity free. It is expected that π' in Theorem 1.1.1(ii), (iii) always has length $f-1$ (see [BP12, p. 107]) but we only know this when $f = 2$ (in fact we do not have an example of a π' of length ≥ 2). Note also that, although one can optimistically hope that π' only depends on $\overline{\rho}$ and that π' in Theorem 1.1.1(ii) is the semisimplification of π' in Theorem 1.1.1(iii), at present we know none of these statements when $f > 1$, even for $f = 2$.

Nevertheless we can prove several results on the irreducible constituents of π. Let I (resp. I_1) be the subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$ of matrices which are upper triangular modulo p (resp. upper unipotent modulo p) and $K_1 \cong 1 + p\mathrm{M}_2(\mathcal{O}_K) \subseteq I_1$ be the subgroup of matrices which are trivial modulo p. Let $Z_1 \cong 1 + p\mathcal{O}_K$ be the center of I_1 (or K_1). We will extensively use the Iwasawa algebra $\Lambda \overset{\text{def}}{=} \mathbb{F}[\![I_1/Z_1]\!]$ which is a (noncommutative) noetherian local ring of Krull dimension $3f$. We denote by $\mathfrak{m}$ its maximal ideal. Since π has a central character, π and any of its subquotients are Λ-modules, and likewise for their linear duals. Since π is admissible, the latter are moreover finitely generated Λ-modules. Recall that a nonzero finitely generated Λ-module M is Cohen–Macaulay of grade $c \geq 0$ if $\mathrm{Ext}^i_\Lambda(M, \Lambda)$ is nonzero if and only if $i = c$.

Theorem 1.1.2. *Assume that $\overline{r}$ is generic, that $r = 1$ and that $\overline{\rho}$ is semisimple.*

(i) *The linear dual $\mathrm{Hom}_{\mathbb{F}}(\pi', \mathbb{F})$ of any nonzero subquotient π' of π is a Cohen–Macaulay Λ-module of grade $2f$.*

(ii) *Any subquotient of π is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle.*

(iii) *For any subquotient π' of π we have*

$$\dim_{\mathbb{F}(\!(X)\!)} D^\vee_\xi(\pi') = |\,\mathrm{JH}(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi'))|,$$

where $D^\vee_\xi(\pi')$ is the cyclotomic (φ, Γ)-module associated to π' in [BHH⁺a, § 2.1.1] and JH *means the set of Jordan–Hölder (or irreducible) constituents.*

(iv) *For any subrepresentations $\pi_1 \subseteq \pi_2$ of π we have a split exact sequence of $\mathrm{GL}_2(\mathcal{O}_K)$-representations*

$$0 \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2/\pi_1) \to 0.$$

(v) *For any subrepresentations $\pi_1 \subseteq \pi_2$ of π and any $n \geq 1$ we have an exact sequence of I-representations*

$$0 \to \pi_1[\mathfrak{m}^n] \to \pi_2[\mathfrak{m}^n] \to (\pi_2/\pi_1)[\mathfrak{m}^n] \to 0,$$

which is split for $n \leq \max\{6, f+1\}$.

Note first that for π itself part (i) was known using [HW22, Prop. A.8] (without assuming $\overline{\rho}$ semisimple) and part (ii) was known by [BHH⁺a, Thm. 1.3.8]. Moreover (iii) was known for subrepresentations π_1 of π by [BHH⁺a, Thm. 3.3.5.3(ii)]. In particular Theorem 1.1.2 was already known for $\overline{\rho}$ irreducible (as π is then also irreducible), and thus the main novelty in Theorem 1.1.2 is that we obtain nontrivial results for *subquotients* of π (when $\overline{\rho}$ is reducible).

When $\overline{\rho}$ is split reducible, (i) is contained in Corollary 3.2.7(ii), (ii) is Corollary 3.2.7(iii), (iii) is contained in Corollary 3.2.7(i) and (iv) is Lemma 3.2.6. Finally (v) is Corollary 3.2.5 (note that the splitness for $n = 1$ directly follows from (iv) since $(-)[\mathfrak{m}] = (-)^{I_1}$). The splitness of the exact sequences in (iv) and in (v) for $n \leq \max\{6, f+1\}$ can be seen as (very weak) evidence for the hope that π is semisimple when $\overline{\rho}$ is.

When $\overline{\rho}$ is non-semisimple, we have the following version of Theorem 1.1.2:

Theorem 1.1.3. *Assume that $\overline{r}$ is generic, that $r = 1$ and that $\overline{\rho}$ is non-semisimple (reducible).*

(i) *The linear dual of any nonzero subquotient of π is a Cohen–Macaulay Λ-module of grade $2f$.*

(ii) *Any subquotient of π is generated by its K_1-invariants.*

The proofs in the non-semisimple case are significantly harder and usually much more technical than in the split case. Part (i) is contained in Corollary 4.4.6 and part (ii) is Theorem 4.4.8(i).

Theorem 1.1.3 is shorter than Theorem 1.1.2 because, in the nonsplit case, if $\pi_1 \subseteq \pi_2$ are nonzero subrepresentations of π the maps $\pi_2^{I_1} \to (\pi_2/\pi_1)^{I_1}$ and $\pi_2^{K_1} \to (\pi_2/\pi_1)^{K_1}$ are not surjective in general (even for $f = 1$). Nonetheless, in [BHH+c] we will completely determine the (semisimple) I-representation $(\pi_2/\pi_1)^{I_1}$ and the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $(\pi_2/\pi_1)^{K_1}$. We will also determine $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_2/\pi_1)$.

Under the same assumptions ($\overline{r}$ generic, $r = 1$) we prove several other results that are not stated above. For instance, *just assuming $\overline{r}$ generic*, we completely determine $\mathrm{gr}_\mathfrak{m}(\pi^\vee)$ as a graded $\mathrm{gr}_\mathfrak{m}(\Lambda)$-module, where $\pi^\vee \overset{\text{def}}{=} \mathrm{Hom}_\mathbb{F}(\pi, \mathbb{F})$ denotes the linear dual of π which is a finitely generated Λ-module, $\mathrm{gr}_\mathfrak{m}(\Lambda) \overset{\text{def}}{=} \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and $\mathrm{gr}_\mathfrak{m}(\pi^\vee) \overset{\text{def}}{=} \bigoplus_{n\geq 0} \mathfrak{m}^n\pi^\vee/\mathfrak{m}^{n+1}\pi^\vee$ (see Theorem 2.1.2 below). This is a key result. Indeed, on the one hand it makes it possible to determine $\mathrm{gr}_\mathfrak{m}((\pi_2/\pi_1)^\vee)$ for any subrepresentations $\pi_1 \subseteq \pi_2$ of π (Corollary 3.2.7(ii) for $\overline{\rho}$ split, [BHH+c] for $\overline{\rho}$ nonsplit with suitable genericity). On the other hand, and most crucially, knowing $\mathrm{gr}_\mathfrak{m}(\pi^\vee)$ is the starting point of *all* the important proofs of this work as we explain now.

1.2 Some sketches of proofs

One important question left open in [BHH+a, § 3.3.2] was the precise structure of the graded $\mathrm{gr}_\mathfrak{m}(\Lambda)$-module $\mathrm{gr}_\mathfrak{m}(\pi^\vee)$ (see [BHH+a, Rk. 3.3.2.6(i)]). We answer this question in the next theorem. We need more notation. Recall from [BHH+a, § 3.1] that $\mathrm{gr}_\mathfrak{m}(\Lambda) \cong \bigotimes_{j\in\{0,\dots,f-1\}} \mathbb{F}\langle y_j, z_j, h_j\rangle$ with relations $[y_j, z_j] = h_j$, $[h_j, z_i] = [y_i, h_j] = 0$ for all $i, j \in \{0, \dots, f-1\}$. We let

$$R \overset{\text{def}}{=} \mathrm{gr}_\mathfrak{m}(\Lambda)/(h_j : 0 \leq j \leq f-1) \cong \mathbb{F}[y_j, z_j : 0 \leq j \leq f-1]$$

which is a (graded) commutative polynomial ring. We let $H \overset{\text{def}}{=} \begin{pmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix} \cong I/I_1$, which naturally acts on Λ, $\mathrm{gr}_\mathfrak{m}(\Lambda)$ and R. Recall that the irreducible continuous representations of I over $\mathbb{F}$ factor as characters $\chi : H \to \mathbb{F}^\times$. In [BHH+a, Def. 3.3.1.1] to each $\chi \in \mathrm{JH}(\pi^{I_1})$ we associated an ideal $\mathfrak{a}(\chi)$ of R (containing y_jz_j for all $j \in \{0, \dots, f-1\}$) which is denoted by $\mathfrak{a}(\lambda)$ in the text and recalled in (12) below.

Theorem 1.2.1 (Theorem 2.1.2)**.** *Assume that $\overline{r}$ is generic.*

(i) *We have an isomorphism of graded $\mathrm{gr}_\mathfrak{m}(\Lambda)$-modules with compatible H-action*

$$\mathrm{gr}_\mathfrak{m}(\pi^\vee) \cong \left(\bigoplus_{\chi\in\mathrm{JH}(\pi^{I_1})} \chi^{-1} \otimes_\mathbb{F} \frac{R}{\mathfrak{a}(\chi)} \right)^{\oplus r}.$$

(ii) *The $\mathrm{gr}_\mathfrak{m}(\Lambda)$-module $\mathrm{gr}_\mathfrak{m}(\pi^\vee)$ is Cohen–Macaulay of grade $2f$.*

In particular the graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ together with its compatible H-action is *local*, i.e. depends only on $\overline{\rho}$, and even just on $\overline{\rho}|_{I_K}$. We remark that Theorem 1.2.1 allows us to compute the entire Hilbert polynomial of $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ (cf. [BHH$^+$c]). Note that, although we know the $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ thanks to Theorem 1.2.1(i), we still do not understand the Λ-module $(\pi|_I)^\vee$.

We sketch the proof of Theorem 1.2.1 (which is given in § 2, especially § 2.5). Denote by N the $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module on the right-hand side of (i). First (ii) follows from (i), since N is Cohen–Macaulay by a direct computation, hence the main issue is (i). If M is any finitely generated R-module which is killed by the ideal $(y_j z_j : 0 \le j \le f-1)$ of R (for instance N), we define its characteristic cycle ([BHH$^+$a, Def. 3.3.4.1])

$$\mathcal{Z}(M) \stackrel{\mathrm{def}}{=} \sum_{\mathfrak{q}} \mathrm{length}(M_{\mathfrak{q}})[\mathfrak{q}] \ \in \bigoplus_{\mathfrak{q}} \mathbb{Z}[\mathfrak{q}], \tag{2}$$

where $\mathfrak{q}$ runs through the minimal prime ideals of $R/(y_j z_j : 0 \le j \le f-1)$. As N is Cohen–Macaulay, any nonzero $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-submodule of N has a nonzero cycle (i.e. N is pure). Since by [BHH$^+$a, Thm. 3.3.2.1] we already have a surjection of graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules $N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ (which implies $\mathcal{Z}(N) \ge \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$ in $\bigoplus_{\mathfrak{q}} \mathbb{Z}[\mathfrak{q}]$), to prove (i) it is enough to prove $\mathcal{Z}(N) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$, as $\mathcal{Z}(-)$ is additive on short exact sequences ([BHH$^+$a, Lemma 3.3.4.2]). To show this, we construct a resolution of the Λ-module $(\pi|_I)^\vee$ by a complex of filtered Λ-modules $P_\bullet$ with compatible H-action such that the associated complex $\mathrm{gr}(P_\bullet)$ of $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules satisfies $H_0(\mathrm{gr}(P_\bullet)) \cong N$ and $H_1(\mathrm{gr}(P_\bullet)) = 0$. Such a filtered complex gives rise to a spectral sequence $E_i^s \Longrightarrow H_i(P_\bullet)$ for $i, s \ge 0$ ([LvO96, § III.1]) and using $H_1(\mathrm{gr}(P_\bullet)) = 0$ we prove that $E_0^\infty = E_0^1$. Since $E_0^1 = H_0(\mathrm{gr}(P_\bullet)) \cong N$ and $E_0^\infty \cong \mathrm{gr}(\pi^\vee)$, where $\mathrm{gr}(\pi^\vee)$ is here computed for the quotient filtration on $\pi^\vee$ induced by the surjection $P_0 \twoheadrightarrow \pi^\vee$, we deduce $N \cong \mathrm{gr}(\pi^\vee)$, which implies $\mathcal{Z}(N) = \mathcal{Z}(\mathrm{gr}(\pi^\vee))$. But we have $\mathcal{Z}(\mathrm{gr}(\pi^\vee)) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$ by [BHH$^+$a, Lemma 3.3.4.3], and thus $\mathcal{Z}(N) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$. The construction of $P_\bullet$ with its properties is quite involved and in particular crucially uses the following result (where the Ext^i_{I/Z_1} are computed in the category of smooth representations of I/Z_1 over $\mathbb{F}$).

Proposition 1.2.2 (§ 2.6)**.** *For any smooth character $\chi : I \to \mathbb{F}^\times$ and any $i \ge 0$, $\mathrm{Ext}^i_{I/Z_1}(\chi, \pi) \neq 0$ only if $\chi \in \mathrm{JH}(\pi^{I_1})$, in which case $\dim_{\mathbb{F}} \mathrm{Ext}^i_{I/Z_1}(\chi, \pi) = \binom{2f}{i} r$.*

Theorem 1.2.1 turns out to be a crucial ingredient in the proof that π is of finite length when $r = 1$ and $\overline{\rho}$ is reducible. We assume these two hypothesis from now on, and we present below a unified sketch of proof in the two cases $\overline{\rho}$ split and $\overline{\rho}$ nonsplit, though in the text we found it preferable to separate the two cases (mainly because the nonsplit case is much more technical).

We fix a nonzero subrepresentation $\pi_1 \subseteq \pi$ and let $\pi_2 \stackrel{\mathrm{def}}{=} \pi/\pi_1$. Hence we have an exact sequence of Λ-modules with H-action $0 \to \pi_2^\vee \to \pi^\vee \to \pi_1^\vee \to 0$. The $\mathfrak{m}$-adic filtration on $\pi^\vee$ induces a filtration on $\pi_2^\vee$ and we denote by $\mathrm{gr}(\pi_2^\vee)$ the associated $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module. Just like the definition of the $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module N in Theorem 1.2.1(i) only uses the H-representation π^{I_1} (and *a fortiori* only the $\mathrm{GL}_2(\mathbb{F}_q)$-representation π^{K_1}), we define an explicit quotient N_1 of N which only depends on the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $\pi_1^{K_1}$. In the split case one has

$$N_1 = \bigoplus_{\chi \in \mathrm{JH}(\pi_1^{I_1})} \chi^{-1} \otimes_{\mathbb{F}} \frac{R}{\mathfrak{a}(\chi)}, \tag{3}$$

in particular N_1 is then a direct summand of N and only depends on the H-representation $\pi_1^{I_1}$, but this is no longer true in the nonsplit case if $\pi_1 \neq \pi$ (see Step 2 in the proof of Proposition 4.4.3 together with (75) and Definition 4.2.4). Defining $N_2 \stackrel{\text{def}}{=} \ker(N \twoheadrightarrow N_1)$, we prove that there is a commutative diagram with exact rows of graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules (see Step 1 in the proof of Proposition 3.2.2 for $\overline{\rho}$ split, Step 2 in the proof of Proposition 4.4.3 for $\overline{\rho}$ nonsplit):

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}(\pi_2^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \arrow[r] & 0 \\
0 \arrow[r] & N_2 \arrow[r] \arrow[u, hook] & N \arrow[r] \arrow[u, "\cong"] & N_1 \arrow[r] \arrow[u, two heads] & 0
\end{tikzcd} \tag{4}$$

with injective (resp. surjective) vertical map on the left (resp. right) and where the middle isomorphism is Theorem 1.2.1(i).

The next step is the following theorem:

Theorem 1.2.3 (Proposition 3.2.2, Proposition 4.4.3)**.** *The left vertical injection in (4), hence also the right vertical surjection, are isomorphisms. In particular $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$, $\mathrm{gr}(\pi_2^\vee)$ are Cohen–Macaulay $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules of grade $2f$, and $\pi_1^\vee$, $\pi_2^\vee$ are Cohen–Macaulay Λ-modules of grade $2f$.*

We sketch the proof of Theorem 1.2.3.

The Cohen–Macaulayness of $\pi_1^\vee$, $\pi_2^\vee$ follows from the one of $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$, $\mathrm{gr}(\pi_2^\vee)$ ([LvO96, Prop. III.2.2.4]), which itself follows from the first statement of Theorem 1.2.3 as N_1, N_2 can be checked to be Cohen–Macaulay $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules. Note that, by dévissage and since Λ is Auslander regular, one then deduces from [LvO96, Cor. III.2.1.6] that the linear dual of *any* subquotient of $\pi^\vee$ is Cohen–Macaulay of grade $2f$. In particular this proves Theorem 1.1.2(i) and Theorem 1.1.3(i).

Hence it is enough to prove $N_1 \xrightarrow{\sim} \mathrm{gr}(\pi_1^\vee)$. Since, just like N, the $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module N_1 is pure, by the same argument as for N (see the sentences below (2)) it is enough to prove that $\mathcal{Z}(N_1) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee))$, or equivalently by diagram (4) that $\mathcal{Z}(N_2) = \mathcal{Z}(\mathrm{gr}(\pi_2^\vee))$.

We then use the essential self-duality of π ([HW22, Thm. 8.2] with [BHH+23, Thm. 8.4.1] and [Wan23, Thm. 6.3(i)]): there is a $\mathrm{GL}_2(K)$-equivariant isomorphism $\mathrm{Ext}^{2f}_\Lambda(\pi^\vee, \Lambda) \cong \pi^\vee \otimes_{\mathbb{F}} (\det(\overline{\rho})\omega^{-1})$, where $\mathrm{Ext}^{2f}_\Lambda(\pi^\vee, \Lambda)$ is endowed with the action of $\mathrm{GL}_2(K)$ defined in [Koh17, Prop. 3.2]. Then we can define $\widetilde{\pi}_2 \subseteq \pi$ as the unique $\mathrm{GL}_2(K)$-subrepresentation such that

$$\widetilde{\pi}_2^\vee = \mathrm{im}\left\{\mathrm{Ext}^{2f}_\Lambda(\pi^\vee, \Lambda) \to \mathrm{Ext}^{2f}_\Lambda(\pi_2^\vee, \Lambda)\right\} \otimes_{\mathbb{F}} (\det(\overline{\rho})^{-1}\omega).$$

Since $\widetilde{\pi}_2$ is a subrepresentation of π, we can define a surjection of $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules

$$\widetilde{N}_2 \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)$$

analogous to $N_1 \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$, where $\widetilde{N}_2$ again only depends on the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $\widetilde{\pi}_2^{K_1}$. In particular $\mathcal{Z}(\widetilde{N}_2) \geq \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee))$. Note that by the same argument as in the proof of [BHH+a, Prop. 3.3.5.3(iii)] we have $\mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)) = \mathcal{Z}(\mathrm{gr}(\pi_2^\vee))$. Since $\mathcal{Z}(\mathrm{gr}(\pi_2^\vee)) \geq \mathcal{Z}(N_2)$ by the left injection in (4), we deduce

$$\mathcal{Z}(\widetilde{N}_2) \geq \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)) = \mathcal{Z}(\mathrm{gr}(\pi_2^\vee)) \geq \mathcal{Z}(N_2)$$

and hence it is enough to prove $\mathscr{Z}(\widetilde{N}_2) = \mathscr{Z}(N_2)$.

The equality $\mathscr{Z}(\widetilde{N}_2) = \mathscr{Z}(N_2)$ is the heart of the proof of Theorem 1.2.3 and is particularly subtle in the nonsplit case. In both cases (split or nonsplit) it boils down to determining the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $\widetilde{\pi}_2^{K_1}$ from the $\mathrm{GL}_2(\mathbb{F}_q)$-representation $\pi_1^{K_1}$. For that, we do not know any proof that avoids (φ, Γ)-modules. We have the formula

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) - \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1), \tag{5}$$

where the first equality follows from $\mathscr{Z}(\mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)) = \mathscr{Z}(\mathrm{gr}(\pi_2^\vee))$ with [BHH$^+$a, Prop. 3.3.5.3(i)] and the second from the exactness of the functor $D_\xi^\vee$ ([BHH$^+$a, Thm. 3.1.3.7]). In the split case, using the equalities

$$\begin{aligned}
\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) &= 2^f, \\
\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}_2) &= |\operatorname{JH}(\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\widetilde{\pi}_2))|, \\
\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1) &= |\operatorname{JH}(\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1))|
\end{aligned}$$

(where the first follows from [BHH$^+$a, Thm. 1.3.1] and where the other two are [BHH$^+$a, Prop. 3.3.5.3(ii)]), we manage starting from (5) to determine $\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\widetilde{\pi}_2)$, hence $\widetilde{\pi}_2^{K_1}$ (using the proof of [BP12, Thm. 19.10]), hence $\widetilde{N}_2$, and finally check that $\mathscr{Z}(\widetilde{N}_2) = \mathscr{Z}(N_2)$. In the nonsplit case using the (much harder) equalities

$$\begin{aligned}
\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) &= 2^f, \\
\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}_2) &= |\operatorname{JH}(\widetilde{\pi}_2^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})|, \\
\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1) &= |\operatorname{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})|
\end{aligned}$$

(which all follow from [Wan, Thm. 1.2]) with (5) (and Theorem 4.3.15 in the text applied to both π_1, $\widetilde{\pi}_2$), we can again determine $\widetilde{\pi}_2^{K_1}$ and once more check $\mathscr{Z}(\widetilde{N}_2) = \mathscr{Z}(N_2)$.

We now sketch the proof that π is of finite length (for $\overline{\rho}$ reducible) using Theorem 1.2.3.

Let $\pi_1 \subseteq \pi$ be a nonzero subrepresentation, and let $\pi_1' \subseteq \pi_1$ be the $\mathrm{GL}_2(K)$-subrepresentation generated by $\operatorname{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)$ if $\overline{\rho}$ is split, by $\pi_1^{K_1}$ if $\overline{\rho}$ is nonsplit. We then have ${\pi_1'}^{K_1} \xrightarrow{\sim} \pi_1^{K_1}$ in both cases (using the proof of [BP12, Thm. 19.10] in the split case). The $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-module N_1 in (4) is the same for both π_1 and π_1' since it only depends on the $\mathrm{GL}_2(\mathbb{F}_q)$-representation ${\pi_1'}^{K_1} \cong \pi_1^{K_1}$. By Theorem 1.2.3 we deduce that the natural surjection $N_1 \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1'^\vee)$ is an isomorphism, in particular $\mathfrak{m}^n\pi_1^\vee/\mathfrak{m}^{n+1}\pi_1^\vee \xrightarrow{\sim} \mathfrak{m}^n\pi_1'^\vee/\mathfrak{m}^{n+1}\pi_1'^\vee$ for all $n \geq 0$, hence by dévissage $\pi_1^\vee/\mathfrak{m}^{n+1}\pi_1^\vee \xrightarrow{\sim} \pi_1'^\vee/\mathfrak{m}^{n+1}\pi_1'^\vee$ for $n \geq 0$, hence $\pi_1^\vee \xrightarrow{\sim} \pi_1'^\vee$ or equivalently $\pi_1' \xrightarrow{\sim} \pi_1$.

This first implies that π_1 is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle if $\overline{\rho}$ is split, by its K_1-invariant if $\overline{\rho}$ is nonsplit (since π_1' is). As the quotient of a $\mathrm{GL}_2(K)$-representation generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle (resp. its K_1-invariants) is *a fortiori* also generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle (resp. its K_1-invariants), we have proven Theorem 1.1.2(ii) and Theorem 1.1.3(ii).

We then obtain that π is of finite length, as there are only finitely many $\mathrm{GL}_2(\mathbb{F}_q)$-subrepresentations $\pi_1^{K_1}$ inside the $\mathrm{GL}_2(\mathbb{F}_q)$-representation π^{K_1} (recall the latter is explicitly known and only depends on $\overline{\rho}|_{I_K}$, see [HW18, LMS22] for $\overline{\rho}$ split, [Le19] for $\overline{\rho}$ nonsplit). A more precise calculation inside π^{K_1} gives the more precise statements in Theorem 1.1.1(ii), (iii), though the multiplicity freeness in the nonsplit case is more involved, see Corollary 4.4.10.

So far we have briefly gone over the proofs of Theorem 1.1.1, of Theorem 1.1.2(i), (ii) and of Theorem 1.1.3(i), (ii). We now sketch the proofs of Theorem 1.1.2(iii), (iv), (v).

Since in the split case N_1 in (3) is a direct summand of N, Theorem 1.2.3 implies that the exact sequence of graded $\mathrm{gr}_{\mathfrak{m}}(\Lambda)$-modules $0 \to \mathrm{gr}(\pi_2^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \to 0$ in (4) is split. Then by a dimension count we deduce that the map $\pi[\mathfrak{m}^n] \to (\pi/\pi_1)[\mathfrak{m}^n]$ is surjective for all $n \geq 0$. It is then not difficult to deduce the exactness in Theorem 1.1.2(v). The splitness in *loc. cit.* for $n \leq \max\{6, f+2\}$ comes from the following description of the I-representation $\pi[\mathfrak{m}^n]$ for such n (see Lemma 2.4.2):

$$\pi[\mathfrak{m}^n] \cong \bigoplus_{\chi \in \mathrm{JH}(\pi^{I_1})} \tau_\chi^{(n)}, \tag{6}$$

where the I-representations $\tau_\chi^{(n)}$ (denoted $\tau_\lambda^{(n)}$ in the text) are defined in Lemma 2.4.1. From (6) one deduces $\pi_1[\mathfrak{m}^n] \cong \bigoplus_{\chi \in \mathrm{JH}(\pi_1^{I_1})} \tau_\chi^{(n)}$ – whence the splitting – using the isomorphism $N_1 \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ in Theorem 1.2.3 together with (3) (see the end of the proof of Corollary 3.2.5).

Then the first exact sequence in Theorem 1.1.2(iv) easily follows from the exact sequence in Theorem 1.1.2(v) applied with $n = 1$ (see Lemma 3.2.6). Note that this first exact sequence implies Theorem 1.1.2(iii) by the exactness of $D_\xi^\vee$ ([BHH$^+$a, Thm. 3.1.3.7]) and the case of subrepresentations ([BHH$^+$a, Thm. 3.3.5.3(ii)]). The second exact sequence in Theorem 1.1.2(iv) and its splitness both follow from the first using, as we have seen with $\widetilde{\pi}_2$ above, that if we know $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)$ for a subrepresentation $\pi_1 \subseteq \pi$ when $\overline{\rho}$ is split we also know $\pi_1^{K_1}$, and moreover that $\pi_1^{K_1}$ is a direct summand of π^{K_1}.

Acknowledgements: The results of this work in the non-semisimple case use as a key input results of Yitong Wang [Wan]. Y. H. is partially supported by National Key R&D Program of China 2020YFA0712600, National Natural Science Foundation of China Grants 12288201 and 12425103, National Center for Mathematics and Interdisciplinary Sciences and Hua Loo-Keng Key Laboratory of Mathematics, Chinese Academy of Sciences. F. H. is partially supported by an NSERC grant. S. M. and B. S. are partially supported by the Institut Universitaire de France.

1.3 Notation and preliminaries

We normalize local class field theory so that uniformizers correspond to geometric Frobenius elements. We fix an embedding $\kappa_0 : \mathbb{F}_q \hookrightarrow \mathbb{F}$ and let $\kappa_j \stackrel{\text{def}}{=} \kappa_0 \circ \varphi^j$, where φ is the arithmetic Frobenius on $\mathbb{F}_q$. Given $J \subseteq \{0, \dots, f-1\}$ we define $J^c \stackrel{\text{def}}{=} \{0, 1, \dots, f-1\} \setminus J$. We let $I \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{O}_K^\times & \mathcal{O}_K \\ p\mathcal{O}_K & \mathcal{O}_K^\times \end{pmatrix} \subseteq \mathrm{GL}_2(\mathcal{O}_K)$ denote the (upper) Iwahori subgroup of $\mathrm{GL}_2(K)$, I_1 the pro-p radical of I, Z_1 the center of I_1, and $K_1 \stackrel{\text{def}}{=} 1 + p\mathrm{M}_2(\mathcal{O}_K) \subseteq I_1$. We let $\Gamma \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbb{F}_q) \cong \mathrm{GL}_2(\mathcal{O}_K)/K_1$.

Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. We will say that $\overline{\rho}$ is *n-generic* for some integer $n \geq 0$ if, up to twist, $\overline{\rho}|_{I_K}^{\mathrm{ss}} \ncong \omega \oplus 1$ and either (using the notation of § 1.1)

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega_f^{\sum_{j=0}^{f-1}(r_j+1)p^j} & * \\ & 1 \end{pmatrix} \qquad \text{with } n \leq r_j \leq p-3-n \text{ for all } 0 \leq j \leq f-1 \tag{7}$$

or

$$\overline{\rho}|_{I_K} \cong \begin{pmatrix} \omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j} & \\ & \omega_{2f}^{p^f(\text{same})} \end{pmatrix} \qquad \text{with } \begin{cases} n \leq r_j \leq p-3-n & \text{for } 0 < j \leq f-1, \\ n+1 \leq r_0 \leq p-2-n & \text{for } j=0. \end{cases} \tag{8}$$

In particular, if $\overline{\rho}$ is n-generic then it is n-generic in the sense of [BHH⁺23, Def. 2.3.4] (see also the beginning of [BHH⁺23, §4.1]), and $\overline{\rho}$ is 0-generic precisely when $\overline{\rho}$ is generic in the sense of [BP12, Def. 11.7] (note that the condition $\overline{\rho}|_{I_K}^{\mathrm{ss}} \ncong \omega \oplus 1$, up to twist, precisely rules out that $(r_0, \ldots, r_{f-1}) \in \{(0, \ldots, 0), (p-3, \ldots, p-3)\}$ when $\overline{\rho}$ is reducible).

Attached to a 0-generic $\overline{\rho}$ we have a set $W(\overline{\rho})$ of Serre weights, i.e. irreducible representations of Γ over $\mathbb{F}$, defined in [BDJ10, § 3], and a finite length Γ-representation $D_0(\overline{\rho})$ over $\mathbb{F}$, defined in [BP12, § 13], which is of the form $D_0(\overline{\rho}) = \bigoplus_{\tau \in W(\overline{\rho})} D_{0,\tau}(\overline{\rho})$, where each $D_{0,\tau}(\overline{\rho})$ is indecomposable and multiplicity free with socle the Serre weight τ ([BP12, § 15]).

Suppose that $\overline{\rho}$ is 0-generic. Recall the set $\mathscr{P}$ parametrizing $D_0(\overline{\rho})^{I_1}$, see [Bre14, § 4] (and denoted there by $\mathscr{P}\mathscr{D}$, resp. $\mathscr{P}\mathscr{I}\mathscr{D}$, if $\overline{\rho}$ is reducible, resp. irreducible). Recall also the subset $\mathscr{D} \subseteq \mathscr{P}$ parametrizing (the I_1-invariants of) the set of Serre weights in $W(\overline{\rho})$ (denoted in *loc. cit.* by $\mathscr{D}$ or $\mathscr{I}\mathscr{D}$ if $\overline{\rho}$ is reducible or irreducible respectively). We let $\mathscr{D}^{\mathrm{ss}} \subseteq \mathscr{P}^{\mathrm{ss}}$ denote the corresponding sets for the semisimplification $\overline{\rho}^{\mathrm{ss}}$ of $\overline{\rho}$, so $\mathscr{P} \subseteq \mathscr{P}^{\mathrm{ss}}$ and $\mathscr{D} \subseteq \mathscr{D}^{\mathrm{ss}}$. Note that $\chi \in \mathrm{JH}(D_0(\overline{\rho})^{I_1})$ implies $\chi \neq \chi^s$ by [BP12, Cor. 13.6].

Since we will be using this many times, we recall more precisely that if $\overline{\rho}$ is *reducible*, $\mathscr{P}^{\mathrm{ss}}$ denotes the set of f-tuples $(\lambda_0(x_0), \ldots, \lambda_{f-1}(x_{f-1}))$ such that:

(i) $\lambda_j(x_j) \in \{x_j, x_j+1, x_j+2, p-3-x_j, p-2-x_j, p-1-x_j\}$;

(ii) if $\lambda_j(x_j) \in \{x_j, x_j+1, x_j+2\}$, then $\lambda_{j+1}(x_{j+1}) \in \{x_{j+1}, x_{j+1}+2, p-2-x_{j+1}\}$;

(iii) if $\lambda_j(x_j) \in \{p-3-x_j, p-2-x_j, p-1-x_j\}$, then $\lambda_{j+1}(x_{j+1}) \in \{x_{j+1}+1, p-3-x_{j+1}, p-1-x_{j+1}\}$

and $\mathscr{D}^{\mathrm{ss}}$ is the subset such that $\lambda_j(x_j) \in \{x_j, x_j+1, p-3-x_j, p-2-x_j\}$. Moreover, there exists a unique subset $J_{\overline{\rho}} \subseteq \{0, \ldots, f-1\}$ such that

$$\begin{aligned} \mathscr{D} &= \Big\{\lambda \in \mathscr{D}^{\mathrm{ss}} : \lambda_j(x_j) \in \{x_j+1, p-3-x_j\} \Rightarrow j \in J_{\overline{\rho}}\Big\}, \\ \mathscr{P} &= \Big\{\lambda \in \mathscr{P}^{\mathrm{ss}} : \lambda_j(x_j) \in \{x_j+2, p-3-x_j\} \Rightarrow j \in J_{\overline{\rho}}\Big\}. \end{aligned} \tag{9}$$

In particular, $|W(\overline{\rho})| = 2^{|J_{\overline{\rho}}|}$.

For $\lambda \in \mathscr{P}$ we denote by χ_λ the character of H corresponding to λ. (More precisely, in

[Bre14, § 4] a Serre weight σ_λ is associated to $\lambda \in \mathscr{P}$ and χ_λ is the action of $H = I/I_1$ on the 1-dimensional subspace $\sigma_\lambda^{I_1}$.) Set

$$J_\lambda \stackrel{\text{def}}{=} \{j \in \{0, \dots, f-1\} : \lambda_j(x_j) \in \{x_j + 1, x_j + 2, p - 3 - x_j\}\} \tag{10}$$

and let $\ell(\lambda) \stackrel{\text{def}}{=} |J_\lambda|$. By [BP12, § 11] the map $\lambda \mapsto J_\lambda$ induces a bijection between $\mathscr{D}^{\text{ss}}$ and the set of subsets of $\{0, \dots, f-1\}$. Sometimes we will abuse notation and write $J_\tau \stackrel{\text{def}}{=} J_\lambda$ and $\ell(\tau) \stackrel{\text{def}}{=} \ell(\lambda)$ if $\tau \in W(\overline{\rho}^{\text{ss}})$ is parametrized by $\lambda \in \mathscr{D}^{\text{ss}}$. Given $\lambda \in \mathscr{D}^{\text{ss}}$ with corresponding subset $J = J_\lambda \subseteq \{0, \dots, f-1\}$ we write $\delta(\lambda) \in \mathscr{D}^{\text{ss}}$ for the f-tuple defined by $\delta(\lambda)_j \stackrel{\text{def}}{=} \lambda_{j+1}$ for all $j \in \{0, \dots, f-1\}$, and $\delta(J) \subseteq \{0, \dots, f-1\}$ for the subset corresponding to $\delta(\lambda)$.

As in [BP12, § 1], given f integers $r_0, \dots, r_{f-1} \in \{0, \dots, p-1\}$ we denote by $(r_0, \dots, r_{f-1})$ the Serre weight

$$\mathrm{Sym}^{r_0}\mathbb{F}^2 \otimes_{\mathbb{F}} (\mathrm{Sym}^{r_1}\mathbb{F}^2)^{\mathrm{Fr}} \otimes \cdots \otimes_{\mathbb{F}} (\mathrm{Sym}^{r_{f-1}}\mathbb{F}^2)^{\mathrm{Fr}^{f-1}},$$

where $\mathrm{GL}_2(\mathbb{F}_q)$ acts on $(\mathrm{Sym}^{r_j}\mathbb{F}^2)^{\mathrm{Fr}^j}$ via $\kappa_j : \mathbb{F}_q \hookrightarrow \mathbb{F}$. Following [HW22, § 2], we say that a Serre weight is *m-generic* for some integer $m \geq 0$ if, up to twist, $\sigma \cong (r_0, \dots, r_{f-1})$, where $m \leq r_j \leq p - 2 - m$ for all $j \in \{0, \dots, f-1\}$. We say that an $\mathbb{F}$-valued character χ of I is *m-generic* if $\chi = \sigma^{I_1}$ for some m-generic Serre weight σ. For any smooth character $\chi : I \to \mathbb{F}^\times$ we define $\chi^s \stackrel{\text{def}}{=} \chi(\Pi(\cdot)\Pi^{-1})$ with $\Pi \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. If σ is a Serre weight, we write χ_σ for the character of I/I_1 on σ^{I_1} and $\sigma^{[s]}$ for the unique Serre weight distinct from σ such that $\chi_{\sigma^{[s]}} = \chi_\sigma^s$. We remark that if $\overline{\rho}$ is n-generic, then any $\sigma \in W(\overline{\rho}^{\text{ss}})$ is n-generic, and χ_λ is $(n-1)$-generic for any $\lambda \in \mathscr{P}^{\text{ss}}$ (if $n \geq 1$).

Let $\Lambda \stackrel{\text{def}}{=} \mathbb{F}[\![I_1/Z_1]\!]$, a complete noetherian local ring with maximal ideal $\mathfrak{m} \stackrel{\text{def}}{=} \mathfrak{m}_{I_1/Z_1}$, and let $\mathrm{gr}(\Lambda) \stackrel{\text{def}}{=} \mathrm{gr}_{\mathfrak{m}}(\Lambda)$ be the graded ring associated to Λ with respect to the $\mathfrak{m}$-adic filtration on Λ. The rings Λ and $\mathrm{gr}(\Lambda)$ are Auslander regular (see [BHH+23, Thm. 5.3.4] with [LvO96, Thm. III.2.2.5]). Recall ([BHH+a, § 3.1]) that we have an isomorphism of (noncommutative) algebras

$$\mathrm{gr}(\Lambda) \cong \bigotimes_{j \in \{0, \dots, f-1\}} \mathbb{F}\langle y_j, z_j, h_j \rangle \tag{11}$$

with relations $[y_j, z_j] = h_j$, $[h_j, z_i] = [y_i, h_j] = 0$ for all $i, j \in \{0, \dots, f-1\}$. We use increasing filtrations throughout, i.e. $F_n\Lambda = \mathfrak{m}^{-n}$ for $n \leq 0$, and the degrees of y_j and z_j (resp. h_j) are -1 (resp. -2). Define the graded ideal $J \stackrel{\text{def}}{=} (h_j, y_j z_j : 0 \leq j \leq f-1)$ of $\mathrm{gr}(\Lambda)$. As in [BHH+a, § 3] we define

$$R \stackrel{\text{def}}{=} \mathrm{gr}(\Lambda)/(h_j : 0 \leq j \leq f-1) \cong \mathbb{F}[y_j, z_j : 0 \leq j \leq f-1]$$

which is the largest commutative quotient of $\mathrm{gr}(\Lambda)$. We also define the following quotient of R:

$$\overline{R} \stackrel{\text{def}}{=} \mathrm{gr}(\Lambda)/J \cong R/(y_j z_j : 0 \leq j \leq f-1).$$

We recall from [BHH+a, Def. 3.3.1.1] that given $\lambda \in \mathscr{P}$ we have an associated homogeneous ideal $\mathfrak{a}(\lambda) = (t_0, \dots, t_{f-1})$ of R, where the $t_j = t_j(\lambda)$ are defined as follows:

$$t_j \stackrel{\text{def}}{=} \begin{cases} z_j & \text{if } \lambda_j(x_j) \in \{x_j, p-3-x_j\} \text{ and } j \in J_{\overline{\rho}} \\ y_j & \text{if } \lambda_j(x_j) \in \{x_j+2, p-1-x_j\} \text{ and } j \in J_{\overline{\rho}} \\ y_j z_j & \text{if } \lambda_j(x_j) \in \{x_j, p-1-x_j\} \text{ and } j \notin J_{\overline{\rho}} \\ y_j z_j & \text{if } \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}. \end{cases} \tag{12}$$

Note that $(y_jz_j : 0 \le j \le f-1) \subseteq \mathfrak{a}(\lambda)$, so we often think of $\mathfrak{a}(\lambda)$ as ideal of $\overline{R}$.

Let $H \overset{\text{def}}{=} \begin{pmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix} \cong I/I_1$. We write $\alpha_j : H \to \mathbb{F}^\times$ for the character defined by $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \kappa_j(ad^{-1})$. We recall that for any $j \in \{0, \dots, f-1\}$ the element y_j (resp. z_j, resp. h_j) is an H-eigenvector with associated eigencharacter α_j (resp. α_j^{-1}, resp. the trivial character). Note that H acts on I_1/Z_1 by conjugation and hence on Λ (resp. $\mathrm{gr}(\Lambda)$), preserving the filtration (resp. the grading). This induces H-actions also on R, $\overline{R}$, and $R/\mathfrak{a}(\lambda)$ for any $\lambda \in \mathscr{P}$. We say that a filtered Λ-module M has a *compatible H-action* if it has an H-action that preserves the filtration and such that $h(rm) = h(r)h(m)$ for all $h \in H$, $r \in \Lambda$, and $m \in M$. Similarly we define the notion of a graded $\mathrm{gr}(\Lambda)$-module with compatible H-action.

Suppose that H' is a compact p-adic analytic group and that π_1, π_2 are smooth representations of H' over $\mathbb{F}$. We write $\mathrm{Ext}^i_{H'}(\pi_1, \pi_2)$ for the i-th Ext group computed in the category of smooth representations of H' over $\mathbb{F}$. Dually, the functors $\mathrm{Tor}_i^{\mathbb{F}[\![H']\!]}(\pi_1^\vee, \pi_2^\vee)$ and $\mathrm{Ext}^i_{\mathbb{F}[\![H']\!]}(\pi_1^\vee, \pi_2^\vee)$ are computed in the abelian category of pseudocompact $\mathbb{F}[\![H']\!]$-modules. (See for example [Eme10, § 2].) If σ is a smooth representation of H' over $\mathbb{F}$ we write $\mathrm{Inj}_{H'}\,\sigma$ for the injective envelope of σ in the category of smooth H'-representations over $\mathbb{F}$. If σ has finite length, we write $\mathrm{JH}(\sigma)$ for its set of irreducible constituents up to isomorphism.

Throughout this paper, if R is a filtered (resp. graded) ring, a morphism of filtered (resp. graded) R-modules $f : M \to N$ will always be a *filtered (resp. graded) morphism of degree zero*, i.e. satisfying $f(M_i) \subseteq N_i$ for all $i \in \mathbb{Z}$. For $k \in \mathbb{Z}$, $M(k)$ denotes the filtered (resp. graded) R-module obtained by filtering (resp. grading) M by $F_n(M(k)) \overset{\text{def}}{=} M(n+k)$ (resp. $M(k)_n \overset{\text{def}}{=} M_{n+k}$) for all $n \in \mathbb{Z}$.

If R is any ring and M any left R-module, we recall that $\mathrm{Ext}^i_R(M, R)$ for $i \in \mathbb{Z}_{\ge 0}$ is a right R-module (for $i = 0$ the right R-action is given by $(fr)(m) \overset{\text{def}}{=} f(m)r$ for $r \in R$, $f \in \mathrm{Hom}_R(M, R)$ and $m \in M$) and we use the notation $\mathrm{E}^i_R(M) \overset{\text{def}}{=} \mathrm{Ext}^i_R(M, R)$. If $R = \Lambda$ or $R = \mathrm{gr}(\Lambda)$, we can and will use the anti-involution $g \mapsto g^{-1}$ on I/Z_1 to consider any right R-module (with compatible H-action or not) as a left R-module.

2 Cohen–Macaulayness of $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$

We completely describe $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ for a smooth mod p representation π of $\mathrm{GL}_2(K)$ satisfying assumptions (i), (ii) in [BHH+a, § 3.3.2] and an extra assumption (iv) (defined below). When π is a suitable Hecke eigenspace in the mod p cohomology, we prove that π satisfies (iv) (in addition to (i) and (ii)).

2.1 The theorem

We state the main theorem (Theorem 2.1.2).

Let $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous 0-generic representation as in § 1.3. Let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying assumptions (i), (ii) in [BHH$^+$a, § 3.3.2], i.e.

(i) there exists an integer $r \geq 1$ such that $\pi^{K_1} \cong D_0(\overline{\rho})^{\oplus r}$ as $\mathrm{GL}_2(\mathcal{O}_K)K^\times$-representations, where $K^\times$ acts by $\det(\overline{\rho})\omega^{-1}$ (in particular π is admissible and has central character $\det(\overline{\rho})\omega^{-1}$);

(ii) for any $\lambda \in \mathscr{P}$ we have $[\pi[\mathfrak{m}^3] : \chi_\lambda] = [\pi[\mathfrak{m}] : \chi_\lambda]$.

For later reference we also recall assumption (iii) of [BHH$^+$a, § 3.3.5], *though we will not assume it until section 3*:

(iii) there is a $\mathrm{GL}_2(K)$-equivariant isomorphism of Λ-modules

$$\mathrm{E}^{2f}_\Lambda(\pi^\vee) \cong \pi^\vee \otimes (\det(\overline{\rho})\omega^{-1}),$$

where $\mathrm{E}^{2f}_\Lambda(\pi^\vee)$ is endowed with the $\mathrm{GL}_2(K)$-action defined in [Koh17, Prop. 3.2].

Additional to assumptions (i), (ii) above, we make the following assumption on π:

(iv) for any smooth character $\chi : I \to \mathbb{F}^\times$ and any $i \geq 0$, $\mathrm{Ext}^i_{I/Z_1}(\chi, \pi) \neq 0$ only if $[\pi[\mathfrak{m}] : \chi] \neq 0$, in which case

$$m_i \overset{\mathrm{def}}{=} \dim_{\mathbb{F}} \mathrm{Ext}^i_{I/Z_1}(\chi, \pi) = \binom{2f}{i} r,$$

where $r \geq 1$ is the multiplicity in assumption (i).

Note that we do not assume that $r = 1$ or that $\overline{\rho}$ is semisimple.

Remark 2.1.1. By picking a minimal free resolution of $\pi^\vee$ with compatible H-action over the local ring Λ (cf. Remark 2.3.1(v)), we see that $\mathrm{Tor}^\Lambda_i(\mathbb{F}, \pi^\vee)$ is dual to

$$\mathrm{Ext}^i_\Lambda(\pi^\vee, \mathbb{F}) \cong \mathrm{Ext}^i_{I_1/Z_1}(\mathbb{F}, \pi) \cong \bigoplus_\chi \mathrm{Ext}^i_{I/Z_1}(\chi, \pi),$$

where χ runs over all smooth $\mathbb{F}$-characters of I. From assumption (iv) we deduce that

$$\dim_{\mathbb{F}} \mathrm{Tor}^\Lambda_i(\mathbb{F}, \pi^\vee) = (\dim_{\mathbb{F}} \pi^{I_1}) \binom{2f}{i} \tag{13}$$

(as $\pi[\mathfrak{m}] = \pi^{I_1}$). Decomposing for the action of H, we see moreover that $\mathrm{Ext}^i_{I/Z_1}(\chi, \pi)$ is dual to the χ^{-1}-isotypic piece of $\mathrm{Tor}^\Lambda_i(\mathbb{F}, \pi^\vee)$, hence

$$\mathrm{Tor}^\Lambda_i(\mathbb{F}, \pi^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} (\chi_\lambda^{-1})^{\oplus m_i}.$$

Our aim in this subsection is to prove the following theorem which strengthens [BHH$^+$a, Thm. 3.3.2.1].

Theorem 2.1.2. *Assume that $\overline{\rho}$ is 9-generic and that π satisfies assumptions* (i), (ii) *and* (iv) *above. Then we have an isomorphism of graded* $\mathrm{gr}(\Lambda)$*-modules with compatible H-action*

$$\Big(\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1}\otimes \frac{R}{\mathfrak{a}(\lambda)}\Big)^{\oplus r} \cong \mathrm{gr}_{\mathfrak{m}}(\pi^\vee). \tag{14}$$

In particular, $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ *is a Cohen–Macaulay* $\mathrm{gr}(\Lambda)$*-module of grade $2f$. Moreover,* $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ *is essentially self-dual in the sense that*

$$\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \cong \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)\otimes(\det(\overline{\rho})\omega^{-1}) \tag{15}$$

as $\mathrm{gr}(\Lambda)$*-modules (without grading) with compatible H-action.*

Remark 2.1.3. The fact that $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ is Cohen–Macaulay as $\mathrm{gr}(\Lambda)$-module implies that $\pi^\vee$ is Cohen–Macaulay as Λ-module [LvO96, Prop. III.2.2.4]. But this was already known by (the proof of) [HW22, Prop. A.8] when $r=1$.

Remark 2.1.4. The isomorphism (14) together with the proof of Corollary 2.3.4 show that the isomorphism (15) cannot respect the grading, even up to shift. Namely, $\mathbb{F}\otimes_{\mathrm{gr}(\Lambda)}\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$ is not supported in just one degree.

The proof of Theorem 2.1.2 will be given in § 2.5. In Proposition 2.6.2 we verify that a globally defined $\pi=\pi(\overline{\rho})$ satisfies assumption (iv) (see § 2.6 below for details). We note that some cases of assumption (iv) were established in [HW22, Prop. 10.10, Cor. 10.11] when $\overline{\rho}$ is nonsplit reducible.

2.2 Preliminaries on filtered and graded modules

Following [LvO96, § I.6], a finitely generated filtered Λ-module L is called *filt-free* if it is free as a Λ-module with basis $(e_i)_{1\le i\le n}$ having the property that there exists a family $(k_i)_{1\le i\le n}$ of integers such that

$$F_kL=\bigoplus_{1\le i\le n}(F_{k-k_i}\Lambda)e_i,\quad \forall\ k\in\mathbb{Z}.$$

For convenience, we call $(e_i)_{1\le i\le n}$ a *filt-basis* of L. Equivalently, L is filt-free if and only if $L\cong\bigoplus_{i=1}^n\Lambda(-k_i)$ for some integers k_i. (We remark that [LvO96] add the condition $e_i\notin F_{k_i-1}L$, but this is automatic over a separated ring, and should not be demanded otherwise because of [LvO96, Lemma I.6.2(1)].)

If L is a filt-free module and L' is a submodule which is itself a free Λ-module, then L', equipped with the induced filtration, need not be filt-free in general, even if L' is a direct summand of L as Λ-modules (see Remark 2.2.2). However, we will see that this is true in some special cases (see Lemma 2.2.3).

Remark 2.2.1. Consider the filt-free module $L=\Lambda(0)\oplus\Lambda(-2)$, with filt-basis (e_1,e_2). Let

$e' = xe_1 + e_2$, with $x \in \Lambda$ and $L' \overset{\text{def}}{=} \Lambda e'$. Then L' is a direct summand of L as a Λ-module. One checks that, equipped with the induced filtration L' is isomorphic to $\Lambda(-2)$, and $\mathrm{gr}(L')$ is a direct summand of $\mathrm{gr}(L)$.

However, if we take $e'' = e_1 + xe_2$ with $x \in \mathfrak{m} \backslash \mathfrak{m}^2$ and $L'' \overset{\text{def}}{=} \Lambda e''$ equipped with the induced filtration, then the morphism $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}(L'') \to \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}(L)$ is zero. Note that L'' is still filt-free (isomorphic to $\Lambda(-1)$).

Remark 2.2.2. Suppose $L = \Lambda(0) \oplus \Lambda(0) \oplus \Lambda(-2)$, with a filt-basis (e_1, e_2, e_3). Let L' be the submodule generated by $f_1 \overset{\text{def}}{=} e_1 + Y_0 e_3$ and $f_2 \overset{\text{def}}{=} e_2 + Z_0 e_3$, with induced filtration, where $Y_0, Z_0 \in \mathfrak{m} \setminus \mathfrak{m}^2$ with $\mathrm{gr}(Y_0) = y_0$, $\mathrm{gr}(Z_0) = z_0$. Then it is easy to check that L' is a direct summand as Λ-module, which is not filt-free because $F_1 L' = L'$, $F_0 L' = \mathfrak{m} L'$ but $F_{-1} L'$ is strictly bigger than $\mathfrak{m}^2 L'$ (it contains $Z_0 f_1 - Y_0 f_2$).

Recall that, if A is a noetherian domain, then the nonzero elements form an Ore set and we can talk about its skew field of fractions ([GW04, Thm. 6.8]). Therefore, any finitely generated A-module has a generic rank. In particular, this applies to the case $A = \mathrm{gr}(\Lambda)$ or $A = \Lambda$. Moreover, if L is a filtered Λ-module with a good filtration, then $\mathrm{gr}(L)$ has a generic rank that is independent of the choice of good filtration. (This can be proved just as in the proof of [Bjö89, Prop. 3.3], cf. the proof of [BHH$^+$a, Lemma 3.3.4.3].)

The next criterion reflects some features of Remark 2.2.1.

Lemma 2.2.3. *Let L be a filt-free Λ-module with compatible H-action. Assume that L admits a direct sum decomposition of filtered Λ-modules $L = L' \oplus L''$ compatible with H-action, with the following properties:*

(i) *As filtered Λ-modules we have*

$$L' \cong \bigoplus_{i=1}^{m} \Lambda(-k_i), \quad L'' \cong \bigoplus_{j=m+1}^{n} \Lambda(-\ell_j)$$

with $k_i \geq \ell_j$ for any pair (i, j).

(ii) *As H-modules,* $\mathrm{JH}(\mathbb{F} \otimes_\Lambda L') \cap \mathrm{JH}(\mathbb{F} \otimes_\Lambda L'') = \emptyset$.

Assume that P is an H-stable direct summand of L such that the composition

$$\mathbb{F} \otimes_\Lambda P \hookrightarrow \mathbb{F} \otimes_\Lambda L \twoheadrightarrow \mathbb{F} \otimes_\Lambda L' \tag{16}$$

is an isomorphism, where the second morphism is induced by the projection $L = L' \oplus L'' \twoheadrightarrow L'$. Then P, equipped with the induced filtration, is filt-free and we have an equality $\mathrm{gr}(P) = \mathrm{gr}(L')$ *inside* $\mathrm{gr}(L)$.

Remark 2.2.4. Keep the notation of Lemma 2.2.3. Under hypothesis (ii), the composition (16) is automatically an isomorphism provided that $\mathbb{F} \otimes_\Lambda P \cong \mathbb{F} \otimes_\Lambda L'$ as H-modules.

Proof. Let $(e_1, \dots, e_m)$ be a filt-basis of L' with e_i of degree k_i, and similarly $(e_{m+1}, \dots, e_n)$ a filt-basis of L'' with e_j of degree l_j. We may require that each e_i is an eigenvector of H ($1 \le i \le n$), as H preserves degrees. By Nakayama's lemma, the surjectivity of (16) implies that the composition $\widetilde{\phi} : P \hookrightarrow L \twoheadrightarrow L'$ is also surjective. Since L' is free, P splits as $L' \oplus N'$ for some submodule N' of P, but the injectivity of (16) implies that $\mathbb{F} \otimes_\Lambda N' = 0$, hence $N' = 0$ by Nakayama's lemma again. We deduce that $\widetilde{\phi}$ is an isomorphism and that P is free of rank m. Hence, $L = P \oplus L''$ and we may write uniquely

$$e_i = f_i + g_i, \quad 1 \le i \le m,$$

where $f_i \in P$ and $g_i \in L''$. Since P is H-stable, it follows that f_i, g_i are eigenvectors of H with the same eigencharacter as e_i. Condition (ii) then forces that $g_i \in \mathfrak{m}L''$ for $1 \le i \le m$.

We claim that $f_i \in F_{k_i}L$ but $f_i \notin F_{k_i-1}L$. Indeed, we have

$$F_{k_i}L = F_{k_i}L' \oplus F_{k_i}L'' = F_{k_i}L' \oplus \Big(\bigoplus_{j=m+1}^{n} (F_{k_i-l_j}\Lambda)e_j\Big) \supseteq F_{k_i}L' \oplus L''$$

as $k_i \ge l_j$ for any pair (i, j) by hypothesis (i), hence $f_i \in F_{k_i}L$. On the other hand,

$$F_{k_i-1}L = F_{k_i-1}L' \oplus \Big(\bigoplus_{j=m+1}^{n} (F_{k_i-l_j-1}\Lambda)e_j\Big) \supseteq F_{k_i-1}L' \oplus \mathfrak{m}L'' \tag{17}$$

thus $f_i \notin F_{k_i-1}L$ because $e_i \notin F_{k_i-1}L'$ by choice. This proves the claim.

Now, since P is equipped with the induced filtration from L, the claim implies that $f_i \in F_{k_i}P$ but $f_i \notin F_{k_i-1}P$. On the other hand, since $\bigoplus_{j=m+1}^n \mathfrak{m}e_j \subseteq F_{k_i-1}L$ by (17), we have $g_i \in F_{k_i-1}L$ and the associated principal part of f_i equals that of e_i. Since $\mathrm{gr}(L')$ is generated by the principal parts of $(e_i)_{1\le i\le m}$, we obtain an inclusion $\mathrm{gr}(L') \subseteq \mathrm{gr}(P)$. However, since P has rank m, the generic rank of $\mathrm{gr}(P)$ is also equal to m as observed above, hence by Lemma 2.2.5 below (applied with $A = \mathrm{gr}(\Lambda)$ and $M = \mathrm{gr}(L)$) we deduce an equality $\mathrm{gr}(P) = \mathrm{gr}(L')$. In particular, $\mathrm{gr}(P)$ is gr-free (see [LvO96, § I.4.1]), and consequently P is filt-free by [LvO96, Lemma I.6.4(3)]. ☐

Lemma 2.2.5. *Let A be a noetherian domain and M be a finite free A-module. Assume that there exist A-submodules $M' \subseteq M''$ of M such that*

(i) *M' is a direct summand of M;*

(ii) *M' and M'' have the same generic rank.*

Then $M' = M''$.

Proof. By (i) we have $M = M' \oplus C$ for some A-submodule C of M. Since $M' \subseteq M''$, it is easy to check that

$$M'' = M' \oplus (M'' \cap C).$$

We need to prove that $M'' \cap C = 0$. If this were not the case, then $M'' \cap C$ would have a nonzero generic rank (as M is free, hence torsion-free), and the generic rank of M'' would be strictly greater than that of M', which contradicts (ii). ☐

The following lemma will be useful later.

Lemma 2.2.6. *Let $\phi : P \to L$ be a morphism between two free Λ-modules of finite rank. Assume that $\overline{\phi} : \mathbb{F} \otimes_\Lambda P \to \mathbb{F} \otimes_\Lambda L$ is injective. Then ϕ is also injective and identifies P with a direct summand of L.*

The same statement holds if P and L are two gr-free $\mathrm{gr}(\Lambda)$-modules of finite rank and ϕ is a graded morphism.

Proof. The first statement is a special case of [BH93, Lemma 1.3.4(b)] whose proof extends to the noncommutative noetherian local ring Λ.

The proof in the graded case is similar, noting that $\mathrm{gr}(\Lambda)$ is a graded local ring (supported in degrees ≤ 0). □

Suppose that $R = \bigoplus_{d \leq 0} R_d$ is a negatively graded ring and that M is a graded R-module (here R is not necessarily the ring of § 1.3). Working in the category of graded R-modules (with graded morphisms of degree 0), for any $n \in \mathbb{Z}$ we can form the quotient object $M_{\geq n} \overset{\text{def}}{=} M / \bigoplus_{d<n} M_d$, and moreover the functor $M \mapsto M_{\geq n}$ is exact. This construction applies in particular to graded abelian groups (i.e. $R = \mathbb{Z}$ supported in degree 0). If N is any graded right R-module, then $N \otimes_R M$ is naturally a graded abelian group, where $(N \otimes_R M)_d$ is generated by all $n \otimes m$ with $n \in N_i$, $m \in M_j$, $i + j = d$ [LvO96, § I.4.1]. As the functor that forgets the grading is exact, we see (for example by [Wei94, Ex. 2.4.2]) that the usual Tor functors $\mathrm{Tor}_i^R(N, M)$ are naturally graded abelian groups.

Lemma 2.2.7. *Suppose that $n, i \geq 0$ and that N is supported in degree 0.*

(i) *We have a canonical isomorphism $(N \otimes_R M)_{\geq n} \cong N \otimes_R (M_{\geq n})$ of graded abelian groups.*

(ii) *If $M \to M'$ is a morphism in the category of graded R-modules inducing an isomorphism $M_{\geq n} \xrightarrow{\sim} M'_{\geq n}$, then the natural map $\mathrm{Tor}_i^R(N, M)_{\geq n} \to \mathrm{Tor}_i^R(N, M')_{\geq n}$ of graded abelian groups is an isomorphism.*

Proof. (i) By assumption, $N \otimes_R (\bigoplus_{d<n} M_d)$ is supported in degrees $< n$ and $N \otimes_R (M_{\geq n})$ is supported in degrees $\geq n$. By exactness of the functor $M \mapsto M_{\geq n}$, the natural map $N \otimes_R M \twoheadrightarrow N \otimes_R (M_{\geq n})$ induces an isomorphism $(N \otimes_R M)_{\geq n} \xrightarrow{\sim} N \otimes_R (M_{\geq n})$, as desired.

(ii) We first show that if $M_{\geq n} = 0$, then $\mathrm{Tor}_i^R(N, M)_{\geq n} = 0$ for all i. As N is supported in degree 0 and R is negatively graded, we can pick a graded free resolution $\cdots \to F_1 \to F_0 \to N \to 0$ that is supported in degrees ≤ 0. By exactness of the functor $(\cdot)_{\geq n}$, the group $\mathrm{Tor}_i^R(N, M)_{\geq n}$ is computed as the i-th homology of the complex $(F_\bullet \otimes_R M)_{\geq n}$, which vanishes because $F_\bullet \otimes_R M$ is supported in degrees $< n$ by assumption on M.

If now $f : M \to M'$ induces an isomorphism in degrees $\geq n$, then we get exact sequences $0 \to X \to M \to Y \to 0$ and $0 \to Y \to M' \to Z \to 0$ such that the composition $M \to Y \to M'$ equals f and $X_{\geq n} = Z_{\geq n} = 0$. By the previous paragraph and exactness of the functor $(\cdot)_{\geq n}$ we obtain isomorphisms $\mathrm{Tor}_i^R(N, M)_{\geq n} \xrightarrow{\sim} \mathrm{Tor}_i^R(N, Y)_{\geq n} \xrightarrow{\sim} \mathrm{Tor}_i^R(N, M')_{\geq n}$ for any i, which completes the proof. □

2.3 Some homological arguments

We construct different kind of resolutions of Λ-modules or $\mathrm{gr}(\Lambda)$-modules.

For convenience, we recall some definitions and useful facts in the following remark.

Remark 2.3.1. Let M (resp. N) be a finitely generated Λ-module (resp. $\mathrm{gr}(\Lambda)$-module).

(i) A free resolution $P_\bullet$ of M is called *minimal* if the transition maps in the induced complex $\mathbb{F}\otimes_\Lambda P_\bullet$ are all zero. A standard argument shows that $P_\bullet$ is minimal if and only if $\mathrm{rk}_\Lambda(P_i) = \dim_{\mathbb{F}} \mathrm{Tor}_i^\Lambda(\mathbb{F}, M)$ for each $i \geq 0$. Using that $(\Lambda, \mathfrak{m})$ is a noetherian local ring, the same argument as in [BH93, § 1.3] shows that minimal free resolutions $P_\bullet$ of M exist and that each term P_i is finitely generated. Similarly, we define a minimal gr-free resolution $G_\bullet$ of N and show that $G_\bullet$ is minimal if and only if $\mathrm{rk}_{\mathrm{gr}(\Lambda)} G_i = \dim_{\mathbb{F}} \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N)$ for each $i \geq 0$. As $\mathrm{gr}(\Lambda)$ is a noetherian graded local ring, minimal gr-free resolutions $G_\bullet$ of N exist and each term G_i is finitely generated.

(ii) Suppose that M carries a good filtration and let $\mathrm{gr}(M)$ be the associated graded $\mathrm{gr}(\Lambda)$-module. Let $G_\bullet$ be a finite gr-free resolution of $\mathrm{gr}(M)$. By [LvO96, Cor. I.7.2.9], it can be "lifted" to a (strict) finite filt-free resolution $P_\bullet$ of M, i.e. $\mathrm{gr}(P_\bullet) \cong G_\bullet$. By (i), we see that $P_\bullet$ is minimal if and only if the following two conditions are satisfied: $G_\bullet$ is minimal and $\dim_{\mathbb{F}} \mathrm{Tor}_i^\Lambda(\mathbb{F}, M) = \dim_{\mathbb{F}} \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}(M))$.

(iii) Suppose that M carries a good filtration. Let $P_\bullet$ be a minimal free resolution of M (as Λ-module). Using [LvO96, Prop. I.6.6] we can always endow each P_i with a good filtration such that $P_\bullet$ becomes a filtered complex (with each transition map having degree 0), but in general $P_\bullet$ is not strict. (In fact, the filtration can be chosen such that $P_\bullet$ is strict or filt-free, but in general not both by (ii).)

(iv) If M carries a good filtration, then $\mathrm{Tor}_i^\Lambda(\mathbb{F}, M)$ (and more generally $\mathrm{Tor}_i^\Lambda(\Lambda/\mathfrak{m}^n, M)$ for any $n \geq 0$) carries a canonical and functorial good filtration as a Λ-module. If $P_\bullet \to M \to 0$ is any strict filt-free resolution of M, then the canonical filtration on $\mathrm{Tor}_i^\Lambda(\mathbb{F}, M)$ is the one induced by the complex $\mathbb{F} \otimes_\Lambda P_\bullet$, with each term carrying the tensor product filtration. See section A for more details.

(v) Suppose that M (or N) carries a compatible H-action. Then we can require the above minimal free resolutions to carry a compatible H-action. We only prove (i) for M. By assumption we may view M as an $\mathbb{F}[\![I/Z_1]\!]$-module. Since $\mathbb{F}[\![I/Z_1]\!]$ is a noetherian semi-local ring with Jacobson radical $\mathfrak{J}$ (say), we can show as in [BH93, § 1.3] that minimal *projective* resolutions of M exist (by taking projective covers at each step), where a resolution $P_\bullet$ by $\mathbb{F}[\![I/Z_1]\!]$-modules is called "minimal" if the transition maps are all zero modulo $\mathfrak{J}$. Note that $\mathbb{F}[\![I/Z_1]\!]$ is finite free over Λ and that $\mathfrak{J} = \mathfrak{m}\mathbb{F}[\![I/Z_1]\!]$. Hence, restricting to Λ we obtain a minimal free resolution of M by Λ-modules with compatible H-action.

Denote by N the left-hand side of (14), i.e. $N \overset{\mathrm{def}}{=} \big(\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)\big)^{\oplus r}$. We first prove that N enjoys a property analogous to assumption (iv) in § 2.1. Note that $[\pi[\mathfrak{m}] : \chi] \neq 0$ if and only if $\chi = \chi_\lambda$ for some $\lambda \in \mathscr{P}$.

Recall from (11) that

$$\mathrm{gr}(\Lambda) \cong \bigotimes_{j=0}^{f-1} \mathrm{gr}(\Lambda)_j, \tag{18}$$

where $\mathrm{gr}(\Lambda)_j$ is the subalgebra generated by h_j, y_j, z_j (it is denoted by $\mathbb{F}\langle y_j, z_j, h_j\rangle$ in (11) and by $U(\overline{\mathfrak{g}}_j)$ in [BHH+23, § 5.3] or [HW22, § 9.2]). Below, we denote by $\mathfrak{b}(\lambda)$ the preimage of the ideal $\mathfrak{a}(\lambda)$ of (12) in $\mathrm{gr}(\Lambda)$, namely

$$\mathfrak{b}(\lambda) = (t_j, h_j : 0 \le j \le f-1).$$

For $n \ge 1$ let $\mathcal{I}^{(n)}$ denote the H-stable graded ideal $(y_j^n, z_j^n, h_j : 0 \le j \le f-1)$ of $\mathrm{gr}(\Lambda)$. By abuse of notation, we also write $\mathcal{I}^{(n)}$ for its image $(y_j^n, z_j^n : 0 \le j \le f-1)$ in R. We let $\mathcal{I} \stackrel{\text{def}}{=} \mathcal{I}^{(3)}$.

Lemma 2.3.2. *There exists a minimal gr-free resolution $G_\bullet$ with compatible H-action of $N/\mathcal{I}N$, which admits an H-stable subcomplex $G'_\bullet$ that is a minimal gr-free resolution of N. The induced map $H_0(G'_\bullet) \to H_0(G_\bullet)$ is the natural map $N \to N/\mathcal{I}N$. Moreover, we have a decomposition $G_\bullet = G'_\bullet \oplus G''_\bullet$ of graded* $\mathrm{gr}(\Lambda)$*-modules with compatible H-action (which may not be respected by the transition maps).*

By minimality we deduce that $\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N) \cong \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} G'_i$ and likewise for $N/\mathcal{I}N$. We deduce:

Corollary 2.3.3. *The natural morphism $N \to N/\mathcal{I}N$ induces injective graded morphisms with compatible H-action*

$$\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N) \hookrightarrow \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N/\mathcal{I}N)$$

for $i \ge 0$.

Proof of Lemma 2.3.2. This is essentially done in [HW22, § 9.1, 9.2]. We recall the argument in our notation. By decomposing N and twisting, it suffices to prove this when N is replaced by $\mathrm{gr}(\Lambda)/\mathfrak{b}$ and $N/\mathcal{I}N$ is replaced by $\mathrm{gr}(\Lambda)/(\mathfrak{b} + \mathcal{I})$, where $\mathfrak{b}$ is a homogeneous ideal of $\mathrm{gr}(\Lambda)$ of the form $(t_j, h_j : 0 \le j \le f-1)$ with $t_j \in \{y_j, z_j, y_j z_j\}$. Define ideals $\mathfrak{b}_j \stackrel{\text{def}}{=} (t_j, h_j)$ and $\mathcal{I}_j \stackrel{\text{def}}{=} (y_j^3, z_j^3, h_j)$ of $\mathrm{gr}(\Lambda)_j$. We have graded isomorphisms with compatible H-action:

$$\frac{\mathrm{gr}(\Lambda)}{\mathfrak{b}} \cong \bigotimes_{j=0}^{f-1} \frac{\mathrm{gr}(\Lambda)_j}{\mathfrak{b}_j}, \quad \frac{\mathrm{gr}(\Lambda)}{\mathfrak{b} + \mathcal{I}} \cong \bigotimes_{j=0}^{f-1} \frac{\mathrm{gr}(\Lambda)_j}{\mathfrak{b}_j + \mathcal{I}_j}.$$

By Lemmas 9.8–9.10 of [HW22] we have a minimal gr-free resolution of $\mathrm{gr}(\Lambda)_j/(\mathfrak{b}_j + \mathcal{I}_j)$ with compatible H-action:

$$0 \to G_3^{(j)} \to G_2^{(j)} \to G_1^{(j)} \to G_0^{(j)} \to \frac{\mathrm{gr}(\Lambda)_j}{\mathfrak{b}_j + \mathcal{I}_j} \to 0,$$

depending on t_j. Without recalling the transition maps, if $t_j = y_j$, then

$$\begin{aligned} G_3^{(j)} &= \mathrm{gr}(\Lambda)_j(6)_{\alpha_j^{-2}}, \\ G_2^{(j)} &= \boxed{\mathrm{gr}(\Lambda)_j(3)_{\alpha_j}} \oplus \mathrm{gr}(\Lambda)_j(4)_{\alpha_j^{-2}} \oplus \mathrm{gr}(\Lambda)_j(5)_{\alpha_j^{-3}}, \\ G_1^{(j)} &= \boxed{\mathrm{gr}(\Lambda)_j(1)_{\alpha_j} \oplus \mathrm{gr}(\Lambda)_j(2)_1} \oplus \mathrm{gr}(\Lambda)_j(3)_{\alpha_j^{-3}}, \\ G_0^{(j)} &= \boxed{\mathrm{gr}(\Lambda)_j(0)_1}, \end{aligned}$$

where the final subscript indicates the H-action and where the boxed terms indicate a subcomplex $G_i'^{(j)}$ that is a minimal gr-free resolution of $\mathrm{gr}(\Lambda)_j/\mathfrak{b}_j$. If $t_j = z_j$, then the terms have the same form, but the characters of H are replaced by their inverses. If $t_j = y_j z_j$, then

$$\begin{aligned} G_3^{(j)} &= \mathrm{gr}(\Lambda)_j(6)_{\alpha_j^2} \oplus \mathrm{gr}(\Lambda)_j(6)_{\alpha_j^{-2}}, \\ G_2^{(j)} &= \mathrm{gr}(\Lambda)_j(5)_{\alpha_j^3} \oplus \mathrm{gr}(\Lambda)_j(4)_{\alpha_j^2} \oplus \boxed{\mathrm{gr}(\Lambda)_j(4)_1} \oplus \mathrm{gr}(\Lambda)_j(4)_{\alpha_j^{-2}} \oplus \mathrm{gr}(\Lambda)_j(5)_{\alpha_j^{-3}}, \\ G_1^{(j)} &= \mathrm{gr}(\Lambda)_j(3)_{\alpha_j^3} \oplus \boxed{\mathrm{gr}(\Lambda)_j(2)_1 \oplus \mathrm{gr}(\Lambda)_j(2)_1} \oplus \mathrm{gr}(\Lambda)_j(3)_{\alpha_j^{-3}}, \\ G_0^{(j)} &= \boxed{\mathrm{gr}(\Lambda)_j(0)_1}. \end{aligned}$$

By the Künneth formula (see e.g. [Wei94, Thm. 3.6.3]) we can take $G_\bullet$ (resp. $G'_\bullet$) to be the tensor product of the complexes $G_\bullet^{(j)}$ (resp. $G_\bullet'^{(j)}$) for $0 \le j \le f-1$. These complexes are still minimal resolutions, since the transition maps are defined by elements lying in the unique maximal graded ideal of $\mathrm{gr}(\Lambda)$. $\square$

Corollary 2.3.4. *The graded right* $\mathrm{gr}(\Lambda)$*-module* $\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(N)$ *is supported in degrees* $\le 4f$*, and* $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(N)$ *is supported in degrees* d *with* $3f \le d \le 4f$*.*

Proof. We may again replace N by $\mathrm{gr}(\Lambda)/\mathfrak{b}$, where $\mathfrak{b} = (t_j, h_j : 0 \le j \le f-1)$ as in the proof of Lemma 2.3.2. By the same proof, we know that $\mathrm{gr}(\Lambda)/\mathfrak{b}$ has a gr-free resolution of length $2f$ with degree-$2f$ term $\bigotimes_{j=0}^{f-1} G_2'^{(j)} \cong \mathrm{gr}(\Lambda)(3(f-d)+4d)$, where $d = |\{j : t_j = y_j z_j\}|$. Hence $\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\mathfrak{b})$ is a quotient of $\mathrm{gr}(\Lambda)(-3(f-d)-4d)$, which is supported in degrees $\le 3(f-d)+4d \le 4f$. Likewise, $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(\Lambda)/\mathfrak{b})$ is a quotient of $\mathbb{F}(-3(f-d)-4d)$ as graded vector spaces, which is supported in degree $3(f-d)+4d \in [3f, 4f]$. $\square$

Lemma 2.3.5. *For each* $i \ge 0$ *we have an isomorphism of* H*-modules*

$$\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N) \cong \bigoplus_{\lambda \in \mathscr{P}} (\chi_\lambda^{-1})^{\oplus m_i},$$

(see assumption (iv) *in § 2.1 for* m_i*) so in particular,* $\dim_{\mathbb{F}} \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N) = \dim_{\mathbb{F}} \mathrm{Tor}_i^{\Lambda}(\mathbb{F}, \pi^\vee)$*. Moreover, as graded* $\mathbb{F}$*-vector space* $\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N)$ *is supported in degrees* $[-2i, -i]$*.*

Proof. Clearly, we may assume $r = 1$ so that $m_i = \binom{2f}{i}$ for $0 \le i \le 2f$.

Going back to the minimal gr-free resolution $G'_\bullet$ of N in the proof of Lemma 2.3.2, we obtain

$$\operatorname{Tor}_i^{\operatorname{gr}(\Lambda)_j}(\mathbb{F}, \operatorname{gr}(\Lambda)_j/\mathfrak{b}_j) \cong \mathbb{F} \otimes_{\operatorname{gr}(\Lambda)} G_i'^{(j)} \cong \begin{cases} \mathbb{F}(0)_1 & \text{if } i = 0, \\ \mathbb{F}(d_j)_{\chi_{t_j}} \oplus \mathbb{F}(2)_1 & \text{if } i = 1, \\ \mathbb{F}(d_j + 2)_{\chi_{t_j}} & \text{if } i = 2, \end{cases} \tag{19}$$

where $\mathfrak{b}_j \stackrel{\text{def}}{=} (t_j, h_j)$, χ_{t_j} denotes the character of H acting on t_j, and $d_j = 2$ (resp. $d_j = 1$) if $t_j = y_j z_j$ (resp. $t_j \in \{y_j, z_j\}$). In particular, we see that there is an isomorphism of graded H-modules

$$\operatorname{Tor}_i^{\operatorname{gr}(\Lambda)_j}(\mathbb{F}, \operatorname{gr}(\Lambda)_j/\mathfrak{b}_j) \cong \bigwedge^i \operatorname{Tor}_1^{\operatorname{gr}(\Lambda)_j}(\mathbb{F}, \operatorname{gr}(\Lambda)_j/\mathfrak{b}_j).$$

Using Künneth's formula

$$\operatorname{Tor}_i^{\operatorname{gr}(\Lambda)}(\mathbb{F}, \operatorname{gr}(\Lambda)/\mathfrak{b}) \cong \bigoplus_{i_0+\cdots+i_{f-1}=i} \bigotimes_{j=0}^{f-1} \operatorname{Tor}_{i_j}^{\operatorname{gr}(\Lambda)_j}(\mathbb{F}, \operatorname{gr}(\Lambda)_j/\mathfrak{b}_j)$$

and a similar formula for $\bigwedge^i(-)$, we deduce an isomorphism of graded H-modules

$$\operatorname{Tor}_i^{\operatorname{gr}(\Lambda)}(\mathbb{F}, \operatorname{gr}(\Lambda)/\mathfrak{b}) \cong \bigwedge^i \operatorname{Tor}_1^{\operatorname{gr}(\Lambda)}(\mathbb{F}, \operatorname{gr}(\Lambda)/\mathfrak{b}) \cong \bigwedge^i \Big(\bigoplus_{j=0}^{f-1} (\mathbb{F}(d_j)_{\chi_{t_j}} \oplus \mathbb{F}(2)_1)\Big) \tag{20}$$

for $i \geq 0$.

For fixed $\lambda \in \mathscr{P}$ we now prove that

$$\dim_{\mathbb{F}} \operatorname{Hom}_H(\chi_\lambda^{-1}, \operatorname{Tor}_i^{\operatorname{gr}(\Lambda)}(\mathbb{F}, N)) \geq \binom{2f}{i}. \tag{21}$$

This will finish the proof of the lemma, as from (20) we know that

$$\dim_{\mathbb{F}} \operatorname{Tor}_i^{\operatorname{gr}(\Lambda)}(\mathbb{F}, N) = \binom{2f}{i} |\mathscr{P}| \tag{22}$$

(so the inequality in (21) is an equality).

Let $d_1 \stackrel{\text{def}}{=} f + |\{j : t_j = y_j z_j\}|$ and $d_2 \stackrel{\text{def}}{=} |\{j : t_j \in \{y_j, z_j\}\}|$, so $d_1 + d_2 = 2f$. We claim that for each subset $S \subseteq \{0, \ldots, f-1\}$ such that $t_j \in \{y_j, z_j\}$ for all $j \in S$ (thus $i_2 \stackrel{\text{def}}{=} |S| \leq d_2$),

$$\dim_{\mathbb{F}} \operatorname{Hom}_H\left(\chi_\lambda^{-1}, \operatorname{Tor}_i^{\operatorname{gr}(\Lambda)}(\mathbb{F}, \chi_{\lambda'}^{-1} \otimes \operatorname{gr}(\Lambda)/\mathfrak{b}(\lambda'))\right) = \binom{d_1}{i_1}, \tag{23}$$

where $i_1 \stackrel{\text{def}}{=} i - i_2$ and $\lambda' \in \mathscr{P}$ is the unique element such that

$$\chi_\lambda^{-1} = \chi_{\lambda'}^{-1} \prod_{j \in S} \chi_{t_j}^{-1}. \tag{24}$$

(The existence of $\lambda' \in \mathscr{P}$ is ensured by Lemma 2.3.6(i) below.) Summing up (23) over all S and using the binomial identity

$$\binom{2f}{i} = \sum_{i_1+i_2=i} \binom{d_1}{i_1}\binom{d_2}{i_2},$$

we deduce (21) from the claim.

To prove the claim, we write $\mathfrak{a}(\lambda') = (t'_j : 0 \le j \le f-1)$. By Lemma 2.3.6(i) below, we have $t'_j = y_j z_j / t_j$ for $j \in S$, and $t_j = t'_j$ otherwise. Namely, $\chi_{t'_j} = \chi_{t_j}^{-1}$ for $j \in S$. Noting that H acts trivially on $y_j z_j$, we easily obtain (23) from (20) and (24).

The equality of dimensions in the statement follows from (22) and (13). The final statement of the lemma follows from a direct analysis of $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} G'_i$ (or by reducing to $i = 1$ by (20)). $\square$

Lemma 2.3.6. *Suppose that $\lambda \in \mathscr{P}$ and let $\mathfrak{a}(\lambda) = (t_j : 0 \le j \le f-1)$ as in (12).*

(i) *If $S \subseteq \{0, \ldots, f-1\}$ is a subset such that $t_j \in \{y_j, z_j\}$ for all $j \in S$, then there exists a unique element $\lambda' \in \mathscr{P}$ such that $\chi_\lambda = \chi_{\lambda'} \prod_{j \in S} \chi_{t_j}$. Moreover, if we write $\mathfrak{a}(\lambda') = (t'_j : 0 \le j \le f-1)$, then $t'_j = y_j z_j / t_j$ for $j \in S$ and $t'_j = t_j$ for $j \notin S$.*

(ii) *Suppose that $\overline{\rho}$ is $(m+1)$-generic. Then $\chi_\lambda(\prod_{j=0}^{f-1} \alpha_j^{i_j}) = \chi_\mu$ for some $\mu \in \mathscr{P}$ and some integers i_j with $|i_j| \le m$ for all j if and only if $|i_j| \le 1$ for all j and $i_j = -1$ (resp. $i_j = 1$) implies $t_j = y_j$ (resp. $t_j = z_j$).*

Proof. (i) For the uniqueness of λ' we need to show that if $\chi_{\lambda'} = \chi_{\lambda''}$ with $\lambda', \lambda'' \in \mathscr{P}$, then $\lambda' = \lambda''$. This follows from [HW22, Lemma 2.1] (noting that $\chi_\mu \neq \chi_\mu^s$ for any $\mu \in \mathscr{P}$).

For the existence of λ' and the last statement, we may assume $S \neq \emptyset$, otherwise we just take $\lambda' = \lambda$. By induction we may assume $|S| = 1$, in which case the result follows from [BHH+a, Rk. 3.3.1.2].

(ii) First note that the "if" part holds by (i), and it remains to prove "only if". As $\overline{\rho}$ is $(m+1)$-generic we can write $\overline{\rho}|_{I_K}$ as in (7) or (8) with $n = m+1$. We deduce that $\lambda_j(r_j), \mu_j(r_j) \in [m+1, p-2-m]$ from the definition of the set $\mathscr{P}$ [Bre14, § 4]. By [Bre14, § 4] we know that for $a, d \in \mathbb{F}_q^\times$ we have

$$\chi_\lambda\Big(\begin{pmatrix} a & \\ & d \end{pmatrix}\Big) = a^{\sum_{j=0}^{f-1} \lambda_j(r_j) p^j} (ad)^{e_\lambda}$$

for some integer $e_\lambda \overset{\text{def}}{=} e(\lambda)(r_0, \ldots, r_{f-1})$ (where the polynomial $e(\lambda)$ is defined in *loc. cit.*). We remark that $e(\lambda)$ and χ_λ can be defined for any f-tuple λ satisfying $\sum_{j=0}^{f-1} \lambda_j(0) \equiv 0 \pmod 2$ (this condition is missing in [HW22], § 2).

Thus the equality $\chi_\lambda(\prod_{j=0}^{f-1} \alpha_j^{i_j}) = \chi_\mu$ is equivalent to the two congruences

$$\begin{aligned} \sum_{j=0}^{f-1} \lambda_j(r_j) p^j + e_\lambda + \sum_{j=0}^{f-1} i_j p^j &\equiv \sum_{j=0}^{f-1} \mu_j(r_j) p^j + e_\mu \pmod{p^f - 1}, \\ e_\lambda - \sum_{j=0}^{f-1} i_j p^j &\equiv e_\mu \pmod{p^f - 1}. \end{aligned} \tag{25}$$

By subtracting, we obtain

$$\sum_{j=0}^{f-1} (\lambda_j(r_j) + i_j) p^j \equiv \sum_{j=0}^{f-1} (\mu_j(r_j) - i_j) p^j \pmod{p^f - 1}.$$

Under the genericity condition, the integers $\lambda_j(r_j) + i_j$, $\mu_j(r_j) - i_j$ (for $0 \le j \le f-1$) lie in the interval $[1, p-2]$. Therefore,

$$\lambda_j(r_j) + i_j = \mu_j(r_j) - i_j \qquad \text{for all } 0 \le j \le f-1. \tag{26}$$

In particular,

$$\lambda_j(r_j) \equiv \mu_j(r_j) \pmod 2 \qquad \text{for all } 0 \le j \le f-1. \tag{27}$$

On the other hand, from (25), the definition of $e(\lambda)$ and (26) we easily deduce that the polynomial $\lambda_{f-1}(x_{f-1}) - \mu_{f-1}(x_{f-1})$ is constant, and hence by (27) that $\lambda_{f-1}(x_{f-1}) - \mu_{f-1}(x_{f-1}) \in \{0, \pm 2\}$. By the definition of $\mathscr{P}$ we deduce by descending induction and (27) that $\lambda_j(x_j) - \mu_j(x_j) \in \{0, \pm 2\}$ for all j. Therefore, by (26), $|i_j| \le 1$ for all j. Assume first that $j > 0$ or that $\overline{\rho}$ is reducible. If $i_j = 1$, then $\lambda_j(x_j) = x_j$ or $\lambda_j(x_j) = p - 3 - x_j$, so $t_j = z_j$. (If $\overline{\rho}$ is nonsplit reducible, note that $\mu_j(x_j) = x_j + 2$ in the first case, so $j \in J_{\overline{\rho}}$ in either case.) Similarly, if $i_j = -1$, then $\lambda_j(x_j) = x_j + 2$ or $\lambda_j(x_j) = p - 1 - x_j$, so $t_j = y_j$. (Again, $j \in J_{\overline{\rho}}$ if $\overline{\rho}$ is nonsplit reducible.) If $j = 0$ and $\overline{\rho}$ is irreducible, the argument is similar. $\square$

Recall that just before Lemma 2.3.2 we defined $\mathcal{I}^{(n)} = (y_j^n, z_j^n, h_j : 0 \le j \le f-1)$, an H-stable graded ideal of $\mathrm{gr}(\Lambda)$.

Lemma 2.3.7. *Suppose that $n \ge 1$ and that $\overline{\rho}$ is $(2n-1)$-generic. For each character χ of H such that $[N/\mathcal{I}^{(n)}N : \chi] \ne 0$, we have $[N/\mathcal{I}^{(n)}N : \chi] = r$.*

Proof. It is equivalent to prove that $N'/\mathcal{I}^{(n)}N'$ is multiplicity free, where $N' \overset{\text{def}}{=} \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)$. We have $R/(\mathfrak{a}(\lambda) + \mathcal{I}^{(n)}) = \mathbb{F}[y_j, z_j : 0 \le j \le f-1]/(t_j, y_j^n, z_j^n : 0 \le j \le f-1)$ and hence the characters of H occurring in $\chi_\lambda^{-1} \otimes R/(\mathfrak{a}(\lambda) + \mathcal{I}^{(n)})$ are given by $\chi_\lambda^{-1}(\prod_{j=0}^{f-1} \alpha_j^{i_j})$, where $|i_j| \le n-1$ and $i_j \le 0$ if $t_j = y_j$ (resp. $i_j \ge 0$ if $t_j = z_j$). Suppose that $N'/\mathcal{I}^{(n)}N'$ fails to be multiplicity free. Then there are $\lambda, \mu \in \mathscr{P}$ and integers i_j, ℓ_j in $[-(n-1), n-1]$ such that $\chi_\lambda^{-1}(\prod_{j=0}^{f-1} \alpha_j^{i_j}) = \chi_\mu^{-1}(\prod_{j=0}^{f-1} \alpha_j^{\ell_j})$ and $(\lambda, \underline{i}) \ne (\mu, \underline{\ell})$. By symmetry we may assume that $\ell_{j_0} > i_{j_0}$ for some j_0. For $0 \le j \le f-1$ let t_j (resp. t'_j) be associated to λ (resp. μ) as in (12). From Lemma 2.3.6(ii) applied to $\chi_\lambda(\prod_{j=0}^{f-1} \alpha_j^{\ell_j - i_j}) = \chi_\mu$ with $m = 2n-2$ we obtain that $\ell_{j_0} - i_{j_0} = 1$ and $t_{j_0} = z_{j_0}$. Applying the same lemma with the roles of λ and μ interchanged, we also get $t'_{j_0} = y_{j_0}$. By above this implies that $i_{j_0} \ge 0 \ge \ell_{j_0}$, contradicting that $\ell_{j_0} > i_{j_0}$. $\square$

2.4 The Iwahori representation τ

We define a finite-dimensional subrepresentation $\tau = \tau^{(3)}$ of $\pi|_I$ and prove a crucial injectivity result on the level of Tor groups in Proposition 2.4.9.

Lemma 2.4.1. *Suppose that $1 \le n \le p$. There exists a finite-dimensional smooth representation $\tau^{(n)}$ of I over $\mathbb{F}$ such that*

$$\mathrm{gr}_{\mathfrak{m}}((\tau^{(n)})^\vee) \cong N/\mathcal{I}^{(n)}N$$

as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action. More precisely, $\tau^{(n)} \cong (\bigoplus_{\lambda \in \mathscr{P}} \tau_\lambda^{(n)})^{\oplus r}$, where $\tau_\lambda^{(n)}$ satisfies

$$\mathrm{gr}_{\mathfrak{m}}((\tau_\lambda^{(n)})^\vee) \cong \chi_\lambda^{-1} \otimes R/(\mathcal{I}^{(n)} + \mathfrak{a}(\lambda))$$

as graded $\mathrm{gr}(\Lambda)$*-modules with compatible* H*-action. In particular,* $\mathrm{soc}_I(\tau_\lambda^{(n)}) = \tau_\lambda^{(n)}[\mathfrak{m}] \cong \chi_\lambda$ *for all* $\lambda \in \mathscr{P}$.

Proof. It suffices to show the existence of $\tau_\lambda^{(n)}$ for each $\lambda \in \mathscr{P}$, which follows by a similar argument as in [HW22, Prop. 9.15] (which considers $n = 3$, using slightly different notation). For convenience of the reader, we recall the argument below.

By [Hu10, Lemma 2.15(i)], for $0 \le s \le p-1$, there exists a unique I-representation which is trivial on K_1, uniserial of length $s+1$ and whose socle filtration has graded pieces $\mathbf{1}, \alpha_i^{-1}, \dots, \alpha_i^{-s}$; we denote this representation by $E_i^-(s)$. For example, $E_0^-(s)$ is just the restriction to I of the Serre weight $(s, 0, \dots, 0)$ twisted by η^{-1}, where η is the character of H acting on $(s, 0, \dots, 0)^{I_1}$. By taking a conjugate action by $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$, we obtain an I-representation $E_i^+(s)$ which is uniserial of length $s+1$ and whose socle filtration has graded pieces $\mathbf{1}, \alpha_i, \dots, \alpha_i^s$. It is direct to check that

$$\mathrm{gr}_\mathfrak{m}(E_i^-(s)^\vee) \cong \mathbb{F}[y_i, z_i]/(y_i^{s+1}, z_i), \quad \mathrm{gr}_\mathfrak{m}(E_i^+(s)^\vee) \cong \mathbb{F}[y_i, z_i]/(y_i, z_i^{s+1}),$$

where $\mathbb{F}[y_i, z_i]$ is viewed as a $\mathrm{gr}(\Lambda)$-module via the natural quotient map. Moreover, the amalgamated sum $E_i^-(s) \oplus_\mathbf{1} E_i^+(s) \overset{\mathrm{def}}{=} (E_i^+(s) \oplus E_i^-(s))/\mathbf{1}$ satisfies

$$\mathrm{gr}_\mathfrak{m}\left((E_i^-(s) \oplus_\mathbf{1} E_i^+(s))^\vee\right) \cong \mathbb{F}[y_i, z_i]/(y_i^{s+1}, y_i z_i, z_i^{s+1}).$$

Recall that $\mathfrak{a}(\lambda) = (t_i : 0 \le i \le f-1)$ with $t_i \in \{y_i, z_i, y_i z_i\}$. Define $W_{\lambda,i}$ to be

$$W_{\lambda,i} \overset{\mathrm{def}}{=} \begin{cases} E_i^+(n-1) & \text{if } t_i = y_i, \\ E_i^-(n-1) & \text{if } t_i = z_i, \\ E_i^-(n-1) \oplus_\mathbf{1} E_i^+(n-1) & \text{if } t_i = y_i z_i, \end{cases}$$

and $\tau_\lambda^{(n)} \overset{\mathrm{def}}{=} \chi_\lambda \otimes (\bigotimes_{i=0}^{f-1} W_{\lambda,i})$, where all tensor products in this proof are taken over $\mathbb{F}$.

We claim that $\mathrm{gr}_\mathfrak{m}((\tau_\lambda^{(n)})^\vee) \cong \chi_\lambda^{-1} \otimes R/(\mathcal{I}^{(n)} + \mathfrak{a}(\lambda))$ as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action. For simplicity we write $M_i \overset{\mathrm{def}}{=} (W_{\lambda,i})^\vee$ and $M \overset{\mathrm{def}}{=} \bigotimes_{i=0}^{f-1} M_i$. Denote by $C_\bullet M$ the tensor product filtration on M, namely

$$C_{-d}M := \sum_{d_0 + \cdots + d_{f-1} = d} \bigotimes_{i=0}^{f-1} \mathfrak{m}^{d_i} M_i \quad \text{for } d \ge 0.$$

Then $\mathrm{gr}_{C_\bullet}(M) \cong \bigotimes_{i=0}^{f-1} \mathrm{gr}_\mathfrak{m}(M_i) \cong R/(\mathcal{I}^{(n)} + \mathfrak{a}(\lambda))$ by construction of M_i. By [AJL83, Lemma 1.1(i)], we have an inclusion $\mathfrak{m}^d M \subseteq C_{-d}M$, which induces a morphism of graded $\mathrm{gr}(\Lambda)$-modules

$$\phi : \mathrm{gr}_\mathfrak{m}(M) \to \mathrm{gr}_{C_\bullet}(M).$$

To prove the claim it suffices to prove that ϕ is an isomorphism, equivalently a surjection for dimension reasons. It is clear that $\mathfrak{m}^0 M = C_0(M) = M$, so ϕ_0 (the degree 0 part of ϕ) is surjective. Since $\mathrm{gr}_{C_\bullet}(M)$ is generated by its degree 0 part, we conclude by Nakayama's lemma.

The last statement easily follows from this. $\square$

By [BHH$^+$a, Thm. 3.3.2.1] we have a surjection $N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ of graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action.

Lemma 2.4.2. *Suppose that $\overline{\rho}$ is $(2n-1)$-generic. There exists an I-equivariant embedding $\tau^{(n)} \hookrightarrow \pi|_I$ such that the composite of the induced maps*

$$N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\tau^{(n)})^\vee \cong N/\mathcal{I}^{(n)}N$$

is identified with the natural quotient map $N \twoheadrightarrow N/\mathcal{I}^{(n)}N$. In particular, the surjections $N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}((\tau^{(n)})^\vee)$ are isomorphisms in degrees $\geq -(n-1)$ and $\tau^{(n)}[\mathfrak{m}^n] = \pi[\mathfrak{m}^n]$.

Proof. (Note that the proof of the first statement is the same as that of [HW22, Prop. 10.20].) From the last assertion of Lemma 2.4.1 we know that $\tau^{(n)}[\mathfrak{m}]$ is isomorphic to $\pi[\mathfrak{m}] = (\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda)^{\oplus r}$, and we may choose such an isomorphism $i : \tau^{(n)}[\mathfrak{m}] \xrightarrow{\sim} \pi[\mathfrak{m}]$ that makes the diagram

$$\begin{tikzcd} N_0 \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi^\vee/\mathfrak{m}\pi^\vee \arrow[d, "i^\vee"', "\cong"] \\ (N/\mathcal{I}^{(n)}N)_0 \arrow[r, "\cong"] & (\tau^{(n)})^\vee/\mathfrak{m}(\tau^{(n)})^\vee \end{tikzcd} \tag{28}$$

commute, where $(-)_0$ denotes the degree 0 part of a graded module. Lemma 2.3.7 implies that

$$\mathrm{JH}(\tau^{(n)}/\tau^{(n)}[\mathfrak{m}]) \cap \mathrm{JH}(\pi[\mathfrak{m}]) = \emptyset. \tag{29}$$

By (29) and assumption (iv) on π, we have in particular $\mathrm{Ext}^i_{I/Z_1}(\chi, \pi) = 0$ for $\chi \in \mathrm{JH}(\tau^{(n)}/\tau^{(n)}[\mathfrak{m}])$ and $i = 0, 1$, hence $\mathrm{Ext}^i_{I/Z_1}(\tau^{(n)}/\tau^{(n)}[\mathfrak{m}], \pi) = 0$ for $i = 0, 1$ by dévissage. We then deduce an isomorphism

$$\mathrm{Hom}_I(\tau^{(n)}, \pi) \xrightarrow{\sim} \mathrm{Hom}_I(\tau^{(n)}[\mathfrak{m}], \pi),$$

so the above embedding $i : \tau^{(n)}[\mathfrak{m}] \cong \pi[\mathfrak{m}] \hookrightarrow \pi$ extends uniquely to an I-equivariant morphism $i' : \tau^{(n)} \to \pi|_I$ which must be injective (being injective on the socle). By the commutativity in (28) it is easy to see that i' satisfies the required condition (as N is generated by N_0).

We get the isomorphism in degrees $\geq -(n-1)$ since h_j kills N, and this implies $\tau^{(n)}[\mathfrak{m}^n] = \pi[\mathfrak{m}^n]$ for dimension reasons. □

Corollary 2.4.3. *Suppose that $\overline{\rho}$ is $(2n-1)$-generic. Then*

(i) *the I-representation $\bigoplus_{\lambda\in\mathscr{P}} \tau_\lambda^{(n)}$ multiplicity free, and*

(ii) *all Jordan–Hölder factors of $\pi[\mathfrak{m}^n] = \tau^{(n)}[\mathfrak{m}^n]$ occur with multiplicity r.*

Proof. Note that the genericity condition implies $n \leq p$, so $\tau_\lambda^{(n)}$ is well-defined by Lemma 2.4.1. By Lemma 2.4.1 again we have $\tau^{(n)} \cong (\bigoplus_{\lambda\in\mathscr{P}} \tau_\lambda^{(n)})^{\oplus r}$ and $\mathrm{gr}_{\mathfrak{m}}((\tau^{(n)})^\vee) \cong N/\mathcal{I}^{(n)}N$, so (i) follows from Lemma 2.3.7. By the last assertion of Lemma 2.4.2 we have $\pi[\mathfrak{m}^n] = \tau^{(n)}[\mathfrak{m}^n]$, so (ii) follows from $\tau^{(n)} \cong (\bigoplus_{\lambda\in\mathscr{P}} \tau_\lambda^{(n)})^{\oplus r}$ and (i). □

Corollary 2.4.4. *Suppose that $\overline{\rho}$ is $(2n-1)$-generic. Then $\pi[\mathfrak{m}^n]$ is isomorphic to the largest subrepresentation V of $\mathrm{Inj}_{I/Z_1}(\pi^{I_1})[\mathfrak{m}^n]$ containing π^{I_1} such that $[V:\chi]=r$ if χ occurs in π^{I_1}.*

Proof. Since $\pi|_I \hookrightarrow \mathrm{Inj}_{I/Z_1}(\mathrm{soc}_I(\pi)) = \mathrm{Inj}_{I/Z_1}(\pi^{I_1})$, we have an injection $\pi[\mathfrak{m}^n] \hookrightarrow \mathrm{Inj}_{I/Z_1}(\pi^{I_1})[\mathfrak{m}^n]$. As $\overline{\rho}$ is $(2n-1)$-generic, we have $[\pi[\mathfrak{m}^n]:\chi]=r$ for all $\chi \in \mathrm{JH}(\pi^{I_1})$ by Corollary 2.4.3(ii). Conversely, suppose that there is an I-representation V such that $\pi^{I_1} \subseteq V \subseteq \mathrm{Inj}_{I/Z_1}(\pi^{I_1})[\mathfrak{m}^n]$ and $[V:\chi]=r$ for all $\chi \in \mathrm{JH}(\pi^{I_1})$. In particular we have $\mathrm{JH}(V/\pi^{I_1}) \cap \mathrm{JH}(\pi^{I_1}) = \emptyset$. As in the proof of Lemma 2.4.2, we deduce that the inclusion $\pi^{I_1} \hookrightarrow \pi$ extends to a necessarily injective morphism $V \hookrightarrow \pi$. Since V is killed by $\mathfrak{m}^n$ by assumption, we have $V \hookrightarrow \pi[\mathfrak{m}^n] \subseteq \pi$. This proves the maximality of $\pi[\mathfrak{m}^n]$. □

Let $\tau \overset{\mathrm{def}}{=} \tau^{(3)}$ denote the representation defined in Lemma 2.4.1 for $n=3$ (well-defined as $p>2$), so $\mathrm{gr}_{\mathfrak{m}}(\tau^\vee) \cong N/\mathcal{I}N$ as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action, where we recall that $\mathcal{I} = \mathcal{I}^{(3)}$ (see above Lemma 2.3.2).

Recall from Lemma 2.3.2 the minimal gr-free resolution $G_\bullet$ of $\mathrm{gr}_{\mathfrak{m}}(\tau^\vee) \cong N/\mathcal{I}N$ which decomposes as $G_\bullet = G'_\bullet \oplus G''_\bullet$, with $G'_\bullet$ being a minimal gr-free resolution of N. More precisely, recall that $\tau^\vee \cong (\bigoplus_{\lambda\in\mathscr{P}} \tau_\lambda^\vee)^{\oplus r}$ and by construction $G_\bullet = \bigoplus_{\lambda\in\mathscr{P}} G_{\lambda,\bullet}$, where $G_{\lambda,\bullet}$ is a minimal gr-free resolution of $\mathrm{gr}_{\mathfrak{m}}(\tau_\lambda^\vee)$ with compatible H-action for each $\lambda \in \mathscr{P}$. By [LvO96, Cor. I.7.2.9] we can lift $G_{\lambda,\bullet}$ to a (strict) filt-free resolution $L_{\lambda,\bullet}$ of $\tau_\lambda^\vee$. By Remark 2.3.1(v), we may and will also require that $L_{\lambda,\bullet}$ carries a compatible H-action. Then $L_\bullet \overset{\mathrm{def}}{=} \bigoplus_{\lambda\in\mathscr{P}} L_{\lambda,\bullet}$ is a (strict) filt-free resolution of $\tau^\vee$ with compatible H-action.

Lemma 2.4.5. *For any $i \geq 0$ there exists a decomposition $L_i = L'_i \oplus L''_i$ as filt-free Λ-modules with compatible H-action that reduces to $G_i = G'_i \oplus G''_i$ on graded pieces.*

Note that we do not require that the map $L_i \to L_{i-1}$ sends L'_i to L'_{i-1}.

Proof. We fix i. Lift G'_i and G''_i to filt-free Λ-modules F'_i and F''_i with compatible H-action. Then L_i and $F'_i \oplus F''_i$ are two filt-free Λ-modules that lift G_i, so by [LvO96, Lemma I.6.2(6)] there exists a filtered morphism $f : L_i \to F'_i \oplus F''_i$ that lifts the given isomorphism $G_i = G'_i \oplus G''_i$. As in Remark 2.3.1(v), we may demand in addition that f is H-equivariant. By [LvO96, Thm. I.4.2.4(5)] the map f is a strict isomorphism, so we may define L'_i and L''_i as pre-images of F'_i and F''_i in L_i. □

Lemma 2.4.6. *Suppose that $\overline{\rho}$ is 5-generic. With the above notation, $L_\bullet$ is also a minimal free resolution of $\tau^\vee$. Moreover, for $i \in \{0,1,2\}$, $L_i = L'_i \oplus L''_i$ defined in Lemma 2.4.5 satisfies conditions (i), (ii) of Lemma 2.2.3.*

Proof. For the first claim it suffices to prove the minimality of $L_{\lambda,\bullet}$ for each $\lambda \in \mathscr{P}$. This is proven in [HW22, Prop. 9.21]. We remark that the proof reduces to the case χ_λ is trivial (by twisting), so does not require any genericity condition on χ_λ; it rather requires $p \geq 7$ to verify the property (**Min**) in *loc. cit.* which guarantees that [HW22, Lemma A.11] applies.

Since $\mathrm{gr}_{\mathfrak{m}}(\Lambda(k)) \cong \mathrm{gr}(\Lambda)(k)$ and $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_{\mathfrak{m}}(M) \cong \mathbb{F} \otimes_\Lambda M$ for any filt-free Λ-module M with compatible H-action, it remains to check the analogues of conditions (i), (ii) for $G_i = G_i' \oplus G_i''$.

Suppose that $i = 2$. It is easy to see that if $\mathrm{gr}(\Lambda)(k)$ occurs in G_2' as a direct summand, then $k \in \{2,3,4\}$, while if it occurs in G_2'' then $k \geq 4$. Hence condition (i) holds. On the other hand, the characters of H occurring in $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} G_2'$ are of the form $\chi_\lambda^{-1}(\prod_{j=0}^{f-1} \alpha_j^{\varepsilon_j'})$, where $\lambda \in \mathscr{P}$ and $|\varepsilon_j'| \leq 1$ for all j, and $\varepsilon_j' = 1$ (resp. $\varepsilon_j' = -1$) implies $t_j = y_j$ (resp. $t_j = z_j$). Similarly, the characters of H occurring in $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} G_2''$ are of the form $\chi_\mu^{-1}(\prod_{j=0}^{f-1} \alpha_j^{\varepsilon_j''})$, where $\mu \in \mathscr{P}$, $|\varepsilon_j''| \leq 3$ for all j and $|\varepsilon_j''| \geq 2$ for at least one j. (In fact, also at most two ε_j' are nonzero, and likewise for the ε_j''.) Then Lemma 2.3.6(ii) (applied to $\chi_\lambda(\prod_{j=0}^{f-1} \alpha_j^{\varepsilon_j''-\varepsilon_j'}) = \chi_\mu$ with $m = 4$; here we use that $\overline{\rho}$ is 5-generic) implies that for some j we have $(\varepsilon_j', \varepsilon_j'', t_j) = (1, 2, z_j)$ or $(\varepsilon_j', \varepsilon_j'', t_j) = (-1, -2, y_j)$ but this contradicts the information about t_j above. Therefore condition (ii) holds.

The cases $i = 0$ and $i = 1$ are similar but easier. $\square$

Remark 2.4.7. The second statement in Lemma 2.4.6 need not be true for $i \gg 0$. Fortunately, for the proof of Theorem 2.1.2 below we only need to treat the terms L_i for $i \in \{0, 1, 2\}$.

The following is a consequence of the first assertion of Lemma 2.4.6.

Corollary 2.4.8. *Suppose that $\overline{\rho}$ is 5-generic. For any $i \geq 0$ there is a canonical isomorphism*

$$\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}_{\mathfrak{m}}(\tau^\vee)) \cong \mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)).$$

(Here, $\mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)$ carries the canonical filtration, cf. Remark 2.3.1(iv).)

Proof. Using the spectral sequence introduced in the proof of Proposition 2.4.9 below, we know that $\mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee))$ is isomorphic to a subquotient of $\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}_{\mathfrak{m}}(\tau^\vee))$. But

$$\dim_{\mathbb{F}} \mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)) = \dim_{\mathbb{F}} \mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee) = \dim_{\mathbb{F}} \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}_{\mathfrak{m}}(\tau^\vee)),$$

where the second equality follows from the first assertion of Lemma 2.4.6 and the minimality of $G_\bullet$ (see Remark 2.3.1(ii)), which concludes the proof. $\square$

Next, we compare $\mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee)$ and $\mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)$. Recall that by Lemma 2.4.2 we have a surjection of $\mathbb{F}[\![I/Z_1]\!]$-modules $\pi^\vee \twoheadrightarrow \tau^\vee$, provided $\overline{\rho}$ is 5-generic.

Proposition 2.4.9. *Assume that $\overline{\rho}$ is 9-generic. The natural morphism*

$$\mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee) \to \mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)$$

is injective for any $0 \leq i \leq 2$.

Proof. Let $\varphi_i : \mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee) \to \mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)$ denote the natural morphism. It suffices to prove the following statement: there exist separated filtrations on the finite-dimensional $\mathbb{F}$-vector spaces $\mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee)$ and $\mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)$, with respect to which φ_i becomes a filtered morphism and such that the induced graded morphism $\mathrm{gr}(\varphi_i)$ is injective. To show this, we use a spectral sequence which computes $\mathrm{gr}(\mathrm{Tor}_i^{\Lambda}(\mathbb{F},-))$ using $\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F},\mathrm{gr}(-))$, analogous to the one introduced in the proof of [BHH$^+$a, Prop. 3.3.4.6].

Starting from a minimal gr-free resolution of $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$, by Remark 2.3.1(ii) we can lift it to a filt-free resolution of $\pi^\vee$, say $M_\bullet$. Tensoring with $\mathbb{F}$, we obtain a filtered complex $\mathbb{F}\otimes_\Lambda M_\bullet$ and we pass to the associated graded complex, $\mathrm{gr}(\mathbb{F}\otimes_\Lambda M_\bullet)$. As in the proof of [BHH$^+$a, Prop. 3.3.4.6] (cf. [LvO96, § III.2.2]), we obtain a spectral sequence $\{E_i^r, r\geq 0, i\geq 0\}$, with the following properties:

(a) $E_i^0=\mathrm{gr}(\mathbb{F}\otimes_\Lambda M_i)\cong \mathbb{F}\otimes_{\mathrm{gr}(\Lambda)}\mathrm{gr}(M_i)$ (by [LvO96, Lemma I.6.14]), $E_i^1=\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F},\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$;

(b) for any fixed $r\geq 1$, there is a complex

$$\cdots\to E_1^r\to E_0^r\to 0$$

whose homology gives E_i^{r+1};

(c) for r large enough (depending on i), $E_i^r\cong E_i^\infty=\mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F},\pi^\vee))$.

Note that the filtration on $\mathrm{Tor}_i^\Lambda(\mathbb{F},\pi^\vee)$ is induced from the one on $\mathbb{F}\otimes_\Lambda M_i$, see [LvO96, § III.1, p. 128]. It is in particular separated. Similarly, replacing $\pi^\vee$ by $\tau^\vee$ and using the minimal filt-free resolution $L_\bullet$ of $\tau^\vee$, we have another spectral sequence $\{E_i'^r, r\geq 0, i\geq 0\}$, converging to $\mathrm{Tor}_i^\Lambda(\mathbb{F},\tau^\vee)$. Moreover, using [LvO96, Prop. I.6.5(2)] a standard argument shows that there is a filtered morphism of complexes of Λ-modules with compatible H-actions $M_\bullet\to L_\bullet$ extending $\pi^\vee\twoheadrightarrow\tau^\vee$. Hence by functoriality we obtain a morphism between the spectral sequences:

$$\begin{tikzcd} E_i^r \arrow[r, Rightarrow] \arrow[d] & \mathrm{Tor}_i^\Lambda(\mathbb{F},\pi^\vee) \arrow[d, "\varphi_i"] \\ E_i'^r \arrow[r, Rightarrow] & \mathrm{Tor}_i^\Lambda(\mathbb{F},\tau^\vee) \end{tikzcd} \tag{30}$$

and that φ_i is a filtered morphism with respect to the canonical filtrations on $\mathrm{Tor}_i^\Lambda(\mathbb{F},\pi^\vee)$ and $\mathrm{Tor}_i^\Lambda(\mathbb{F},\tau^\vee)$. Note that the bottom spectral sequence degenerates at the page $r=1$, by Corollary 2.4.8. As explained above, it suffices to show that the natural map

$$\mathrm{gr}(\varphi_i):\ E_i^\infty=\mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F},\pi^\vee))\to\mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F},\tau^\vee))=E_i'^\infty$$

is injective for $0\leq i\leq 2$.

Step 1. Suppose $i=0$. Then the natural surjection

$$E_0^1=\mathbb{F}\otimes_{\mathrm{gr}(\Lambda)}\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)\twoheadrightarrow\mathbb{F}\otimes_{\mathrm{gr}(\Lambda)}\mathrm{gr}_{\mathfrak{m}}(\tau^\vee)=E_0'^1$$

is an isomorphism by Lemma 2.4.2. We then have a commutative diagram

$$\begin{tikzcd} E_0^1 \arrow[r, two heads] \arrow[d, "\cong"] & E_0^\infty \arrow[d] \\ E_0'^1 \arrow[r, "\cong"] & E_0'^\infty \end{tikzcd}$$

where the bottom map is an isomorphism by Corollary 2.4.8. It follows that the top map and the natural map $E_0^\infty \to E_0'^\infty$ are both isomorphisms.

Step 2. Suppose $i = 1$. By the previous step we know that the map $E_0^1 \twoheadrightarrow E_0^\infty$ is an isomorphism, so the map $E_1^r \to E_0^r$ is zero for all $r \geq 1$. Hence we get a natural surjection $E_1^r \twoheadrightarrow E_1^{r+1}$ for $r \geq 1$ and, in particular, $E_1^1 \twoheadrightarrow E_1^\infty$. On the other hand, let $V_i' \overset{\text{def}}{=} \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N)$ for any $i \geq 0$. Corollary 2.3.3 and the isomorphism $\mathrm{gr}_{\mathfrak{m}}(\tau^\vee) \cong N/\mathcal{I}N$ (Lemma 2.4.2) imply that the composition

$$V_i' \to E_i^1 \to E_i'^1$$

is injective for any $i \geq 0$. We obtain a commutative diagram

$$\begin{tikzcd}
V_1' \arrow[dr] \arrow[drr] \arrow[ddr, hook] & & \\
& E_1^1 \arrow[r, two heads] \arrow[d] & E_1^\infty \arrow[d] \\
& E_1'^1 \arrow[r, "\cong"] & E_1'^\infty
\end{tikzcd}$$

where we use again Corollary 2.4.8 (for $i = 1$) for the bottom isomorphism. Therefore, the top diagonal map $V_1' = \mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N) \to \mathrm{gr}(\mathrm{Tor}_1^\Lambda(\mathbb{F}, \pi^\vee)) = E_1^\infty$ is injective. For dimension reasons (Lemma 2.3.5), it is actually an isomorphism, hence the vertical map $E_1^\infty \to E_1'^\infty$ is injective.

Step 3. Suppose $i = 2$. We cannot use exactly the same argument as in Step 2, since we do not (yet) know that the map $E_1^1 \twoheadrightarrow E_1^\infty$ is an isomorphism, but fortunately it suffices to prove this in graded degrees ≥ -4 as we now explain. Recall the exact functor $M \mapsto M_{\geq -4}$ for a graded $\mathrm{gr}(\Lambda)$-module M introduced just before Lemma 2.2.7. By Lemma 2.4.2 with $n = 5$ we know that the natural surjection $N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ is an isomorphism in degrees ≥ -4; here we use the assumption that $\overline{\rho}$ is 9-generic. The same is then true for the induced map of graded vector spaces $V_1' = \mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N) \to \mathrm{Tor}_1^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) = E_1^1$ by Lemma 2.2.7(ii). The diagram in Step 2 implies that the surjection $E_1^1 \twoheadrightarrow E_1^\infty$ is an isomorphism in degrees ≥ -4. Consider now the truncation in degrees ≥ -4 of the spectral sequences associated to the above filtered complexes, which have terms $(E_i^r)_{\geq -4}$ and $(E_i'^r)_{\geq -4}$. Exactly the same argument as in Step 2 (truncated in degrees ≥ -4) gives us a map $\alpha : (V_2')_{\geq -4} \to (E_2^\infty)_{\geq -4}$ fitting into a diagram

$$\begin{tikzcd}
(V_2')_{\geq -4} \arrow[r, "\alpha"] & (E_2^\infty)_{\geq -4} \arrow[r, "\beta"] & (E_2'^\infty)_{\geq -4} \\
V_2' \arrow[u, two heads, "\gamma"] & E_2^\infty \arrow[u, two heads, "\delta"] \arrow[r, "\epsilon"] & E_2'^\infty \arrow[u, two heads]
\end{tikzcd}$$

where the horizontal composition $\beta \circ \alpha$ is injective. In particular, α is injective. As γ is an isomorphism by the last statement in Lemma 2.3.5, as $\dim_{\mathbb{F}} V_2' = \dim_{\mathbb{F}} E_2^\infty$ (again by Lemma 2.3.5) we deduce that α and δ are isomorphisms. Therefore β is injective, so ϵ is injective, as desired. $\square$

2.5 Proof of the theorem

We prove Theorem 2.1.2, using our Tor injectivity result (Proposition 2.4.9).

Proof of Theorem 2.1.2. We first show that N is Cohen–Macaulay and is essentially self-dual of grade $2f$ (in the sense that $\mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(N) \cong N \otimes (\det(\overline{\rho})\omega^{-1})$). Write again $\mathfrak{b}(\lambda) = (t_j, h_j : 0 \le j \le f-1)$. By [Lev92, Thm. 4.3] we know that if M is a finitely generated module over an Auslander–Gorenstein ring R and $f : M \to M$ is injective R-linear, then $j_R(M/f(M)) \ge j_R(M) + 1$, where $j_R(-) \stackrel{\text{def}}{=} \min\{i : \mathrm{Ext}^i_R(-, R) \neq 0\}$ denotes the grade. We apply this inductively with the central regular sequence $h_0, \dots, h_{f-1}$ and then $t_0, \dots, t_{f-1}$ (and $M = \mathrm{gr}(\Lambda)$) to deduce that $j_{\mathrm{gr}(\Lambda)}(N) \ge 2f$. By [BHH+a, Prop. 3.3.1.10] we deduce that $j_{\mathrm{gr}(\Lambda)}(N) = 2f$ and the essential self-duality holds. In Lemma 2.3.2 we constructed a free resolution of N of length $2f$, hence $\mathrm{E}^i_{\mathrm{gr}(\Lambda)}(N) = 0$ for $i > 2f$ and N is Cohen–Macaulay.

Recall that we already have a surjection $N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ by [BHH+a, Thm. 3.3.2.1]. In particular, we have $\mathcal{Z}(N) \ge \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$, where the characteristic cycle is defined in [BHH+a, § 3.3.4]. (This is just the usual cycle as $\mathrm{gr}(\Lambda)/J$-module, since the modules are annihilated by J here.) As N is essentially self-dual, it is pure by [LvO96, Prop. III.4.2.8(1)], so any of its nonzero submodules is of grade $2f$ over $\mathrm{gr}(\Lambda)$ and hence of grade 0 over $\mathrm{gr}(\Lambda)/J$ by the second statement in [BHH+a, Lemma 3.3.1.9]. In particular, any nonzero submodule of N has a nonzero cycle. Therefore, to prove the injectivity of $N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$, it suffices to prove that $\mathcal{Z}(N) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$.

Let $P_\bullet$ be a minimal free resolution of $(\pi|_I)^\vee$ with compatible H-action, see Remark 2.3.1. Note that initially $P_\bullet$ is not yet given a filtration.

Step 1. It suffices to prove that there exists a good filtration on each P_i, such that $P_\bullet$ becomes a complex of filtered Λ-modules, satisfying the following properties:

(a) the associated graded complex $\mathrm{gr}(P_\bullet)$ is exact in degree 1, i.e. $H_1(\mathrm{gr}(P_\bullet)) = 0$;

(b) there is an isomorphism of graded $\mathrm{gr}(\Lambda)$-modules $H_0(\mathrm{gr}(P_\bullet)) \cong N$.

Indeed, we may associate to the filtered complex $P_\bullet$ a convergent spectral sequence, say $\{E^r_i, r \ge 0, i \ge 0\}$, as in [LvO96, § III.1], such that $E^0_i = \mathrm{gr}(P_i)$, $E^1_i = H_i(\mathrm{gr}(P_\bullet))$ and

$$E^r_i \Longrightarrow H_i(P_\bullet)$$

for a suitable good filtration on $H_i(P_\bullet)$, namely the abutment filtration. Condition (a) means that $E^1_1 = 0$, which implies (using the property analogous to (b) in the proof of Proposition 2.4.9) that $E^r_1 = 0$ and $E^{r+1}_0 = E^r_0$ for $r \ge 1$, in particular that $E^\infty_0 = E^1_0$. On the one hand, $E^\infty_0 \cong \mathrm{gr}(H_0(P_\bullet)) = \mathrm{gr}(\pi^\vee)$ for *some* good filtration on $\pi^\vee$ (the one induced from P_0). On the other hand, $E^1_0 = H_0(\mathrm{gr}(P_\bullet)) \cong N$ by condition (b), so $N \cong \mathrm{gr}(\pi^\vee)$ as graded $\mathrm{gr}(\Lambda)$-modules. In particular, we have $\mathcal{Z}(N) = \mathcal{Z}(\mathrm{gr}(\pi^\vee))$ and we conclude by the discussion preceding Step 1, as $\mathcal{Z}(\mathrm{gr}(\pi^\vee)) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee))$ by [BHH+a, Lemma 3.3.4.3].

Step 2. Recall that $L_\bullet$ denotes a minimal filt-free resolution of $\tau^\vee$ with compatible H-action (cf. Lemma 2.4.6). As in Remark 2.3.1(v), we can extend the morphism $\pi^\vee \twoheadrightarrow \tau^\vee$ to a morphism of complexes of Λ-modules with compatible H-actions

$$\phi_\bullet : P_\bullet \to L_\bullet.$$

Using that $P_\bullet$ and $L_\bullet$ are minimal, Proposition 2.4.9 implies that

$$\mathbb{F} \otimes_\Lambda P_i \cong \mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee) \to \mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee) \cong \mathbb{F} \otimes_\Lambda L_i$$

is injective for $0 \le i \le 2$. By Lemma 2.2.6, we deduce that ϕ_i is injective and identifies P_i with a direct summand of L_i as Λ-modules for $0 \le i \le 2$. For $0 \le i \le 2$ we equip P_i with the induced filtration from L_i. For $i > 2$ we initially give P_i an arbitrary good filtration and shift it inductively using [LvO96, Prop. I.6.6] so that all transition morphisms in $P_\bullet$ are filtered (of degree 0). Then $P_\bullet$ is a complex of filtered Λ-modules. (We can further shift the filtration on P_i so that the morphisms ϕ_i are also filtered, but we do not need this in what follows.)

On the other hand, in Lemma 2.3.2 we have decomposed $\mathrm{gr}(L_\bullet) = G_\bullet = G'_\bullet \oplus G''_\bullet$ as graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action, where $G'_\bullet$ is a subcomplex. From Lemma 2.4.5 we also get a decomposition $L_i = L'_i \oplus L''_i$ as filt-free Λ-modules with compatible H-action.

Step 3. Suppose that $i \in \{0, 1, 2\}$. We prove that P_i is filt-free and that inside $\mathrm{gr}(L_i)$ the injective map ϕ_i induces an equality

$$\mathrm{gr}(P_i) = \mathrm{gr}(L'_i)\ (= G'_i).$$

By Step 2 we know that ϕ_i identifies P_i with a direct summand of L_i. As $\mathbb{F} \otimes_\Lambda P_i \cong \mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee)$ and $\mathbb{F} \otimes_\Lambda L'_i \cong \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}(L'_i) \cong \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, N)$, we deduce by Lemma 2.3.5 and Remark 2.1.1 that $\mathbb{F} \otimes_\Lambda P_i \cong \mathbb{F} \otimes_\Lambda L'_i$ as H-modules. By Lemma 2.4.6, the decomposition $L_i = L'_i \oplus L''_i$ satisfies conditions (i) and (ii) of Lemma 2.2.3. Hence, by Lemma 2.2.3 and Remark 2.2.4 we deduce the claim.

Finally, as $G'_2 \to G'_1 \to G'_0 \to N \to 0$ is an exact sequence of graded $\mathrm{gr}(\Lambda)$-modules (as $G'_\bullet$ is a resolution of N), the equality $G'_i = \mathrm{gr}(P_i)$ for $i \in \{0, 1, 2\}$ implies (a), (b) in Step 1. □

Corollary 2.5.1. *Suppose that $\overline{\rho}$ is 9-generic.*

(i) *For any $i \ge 0$ there is a canonical isomorphism compatible with H-action*

$$\mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}_\mathfrak{m}(\pi^\vee)) \cong \mathrm{gr}(\mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee)).$$

(Here, $\mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee)$ carries the canonical filtration, cf. Remark 2.3.1(iv).)

(ii) *The natural morphism*

$$\mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi^\vee) \to \mathrm{Tor}_i^\Lambda(\mathbb{F}, \tau^\vee)$$

is injective for any $i \ge 0$.

Proof. (i) The proof is exactly as the proof of Corollary 2.4.8, using Lemma 2.3.5 together with Theorem 2.1.2 instead of Lemma 2.4.6 to check that both spaces have the same dimension.

(ii) Consider again the morphism of spectral sequences (30) of the proof of Proposition 2.4.9. By part (i) and Corollary 2.4.8, both spectral sequences degenerate at the page $r = 1$. The map $E_1^r \to E_1'^r$ is injective by Corollary 2.3.3 together with Theorem 2.1.2, hence the claim follows (cf. the first paragraph of the proof of Proposition 2.4.9). □

2.6 Verifying assumption (iv)

We prove that a globally defined π satisfies assumption (iv).

We first recall our global setup and refer the reader to [BHH$^+$23, § 8.1] for more details. We fix a totally real number field F with ring of integers $\mathcal{O}_F$ and let S_p denote the set of places of F above p. We assume that F is unramified at all places in S_p. For each finite place w of F we denote by F_w the completion of F at w, by $\mathcal{O}_{F_w}$ its ring of integers and by Frob_w a choice of a geometric Frobenius element of $\mathrm{Gal}(\overline{F}_w/F_w)$. We fix a quaternion algebra D over F, with center F such that D splits at all places in S_p and at most one infinite place. We let S_D denote the set of places of F where D ramifies. We fix a maximal order $\mathcal{O}_D$ in D and isomorphisms $(\mathcal{O}_D)_w \overset{\mathrm{def}}{=} \mathcal{O}_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_w} \xrightarrow{\sim} \mathrm{M}_2(\mathcal{O}_{F_w})$ for $w \notin S_D$.

We fix a continuous representation $\overline{r} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\mathbb{F})$ and let $S_{\overline{r}}$ denote the set of places where $\overline{r}$ ramifies. We write $\overline{r}_w$ for $\overline{r}|_{\mathrm{Gal}(\overline{F}_w/F_w)}$. We assume that:

- $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\sqrt[p]{1}))}$ is absolutely irreducible;
- for all $w \in S_p$, $\overline{r}_w$ is 0-generic (so $S_p \subseteq S_{\overline{r}}$);
- for all $w \in (S_D \cup S_{\overline{r}}) \setminus S_p$ the universal framed deformation ring of $\overline{r}_w$ is formally smooth over $W(\mathbb{F})$.

If D splits at exactly one infinite place (the "indefinite case"), we make the following choices. Given a compact open subgroup V of $(D \otimes_F \mathbb{A}_F^\infty)^\times$ (where $\mathbb{A}_F^\infty$ denotes the finite adèles of F) we first let X_V denote the smooth projective Shimura curve over F associated to V constructed with the convention "$\varepsilon = -1$" (see [BD14, § 3.1] and [BDJ10, § 2]). We choose:

(i) a finite place $w_1 \notin S_D \cup S_{\overline{r}}$ such that (see [EGS15, §§6.2, 6.5]):

  (a) $\mathrm{Norm}(w_1)$ is not congruent to 1 mod p;
  (b) the ratio of the eigenvalues of $\overline{r}(\mathrm{Frob}_{w_1})$ is not in $\{1, \mathrm{Norm}(w_1), \mathrm{Norm}(w_1)^{-1}\}$;
  (c) for any nontrivial root of unity ζ in a quadratic extension of F, $w_1 \nmid (\zeta + \zeta^{-1} - 2)$;

(ii) a finite set S of finite places of F such that:

  (a) $S_D \cup S_{\overline{r}} \subseteq S$ and $w_1 \notin S$;
  (b) for all $w \in S \backslash S_p$ the framed deformation ring $R_{\overline{r}_w^\vee}$ of $\overline{r}_w^\vee$ is formally smooth over $W(\mathbb{F})$;

(iii) compact open subgroups $V = \prod_w V_w \subseteq U = \prod_w U_w$ of $(\mathcal{O}_D)_w^\times$ such that:

  (a) $U_w = (\mathcal{O}_D)_w^\times$ for $w \notin S \cup \{w_1\}$ or $w \in S_p$;
  (b) U_{w_1} is contained in the subgroup of $(\mathcal{O}_D)_{w_1}^\times \cong \mathrm{GL}_2(\mathcal{O}_{F_{w_1}})$ of matrices that are upper-triangular unipotent mod w_1;
  (c) $V_w = U_w$ for $w \notin S_p$ and $V_w \subseteq 1 + p\,\mathrm{M}_2(\mathcal{O}_{F_w})$, $V_w \lhd (\mathcal{O}_D)_w^\times$ for $w \in S_p$;

(d) we have

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\Big(\overline{r}, H^1_{\text{ét}}(X_V \times_F \overline{F}, \mathbb{F})\Big) \neq 0. \tag{31}$$

If D splits at no infinite places (the “definite case”) we make the same choices as (i)–(iii) above, replacing (31) by the condition $S(V,\mathbb{F})[\mathfrak{m}] \neq 0$, where:

- $S(V,\mathbb{F}) \stackrel{\text{def}}{=} \{f : D^\times\backslash(D\otimes_F \mathbb{A}_F^\infty)^\times/V \to \mathbb{F}\}$;
- $\mathfrak{m}$ is generated by $T_w - S_w\mathrm{tr}(\overline{r}(\mathrm{Frob}_w))$, $\mathrm{Norm}(w) - S_w\det(\overline{r}(\mathrm{Frob}_w))$ for $w \notin S\cup\{w_1\}$ such that $V_w = (\mathcal{O}_D)_w^\times$, with T_w, S_w acting on $S(V,\mathbb{F})$ (via right translation on functions) by $V\begin{pmatrix}\varpi_w & 0\\ 0 & 1\end{pmatrix}V$, $V\begin{pmatrix}\varpi_w & 0\\ 0 & \varpi_w\end{pmatrix}V$ respectively (where ϖ_w is any choosen uniformizer of F_w).

Fix now a place $v \in S_p$. For each $w \in S_p\setminus\{v\}$ we fix a Serre weight $\sigma_w \in W(\overline{r}_w^\vee)$ and write $K \stackrel{\text{def}}{=} F_v$, $\overline{\rho} \stackrel{\text{def}}{=} \overline{r}_v^\vee$. We define the admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ (which is nonzero by (31) above):

$$\pi(\overline{\rho}) \stackrel{\text{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{U^v/V^v}\Big(\bigotimes_{w\in S_p\setminus\{v\}}\sigma_w, \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}\big(\overline{r}, H^1_{\text{ét}}(X_{V^vV_v}\times_F\overline{F},\mathbb{F})\big)\Big) \quad \text{in the indefinite case,}$$

$$\pi(\overline{\rho}) \stackrel{\text{def}}{=} \varinjlim_{V_v} \mathrm{Hom}_{U^v/V^v}\Big(\bigotimes_{w\in S_p\setminus\{v\}}\sigma_w, S(V^vV_v,\mathbb{F})[\mathfrak{m}]\Big) \quad \text{in the definite case,}$$

where the limit is over all compact open subgroups $V_v \lhd (\mathcal{O}_D)_v^\times \cong \mathrm{GL}_2(\mathcal{O}_K)$ which are contained in $1 + p\,\mathrm{M}_2(\mathcal{O}_K)$. We caution the reader that, despite the notation, the representation $\pi(\overline{\rho})$ *a priori depends on all of our global choices and not just on* $\overline{\rho}$.

We now check that, when $\overline{\rho}$ is 12-generic, the globally defined representation $\pi = \pi(\overline{\rho})$ satisfies assumption (iv) of § 2.1. For this, we fix a patched module $\mathbb{M}_\infty$ over a suitable formally smooth local $\mathcal{O}$-algebra R_∞ as in [CEG+16] (see also [BHH+23, § 8.4]) where $\mathcal{O} \stackrel{\text{def}}{=} W(\mathbb{F})$, such that

$$\mathbb{M}_\infty \otimes_{R_\infty} \mathbb{F} \cong \pi^\vee. \tag{32}$$

We do not recall the construction and properties of $\mathbb{M}_\infty$ here but we refer the reader to [CEG+18, § 3.1] and item (ii) in the proof of [BHH+23, Thm. 8.4.1].

In fact, we will consider the fixed central character version of $\mathbb{M}_\infty$, see [CEG+16, § 4.22]. This amounts to taking the maximal quotient of $\mathbb{M}_\infty$ on which the centre Z of $\mathrm{GL}_2(K)$ acts via a fixed character $\zeta : Z \to \mathcal{O}^\times$ lifting that of $\pi^\vee$. In particular, setting

$$M_\infty(\sigma) \stackrel{\text{def}}{=} \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]}(\mathbb{M}_\infty, \sigma^\vee)^\vee$$

for any continuous $\mathrm{GL}_2(\mathcal{O}_K)$-representation σ on a finitely generated $\mathcal{O}$-module with central character ζ^{-1}, we obtain a patching functor M_∞ as in [EGS15, § 6] or [BHH+23, § 8.1]. Here, for a linear-topological $\mathcal{O}$-module A, $A^\vee$ denotes the Pontryagin dual $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(A, \mathcal{O}[\frac{1}{p}]/\mathcal{O})$ with compact-open topology. We recall that $M_\infty(\sigma)$ is a finitely generated R_∞-module. For convenience, below we assume that the action of Z_1 on $\mathbb{M}_\infty$ is trivial; this can be achieved up to twist (as Z_1 acts trivially on π).

Lemma 2.6.1. *Suppose that $\mathbb{M}_\infty$ is flat over R_∞. For any finite-dimensional smooth $\mathrm{GL}_2(\mathcal{O}_K)$-representation W over $\mathbb{F}$ and any integer $i \geq 0$, there are natural isomorphisms*

$$\mathrm{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, M_\infty(W)) \cong \mathrm{Tor}_i^{\Lambda'}(W, \pi^\vee) \cong \mathrm{Ext}^i_{\Lambda'}(W, \pi)^\vee,$$

where $\overline{R}_\infty \overset{\mathrm{def}}{=} R_\infty \otimes_{\mathcal{O}} \mathbb{F}$ and $\Lambda' \overset{\mathrm{def}}{=} \mathbb{F}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$.

Whenever necessary, e.g. in $\mathrm{Tor}_i^{\Lambda'}(W, \pi^\vee)$ in Lemma 2.6.1, we consider W as *right* Λ'-module via the inversion on $\mathrm{GL}_2(\mathcal{O}_K)/Z_1$.

Proof. Note that $\overline{R}_\infty$ is a regular local $\mathbb{F}$-algebra whose maximal ideal is generated by a regular sequence, say $\underline{y}$. By [CEG+18, § 3.1], $\mathbb{M}_\infty$ is projective as a pseudocompact $\mathcal{O}[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$-module, hence $\overline{\mathbb{M}}_\infty \overset{\mathrm{def}}{=} \mathbb{M}_\infty \otimes_{\mathcal{O}} \mathbb{F}$ is projective as a pseudocompact Λ'-module. Since $\overline{\mathbb{M}}_\infty \otimes_{\overline{R}_\infty} \mathbb{F} \cong \pi^\vee$, we obtain a Koszul complex $K_\bullet(\underline{y}, \overline{\mathbb{M}}_\infty) = \overline{\mathbb{M}}_\infty \otimes_{\overline{R}_\infty} K_\bullet(\underline{y})$ of $\overline{R}_\infty$-modules whose homology in degree 0 gives $\pi^\vee$. Since $\overline{\mathbb{M}}_\infty$ is flat over $\overline{R}_\infty$ by assumption, $K_\bullet(\underline{y}, \overline{\mathbb{M}}_\infty)$ provides a resolution of $\pi^\vee$ by projective pseudocompact Λ'-modules.

We claim that we have a canonical isomorphism $W \otimes_{\Lambda'} \overline{\mathbb{M}}_\infty \cong M_\infty(W)$ of $\overline{R}_\infty$-modules. Working in the category of pseudocompact Λ'-modules (resp. $\mathbb{F}$-modules) we have by [Bru66, Lemma 2.4] that

$$\mathrm{Hom}^{\mathrm{cont}}_{\Lambda'}(\overline{\mathbb{M}}_\infty, \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(W, \mathbb{F})) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{F}}(W \mathbin{\widehat{\otimes}}_{\Lambda'} \overline{\mathbb{M}}_\infty, \mathbb{F}), \tag{33}$$

where every space of continuous homomorphisms carries the discrete topology, and clearly this isomorphism is $\overline{R}_\infty$-equivariant. As W is a finitely presented Λ'-module, we have

$$W \mathbin{\widehat{\otimes}}_{\Lambda'} \overline{\mathbb{M}}_\infty \cong W \otimes_{\Lambda'} \overline{\mathbb{M}}_\infty \tag{34}$$

by [Bru66, Lemma 2.1]. The claim follows by dualizing (33).

By the Koszul resolution of $\pi^\vee$ above, we see that $\mathrm{Tor}_i^{\Lambda'}(W, \pi^\vee)$ is computed as the i-th homology group of

$$W \otimes_{\Lambda'} K_\bullet(\underline{y}, \overline{\mathbb{M}}_\infty) = K_\bullet(\underline{y}, W \otimes_{\Lambda'} \overline{\mathbb{M}}_\infty),$$

which is precisely the Koszul complex of $W \otimes_{\Lambda'} \overline{\mathbb{M}}_\infty \cong M_\infty(W)$ as $\overline{R}_\infty$-module, and hence also computes $\mathrm{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, M_\infty(W))$.

The second isomorphism is a general fact, by using [Bru66, Cor. 2.6] and noting that

$$\mathrm{Ext}^i_{\Lambda'}(\pi^\vee, W^\vee)^\vee \cong \mathrm{Ext}^i_{\Lambda'}(W, \pi)^\vee. \qquad \square$$

Proposition 2.6.2. *If $\overline{\rho}$ is 12-generic, then assumption* (iv) *holds for $\pi = \pi(\overline{\rho})$. As a consequence, Theorem 2.1.2 holds for π.*

Proof. Under the genericity condition, $\mathbb{M}_\infty$ is flat over R_∞ by [BHH+23, Thm. 8.4.3] (for $\overline{\rho}$ semisimple), [HW22, Thm. 8.15] (for $\overline{\rho}$ nonsplit reducible and $r = 1$) and [Wan23, Thm. 6.3] (for $\overline{\rho}$ nonsplit reducible and general r). If $\chi : I \to \mathbb{F}^\times$ is a smooth character, then by Lemma 2.6.1

and Frobenius reciprocity we have

$$\mathrm{Ext}^i_{I/Z_1}(\chi,\pi) \cong \mathrm{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, M_\chi)^\vee,$$

where we write

$$M_\chi \overset{\text{def}}{=} M_\infty(\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![I]\!]}(\mathbb{M}_\infty, \chi^\vee)^\vee.$$

If $\chi \notin \mathrm{JH}(\pi^{I_1})$, then $M_\chi = 0$ by dévissage and [Bre14, Prop. 4.2], as $M_\infty(\sigma) = 0$ if σ is a Serre weight that is not in $W(\overline{\rho})$, so we are done. Otherwise, $\chi = \chi_\lambda$ for some $\lambda \in \mathscr{P}$. Let $\mathcal{I}_\chi \subseteq \overline{R}_\infty$ be the annihilator of M_χ. By [BHH+23, Prop. 8.2.3] if $\overline{\rho}$ is semisimple, [Wan23, Prop. 6.1] if $\overline{\rho}$ is nonsplit reducible, M_χ is free of rank r over $\overline{R}_\infty/\mathcal{I}_\chi$, which is isomorphic to $\overline{R}_\infty^{(1,0),\tau}$ of *loc. cit.*, where τ is the inertial type corresponding to $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi)$. By [EGS15, Thm. 7.2.1], $\overline{R}_\infty/\mathcal{I}_\chi$ is a local complete intersection ring. Since $\mathbb{M}_\infty$ is a finite projective $S_\infty[\![\mathrm{GL}_2(\mathcal{O}_K)/Z_1]\!]$-module, where S_∞ is a certain $\mathcal{O}$-subalgebra of R_∞ in the patching construction (see the proof of [CEG+16, Lemma 4.18]), M_χ is a finite free $\overline{S}_\infty \overset{\text{def}}{=} S_\infty \otimes_{\mathcal{O}} \mathbb{F}$-module. Hence

$$\dim(\overline{R}_\infty) - \dim(\overline{R}_\infty/\mathcal{I}_\chi) = \dim(\overline{R}_\infty) - \dim(\overline{S}_\infty) = 2f,$$

where the last equality follows from [BHH+23, (81)] (note that the assumption $\overline{\rho}$ semisimple there is unnecessary, see e.g. the proof of [Wan23, Thm. 6.3(i)]). We deduce from [BH93, Thm. 2.3.3(c)] that $\mathcal{I}_\chi$ is generated by a regular sequence in $\overline{R}_\infty$ of length $2f$, say $\underline{a}$. Also note that $\overline{R}_\infty$ is a regular local $\mathbb{F}$-algebra whose maximal ideal is generated by a regular sequence, say $\underline{y}$.

By [BH93, Thm. 2.3.9] applied to $S = \overline{R}_\infty$, $\mathbf{a} = \underline{a}$ and $\mathbf{y} = \underline{y}$, $H_i(K_\bullet(\underline{y}, \overline{R}_\infty/\mathcal{I}_\chi))$ is isomorphic to $\bigwedge^i(\mathbb{F}^{\oplus 2f})$ for any $i \geq 0$, hence has dimension $\binom{2f}{i}$ over $\mathbb{F}$ (recall $\overline{R}_\infty/(\underline{y}) = \mathbb{F}$). Since M_χ is free of rank r over $\overline{R}_\infty/\mathcal{I}_\chi$, we have

$$K_\bullet(\underline{y}, M_\chi) \cong \big(K_\bullet(\underline{y}, \overline{R}_\infty/\mathcal{I}_\chi)\big)^{\oplus r}.$$

Taking homology we obtain $\dim_{\mathbb{F}} \mathrm{Tor}_i^{\overline{R}_\infty}(\mathbb{F}, M_\chi) = \binom{2f}{i} r = m_i$, as desired. $\square$

3 Finite length in the split reducible case

We prove that a smooth mod p representation π of $\mathrm{GL}_2(K)$ satisfying assumptions (i)–(iv) of § 2.1 with $r = 1$ has finite length when the underlying Galois representation $\overline{\rho}$ is split reducible. We also establish several structural results on π as an I- and $\mathrm{GL}_2(\mathcal{O}_K)$-representation.

We assume that $\overline{\rho} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ is split reducible and 0-generic. Throughout this section, π is an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying assumptions (i)–(iv) of § 2.1. As seen in § 2.6, recall that $\pi = \pi(\overline{\rho})$ as defined in § 2.6 satisfies assumption (iv) for any $r \geq 1$. From [BHH+23, Thm. 1.9 and Thm. 1.5] one deduces that $\pi(\overline{\rho})$ satisfies assumptions (i) and (ii) of § 2.1. (See also [BHH+a, Cor. 3.4.2.2, Thm. 3.4.4.1] for similar results in the definite case.) It also satisfies assumption (iii) (for any $r \geq 1$) by [HW22, Thm. 8.2] with [BHH+23, Thm. 8.4.1].

We now assume moreover that π is minimal, i.e. $r = 1$ in assumptions (i) *and* (iv)*.*

3.1 Preliminaries

Given a character $\psi : I \to \mathbb{F}^\times$ satisfying $\psi \neq \psi^s$, the Jordan–Hölder factors of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \psi^s$ are parametrized by some subsets of a suitable set $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}(x_0, \dots, x_{f-1})$ with $|\mathcal{P}| = 2^f$, see [BP12, Lemma 2.2] (not to be confused with the set $\mathscr{P}$ of §1.3!). Again by [BP12, Lemma 2.2], if ψ is 1-generic (actually this condition can be slightly weakened), then the above parametrization is bijective with $\mathcal{P}$.

For $\xi \in \mathcal{P}$ set (following [BP12, § 19])

$$\mathcal{S}(\xi) \stackrel{\text{def}}{=} \{j \in \{0, \dots, f-1\} : \xi_j(x_j) \in \{x_j - 1, p - 1 - x_j\}\}. \tag{35}$$

We remark that the set

$$\delta(\mathcal{S}(\xi)) = \{j \in \{0, \dots, f-1\} : \xi_j(x_j) \in \{p - 2 - x_j, p - 1 - x_j\}\}, \tag{36}$$

is denoted by $J(\xi)$ in [BP12, § 2], [HW22, § 3], but for our purposes $\mathcal{S}(\xi)$ will be more convenient. The function $\xi \mapsto \mathcal{S}(\xi)$ induces a bijection between $\mathcal{P}$ and the set of subsets of $\{0, \dots, f-1\}$. In this way, any Jordan–Hölder factor of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \psi^s$ is parametrized by a subset of $\{0, \dots, f-1\}$ and, if ψ is 1-generic, this parametrization is a bijection.

Remark 3.1.1. In the following we will usually talk about a Jordan–Hölder factor of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$ (rather than $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi^s$) parametrized by an element $\xi \in \mathcal{P}$, by which we mean the Jordan–Hölder factor of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \psi^s$ parametrized by ξ in the case where $\psi = \chi^s$. With this convention, $\emptyset$ (resp. $\{0, 1, \dots, f-1\}$) corresponds to the socle (resp. cosocle) of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$. Concretely, if $\chi(\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)) = a^s \eta(ad)$ for some character $\eta : \mathbb{F}_q^\times \to \mathbb{F}^\times$ and integer $s = \sum_{j=0}^{f-1} p^j s_j$ with $0 \le s_j \le p-1$, then $\xi \in \mathcal{P}$ corresponds to the Jordan–Hölder factor $\xi^c(s_0, \dots, s_{f-1}) \otimes \det^{e(\xi^c)(s_0, \dots, s_{f-1})} \eta$ (provided $0 \le \xi_i^c(s_i) \le p-1$ for all i), where $\xi^c \stackrel{\text{def}}{=} \xi(p-1-x_0, \dots, p-1-x_{f-1})$. (We remark that $\xi^c \in \mathcal{P}$ and that $\mathcal{S}(\xi^c) = \mathcal{S}(\xi)^c$.)

If $\sigma \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$ is the Serre weight corresponding to $\xi \in \mathcal{P}$ (via Remark 3.1.1), we also write $\mathcal{S}(\sigma) = \mathcal{S}(\xi)$.

Assume that $\overline{\rho}$ is 0-generic. Recall from § 1.3 that we have a decomposition

$$D_0(\overline{\rho}) = \bigoplus_{\tau \in W(\overline{\rho})} D_{0,\tau}(\overline{\rho}) = \bigoplus_{i=0}^{f} D_0(\overline{\rho})_i,$$

where $D_0(\overline{\rho})_i \stackrel{\text{def}}{=} \bigoplus_{\ell(\tau)=i} D_{0,\tau}(\overline{\rho})$. Recall also the set $\mathscr{P}$ from § 1.3. We have an involution $\lambda \mapsto \lambda^*$ of $\mathscr{P}$ defined in [BHH$^+$a, § 3.3.1]. By [BHH$^+$a, Lemma 3.3.1.7] we deduce:

Corollary 3.1.2. *The map $\chi_\lambda \mapsto \chi_{\lambda^*}$ induces a bijection between $\mathrm{JH}_H(D_0(\overline{\rho})_i^{I_1})$ and $\mathrm{JH}_H(D_0(\overline{\rho})_{f-i}^{I_1})$.*

Lemma 3.1.3. *Suppose that $\lambda \in \mathscr{P}$. Then χ_λ occurs in $D_{0,\tau}(\overline{\rho})^{I_1}$, where $\tau \in W(\overline{\rho})$ is determined by $J_\tau = J_\lambda$. Moreover, as a Jordan–Hölder factor of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda$, τ is parametrized (via Remark 3.1.1 and* (35)*) by the following subset of $\{0, \dots, f-1\}$:*

$$X^{\mathrm{ss}}(\lambda) \stackrel{\text{def}}{=} \{j : \lambda_j(x_j) \in \{x_j, x_j + 1, p - 2 - x_j, p - 3 - x_j\}\}. \tag{37}$$

We will prove a more general version of Lemma 3.1.3 below, see Lemma 4.1.1.

3.2 Finite length

We prove that π is of finite length (as $\mathrm{GL}_2(K)$-representation) and some structural results on π as an I-representation.

Recall from § A that if M is a finitely generated (left) Λ-module equipped with a good filtration, then the right Λ-module $\mathrm{E}^i_\Lambda(M)$ carries a canonical and functorial good filtration. If furthermore M has grade j we obtain a canonical injection $0 \to \mathrm{gr}(\mathrm{E}^j_\Lambda(M)) \to \mathrm{E}^j_{\mathrm{gr}(\Lambda)}(\mathrm{gr}(M))$ of graded $\mathrm{gr}(\Lambda)$-modules, which is an isomorphism if $\mathrm{gr}(M)$ is Cohen–Macaulay, see [BHH+a, Prop. 3.3.4.6] (see also [BE90, Prop. 5.6]).

Applying the above paragraph to $M = \pi^\vee$ with its $\mathfrak{m}$-adic filtration (where we recall that π is assumed to satisfy assumptions (i)–(iv)), we deduce using the second assertion of Theorem 2.1.2 a canonical isomorphism

$$\mathrm{gr}(\mathrm{E}^{2f}_\Lambda(\pi^\vee)) \xrightarrow{\sim} \mathrm{E}^{2f}_{\mathrm{gr}(\Lambda)}(\mathrm{gr}_\mathfrak{m}(\pi^\vee)). \tag{38}$$

Since all these constructions are canonical, one can check that both terms are endowed with an action of H and that the above isomorphism is H-equivariant.

Remark 3.2.1. Recall that assumption (iii) says that there is a $\mathrm{GL}_2(K)$-equivariant isomorphism of Λ-modules $\mathrm{E}^{2f}_\Lambda(\pi^\vee) \cong \pi^\vee \otimes (\det(\overline{\rho})\omega^{-1})$. By Remark 2.1.4 and the isomorphism (38), we see that the canonical filtration on $\mathrm{E}^{2f}_\Lambda(\pi^\vee)$ does not correspond to the $\mathfrak{m}$-adic filtration on $\pi^\vee \otimes (\det(\overline{\rho})\omega^{-1})$ under the isomorphism.

We denote again by N the graded module defined in § 2.3 (with $r = 1$), namely

$$N = \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{R}{\mathfrak{a}(\lambda)}.$$

By Theorem 2.1.2 and our assumptions on π, we have $\mathrm{gr}_\mathfrak{m}(\pi^\vee) \cong N$ provided $\overline{\rho}$ is 9-generic.

Recall that in [BHH+a, § 2.1.1] and [BHH+a, Thm. 3.1.3.7] we generalized the Colmez functor from $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\mathrm{GL}_2(K)$ by associating to any smooth admissible representation π' of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ which lies in the abelian category $\mathcal{C}$ of [BHH+a, § 3.1.2] a (finite-dimensional étale cyclotomic) (φ, Γ)-module $D^\vee_\xi(\pi')$ over $\mathbb{F}(\!(X)\!) \cong \mathbb{F}[\![\mathbb{Z}_p]\!][1/([1]-1)]$. The functor $D^\vee_\xi$ is contravariant and exact by [BHH+a, Thm. 3.1.3.7]. For instance, if the action of $\mathrm{gr}(\Lambda)$ on $\mathrm{gr}_\mathfrak{m}(\pi'^\vee)$ factors through its quotient $\overline{R}$ of § 1.3, then π' lies in $\mathcal{C}$. In particular the representation π and its subquotients all lie in $\mathcal{C}$ (assumption (ii) implies that $\mathrm{gr}_\mathfrak{m}(\pi^\vee)$ is killed by the ideal $J \subseteq \mathrm{gr}(\Lambda)$ by the proof of [BHH+23, Cor. 5.3.5]). This allows us to use the functor $D^\vee_\xi$ in the following proof.

Proposition 3.2.2. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. Let $0 \subsetneq \pi_1 \subsetneq \pi$ be a subrepresentation of π and let $\pi_2 \overset{\mathrm{def}}{=} \pi/\pi_1$. Then both $\mathrm{gr}_\mathfrak{m}(\pi_1^\vee)$ and $\mathrm{gr}_F(\pi_2^\vee)$ are Cohen–Macaulay $\mathrm{gr}(\Lambda)$-modules of grade $2f$, where F denotes the filtration induced from $\pi^\vee$. In particular, $\pi_1^\vee$ and $\pi_2^\vee$ are Cohen–Macaulay Λ-modules of grade $2f$.*

We note that $\overline{\rho}$ is in particular $(2f-1)$-generic, so we may apply [BHH$^+$a, § 3.3.5] in the proof.

Proof. Let

$$\tau \overset{\text{def}}{=} \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi) = \bigoplus_{\sigma\in W(\overline{\rho})} \sigma,$$
$$\tau_1 \overset{\text{def}}{=} \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1), \quad \tau_2 \overset{\text{def}}{=} \tau/\tau_1.$$

Then $\tau_2 \hookrightarrow \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2)$ (note that *a priori* this might be a strict inclusion).

Recall that $\pi^{K_1} = \bigoplus_{\sigma\in W(\overline{\rho})} D_0(\overline{\rho})$ by assumption (i) in § 2.1 (with $r=1$). By the proof of [BP12, Thm. 19.10] we have $D_{0,\sigma}(\overline{\rho}) \subseteq \pi_1^{K_1}$ for any Serre weight $\sigma \subseteq \tau_1$. It follows that $\pi_1^{K_1} = \bigoplus_{\sigma\subseteq\tau_1} D_{0,\sigma}(\overline{\rho})$. As $\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ preserves $\pi_1^{I_1}$, $(\pi_1^{I_1} \hookrightarrow \pi_1^{K_1})$ is a direct summand of $(D_1(\overline{\rho}) \hookrightarrow D_0(\overline{\rho}))$ as a diagram, so we deduce from [BP12, Thm. 15.4] that $\pi_1^{K_1} = \bigoplus_{i\in\Sigma} D_0(\overline{\rho})_i$ for some $\Sigma \subseteq \{0,1,\dots,f\}$. In particular, the direct sum decomposition $\tau = \tau_1 \oplus \tau_2$ induces a decomposition of $\pi^{K_1} = D_0(\overline{\rho})$ of the form:

$$D_0(\overline{\rho}) = D_0(\overline{\rho})^{(1)} \oplus D_0(\overline{\rho})^{(2)}$$

with $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})^{(i)}) = \tau_i$. This in turn induces a decomposition $\mathscr{P} = \mathscr{P}_1 \sqcup \mathscr{P}_2$, hence a decomposition $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \cong N = N_1 \oplus N_2$, with $N_i \overset{\text{def}}{=} \bigoplus_{\lambda\in\mathscr{P}_i} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)$. By construction, the degree 0 part of N_1 is dual to $\pi_1^{I_1}$ and the degree 0 part of N_2 is dual to $\bigoplus_{i\notin\Sigma} D_0(\overline{\rho})_i^{I_1}$ (as follows from the proof of [BHH$^+$a, Thm. 3.3.2.1]).

Step 1. Consider the induced short exact sequence

$$0 \to \mathrm{gr}_F(\pi_2^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \to 0,$$

where F is the filtration on $\pi_2^\vee$ induced from the $\mathfrak{m}$-adic filtration on $\pi^\vee$. The composite morphism $N_2 \hookrightarrow N \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ is identically zero, as N_2 is generated by its degree 0 part, which is sent to zero in $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$. So we get an induced commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}_F(\pi_2^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \arrow[r] & 0 \\
0 \arrow[r] & N_2 \arrow[r] \arrow[u, hook] & N \arrow[r] \arrow[u, "\cong"] & N_1 \arrow[r] \arrow[u, two heads] & 0
\end{tikzcd} \tag{39}$$

with injective (resp. surjective) vertical map on the left (resp. right). Thus

$$\mathcal{Z}(N_1) \geq \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)), \quad \mathcal{Z}(N_2) \leq \mathcal{Z}(\mathrm{gr}_F(\pi_2^\vee)), \tag{40}$$

where we use here the characteristic cycle of $\overline{R}$-modules defined in (2) (see [BHH$^+$a, § 3.3.4]).

Step 2. We show that $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ and $\mathrm{gr}_F(\pi_2^\vee)$ are Cohen–Macaulay.

Recall that by assumption π satisfies assumption (iii) in § 2.1, namely $\mathrm{E}_\Lambda^{2f}(\pi^\vee) \cong \pi^\vee \otimes \eta$ as $\mathrm{GL}_2(K)$-representations, where $\eta \overset{\text{def}}{=} \det(\overline{\rho})\omega^{-1}$. As in the proof of [BHH$^+$a, Prop. 3.3.5.3(iii)] we may construct a subrepresentation $\widetilde{\pi}_2 \subseteq \pi$ such that $\mathscr{Z}(\mathrm{gr}(\pi_2^\vee)) = \mathscr{Z}(\mathrm{gr}(\widetilde{\pi}_2^\vee))$ (with respect to any good filtrations by [BHH$^+$a, Lemma 3.3.4.3]) and consequently by [BHH$^+$a, Prop. 3.3.5.3(i)]:

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}_2). \tag{41}$$

Concretely, the $\mathrm{GL}_2(K)$-representation $\widetilde{\pi}_2$ is defined by dualizing (and untwisting) the exact sequence

$$0 \to \mathrm{E}_\Lambda^{2f}(\pi_1^\vee) \to \mathrm{E}_\Lambda^{2f}(\pi^\vee) \to \widetilde{\pi}_2^\vee \otimes \eta \to 0. \tag{42}$$

The first two terms carry their canonical filtrations (§ A) and the morphism between them is strict by Lemma A.5. We give $\widetilde{\pi}_2^\vee \otimes \eta$ the induced filtration, so that the induced sequence of their graded modules is again exact. We consider the following commutative diagram with exact rows of graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action, where the upper vertical maps are explained above and the lower vertical maps arise from Step 1:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}(\mathrm{E}_\Lambda^{2f}(\pi_1^\vee)) \arrow[r] \arrow[d, hook] & \mathrm{gr}(\mathrm{E}_\Lambda^{2f}(\pi^\vee)) \arrow[r] \arrow[d, "\cong"] & \mathrm{gr}(\widetilde{\pi}_2^\vee \otimes \eta) \arrow[r] \arrow[dd, dashed] & 0 \\
0 \arrow[r] & \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)) \arrow[r] \arrow[d, hook] & \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)) \arrow[d, equal] & & \\
0 \arrow[r] & \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_1) \arrow[r] & \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N) \arrow[r] & \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_2) \arrow[r] & 0.
\end{tikzcd}$$

The surjection on the right gives us surjections of H-modules $\mathrm{gr}(\widetilde{\pi}_2^\vee \otimes \eta) \twoheadrightarrow \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_2) \twoheadrightarrow \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_2)$, where the final graded $\mathbb{F}$-vector space is supported in degrees $[3f, 4f]$ by Corollary 2.3.4 (noting that N_2 is a direct factor of N). In particular, by the semisimplicity of $\mathbb{F}[H]$, we deduce that $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_2)$ is a subquotient of $F_{4f}(\widetilde{\pi}_2^\vee \otimes \eta)/F_{3f-1}(\widetilde{\pi}_2^\vee \otimes \eta)$ as H-modules. The same corollary applied to N implies that $\mathrm{gr}(\mathrm{E}_\Lambda^{2f}(\pi^\vee))$ is supported in degrees $\leq 4f$, so $F_{4f}(\mathrm{E}_\Lambda^{2f}(\pi^\vee)) = \mathrm{E}_\Lambda^{2f}(\pi^\vee)$. Hence $F_{4f}(\widetilde{\pi}_2^\vee \otimes \eta) = \widetilde{\pi}_2^\vee \otimes \eta$ by (42), so $\mathfrak{m}^{f+1}\widetilde{\pi}_2^\vee \otimes \eta \subseteq F_{3f-1}(\widetilde{\pi}_2^\vee \otimes \eta)$. It follows from all this that $\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_2)$ is a subquotient of $(\widetilde{\pi}_2^\vee/\mathfrak{m}^{f+1}\widetilde{\pi}_2^\vee) \otimes \eta$, or equivalently of $\bigoplus_{i=0}^f \mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)_i \otimes \eta$, as H-modules.

We have $\mathrm{E}_{\mathrm{gr}(\Lambda)}^{2f}(N_2) \otimes \eta^{-1} \cong N_2'$ as $\mathrm{gr}(\Lambda)$-modules (without grading), where $N_2' \overset{\mathrm{def}}{=} \bigoplus_{\lambda \in \mathscr{P}_2^*} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda)$, by [BHH$^+$a, Prop. 3.3.1.10]. Corollary 3.1.2 implies that $(N_2')_0$ is dual to $\bigoplus_{i \notin \Sigma} D_0(\overline{\rho})_{f-i}^{I_1}$. On the other hand, as at the beginning of the proof, we have $\widetilde{\pi}_2^{K_1} = \bigoplus_{i \in \Sigma'} D_0(\overline{\rho})_i$ for some $\Sigma' \subseteq \{0, 1, \ldots, f\}$. Let $\widetilde{N}_2$ be the direct summand of N such that its degree 0 part is dual to $\widetilde{\pi}_2^{I_1} = \bigoplus_{i \in \Sigma'} D_0(\overline{\rho})_i^{I_1}$. Then as before we have a surjection $\widetilde{N}_2 \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)$. From the previous paragraph, $(N_2')_0 \cong \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} N_2'$ is a subquotient of $\bigoplus_{i=0}^f (\widetilde{N}_2)_{-i}$ as H-modules. But $\bigoplus_{i=0}^f N_{-i}$ is multiplicity free as H-module by Lemma 2.3.7 (with $n = f+1$ and $r = 1$, using that $\overline{\rho}$ is $(2f+1)$-generic) and $(N_2')_0 \subseteq N_0$, so we deduce that $(N_2')_0 \subseteq (\widetilde{N}_2)_0$ as H-modules (do not confuse the graded piece N_i of N for $i = 1, 2$ with the submodules N_1, N_2 of N defined just before Step 1!). Dually, $\bigoplus_{i \in \Sigma'} D_0(\overline{\rho})_i^{I_1}$ surjects onto $\bigoplus_{i \notin \Sigma} D_0(\overline{\rho})_{f-i}^{I_1}$ as H-modules. In particular, $\Sigma' \supseteq f - \Sigma^c$, i.e.

$$\widetilde{\pi}_2^{K_1} = \bigoplus_{i \in \Sigma'} D_0(\overline{\rho})_i \supseteq \bigoplus_{i \notin \Sigma} D_0(\overline{\rho})_{f-i}. \tag{43}$$

Taking $\mathrm{GL}_2(\mathcal{O}_K)$-socles we get

$$\begin{aligned}\ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\widetilde{\pi}_2)) = \sum_{i\in\Sigma'} \ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})_i)) &\geq \sum_{i\notin\Sigma} \ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})_{f-i})) \\ &= \sum_{i\notin\Sigma} \ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})_i)) \\ &= \ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)) - \ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)).\end{aligned} \tag{44}$$

By (41) and exactness of the functor $D_\xi^\vee$ we know that

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) - \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1)$$

and hence by [BHH$^+$a, Prop. 3.3.5.3(ii)] that equality has to hold in (44) and hence in (43). By taking I_1-invariants in (43) we deduce that $N_2' = \widetilde{N}_2$.

Consider

$$\mathcal{Z}(\mathrm{gr}_F(\pi_2^\vee)) \geq \mathcal{Z}(N_2) = \mathcal{Z}(N_2') = \mathcal{Z}(\widetilde{N}_2) \geq \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)), \tag{45}$$

where the first inequality is equation (40), the first equality comes from [BHH$^+$a, Thm. 3.3.4.5], the second equality holds as $N_2' = \widetilde{N}_2$, and the final inequality comes from $\widetilde{N}_2 \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee)$. As $\mathcal{Z}(\mathrm{gr}(\pi_2^\vee)) = \mathcal{Z}(\mathrm{gr}(\widetilde{\pi}_2^\vee))$, we deduce that equality holds in (45), so $\mathcal{Z}(N_2) = \mathcal{Z}(\mathrm{gr}_F(\pi_2^\vee))$ and hence also $\mathcal{Z}(N_1) = \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee))$ by the additivity of $\mathcal{Z}$ in short exact sequences (recalling diagram (39)). Since N_1 is pure, any of its nonzero submodules has a nonzero cycle, hence the surjection $N_1 \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ must be an isomorphism and consequently $\mathrm{gr}_F(\pi_2^\vee) \cong N_2$ by Step 1. This implies that $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \cong N_1$ and $\mathrm{gr}_F(\pi_2^\vee) \cong N_2$ are Cohen–Macaulay, as N is Cohen–Macaulay and the N_i are direct summands of N. Hence $\pi_1^\vee$ and $\pi_2^\vee$ are Cohen–Macaulay, because if a finitely generated Λ-module M admits a good filtration such that the associated graded module is Cohen–Macaulay, then M itself is Cohen–Macaulay as a consequence of [LvO96, Prop. III.2.2.4]. ☐

Theorem 3.2.3. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic.*

(i) *Any subrepresentation of π is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle.*

(ii) $\ell_{\mathrm{GL}_2(K)}(\pi) \leq f+1$.

Note that part (i) for π itself was proved in [BHH$^+$a, Thm. 3.3.5.5] under a slightly weaker genericity assumption.

Proof. Let π_1 be a subrepresentation of π, and π_1' be the subrepresentation of π_1 generated by $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)$. In particular, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1) = \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1')$. We then have $\pi_1^{K_1} = \pi_1'^{K_1} = \bigoplus_{i\in\Sigma} D_0(\overline{\rho})_i$ for a unique subset $\Sigma \subseteq \{0, 1, \dots, f\}$, cf. the second paragraph of the proof of Proposition 3.2.2. In particular, $\pi_1^{I_1} = \pi_1'^{I_1}$, so the proof of Proposition 3.2.2 applies to π_1' and shows that the composition of the graded morphisms

$$N_1 \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1'^\vee)$$

is an isomorphism. Hence, we deduce $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) = \mathrm{gr}_{\mathfrak{m}}(\pi_1'^\vee)$, from which we deduce $\pi_1^\vee/\mathfrak{m}^n \xrightarrow{\sim} \pi_1'^\vee/\mathfrak{m}^n$ for all $n \geq 1$ for dimension reasons and hence $\pi_1 = \pi_1'$. This proves (i).

To prove (ii), it suffices to show that any finite ascending chain of $\mathrm{GL}_2(K)$-subrepresentations $0 = \pi_0 \subsetneq \pi_1 \subsetneq \cdots \subsetneq \pi_\ell = \pi$ has length $\ell \leq f+1$. As seen above we can write $\pi_j^{K_1} = \bigoplus_{i\in\Sigma_j} D_0(\overline{\rho})_i$ for unique subsets $\varnothing = \Sigma_0 \subseteq \cdots \subseteq \Sigma_\ell = \{0,1,\dots,f\}$. Since $\pi_j^{K_1}$ contains $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_j)$, we deduce from (i) that $\Sigma_j \subsetneq \Sigma_{j+1}$ for all $0 \leq j < \ell$, so indeed $\ell \leq f+1$. □

We now note further consequences of Proposition 3.2.2.

Corollary 3.2.4. *Keep the notation of Proposition 3.2.2 and suppose that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic.*

(i) *The $\mathfrak{m}$-adic filtration on $\pi^\vee$ induces the $\mathfrak{m}$-adic filtration on $\pi_2^\vee$.*

(ii) *The induced sequence*

$$0 \to \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \to 0$$

of graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action is split exact. More precisely,

$$\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \cong \bigoplus_{\lambda\in\mathscr{P}_1} \chi_\lambda^{-1} \otimes \frac{R}{\mathfrak{a}(\lambda)}$$

and

$$\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \cong \bigoplus_{\lambda\in\mathscr{P}\setminus\mathscr{P}_1} \chi_\lambda^{-1} \otimes \frac{R}{\mathfrak{a}(\lambda)},$$

where $\mathscr{P}_1 \subseteq \mathscr{P}$ corresponds to $\pi_1^{I_1} \subseteq \pi^{I_1}$ (see § 1.3).

Proof. We keep the notation of the proof of Proposition 3.2.2.

(i) By the isomorphism $\mathrm{gr}_F(\pi_2^\vee) \cong N_2$ proved in Step 2 of the proof of Proposition 3.2.2, $\mathrm{gr}_F(\pi_2^\vee)$ is generated by its degree 0 part $\mathrm{gr}_F(\pi_2^\vee)_0$ as a $\mathrm{gr}(\Lambda)$-module. Since $\mathfrak{m}^n\pi_2^\vee \subseteq \pi_2^\vee \cap \mathfrak{m}^n\pi^\vee = F_{-n}\pi_2^\vee$, we have the natural morphism

$$\kappa : \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \to \mathrm{gr}_F(\pi_2^\vee) \cong N_2,$$

which is surjective in degree 0 as $\mathfrak{m}^0\pi_2^\vee = F_0\pi_2^\vee$ $(= \pi_2^\vee)$. Since N_2 is generated by its degree 0 part, κ is surjective and it follows from [LvO96, Thm. I.4.2.4(5)] (applied with $L = M = \pi_2^\vee$ and $N = 0$) that $\mathfrak{m}^n\pi_2^\vee = F_{-n}\pi_2^\vee$ for all $n \geq 0$.

Part (ii) follows, since the sequence $0 \to N_2 \to N \to N_1 \to 0$ is split exact by construction. □

Corollary 3.2.5. *Suppose that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. Let $\pi_1 \subseteq \pi_2$ be subrepresentations of π. Then for any $n \geq 1$, the sequence of Λ-modules*

$$0 \to \pi_1[\mathfrak{m}^n] \to \pi_2[\mathfrak{m}^n] \to (\pi_2/\pi_1)[\mathfrak{m}^n] \to 0$$

is exact. Moreover, the sequence splits as I-representations if $\overline{\rho}$ is also $(2n-1)$-generic.

Proof. We first treat the special case $\pi_2 = \pi$. Then we trivially have $0 \to \pi_1[\mathfrak{m}^n] \to \pi[\mathfrak{m}^n] \to (\pi/\pi_1)[\mathfrak{m}^n]$. The final map is surjective for dimension reasons because $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \cong \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \oplus \mathrm{gr}_{\mathfrak{m}}((\pi/\pi_1)^\vee)$ by Corollary 3.2.4(ii). In particular, for any subrepresentation π_1 of π we obtain

$$\dim_{\mathbb{F}}((\pi/\pi_1)[\mathfrak{m}^n]) = \dim_{\mathbb{F}}(\pi[\mathfrak{m}^n]) - \dim_{\mathbb{F}}(\pi_1[\mathfrak{m}^n]). \tag{46}$$

Now we treat the general case. Since $\pi[\mathfrak{m}^n] \to (\pi/\pi_2)[\mathfrak{m}^n]$ is surjective by the last paragraph, the morphism

$$(\pi/\pi_1)[\mathfrak{m}^n] \to (\pi/\pi_2)[\mathfrak{m}^n]$$

is also surjective, and hence the sequence

$$0 \to (\pi_2/\pi_1)[\mathfrak{m}^n] \to (\pi/\pi_1)[\mathfrak{m}^n] \to (\pi/\pi_2)[\mathfrak{m}^n] \to 0$$

is exact. Applying (46) to π_1 and π_2, we deduce

$$\dim_{\mathbb{F}}((\pi_2/\pi_1)[\mathfrak{m}^n]) = \dim_{\mathbb{F}}(\pi_2[\mathfrak{m}^n]) - \dim_{\mathbb{F}}(\pi_1[\mathfrak{m}^n]),$$

from which the first assertion follows.

For the last assertion, it suffices to show that $\pi_1[\mathfrak{m}^n]$ is a direct summand of $\pi[\mathfrak{m}^n]$ (hence is also a direct summand of $\pi_2[\mathfrak{m}^n]$ as in the proof of Lemma 2.2.5). As $\overline{\rho}$ is $(2n-1)$-generic we note that $\pi[\mathfrak{m}^n] = \tau^{(n)}[\mathfrak{m}^n]$ by Lemma 2.4.2, where $\tau^{(n)} = \bigoplus_{\lambda \in \mathscr{P}} \tau_\lambda^{(n)}$ is the subrepresentation of $\pi|_I$ from Lemma 2.4.1. Let $\mathscr{P}_1 \subseteq \mathscr{P}$ be the subset as in the proof of Proposition 3.2.2 and put

$$\tau_1^{(n)} \overset{\mathrm{def}}{=} \bigoplus_{\lambda \in \mathscr{P}_1} \tau_\lambda^{(n)}, \quad N_1 \overset{\mathrm{def}}{=} \bigoplus_{\lambda \in \mathscr{P}_1} \chi_\lambda^{-1} \otimes \frac{R}{\mathfrak{a}(\lambda)}.$$

It suffices to show that $\pi_1[\mathfrak{m}^n] = \tau_1^{(n)}[\mathfrak{m}^n]$, or equivalently (as $\pi[\mathfrak{m}^n]$ is multiplicity free) that these Λ-modules (with compatible H-action) have the same graded modules. This follows from the isomorphism $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \cong N_1$ established in the proof of Proposition 3.2.2, noting that

$$\mathrm{gr}_{\mathfrak{m}}((\tau_1^{(n)})^\vee/\mathfrak{m}^n) = \mathrm{gr}_{\mathfrak{m}}((\tau_1^{(n)})^\vee)/\overline{\mathfrak{m}}^n = N_1/\overline{\mathfrak{m}}^n$$

by the proof of Lemma 2.4.2, where $\overline{\mathfrak{m}}$ denotes the unique maximal graded ideal of $\mathrm{gr}(\Lambda)$. $\square$

Lemma 3.2.6. *Suppose that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. Let $\pi_1 \subseteq \pi_2$ be subrepresentations of π. Then the natural sequence*

$$0 \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2) \to \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2/\pi_1) \to 0 \tag{47}$$

is exact.

Proof. By the second paragraph of the proof of Proposition 3.2.2 there exist two subsets $\Sigma_1 \subseteq \Sigma_2$ of $\{0, \ldots, f\}$ such that for $j \in \{1, 2\}$,

$$\pi_j^{K_1} = \bigoplus_{i \in \Sigma_j} D_0(\overline{\rho})_i, \quad \pi_j^{I_1} = \bigoplus_{i \in \Sigma_j} D_0(\overline{\rho})_i^{I_1}.$$

Setting $\pi' \overset{\text{def}}{=} \pi_2/\pi_1$, we deduce that $\pi'^{I_1} \cong \bigoplus_{i\in\Sigma_2\setminus\Sigma_1} D_0(\overline{\rho})_i^{I_1}$ by Corollary 3.2.5 (applied with $n=1$), and also that there exists an embedding $\bigoplus_{i\in\Sigma_2\setminus\Sigma_1} D_0(\overline{\rho})_i \hookrightarrow \pi'^{K_1}$. This in particular implies

$$S \overset{\text{def}}{=} \bigoplus_{i\in\Sigma_2\setminus\Sigma_1} \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})_i) \hookrightarrow \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi').$$

We need to prove that it is an isomorphism. If not, then there exists some Serre weight σ such that $\sigma \oplus S \hookrightarrow \pi'|_{\mathrm{GL}_2(\mathcal{O}_K)}$, hence also $\sigma \oplus \big(\bigoplus_{i\in\Sigma_2\setminus\Sigma_1} D_0(\overline{\rho})_i\big) \hookrightarrow \pi'|_{\mathrm{GL}_2(\mathcal{O}_K)}$, which contradicts the structure of π'^{I_1}. $\square$

Corollary 3.2.7. *Suppose that $\overline{\rho}$ is $\max\{9,2f+1\}$-generic. Suppose π' is any subquotient of π.*

(i) *We have $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi') = \ell(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi'))$. In particular, if $\pi' \neq 0$, then $D_\xi^\vee(\pi')$ is nonzero.*

(ii) *Let $\mathscr{P}' \subseteq \mathscr{P}$ correspond to $(\pi')^{I_1}$ (such a subset exists by Corollary 3.2.5 with $n=1$). Then the natural map*

$$\bigoplus_{\lambda\in\mathscr{P}'} \chi_\lambda^{-1} \otimes R/\mathfrak{a}(\lambda) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi'^\vee)$$

of graded $\mathrm{gr}(\Lambda)$-modules with compatible H-action is an isomorphism. In particular, $\mathrm{gr}_{\mathfrak{m}}(\pi'^\vee)$ (resp. $\pi'^\vee$) is Cohen–Macaulay of grade $2f$.

(iii) *π' is generated by its $\mathrm{GL}_2(\mathcal{O}_K)$-socle.*

(iv) *π itself is multiplicity free (of length $\leq f+1$).*

(v) *We have an isomorphism $\mathrm{E}_\Lambda^{2f}(\pi'^\vee) \cong \pi''^\vee \otimes (\det(\overline{\rho})\omega^{-1})$ as Λ-modules with compatible actions of $\mathrm{GL}_2(K)$, where π'' is another subquotient of π, uniquely determined (by part (iv)) by*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi'') \cong \bigoplus_{i\in\Sigma'} \mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_0(\overline{\rho})_{f-i}).$$

Proof. (i) Choose $\pi_1 \subseteq \pi_2 \subseteq \pi$ such that $\pi' \cong \pi_2/\pi_1$. By [BHH+a, Prop. 3.3.5.3(ii)] the assertion holds for π_1 and π_2, so we conclude by the exactness of $D_\xi^\vee(-)$ ([BHH+a, Thm. 3.1.3.7]) combined with Lemma 3.2.6.

(ii) Let π_1, π_2 be as in (i). Let $\mathscr{P}_1 \subseteq \mathscr{P}_2 \subseteq \mathscr{P}$ be the subsets corresponding to $\pi_1 \subseteq \pi_2$ (see § 1.3), so $\mathscr{P}' = \mathscr{P}_2 \setminus \mathscr{P}_1$ by the proof of Proposition 3.2.2. Let $N_1 \subseteq N_2$ (resp. N') be the direct summands of N determined by $\mathscr{P}_1 \subseteq \mathscr{P}_2$ (resp. $\mathscr{P}'$). As in Step 1 of the proof of Proposition 3.2.2 we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}_F(\pi'^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \arrow[r] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[u] & N_2 \arrow[r] \arrow[u] & N_1 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

with exact rows, where F is the filtration on $\pi'^\vee$ induced from the $\mathfrak{m}$-adic filtration on $\pi_2^\vee$, and by Step 2 of the proof of Proposition 3.2.2, the second and third vertical arrows are isomorphisms, hence so is the first. As $0 \to \pi_1^{I_1} \to \pi_2^{I_1} \to \pi'^{I_1} \to 0$ is exact, we conclude that F is the $\mathfrak{m}$-adic filtration exactly as at the end of the proof of Corollary 3.2.4(i).

(iii) Let π_1, π_2 be as in (i). The assertion holds for subrepresentations of π by Theorem 3.2.3(i), so π_2 is generated by $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2)$. Thus π' is generated by the image of $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_2)$ in π', which is contained in $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi')$ (even equal by Lemma 3.2.6).

(iv) It is clear by the exact sequence (47) in Lemma 3.2.6, since $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi)$ is multiplicity free.

(v) If π' is a quotient of π, this is established in Step 2 of the proof of Proposition 3.2.2. In general, if $\pi_1 \subseteq \pi_2 \subseteq \pi$ such that $\pi' \cong \pi_2/\pi_1$, then we get an exact sequence $0 \to \pi' \to \pi/\pi_1 \to \pi/\pi_2 \to 0$ and hence an exact sequence

$$0 \to \mathrm{E}_\Lambda^{2f}(\pi'^\vee) \otimes \eta \to \mathrm{E}_\Lambda^{2f}((\pi/\pi_1)^\vee) \otimes \eta \to \mathrm{E}_\Lambda^{2f}((\pi/\pi_2)^\vee) \otimes \eta \to 0,$$

as $\pi'^\vee$ is Cohen–Macaulay by part (ii) and where $\eta \overset{\mathrm{def}}{=} \det(\overline{\rho})\omega^{-1}$. Then the claim follows from Lemma 3.2.6 and the known case for quotient representations (cf. Step 2 of the proof of Proposition 3.2.2). □

4 Finite length in the nonsplit reducible case

We prove that a smooth mod p representation π of $\mathrm{GL}_2(K)$ satisfying assumptions (i)–(iv) of § 2.1 with $r = 1$ has finite length when the underlying Galois representation $\overline{\rho}$ is *nonsplit* reducible. We also establish several structural results on π as an I- and $\mathrm{GL}_2(\mathcal{O}_K)$-representation.

Unless otherwise stated, we assume that $\overline{\rho}$ is nonsplit reducible and 0-generic. We let π be an admissible smooth representation of $\mathrm{GL}_2(K)$ over $\mathbb{F}$ satisfying assumptions (i)–(iv) of § 2.1. Recall from § 2.6 that $\pi = \pi(\overline{\rho})$ as defined in § 2.6 satisfies assumption (iv) for any $r \geq 1$. From [Wan23, Thm. 6.3(ii)] together with [BHH+23, Prop. 6.4.6] (the hypotheses of the latter being checked in the proof of [Wan23, Thm. 5.1]) one deduces that $\pi(\overline{\rho})$ satisfies assumptions (i) and (ii) of § 2.1, for any $r \geq 1$. It also satisfies assumption (iii) (again, for any $r \geq 1$) by [HW22, Thm. 8.2] with [Wan23, Thm. 6.3(i)].

As in § 3 we assume that $r = 1$ in assumptions (i) and (iv) throughout.

4.1 Preliminaries on Serre weights

We collect a number of results on the combinatorics of Serre weights and injective envelopes.

Recall from § 1.3 that $D_0(\overline{\rho}) = \bigoplus_{\sigma \in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})$, and from [BP12, § 13] that $D_{0,\sigma}(\overline{\rho})$ is maximal (for the inclusion) with respect to the two properties $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(D_{0,\sigma}(\overline{\rho})) = \sigma$ and $\mathrm{JH}(D_{0,\sigma}(\overline{\rho})/\sigma) \cap W(\overline{\rho}) = \emptyset$. In particular, $D_{0,\sigma}(\overline{\rho}^{\mathrm{ss}}) \subseteq D_{0,\sigma}(\overline{\rho})$.

We first generalize Lemma 3.1.3 to the case where $\overline{\rho}$ need not be semisimple.

Lemma 4.1.1. *If $\mu \in \mathscr{P}$, then χ_μ occurs in $D_{0,\sigma}(\overline{\rho})^{I_1}$, where $\sigma \in W(\overline{\rho})$ is determined (via* (10)*) by $J_\sigma = J_{\overline{\rho}} \cap J_\mu$. Moreover, as a Jordan–Hölder factor of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu$, σ is parametrized (via Remark 3.1.1 and* (35)*) by the following subset of $\{0, \dots, f-1\}$:*

$$X(\mu) \stackrel{\text{def}}{=} \{j : \mu_j(x_j) \in \{x_j, p-2-x_j, p-3-x_j\}\} \cup \{j \in J_{\overline{\rho}} : \mu_j(x_j) = x_j + 1\}. \tag{48}$$

Proof. The proof goes as in [Hu16, Prop. 2.1] and we only briefly recall it.

Let $\lambda \in \mathscr{D}$ such that $\sigma \in W(\overline{\rho})$ corresponds to λ. It is clear that σ is a subquotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu$, so via Remark 3.1.1 there is a unique f-tuple $\xi \in \mathcal{P}$ such that

$$\xi_j^c(\mu_j(x_j)) = \xi_j(\mu_j^{[s]}(x_j)) = \lambda_j(x_j) \tag{49}$$

for any $j \in \{0, \dots, f-1\}$, where

$$\mu^{[s]} \stackrel{\text{def}}{=} (p-1-\mu_0(x_0), \dots, p-1-\mu_{f-1}(x_{f-1})) \in \mathscr{P}. \tag{50}$$

Here, we used [HW22, Lemma 2.1, Lemma 2.7] to obtain (49) (equality between formal f-tuples).

Note that our convention for $J \mapsto \xi_J$ is shifted by one compared to [BP12, § 2] and [Bre14, § 2]. Using the second equality in (49), [Bre14, Prop. 4.3] (and the formula for $J^{\max}$ in eq. (19) in its proof, replacing λ there by our μ and noting that $\chi_\mu \neq \chi_\mu^s$) gives the following relation

$$\begin{aligned} \xi_j(y_j) \in \{y_j - 1, p-1-y_j\} \iff & \mu_j^{[s]}(x_j) \in \{x_j+1, x_j+2, \underline{p-2-x_j}, p-1-x_j\} \\ \iff & \mu_j(x_j) \in \{x_j, \underline{x_j+1}, p-2-x_j, p-3-x_j\}, \end{aligned} \tag{51}$$

making the convention that an underlined entry is only allowed when $j \in J_{\overline{\rho}}$. We say that a pair $(\xi, \mu) \in \mathcal{P} \times \mathscr{P}$ is *compatible* if (51) holds.

It is straightforward to list all the possibilities of compatible pairs $(\xi, \mu) \in \mathcal{P} \times \mathscr{P}$ and verify that

$$\begin{aligned} \xi_j(\mu_j^{[s]}(x_j)) = \lambda_j(x_j) \in \{x_j+1, p-3-x_j\} \iff & \mu_j^{[s]}(x_j) \in \{x_j+2, \underline{p-2-x_j}, p-3-x_j\} \\ \iff & \mu_j(x_j) \in \{\underline{x_j+1}, x_j+2, p-3-x_j\}. \end{aligned}$$

The left-hand side is equivalent to $j \in J_\sigma = J_\lambda$ and the right-hand side is equivalent to $j \in J_\mu \cap J_{\overline{\rho}}$ by (10). The second part results from (51). $\square$

Let σ be a 1-generic Serre weight. Recall that the set of Jordan–Hölder factors of $\mathrm{Inj}_\Gamma \sigma$ is parametrized by a set of f-tuples denoted by $\mathcal{I} \stackrel{\text{def}}{=} \mathcal{I}(x_0, \dots, x_{f-1})$ in [BP12, § 3] (do not confuse this $\mathcal{I}$ with the ideal $\mathcal{I}$ before Lemma 2.3.2!). Given $\lambda \in \mathcal{I}$ we write

$$\mathcal{S}(\lambda) \stackrel{\text{def}}{=} \{j \in \{0, \dots, f-1\} : \lambda_j(x_j) \in \{x_j \pm 1, p-2-x_j \pm 1\}\}$$

as in [BP12, § 4]. (This notation is consistent with (35), noting that $\mathcal{P} \subseteq \mathcal{I}$.)

The following lemma is true for any 0-generic $\overline{\rho}$.

Lemma 4.1.2. *We have $W(\overline{\rho}^{\mathrm{ss}}) \subseteq \mathrm{JH}(D_0(\overline{\rho}))$.*

Proof. By the construction of $D_0(\overline{\rho})$ (see [BP12, Prop. 13.1]) and [BP12, Prop. 13.4] we have

$$\mathrm{JH}(D_0(\overline{\rho})) = \mathrm{JH}\Big(\bigoplus_{\sigma \in W(\overline{\rho})} \mathrm{Inj}_\Gamma \sigma\Big). \tag{52}$$

Thus it suffices to prove that

$$W(\overline{\rho}^{\mathrm{ss}}) \subseteq \mathrm{JH}\Big(\bigoplus_{\sigma \in W(\overline{\rho})} \mathrm{Inj}_\Gamma \sigma\Big).$$

But it is clear from [BP12, Lemma 3.2, Lemma 11.2] that $W(\overline{\rho}^{\mathrm{ss}}) \subseteq \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma_0)$, where $\sigma_0 \in W(\overline{\rho})$ denotes the unique Serre weight corresponding to $(x_0, \dots, x_{f-1}) \in \mathscr{D}$. $\square$

Recall from [BP12, Cor. 3.12] that given a 0-generic Serre weight σ and $\tau \in \mathrm{JH}(\mathrm{Inj}_\Gamma \sigma)$, there exists a unique finite dimensional Γ-representation $I(\sigma, \tau)$ such that $\mathrm{soc}_\Gamma I(\sigma,\tau) = \sigma$, $\mathrm{cosoc}_\Gamma I(\sigma,\tau) = \tau$ and $[I(\sigma,\tau) : \sigma] = 1$. If σ is 1-generic, [BP12, Cor. 4.11] implies that $I(\sigma,\tau)$ has length $2^{|\mathcal{S}(\lambda)|}$, where $\lambda \in \mathcal{I}$ corresponds to τ. Recall that any $\sigma \in W(\overline{\rho}^{\mathrm{ss}})$ is n-generic if $\overline{\rho}$ is n-generic.

Lemma 4.1.3. *Assume that $\overline{\rho}$ is 0-generic. Let $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ and $\sigma \in W(\overline{\rho})$ be the unique Serre weight determined by $J_\sigma = J_{\overline{\rho}} \cap J_\tau$ (via (10)). Then $\tau \in \mathrm{JH}(D_{0,\sigma}(\overline{\rho}))$ and the Jordan–Hölder factors of $I(\sigma,\tau)$ are exactly the Serre weights $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ satisfying $J_\sigma \subseteq J_{\tau'} \subseteq J_\tau$. In particular, $\ell(\tau') \le \ell(\tau)$ for any $\tau' \in \mathrm{JH}(I(\sigma,\tau))$, with equality if and only if $\tau' = \tau$.*

Proof. The assertion $\tau \in \mathrm{JH}(D_{0,\sigma}(\overline{\rho}))$ follows directly from [BP12, Lemma 15.3] (note that the condition $\ell(\overline{\rho}, \tau) < +\infty$ in *loc. cit.* is satisfied by Lemma 4.1.2). To verify the remaining claim, for any subset $J \subseteq \{0, 1, \dots, f-1\}$ let $\sigma_J \in W(\overline{\rho}^{\mathrm{ss}})$ determined by $J_{\sigma_J} = J$. From [BP12, Cor. 4.11] we deduce that $I(\sigma_\emptyset, \sigma_{\{0,\dots,f-1\}})$ is of length 2^f with constituents all σ_J ($J \subseteq \{0, 1, \dots, f-1\}$). Moreover, the proof of *loc. cit.* (referring to [BP12, Thm. 4.7]) shows that the lattice of submodules is isomorphic to the lattice of ideals of the partially ordered set $(\{0, \dots, f-1\}, \subseteq)$, by sending a submodule M to the ideal $\{J : \sigma_J \in \mathrm{JH}(M)\}$. The claim follows, since $\sigma = \sigma_{J_{\overline{\rho}} \cap J_\tau}$ and $\tau = \sigma_{J_\tau}$. $\square$

Lemma 4.1.4. *Suppose that $\lambda \in \mathscr{P}$ and that $J \overset{\mathrm{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) \in \{x_j, p-1-x_j\}\}$. Then $|J_\lambda| + |J_{\lambda^*}| + |J| = f$, where $\lambda \mapsto \lambda^*$ is the involution of $\mathscr{P}$ defined in [BHH+a, Def. 3.3.1.6].*

Proof. This follows directly from (12) and [BHH+a, Def. 3.3.1.6]. $\square$

4.2 Some commutative algebra

We prove that certain explicit $\overline{R}$-modules are Cohen–Macaulay.

Recall from § 1.3 that $R = \mathbb{F}[y_j, z_j : 0 \le j \le f-1]$ and $\overline{R} = \mathbb{F}[y_j, z_j : 0 \le j \le f-1]/(y_j z_j : 0 \le j \le f-1)$.

Lemma 4.2.1. *Suppose that M is a nonzero finitely generated graded R-module. Then M is Cohen–Macaulay (in the sense of commutative algebra) if and only if $\mathrm{E}^i_R(M) = 0$ for all $i \neq j_R(M)$.*

Proof. Let $\mathfrak{m} = (y_j, z_j : 0 \le j \le f-1)$ denote the unique maximal graded ideal of R. Then M is a Cohen–Macaulay R-module if and only if $M_\mathfrak{m}$ is a Cohen–Macaulay $R_\mathfrak{m}$-module ([BH93, Cor. 2.2.15]) if and only if $\mathrm{E}^i_{R_\mathfrak{m}}(M_\mathfrak{m}) = 0$ for all but one i ([BH93, Cor. 3.5.11], as $R_\mathfrak{m}$ is regular) if and only if $\mathrm{E}^i_R(M) = 0$ for all but one i (using $\mathrm{E}^i_R(M) \otimes_R R_\mathfrak{m} \cong \mathrm{E}^i_{R_\mathfrak{m}}(M_\mathfrak{m})$ and [BH93, Prop. 1.5.15(c)]). By definition, $\mathrm{E}^{j_R(M)}(M) \neq 0$. $\square$

Lemma 4.2.2. *Suppose that $t_j \in \{y_j, z_j, y_j z_j\}$ for $0 \le j \le f-1$. Then the R-module $\overline{R}/(t_0, \dots, t_{f-1})$ is Cohen–Macaulay of grade f.*

Proof. As $\overline{R}/(t_0, \dots, t_{f-1})$ is a Cohen–Macaulay $\mathrm{gr}(\Lambda)$-module of grade $2f$ by the beginning of the proof of Theorem 2.1.2 in § 2.5, the result follows from [BHH$^+$a, Lemma 3.3.1.9]. $\square$

Proposition 4.2.3. *Suppose that $1 \le d \le f$. Let I_d be the homogeneous ideal of $\overline{R}$ generated by all monomials $z_{i_1} \cdots z_{i_d}$ with $0 \le i_1 < \cdots < i_d \le f-1$. Then the R-module $\overline{R}/I_d$ is Cohen–Macaulay of grade f.*

Proof. If $d = 1$ this follows from Lemma 4.2.2, so we suppose $d \ge 2$. Then the ring $\overline{R}/I_d = R/(y_j z_j, z_{i_1} \cdots z_{i_d})$ (all j, all $0 \le i_1 < \cdots < i_d \le f-1$) is the Stanley–Reisner ring $\mathbb{F}[\Delta]$ associated to the simplicial complex Δ whose minimal non-faces $\{y_j, z_j\}$, $\{z_{i_1}, \dots, z_{i_d}\}$ correspond to the generators [BH93, § 5.1]. Thus Δ is the pure $(f-1)$-dimensional simplicial complex with facets $\underline{x} = \{x_0, \dots, x_{f-1}\}$, where $x_j \in \{y_j, z_j\}$, $|\{j : x_j = z_j\}| < d$. For a facet $\underline{x} = \{x_0, \dots, x_{f-1}\}$ let $J(\underline{x}) \stackrel{\text{def}}{=} \{j : x_j = z_j\}$.

We prove that Δ is shellable [BH93, Def. 5.1.11], which implies that $\mathbb{F}[\Delta]$ is a Cohen–Macaulay ring by [BH93, Thm. 5.1.13]. To see this, we order the facets as $\underline{x}^{(0)}, \underline{x}^{(1)}, \dots$ such that $|J(\underline{x}^{(0)})| \le |J(\underline{x}^{(1)})| \le \cdots$ is non-decreasing. Then, using the notation of [BH93, § 5.1], for any $i_0 > 0$ the intersection $\langle \underline{x}^{(0)}, \dots, \underline{x}^{(i_0-1)} \rangle \cap \langle \underline{x}^{(i_0)} \rangle$ is generated by the maximal proper faces of $\underline{x}^{(i_0)}$ that are of the form $\underline{x}^{(i_0)} \setminus \{x_j^{(i_0)}\}$ for some $j \in J(\underline{x}^{(i_0)})$, proving shellability.

Let $S \stackrel{\text{def}}{=} \overline{R}/I_d = \mathbb{F}[\Delta]$, which is graded of dimension f [BH93, Thm. 5.1.4], and let $\mathfrak{m}$ denote the unique maximal graded ideal of R (or its image in S). As S is a Cohen–Macaulay ring, it is also a Cohen–Macaulay R-module [BH93, § 2.1]. We compute

$$j_R(S) = j_{R_\mathfrak{m}}(S_\mathfrak{m}) = \dim R_\mathfrak{m} - \dim_{R_\mathfrak{m}} S_\mathfrak{m} = \dim R - \dim_R S = 2f - f = f,$$

where we used [BH93, Prop. 1.5.15(e)] for the first equality, [BH93, Cor. 3.5.11] for the second equality, and [BH93, Ex. 1.5.25] for the third equality. (Alternatively, it follows from [BH93, Thm. 5.7.3] that $\mathrm{Ext}^f_R(S, R) \neq 0$.) $\square$

Definition 4.2.4. Suppose that J_1, J_2 are disjoint subsets of $\{0, \dots, f-1\}$ and that $d \in \mathbb{Z}$. We define the ideal $I(J_1, J_2, d)$ of $\overline{R}$ as follows: if $d \ge 1$ let $I(J_1, J_2, d)$ be generated by all $\prod_{j \in J_1'} y_j \prod_{j \in J_2'} z_j$ with $J_1' \subseteq J_1$, $J_2' \subseteq J_2$, $|J_1'| + |J_2'| = d$; if $d \le 0$, let $I(J_1, J_2, d) \stackrel{\text{def}}{=} \overline{R}$. Suppose moreover that $t_j \in \{y_j, z_j, y_j z_j\}$ for all $0 \le j \le f-1$, we define the ideal $I(J_1, J_2, d, \underline{t})$ of $\overline{R}$ as $I(J_1, J_2, d) + (t_0, \ldots, t_{f-1})$.

Corollary 4.2.5. *If $d \ge 1$ and $t_j = y_j z_j$ for all $j \in J_1 \sqcup J_2$, the R-module $\overline{R}/I(J_1, J_2, d, \underline{t})$ is Cohen–Macaulay of grade f.*

Proof. Relabel indices so that $J_1 \sqcup J_2 = \{0, \ldots, k-1\}$ for some $1 \le k \le f$. We define the $\mathbb{F}$-algebras $R^{(1)}$ and $\overline{R}^{(1)}$ (resp. $R^{(2)}$ and $\overline{R}^{(2)}$) exactly as we defined R and $\overline{R}$ but using only indices $0 \le j \le k-1$ (resp. $k \le j \le f-1$). Then $R \cong R^{(1)} \otimes_{\mathbb{F}} R^{(2)}$ and $\overline{R}/I(J_1, J_2, d, \underline{t})$ is the tensor product of $M^{(1)} \stackrel{\text{def}}{=} \overline{R}^{(1)}/I(J_1, J_2, d)$ and $M^{(2)} \stackrel{\text{def}}{=} \overline{R}^{(2)}/(t_j : k \le j \le f-1)$ over $\mathbb{F}$. We know that $M^{(1)}$ is a Cohen–Macaulay $R^{(1)}$-module of grade k (by Proposition 4.2.3 if $d \le k$, and by Lemma 4.2.2, taking $t_j = y_j z_j$ for all j, otherwise). By Lemma 4.2.2 the $R^{(2)}$-module $M^{(2)}$ is Cohen–Macaulay of grade $f-k$. By the Künneth formula, we obtain that

$$\mathrm{E}^n_R(\overline{R}/I(J_1, J_2, d, \underline{t})) \cong \bigoplus_{i+j=n} \mathrm{E}^i_{R^{(1)}}(M^{(1)}) \otimes_{\mathbb{F}} \mathrm{E}^j_{R^{(2)}}(M^{(2)}),$$

hence $\overline{R}/I(J_1, J_2, d, \underline{t})$ is a Cohen–Macaulay R-module of grade f. □

If N' is a finitely generated $\overline{R}$-module, we let $m_{\mathfrak{q}}(N') \stackrel{\text{def}}{=} \mathrm{length}_{\overline{R}_{\mathfrak{q}}}(N'_{\mathfrak{q}})$ and define $m(N') \stackrel{\text{def}}{=} \sum_{\mathfrak{q}} m_{\mathfrak{q}}(N')$, which is the total multiplicity of the cycle $\mathcal{Z}(N')$ in (2) (here $\mathfrak{q}$ runs through the minimal prime ideals of $\overline{R}$).

Lemma 4.2.6. *Suppose that $t_j = y_j z_j$ for all $j \in J \stackrel{\text{def}}{=} J_1 \sqcup J_2$. Then*

$$m(\overline{R}/I(J_1, J_2, d, \underline{t})) = 2^{|\{j \in J^c : t_j = y_j z_j\}|} \left(\sum_{i<d} \binom{|J|}{i} \right).$$

Proof. If $d \le 0$, the formula is trivially true, so we suppose $d \ge 1$. Without loss of generality we assume that $J = \{0, \ldots, k-1\}$ for some $1 \le k \le f$ and that $J_1 = \emptyset$. Consider the minimal prime $\mathfrak{q} = (v_0, \ldots, v_{f-1})$ of $\overline{R}$ given by $v_j \in \{y_j, z_j\}$. Write

$$M \stackrel{\text{def}}{=} \overline{R}/I(J_1, J_2, d, \underline{t}) = \mathbb{F}[y_j, z_j : 0 \le j \le f-1]/(y_j z_j, z_{i_1} \cdots z_{i_d}, t_{j'}),$$

where $0 \le j < k \le j' < f$, and $0 \le i_1 < \cdots < i_d \le k-1$. If $v_j = y_j$, then in $M_{\mathfrak{q}}$ the variable z_j is inverted and y_j becomes zero, and vice versa when $v_j = z_j$. It follows that $m_{\mathfrak{q}}(M) = 1$ if $|\{0 \le j \le k-1 : v_j = y_j\}| < d$ and $v_{j'}$ divides $t_{j'}$ for all $k \le j' < f$, whereas $m_{\mathfrak{q}}(M) = 0$ otherwise. The lemma follows by summing over all $\mathfrak{q}$. □

4.3 On the structure of subrepresentations of π

The main result of this section is the description of the K_1-invariants of subrepresentations of π (Theorem 4.3.15). We need several technical results on $\mathrm{GL}_2(\mathcal{O}_K)$-representations induced from certain multiplicity-free I-representations.

4.3.1 Some induced representations of $\mathrm{GL}_2(\mathcal{O}_K)$

We study $\mathrm{GL}_2(\mathcal{O}_K)$-representations induced from certain multiplicity-free I-representations.

Given a character $\chi : I \to \mathbb{F}^\times$ and two subsets $J_1, J_2 \subseteq \{0, \ldots, f-1\}$ such that $J_1 \cap J_2 = \emptyset$, set

$$\chi^{J_1,J_2} \stackrel{\text{def}}{=} \chi \prod_{j\in J_1} \alpha_j^{-1} \prod_{j\in J_2} \alpha_j. \tag{53}$$

Lemma 4.3.1. *There exists a unique I-representation of dimension $2^{|J_1|+|J_2|}$ with socle χ and cosocle χ^{J_1,J_2} such that the d-th socle layer is given by*

$$\bigoplus_{J_1'\subseteq J_1, J_2'\subseteq J_2, |J_1'|+|J_2'|=d} \chi \prod_{j\in J_1'} \alpha_j^{-1} \prod_{j\in J_2'} \alpha_j.$$

We denote it by $W(\chi, \chi^{J_1,J_2})$. Moreover,

(i) $W(\chi, \chi^{J_1,J_2})$ *is multiplicity free;*

(ii) $W(\chi, \chi^{J_1,J_2})$ *is fixed by K_1 if and only if $J_2 = \emptyset$.*

Proof. We first prove uniqueness. By [BHH+23, (42), (43)] and the Poincaré–Birkhoff–Witt theorem, any Jordan–Hölder factor χ' of $(\mathrm{Inj}_{I/Z_1}\chi)[\mathfrak{m}^{n+1}]$ has the form $\chi\alpha_{i_1}^{t_1}\cdots\alpha_{i_m}^{t_m}$, where $m \leq n$, $i_k \in \{0, \ldots, f-1\}$ and $t_k \in \{\pm 1\}$, which is equal to

$$\chi \prod_j \alpha_j^{b_j} \prod_j \alpha_j^{-b_j'}$$

with $b_j \stackrel{\text{def}}{=} |\{k : i_k = j, t_k = 1\}|$ and $b_j' \stackrel{\text{def}}{=} |\{k : i_k = j, t_k = -1\}|$. In particular, χ^{J_1,J_2} occurs in $(\mathrm{Inj}_{I/Z_1}\chi)[\mathfrak{m}^{|J_1|+|J_2|+1}]$. We claim that it occurs with multiplicity one. Indeed, if $\chi' = \chi^{J_1,J_2}$, Lemma 4.3.2 shows that

$$b_j - b_j' = \begin{cases} -1 & \text{if } j \in J_1, \\ 1 & \text{if } j \in J_2, \\ 0 & \text{otherwise.} \end{cases}$$

Using the condition $\sum_{j=0}^{f-1}(b_j + b_j') \leq |J_1| + |J_2|$, we deduce that

$$(b_j, b_j') = \begin{cases} (0,1) & \text{if } j \in J_1, \\ (1,0) & \text{if } j \in J_2, \\ (0,0) & \text{otherwise.} \end{cases}$$

This implies the claim. As a consequence, if $W(\chi, \chi^{J_1,J_2})$ exists, it embeds into $(\mathrm{Inj}_{I/Z_1}\chi)[\mathfrak{m}^{|J_1|+|J_2|+1}]$ and is hence the unique subrepresentation of $(\mathrm{Inj}_{I/Z_1}\chi)[\mathfrak{m}^{|J_1|+|J_2|+1}]$ with cosocle χ^{J_1,J_2}.

For the existence, we may assume $\chi = \mathbf{1}$, and let $E_j^{\pm}$ denote the I-representation $E_j^{\pm}(1)$ constructed in the proof of Lemma 2.4.1 (with $s = 1$). We take

$$W(\mathbf{1}, \mathbf{1}^{J_1,J_2}) \stackrel{\text{def}}{=} \Big(\bigotimes_{j\in J_1} E_j^-\Big) \otimes_{\mathbb{F}} \Big(\bigotimes_{j\in J_2} E_j^+\Big).$$

It is multiplicity free by Lemma 4.3.2. The assertion on the d-th socle layer of $W(\mathbf{1}, \mathbf{1}^{J_1,J_2})$ can be proved as in the proof of Lemma 2.4.1, which shows that the socle filtration of $W(\mathbf{1}, \mathbf{1}^{J_1,J_2})$ corresponds to a suitable tensor product filtration on $W(\mathbf{1}, \mathbf{1}^{J_1,J_2})^\vee$ under duality.

Finally, assertion (i) was established above and (ii) follows from the fact that E_j^- is fixed by K_1 and E_j^+ is not. $\square$

Lemma 4.3.2. *Suppose $p > 3$. Let $\underline{a} = (a_0, \ldots, a_{f-1})$, $\underline{b} = (b_0, \ldots, b_{f-1}) \in \mathbb{Z}^f$. Assume that $a_j \in \{-1, 0, 1\}$ for all j, $\sum_{j=0}^{f-1} |a_j| \geq \sum_{j=0}^{f-1} |b_j|$ and*

$$\sum_{j=0}^{f-1} a_j p^j \equiv \sum_{j=0}^{f-1} b_j p^j \pmod{p^f - 1}.$$

Then $\underline{a} = \underline{b}$.

Proof. Let $|\underline{a}| \overset{\text{def}}{=} \sum_{j=0}^{f-1} |a_j|$. We induct on the pair $(|\underline{a}|, |\underline{b}|)$ with lexicographic order. We fix a pair $(\underline{a}, \underline{b})$ and suppose that the result holds for all $(\underline{a}', \underline{b}') < (\underline{a}, \underline{b})$.

We claim that $|b_j| \leq p - 2$ for all j. If $b_i \geq p - 1$ for some i, we define $\underline{b}' \in \mathbb{Z}^f$ by

$$b'_j \overset{\text{def}}{=} \begin{cases} b_j - p & \text{if } j = i, \\ b_j + 1 & \text{if } j = i + 1, \\ b_j & \text{otherwise.} \end{cases}$$

Then $\sum_{j=0}^{f-1} b_j p^j \equiv \sum_{j=0}^{f-1} b'_j p^j \pmod{p^f - 1}$, and $|\underline{b}'| < |\underline{b}|$ (as $p > 3$). By induction we have $\underline{a} = \underline{b}'$, which implies $|\underline{a}| = |\underline{b}'| < |\underline{b}|$, contradiction. Thus $b_j \leq p - 2$ for all j. In a similar way we get $b_j \geq -(p - 2)$ for all j.

The assumption implies

$$\sum_j (b_j - a_j) p^j \equiv 0 \pmod{p^f - 1}$$

with $|b_j - a_j| \leq p - 1$ for all j, as $|b_j| \leq p - 2$. Then this can happen only when $b_j = a_j$ for all j, or $|b_j - a_j| = p - 1$ for all j. The second possibility cannot happen, because it forces $|b_j| = p - 2$ for all j, which contradicts $|\underline{a}| \geq |\underline{b}|$, as $p > 3$. $\square$

Note that if χ is n-generic (see § 1.3) with $n \geq 2$, then every character occurring in $W(\chi, \chi^{J_1,J_2})$ is $(n - 2)$-generic by Lemma 4.3.1.

Lemma 4.3.3. *Assume that χ is 2-generic. Then the $\mathrm{GL}_2(\mathcal{O}_K)$-representation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi^{J_1,J_2})$ is multiplicity free.*

Proof. It follows from our genericity assumption and [BP12, Lemma 2.2]. $\square$

We recall from § 3.1 that the Jordan–Hölder factors σ of a principal series representation $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi'$ for a 1-generic character $\chi' : I \to \mathbb{F}^\times$ are parametrized by the subsets of $\{0, \ldots, f -$

$1\}$, sending σ to $\mathcal{S}(\sigma)$, such that the socle of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi'$ corresponds to the empty set (see Remark 3.1.1).

We also recall from [HW22, Def. 2.9] some notation and a lemma. Assume first $f > 1$. Given $j \in \{0, \dots, f-1\}$ and $* \in \{+, -\}$ we define the elements $\mu_j^* \in \mathcal{I}$ as follows: $(\mu_j^*)_{j-1}(x_{j-1}) = p - 2 - x_{j-1}$, $(\mu_j^*)_j(x_j) = x_j * 1$ and $(\mu_j^*)_i(x_i) = x_i$ for $i \notin \{j-1, j\}$. If $f = 1$ we define $\mu_0^* \in \mathbb{Z}[x_0]$ by $\mu_0^*(x_0) = p - 2 - (*1) - x_0$. For any $f \geq 1$, if σ is a 0-generic Serre weight corresponding to a tuple $(s_0, \dots, s_{f-1}) \in \{0, \dots, p-1\}^f$ we write $\mu_j^*(\sigma)$ for the Serre weight corresponding to the f-tuple $\mu_j^*((s_0, \dots, s_{f-1})) \otimes \det^{e(\mu_j^*)(s_0, \dots, s_{f-1})}$, where $e(\mu_j^*) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z} x_i$ is defined in [BP12, § 3]. (Note that $\mu_j^-(\sigma)$ is undefined if $f \geq 2$ and $s_j = 0$ and $\mu_j^+(\sigma)$ is undefined if $f = 1$ and $s_j = p-2$.)

Lemma 4.3.4. *Let σ and σ' be two 0-generic Serre weights. If $f = 1$, suppose that σ, σ' are not both isomorphic to $\mathrm{Sym}^{p-2}\mathbb{F}^2 \otimes \eta$ for some η. Then*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathcal{O}_K)/Z_1}(\sigma', \sigma) \neq 0 \iff \mathrm{Ext}^1_\Gamma(\sigma', \sigma) \neq 0 \iff \sigma' \in \{\mu_j^*(\sigma) : 0 \leq j \leq f-1, * \in \{+, -\}\}.$$

Lemma 4.3.4 follows from [HW22, Lemma 2.10] and [Hu10, Prop. 2.21], except when $f = 1$ the proof is incomplete in *loc. cit.*

Proof. If $\sigma' = \mu_j^*(\sigma)$ for some $0 \leq j \leq f-1$ and $* \in \{+, -\}$, it follows from [BP12, Cor. 5.6] that $\mathrm{Ext}^1_\Gamma(\sigma', \sigma) = \mathrm{Ext}^1_{K/Z_1}(\sigma', \sigma) \neq 0$. Conversely, suppose $\mathrm{Ext}^1_{K/Z_1}(\sigma', \sigma) \neq 0$ and we need to prove that $\sigma' = \mu_j^*(\sigma)$ for some $0 \leq j \leq f-1$ and $* \in \{+, -\}$. Using [BP12, Cor. 5.6] and [HW22, Lem. 2.10(i)], it suffices to exclude cases (a) and (c) of [BP12, Cor. 5.6(ii)]. The argument below is taken from the proof of [HW22, Lemma 2.10(i)].

First assume that we are in case (c). Thus, as σ, σ' are 0-generic, we may write

$$\sigma' = (s_0, \dots, s_{f-1}) \otimes \eta, \quad \sigma = (s_0, \dots, s_j - 2, \dots, s_{f-1}) \otimes \eta \det^{p^j},$$

with $2 \leq s_j \leq p-2$ and $0 \leq s_i \leq p-2$ for $i \neq j$. Let $0 \to \sigma \to V \to \sigma' \to 0$ be a nonsplit K/Z_1-extension. Let $w \in V$ be an H-eigenvector of character $\chi_{\sigma'}$ such that its image in σ' spans σ'^{I_1}. We will prove that w is fixed by I_1/Z_1, thus by Frobenius reciprocity we obtain a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant morphism $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\sigma'} \to V$ which must be surjective (as it surjects onto $\operatorname{cosoc}_K V$). But this is impossible by the structure of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_{\sigma'}$ by [BP12, Lemma 2.2].

For $0 \leq i \leq f-1$, consider the operators

$$X_i \overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q} \kappa_0(\lambda)^{-p^i} \begin{pmatrix} 1 & 0 \\ [\lambda] & 1 \end{pmatrix}, \quad Y_i \overset{\text{def}}{=} \sum_{\lambda \in \mathbb{F}_q} \kappa_0(\lambda)^{-p^i} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix},$$

which are viewed as elements of $\mathbb{F}[\![K/Z_1]\!]$. By [BHH+a, Lemma 3.2.2.1], we have $\mathbb{F}[\![\begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}]\!] = \mathbb{F}[\![Y_0, \dots, Y_{f-1}]\!]$ and similarly $\mathbb{F}[\![\begin{pmatrix} 1 & 0 \\ p\mathcal{O}_K & 1 \end{pmatrix}]\!] = \mathbb{F}[\![X_0^p, \dots, X_{f-1}^p]\!]$. Thus, by (the proof of) [BHH+23, Prop. 5.3.3] the elements $\{X_i^p, Y_i : 0 \leq i \leq f-1\}$ topologically generate the maximal ideal of $\mathbb{F}[\![I_1/Z_1]\!]$ and we are left to prove that $X_i^p w = Y_i w = 0$ for all $0 \leq i \leq f-1$.

It is direct to check that X_iw (resp. Y_iw) has H-eigencharacter $\chi_{\sigma'}\alpha_i^{-1}$ (resp. $\chi_{\sigma'}\alpha_i$). On the other hand, as $\chi_\sigma = \chi_{\sigma'}\alpha_j^{-1}$, we have

$$\mathrm{JH}(V|_H) = \mathrm{JH}(\sigma'|_H) = \Big\{\chi_{\sigma'}\prod_{i=0}^{f-1}\alpha_i^{-k_i} : 0 \leq k_i \leq s_i\Big\}.$$

Moreover, $Y_iw \in \sigma$, by our choice of w. However, it is direct to check that $\chi_{\sigma'}\alpha_i^{-(p-1)}$ does not occur in $\mathrm{JH}(V|_H)$ and $\chi_{\sigma'}\alpha_i$ does not occur in $\mathrm{JH}(\sigma|_H)$. Therefore $X_i^{p-1}w = Y_iw = 0$ for all $0 \leq i \leq f-1$, hence also $X_i^p w = 0$, as desired.

Case (a) can be treated by passing to the dual (the dual extension of $\sigma^\vee$ by $\sigma'^\vee$ is as in case (c)). $\square$

Lemma 4.3.5. *Assume that χ is 3-generic. Let $\sigma \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi)$ and let $\sigma' = \mu_j^*(\sigma)$ for some $0 \leq j \leq f-1$ and $* \in \{+,-\}$. Let $J_1, J_2 \subseteq \{0,\dots,f-1\}$ such that $J_1 \cap J_2 = \emptyset$. Assume $\sigma' \in \mathrm{JH}\big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W(\chi,\chi^{J_1,J_2})\big)$. Then $\sigma' \in \mathrm{JH}\big(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi\oplus\chi\alpha_j^{-1}\oplus\chi\alpha_j)\big)$ and exactly one of the following cases happens:*

(i) *$\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi)$, in which case either $\mathcal{S}(\sigma) \sqcup \{j\} = \mathcal{S}(\sigma')$ or $\mathcal{S}(\sigma') \sqcup \{j\} = \mathcal{S}(\sigma)$;*

(ii) *$\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi\alpha_j^{-1})$, in which case $j \in J_1$ and $\mathcal{S}(\sigma) \sqcup \{j\} = \mathcal{S}(\sigma')$;*

(iii) *$\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi\alpha_j)$, in which case $j \in J_2$ and $\mathcal{S}(\sigma') \sqcup \{j\} = \mathcal{S}(\sigma)$.*

Proof. First, it is direct to check that σ' occurs in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}(\chi\oplus\chi\alpha_j^{-1}\oplus\chi\alpha_j)$. The claim on the relation between $\mathcal{S}(\sigma)$ and $\mathcal{S}(\sigma')$ follows directly from the definition of $\mu_j^*(\sigma)$ and (35) in case (i), and from [HW22, Lemmas 3.8, 3.7] in cases (ii) and (iii) respectively. $\square$

Proposition 4.3.6. *Assume that χ is 5-generic.*

(i) *The cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W(\chi,\chi^{J_1,J_2})$ is $\bigoplus_{J_1'\subseteq J_1}\sigma^{J_1',J_2}$, where σ^{J_1',J_2} denotes the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi^{J_1',J_2}$.*

(ii) *Let $\sigma \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi)$ be parametrized by $\mathcal{S}(\sigma)$ and $\tau \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi^{J_1,J_2})$ be parametrized by $\mathcal{S}(\tau)$. Let Q_σ be the unique quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W(\chi,\chi^{J_1,J_2})$ with socle σ (by Lemma 4.3.3). Then $\tau \in \mathrm{JH}(Q_\sigma)$ if and only if*

$$\mathcal{S}(\sigma)\cap J_1 = \emptyset \quad \text{and} \quad \mathcal{S}(\sigma)\sqcup J_1 \subseteq \mathcal{S}(\tau)\cup J_2. \tag{54}$$

Remark 4.3.7. In Proposition 4.3.6(ii), let $V_\tau \subseteq \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}W(\chi,\chi^{J_1,J_2})$ be the unique subrepresentation with cosocle τ (again by Lemma 4.3.3). Then $\tau \in \mathrm{JH}(Q_\sigma)$ if and only if $\sigma \in \mathrm{JH}(V_\tau)$.

Proof. Note that the genericity assumption implies that any $\chi' \in \mathrm{JH}(W(\chi,\chi^{J_1,J_2}))$ is 3-generic.

(i) By Lemma 4.3.1(ii), any I-equivariant morphism $W(\chi,\chi^{J_1,J_2}) \to \sigma'|_I$ (where σ' is a Serre weight) factors through the quotient of K_1-coinvariants

$$W(\chi, \chi^{J_1,J_2}) \twoheadrightarrow W(\chi^{\emptyset,J_2}, \chi^{J_1,J_2}).$$

By Frobenius reciprocity, this implies that the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi^{J_1,J_2})$ is equal to that of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi^{\emptyset,J_2}, \chi^{J_1,J_2})$, so any of its irreducible constituents is of the form σ^{J_1',J_2} for some $J_1' \subseteq J_1$. Conversely, $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi^{J_1,J_2})$ surjects onto $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi^{J_1',J_2}, \chi^{J_1,J_2})$, which surjects onto σ^{J_1',J_2}. (Write $\sigma^{J_1',J_2} = (s_0, \ldots, s_{f-1}) \otimes \eta$ with $0 \leq s_j \leq p-1$. As $s_j \geq 1$ for all j, $W(\chi^{J_1',J_2}, \chi^{J_1,J_2})$ embeds in $\sigma^{J_1',J_2}|_I$ and is identified with the subspace spanned by $x^{s-i}y^i \in \sigma^{J_1',J_2}$ for $i \in \{\sum_{j \in J_1''} p^j : J_1'' \subseteq J_1'\}$, where $s \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j s_j$; see the discussion at the beginning of [BP12, § 17]. As a consequence, σ^{J_1',J_2} occurs in the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi^{J_1',J_2}, \chi^{J_1,J_2})$ by Frobenius reciprocity.)

(ii) Assume τ satisfies condition (54). Let

$$\chi'' \stackrel{\text{def}}{=} \chi^{J_1,J_2}, \quad \chi' \stackrel{\text{def}}{=} \chi^{J_1,\emptyset} = \chi \prod_{j \in J_1} \alpha_j^{-1}.$$

Then $W(\chi, \chi') \hookrightarrow W(\chi, \chi'')$ and $W(\chi, \chi'') \twoheadrightarrow W(\chi', \chi'')$. Let Q_1 be the image of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi')$ in Q_σ and Q_2 be the pushout of Q_σ and $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi', \chi'')$ along $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi'')$.

If $\tau' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi')$ denotes the Jordan–Hölder factor parametrized by $\mathcal{S}(\sigma) \sqcup J_1$, then $\tau' \in \mathrm{JH}(Q_1)$ by repeated use of [HW22, Lemma 3.8], and consequently $\tau' \in \mathrm{JH}(Q_\sigma)$. Since $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi'')$ is multiplicity free by Lemma 4.3.3 and τ' occurs in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi'$, we have $\tau' \in \mathrm{JH}(Q_2)$ by construction of Q_2. Thus $\tau'' \in \mathrm{JH}(Q_2)$, where $\tau'' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi')$ denotes the Jordan–Hölder factor parametrized by $(\mathcal{S}(\sigma) \cup J_2) \sqcup J_1$. By repeated use of [HW22, Lemma 3.7] we deduce that $\tau''' \in \mathrm{JH}(Q_2)$, where $\tau''' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi'')$ denotes the Jordan–Hölder factor parametrized by $(\mathcal{S}(\sigma) \setminus J_2) \sqcup J_1$. As $(\mathcal{S}(\sigma) \setminus J_2) \sqcup J_1 \subseteq \mathcal{S}(\tau)$ by assumption (54), we conclude that $\tau \in \mathrm{JH}(Q_2) \subseteq \mathrm{JH}(Q_\sigma)$.

Conversely, assume $\tau \in \mathrm{JH}(Q_\sigma)$. Let $Q_\sigma^\tau \subseteq Q_\sigma$ be the unique subrepresentation with cosocle τ. We induct on the length $\ell \stackrel{\text{def}}{=} \ell(Q_\sigma^\tau)$. If $\ell = 1$, then $\tau = \sigma$ and $J_1 = J_2 = \emptyset$, so (54) follows. If $\ell \geq 2$, let $\mathcal{E} \subseteq Q_\sigma^\tau$ be a subrepresentation of length 2, namely $\mathcal{E}$ has the form

$$0 \to \sigma \to \mathcal{E} \to \sigma' \to 0$$

for some σ' satisfying $\mathrm{Ext}^1_{K/Z_1}(\sigma', \sigma) \neq 0$. By Lemma 4.3.4, $\sigma' \cong \mu_j^*(\sigma)$ for some $0 \leq j \leq f-1$ and $* \in \{+, -\}$. Define again $\chi'' \stackrel{\text{def}}{=} \chi^{J_1,J_2}$. Let $\chi' = \chi^{J_1',J_2'}$ be the unique character occurring in $W(\chi, \chi'')$ such that $\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi')$. Let $Q_{\sigma'}$ denote the unique quotient of Q_σ with socle σ'. Since $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi'')$ is multiplicity free, it is easy to see that the quotient map

$$\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi'') \twoheadrightarrow Q_{\sigma'}$$

factors through $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi', \chi'')$, namely $Q_{\sigma'}$ is a quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi', \chi'')$.

By Lemma 4.3.5, we have either $J_1' = J_2' = \emptyset$ or $J_1' \sqcup J_2' = \{j\}$. In the first case, since $\mathcal{E}$ is a subquotient of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$, we must have $\mathcal{S}(\sigma) \sqcup \{j\} = \mathcal{S}(\sigma')$ by Lemma 4.3.5 and [BP12, Thm. 2.4]. On the other hand, the inductive hypothesis (applied to $Q_{\sigma'}$) implies $\mathcal{S}(\sigma') \cap J_1 = \emptyset$ and $\mathcal{S}(\sigma') \sqcup J_1 \subseteq \mathcal{S}(\tau) \cup J_2$, from which we conclude. In the second case, we have the following two subcases:

- $J_1' = \{j\}$ and $J_2' = \emptyset$, in which case $\mathcal{S}(\sigma) \sqcup \{j\} = \mathcal{S}(\sigma')$. By the inductive hypothesis, we also have
$$\mathcal{S}(\sigma') \sqcup (J_1 \setminus \{j\}) \subseteq \mathcal{S}(\tau) \cup J_2$$
and hence (54) holds.

- $J_1' = \emptyset$ and $J_2' = \{j\}$, in which case $\mathcal{S}(\sigma) = \mathcal{S}(\sigma') \sqcup \{j\}$. By the inductive hypothesis, we also have
$$\mathcal{S}(\sigma') \sqcup J_1 \subseteq \mathcal{S}(\tau) \cup (J_2 \setminus \{j\})$$
and hence (54) holds. □

Proposition 4.3.8. *Let σ, τ be as in Proposition 4.3.6(ii). Assume $\tau \in \mathrm{JH}(Q_\sigma)$. Then the following are equivalent:*

(i) $\tau \in \mathrm{JH}(Q_\sigma^{K_1})$;

(ii) $\tau \in \mathrm{JH}(\operatorname{Inj}_\Gamma \sigma)$;

(iii) $J_2 \subseteq \mathcal{S}(\sigma)$ *and* $\mathcal{S}(\tau) \cap J_2 = \emptyset$.

Proof. Clearly (i) implies (ii). We now show that (ii) implies (iii). Let τ' be the constituent of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$ that is parametrized by $\mathcal{S}(\tau') = \mathcal{S}(\tau)$. Let $\sigma_\emptyset$ denote the socle of $\operatorname{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi$. Then by the definitions, $\sigma = \lambda(\sigma_\emptyset)$ and $\tau' = \mu'(\sigma_\emptyset)$ for unique $\lambda, \mu' \in \mathcal{P}$ (using again the notation of [HW22, (2.2)]). Moreover, as $\mathcal{S}(\tau) = \mathcal{S}(\tau')$ we can write $\tau = \mu(\sigma_\emptyset)$, where

$$\mu_j(x_j) = \begin{cases} \mu_j'(x_j + 2) & \text{if } j \in J_1, \\ \mu_j'(x_j - 2) & \text{if } j \in J_2, \\ \mu_j'(x_j) & \text{otherwise.} \end{cases} \tag{55}$$

As $\tau \in \mathrm{JH}(\operatorname{Inj}_\Gamma \sigma)$, it corresponds to some $\nu \in \mathcal{I}$ such that $\tau = \nu(\sigma)$. It is direct to check that μ defined in (55) satisfies the condition in [HW22, Lemma 2.1], so by [HW22, Lemmas 2.1, 2.7], we deduce $\mu = \nu \circ \lambda$.

Suppose by contradiction that $J_2 \setminus \mathcal{S}(\sigma) \neq \emptyset$ and we choose $j \in J_2 \setminus \mathcal{S}(\sigma)$. Then $\lambda_j(x_j) \in \{x_j, p-2-x_j\}$, and by (55) and the definition of $\mathcal{P}$ we have

$$\mu_j(x_j) \in \{x_j - 2, x_j - 3, p + 1 - x_j, p - x_j\},$$

contradicting that $\mu = \nu \circ \lambda$ with $\nu \in \mathcal{I}$ (see also the table in the proof of [HW22, Lemma 2.6]). Similarly, suppose that $J_2 \cap \mathcal{S}(\tau) \neq \emptyset$ and we choose $j \in J_2 \cap \mathcal{S}(\tau)$. Then $\mu_j'(x_j) \in \{x_j - 1, p-1-x_j\}$, so by (55) and the definition of $\mathcal{P}$ we have

$$\mu_j(x_j) \in \{x_j - 3, p + 1 - x_j\}, \quad \lambda_j(x_j) \in \{x_j, x_j - 1, p - 1 - x_j, p - 2 - x_j\}.$$

This yields a contradiction as before.

We finally show that (iii) implies (i). Let $Q_\sigma^\tau \subseteq Q_\sigma$ be the unique subrepresentation with cosocle τ (which exists by assumption). It will suffice to show that Q_σ^τ is K_1-invariant, and we will do that by verifying the assumption of [BP12, Cor. 5.7]. Note first that Q_σ^τ is multiplicity free. By the genericity condition (which in particular implies $p > 3$) it will suffice to rule out conditions (a) and (c) of [BP12, Cor. 5.6] for any pair of distinct constituents of Q_σ^τ and for any $0 \le j \le f-1$. Observe that if τ' is a (sufficiently generic) constituent of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi'$, then the constituent σ' described in condition (a) or (c) of [BP12, Cor. 5.6] for some $0 \le j \le f-1$ occurs in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi'\alpha_j^{\pm 1}$ for some choice of sign, and moreover $\mathcal{S}(\sigma') = \mathcal{S}(\tau')$ for the parametrizing sets. It therefore suffices to show that any two distinct constituents of Q_σ^τ have distinct parametrizing sets.

Suppose that $\tau' \in \mathrm{JH}(Q_\sigma^\tau)$ occurs in $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi^{J_1',J_2'})$ for some $J_1' \subseteq J_1$ and $J_2' \subseteq J_2$. By the previous paragraph it is enough to show that $\mathcal{S}(\tau') \cap J_1 = J_1'$ and $\mathcal{S}(\tau') \cap J_2 = J_2''$, where we write $J_i'' \overset{\text{def}}{=} J_i \setminus J_i'$. From $\tau' \in \mathrm{JH}(Q_\sigma)$ and $\tau \in \mathrm{JH}(Q_{\tau'})$ we obtain from Proposition 4.3.6(ii) that

$$\mathcal{S}(\sigma) \sqcup J_1' \subseteq \mathcal{S}(\tau') \cup J_2', \quad \mathcal{S}(\tau') \cap J_1'' = \emptyset, \quad \mathcal{S}(\tau') \sqcup J_1'' \subseteq \mathcal{S}(\tau) \cup J_2''.$$

The first and second statements together show that $\mathcal{S}(\tau') \cap J_1 = J_1'$. The first statement plus $J_2 \subseteq \mathcal{S}(\sigma)$ and the third statement plus $\mathcal{S}(\tau) \cap J_2 = \emptyset$ give $\mathcal{S}(\tau') \cap J_2 = J_2''$, as desired. ☐

4.3.2 Some $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations of π

We apply § 4.3.1 to construct some $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentations of π that will be important in the proof of Theorem 4.3.15.

For $\mu \in \mathscr{P}$ define

$$Y(\mu) \overset{\text{def}}{=} \{j : \mu_j(x_j) \in \{x_j, x_j+1, p-2-x_j, p-3-x_j\}\} \cup J_{\overline{\rho}}^c, \tag{56}$$

$$Z(\mu) \overset{\text{def}}{=} \{j : \mu_j(x_j) \in \{x_j+1, x_j+2, p-1-x_j, p-2-x_j\}\} \cup J_{\overline{\rho}}^c. \tag{57}$$

Note that $Y(\mu)$ (resp. $Z(\mu)$) is exactly the set of j such that $t_j \ne y_j$ (resp. $t_j \ne z_j$) in (12). Here, we recall that $\mu_j(x_j) \in \{x_j+2, p-3-x_j\}$ implies $j \in J_{\overline{\rho}}$ by (9).

Lemma 4.3.9. *Suppose that $\overline{\rho}$ is 2-generic. Let $\mu \in \mathscr{P}$ and $\chi \overset{\text{def}}{=} \chi_\mu$. Let $J_1 \subseteq Y(\mu)$ and $J_2 \subseteq Z(\mu)$ be subsets satisfying $J_1 \cap J_2 = \emptyset$. Then $\mathrm{JH}(W(\chi, \chi^{J_1,J_2})) \cap \mathrm{JH}(\pi^{I_1}) = \{\chi\}$ and there exists a unique (up to scalar) I-equivariant embedding $W(\chi, \chi^{J_1,J_2}) \hookrightarrow \pi|_I$. Moreover,*

(i) *the image of the induced morphism*

$$\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi^{J_1,J_2}) \to \pi|_{\mathrm{GL}_2(\mathcal{O}_K)}$$

has socle $\sigma \in W(\overline{\rho})$, where σ is the Serre weight determined by $J_\sigma = J_{\overline{\rho}} \cap J_\mu$ (via (10));

(ii) *$\sigma \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi)$ and it is parametrized by $X(\mu)$, where $X(\mu)$ is defined in (48).*

Proof. The first claim follows from Lemma 2.3.6(ii) with $m = 1$. The second follows from the fact that $\mathrm{Ext}^i_{I/Z_1}(\chi', \pi) = 0$ for $\chi' \in \mathrm{JH}(W(\chi, \chi^{J_1,J_2})/\chi)$ and $i = 0, 1$ using the first claim together with assumption (iv) imposed on π; see the proof of Lemma 2.4.2.

By Lemma 4.1.1, the image of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi \to \pi$ has socle σ and σ is parametrized by $X(\mu)$. To deduce (i) and (ii), it suffices to prove

$$\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi, \chi^{J_1,J_2})/\chi) \cap W(\overline{\rho}) = \emptyset.$$

This follows from the first claim and the fact that $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi') \cap W(\overline{\rho}) = \emptyset$ for any $\chi' \notin \pi^{I_1}$ by [Bre14, Prop. 4.2]. □

Lemma 4.3.10. *Let $\mu \in \mathscr{P}$ and $\sigma \in W(\overline{\rho})$ be the Serre weight determined by $J_\sigma = J_{\overline{\rho}} \cap J_\mu$. If $\sigma' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu) \cap W(\overline{\rho})$ then $\mathcal{S}(\sigma') \subseteq \mathcal{S}(\sigma) = X(\mu)$. As a consequence, $\sigma \in \mathrm{JH}(Q_{\sigma'})$, where $Q_{\sigma'}$ denotes the unique quotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu$ with socle σ'.*

Proof. Lemma 4.1.1 implies that the image V of the natural map $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu \to D_0(\overline{\rho})$ has socle σ and $\mathrm{JH}(V/\sigma) \cap W(\overline{\rho}) = \emptyset$. From [Bre14, Prop. 4.3] (applied with $\chi = \chi_\mu^s$, noting $\chi \neq \chi^s$) we deduce that $\mathcal{S}(\sigma') \subseteq \mathcal{S}(\sigma) = X(\mu)$. □

Now we consider a special situation. Suppose that $\overline{\rho}$ is 3-generic, so that χ_μ is 2-generic for any $\mu \in \mathscr{P}$. Let $\lambda \in \mathscr{P}^{\mathrm{ss}}$ and denote

$$J_1 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p - 3 - x_j\}, \quad J_2 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = x_j + 2\}. \tag{58}$$

We define an f-tuple $\mu = (\mu_j(x_j))$ by

$$\mu_j(x_j) \stackrel{\text{def}}{=} \begin{cases} p - 1 - x_j & \text{if } j \in J_1, \\ x_j & \text{if } j \in J_2, \\ \lambda_j(x_j) & \text{otherwise.} \end{cases} \tag{59}$$

It is direct to check that $\mu \in \mathscr{P}$, $\chi_\lambda = \chi_\mu \prod_{j \in J_1} \alpha_j^{-1} \prod_{j \in J_2} \alpha_j$ and $|J_\mu| = |J_\lambda| - |J_1| - |J_2|$. It is clear that $J_1 \subseteq Y(\mu)$, $J_2 \subseteq Z(\mu)$ and $J_1 \cap J_2 = \emptyset$. By Lemma 4.3.9 there is a unique embedding $\iota : W(\chi_\mu, \chi_\lambda) \hookrightarrow \pi|_I$. Consider the induced morphism

$$\widetilde{\iota} : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu, \chi_\lambda) \to \pi|_{\mathrm{GL}_2(\mathcal{O}_K)}$$

and let V be its image. By Lemma 4.3.9, $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(V) = \sigma$, where $\sigma \in W(\overline{\rho})$ is the Serre weight determined by $J_\sigma = J_{\overline{\rho}} \cap J_\mu$, so that χ_μ contributes to $D_{0,\sigma}(\overline{\rho})^{I_1}$ by Lemma 4.1.1. Also, let $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ be the Serre weight determined by $J_\tau = J_\lambda$, so that χ_λ contributes to $D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})^{I_1}$ by Lemma 3.1.3. Then τ occurs as a subquotient of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\lambda$, hence also of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu, \chi_\lambda)$ (with multiplicity one by Lemma 4.3.3).

Lemma 4.3.11. *Keep the above notation and assume that $\overline{\rho}$ is 6-generic. We have $I(\sigma, \tau) \subseteq V$ and*

$$\mathrm{JH}(V/I(\sigma, \tau)) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset.$$

Proof. By Lemma 4.3.9(ii), the Jordan–Hölder factor σ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_\mu$ is parametrized by the subset

$$\begin{aligned} X(\mu) &= \{j : \mu_j(x_j) \in \{x_j, \underline{x_j+1}, p-2-x_j, p-3-x_j\}\} \\ &= J_2 \sqcup \{j : \lambda_j(x_j) \in \{x_j, \underline{x_j+1}, p-2-x_j, \underline{p-3-x_j}\}\} \end{aligned} \tag{60}$$

and the Jordan–Hölder factor τ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_\lambda$ is parametrized by

$$X^{\mathrm{ss}}(\lambda) \stackrel{\mathrm{def}}{=} \{j : \lambda_j(x_j) \in \{x_j, x_j+1, p-2-x_j, p-3-x_j\}\}.$$

(As in the proof of Lemma 4.1.1 we use the convention that an underlined entry is only allowed when $j \in J_{\overline{\rho}}$.) We check by the definition of μ and (60) that $X(\mu) \cap J_1 = \emptyset$ and

$$X(\mu) \sqcup J_1 \subseteq X^{\mathrm{ss}}(\lambda) \cup J_2.$$

We then conclude by Proposition 4.3.6(ii) (note that χ_μ is 5-generic) that τ contributes to the image V of $\widetilde{\iota}$. Moreover, since $J_2 \subseteq X(\mu)$ and $X^{\mathrm{ss}}(\lambda) \cap J_2 = \emptyset$, Proposition 4.3.8 implies that $\tau \in V^{K_1}$ and $I(\sigma,\tau)$ is identified with the unique subrepresentation of V with cosocle τ. This proves the first assertion.

As V is multiplicity free, it remains to prove

$$\mathrm{JH}(V) \cap W(\overline{\rho}^{\mathrm{ss}}) \subseteq \mathrm{JH}(I(\sigma,\tau)).$$

Let $\chi' \in \mathrm{JH}(W(\chi_\mu,\chi_\lambda))$ and write $\chi' = \chi_\mu^{J_1',J_2'}$ for $J_1' \subseteq J_1$ and $J_2' \subseteq J_2$. By the definition of $\mathscr{P}^{\mathrm{ss}}$, it is clear that $\chi' = \chi_{\mu'}$ for some $\mu' \in \mathscr{P}^{\mathrm{ss}}$ with

$$X^{\mathrm{ss}}(\mu') = (X^{\mathrm{ss}}(\lambda) \setminus J_1'') \cup J_2'', \tag{61}$$

where $J_1'' \stackrel{\mathrm{def}}{=} J_1 \backslash J_1'$ and $J_2'' \stackrel{\mathrm{def}}{=} J_2 \backslash J_2'$. In particular, χ' is also 5-generic. Let $\tau' \in \mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi')$ be the constituent parametrized by $\mathcal{S}(\tau')$. If $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$, then $\mathcal{S}(\tau') \subseteq X^{\mathrm{ss}}(\mu')$ by Lemma 4.3.10 (applied to $\overline{\rho}^{\mathrm{ss}}$), and so $\mathcal{S}(\tau') \sqcup J_1'' \subseteq X^{\mathrm{ss}}(\lambda) \cup J_2''$ by (61). By Proposition 4.3.6(ii) applied to $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi',\chi_\lambda)$, this implies that $\tau' \in \mathrm{JH}(V_\tau')$, where $V_\tau' \subseteq \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi',\chi_\lambda)$ is the unique subrepresentation with cosocle τ (cf. Remark 4.3.7). Hence τ' occurs in the unique subrepresentation $\widetilde{V}_\tau'$ of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu,\chi_\lambda)$ with cosocle τ. If moreover $\tau' \in \mathrm{JH}(V)$, then τ' has to occur in the image $V_\tau \subseteq V$ of $\widetilde{V}_\tau'$, which is just $I(\sigma,\tau)$ by the previous paragraph. Thus, we obtain $\mathrm{JH}(V) \cap W(\overline{\rho}^{\mathrm{ss}}) \subseteq \mathrm{JH}(V_\tau) = \mathrm{JH}(I(\sigma,\tau))$. □

4.3.3 Generalization of [BP12, § 19]

We generalize [BP12, Lemma 19.7] (Lemma 4.3.13).

Assume that $\overline{\rho}$ is 3-generic so that χ_λ is 2-generic for any $\lambda \in \mathscr{P}^{\mathrm{ss}}$. Let $\lambda \in \mathscr{D}^{\mathrm{ss}}$ which corresponds to $\sigma \in W(\overline{\rho}^{\mathrm{ss}})$ and let $\chi \stackrel{\mathrm{def}}{=} \chi_\sigma = \chi_\lambda$ (see § 1.3). We let

$$\widetilde{R}(\chi) \stackrel{\mathrm{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W\Big(\chi^s, \chi^s \prod_{j=0}^{f-1} \alpha_j\Big). \tag{62}$$

In particular, $\widetilde{R}(\chi)$ is multiplicity free by Lemma 4.3.3. It is isomorphic to the $\mathrm{GL}_2(\mathcal{O}_K)$-representation denoted by $\widetilde{R}(\sigma)$ in [BP12, § 17]. (Indeed, with the notation in *loc. cit.*, $J_\sigma = \{0,\dots,f-1\}$ by our genericity assumption and $W(\chi,\chi\prod_{j=0}^{f-1}\alpha_j^{-1})$ embeds in $\sigma|_I$ and is identified with the subspace spanned by $x^{s-i}y^i \in \sigma$ for $i \in \{\sum_{j\in J} p^j : J \subseteq \{0,\dots,f-1\}\}$, where we have written $\sigma = (s_0,\dots,s_{f-1})\otimes\theta$ and $s \stackrel{\text{def}}{=} \sum_{j=0}^{f-1} p^j s_j$. The representation $\widetilde{R}(\sigma)$ is then the $\mathrm{GL}_2(\mathcal{O}_K)$-subrepresentation of $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)Z}^{\mathrm{GL}_2(K)}\sigma$ generated by $[(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}), W(\chi,\chi\prod_{j=0}^{f-1}\alpha_j^{-1})]$, hence is isomorphic to our $\widetilde{R}(\chi)$ in (62).) Furthermore, in *loc. cit.* is defined a subrepresentation of $\widetilde{R}(\chi)$, denoted by $R(\chi)$, whose Jordan–Hölder factors consist precisely of all the *special* ones, cf. [BP12, Def. 17.2]. We remark that $R(\chi)$ is the unique subrepresentation of $\widetilde{R}(\chi)$ whose cosocle is the socle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi^s\prod_{j=0}^{f-1}\alpha_j$.

We recall the following result from [BP12, Lemmas 19.5, 19.7]. Let $\delta(\sigma) \in W(\overline{\rho}^{\mathrm{ss}})$ be the Serre weight corresponding to $\delta(\lambda) \in \mathscr{D}^{\mathrm{ss}}$ (see § 1.3 for the definition of $\delta(\lambda)$) and recall that $\sigma^{[s]}$ is defined in § 1.3.

Proposition 4.3.12. *There is a unique quotient $Q(\overline{\rho}^{\mathrm{ss}},\sigma^{[s]})$ of $R(\chi)$ such that:*

- $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} Q(\overline{\rho}^{\mathrm{ss}},\sigma^{[s]}) \subseteq \bigoplus_{\sigma'\in W(\overline{\rho}^{\mathrm{ss}})}\sigma'$;
- $Q(\overline{\rho}^{\mathrm{ss}},\sigma^{[s]})$ *contains* $I(\delta(\sigma),\sigma^{[s]})$*, the unique subrepresentation of* $\mathrm{Inj}_\Gamma\,\delta(\sigma)$ *with cosocle* $\sigma^{[s]}$ *in which* $\delta(\sigma)$ *occurs with multiplicity one.*

Moreover, we have $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} Q(\overline{\rho}^{\mathrm{ss}},\sigma^{[s]}) = \delta(\sigma)$ *and* $Q(\overline{\rho}^{\mathrm{ss}},\sigma^{[s]})$ *contains* $D_{0,\delta(\sigma)}(\overline{\rho}^{\mathrm{ss}})$.

For $J \subseteq \{0,\dots,f-1\}$, we define

$$\widetilde{R}_J(\chi) \stackrel{\text{def}}{=} \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi^s,\chi^s\prod_{j\in J}\alpha_j) \hookrightarrow \widetilde{R}(\chi)$$

and $R_J(\chi) \stackrel{\text{def}}{=} \widetilde{R}_J(\chi)\cap R(\chi)$. The following result slightly strengthens Proposition 4.3.12.

Lemma 4.3.13. *Let $J \subseteq \{0,\dots,f-1\}$ be a subset satisfying*

$$J \supseteq \{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\}.$$

Then $D_{0,\delta(\sigma)}(\overline{\rho}^{\mathrm{ss}})$ is contained in the image of $R_J(\chi) \hookrightarrow R(\chi) \twoheadrightarrow Q(\overline{\rho}^{\mathrm{ss}},\sigma^{[s]})$. In particular, the unique quotient of $R_J(\chi)$ (or of $\widetilde{R}_J(\chi)$) with socle $\delta(\sigma)$ contains $D_{0,\delta(\sigma)}(\overline{\rho}^{\mathrm{ss}})$.

Proof. Clearly we may assume $J = \{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\}$. Applying Lemma 3.1.3 to $\lambda^{[s]}$ (same notation as (50)), and remembering that $\lambda \in \mathscr{D}^{\mathrm{ss}}$, we conclude that $X^{\mathrm{ss}}(\lambda^{[s]}) = J$ parametrizes $\delta(\sigma)$ (as a constituent of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_\lambda^s$). Let $\xi \in \mathcal{P}$ correspond to $\delta(\sigma)$, so that $\mathcal{S}(\xi) = J$. Define $\mu_\xi \in \mathcal{I}$ as follows (cf. [BP12, § 19]):

$$\mu_{\xi,j}(y_j) \stackrel{\text{def}}{=} \begin{cases} p-1-y_j & \text{if } \xi_j(x_j) \in \{x_j-1, x_j\}, \\ p-3-y_j & \text{if } \xi_j(x_j) \in \{p-2-x_j, p-1-x_j\}. \end{cases}$$

Write $\sigma = (s_0, \dots, s_{f-1}) \otimes \theta$, so $\delta(\sigma) = (\xi_0(s_0), \dots, \xi_{f-1}(s_{f-1})) \otimes \det^{e(\xi)(s_0,\dots,s_{f-1})}\theta$. By [BP12, Lemma 19.2], $D_{0,\delta(\sigma)}(\overline{\rho}^{\mathrm{ss}})$ is equal to $I(\delta(\sigma), \tau)$, where

$$\tau = \mu_\xi(\delta(\sigma)) \overset{\mathrm{def}}{=} \big(\mu_{\xi,0}(\xi_0(s_0)), \dots, \mu_{\xi,f-1}(\xi_{f-1}(s_{f-1}))\big) \otimes \det^{e(\mu_\xi\circ\xi)(s_0,\dots,s_{f-1})}\theta.$$

By Proposition 4.3.12, τ occurs in $Q(\overline{\rho}^{\mathrm{ss}}, \sigma^{[s]})$, hence also in $R(\chi)$.

To conclude, it suffices to prove that τ occurs in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi^s \prod_{j\in J} \alpha_j$. By [BP12, Lemma 17.12(i)], it is equivalent to proving that J equals

$$J(\mu_\xi \circ \xi) \overset{\mathrm{def}}{=} \{j \in \{0, \dots, f-1\} : (\mu_\xi \circ \xi)(x_j) \in \{x_j - 2, p - x_j\}\}.$$

By the definition of μ_ξ, we see that $J(\mu_\xi \circ \xi)$ is exactly $\{j : \xi_j(x_j) \in \{x_j - 1, p - 1 - x_j\}\} = \mathcal{S}(\xi)$, which equals J by above. $\square$

4.3.4 K_1- and I_1-invariants of subrepresentations of π

We describe the K_1-invariants, I_1-invariants and the $\mathrm{GL}_2(\mathcal{O}_K)$-socles of subrepresentations of π.

By [Hu16, Prop. 5.2], the Γ-representation $D_0(\overline{\rho})$ admits a unique filtration

$$0 = D_0(\overline{\rho})_{\le -1} \subsetneq D_0(\overline{\rho})_{\le 0} \subsetneq \cdots \subsetneq D_0(\overline{\rho})_{\le i} \subsetneq \cdots \subsetneq D_0(\overline{\rho})_{\le f} = D_0(\overline{\rho}) \tag{63}$$

such that for any $0 \le i \le f$,

$$D_0(\overline{\rho})_i \overset{\mathrm{def}}{=} D_0(\overline{\rho})_{\le i}/D_0(\overline{\rho})_{\le i-1}$$

is a subrepresentation of $D_0(\overline{\rho}^{\mathrm{ss}})_i \overset{\mathrm{def}}{=} \bigoplus_{\tau\in W(\overline{\rho}^{\mathrm{ss}}),\ell(\tau)=i} D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$ and

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)} D_0(\overline{\rho})_i = \bigoplus_{\tau\in W(\overline{\rho}^{\mathrm{ss}}),\ \ell(\tau)=i} \tau. \tag{64}$$

By construction, $D_0(\overline{\rho})_{\le i}$ is the largest Γ-subrepresentation of $D_0(\overline{\rho})$ not containing any $\tau \in W(\overline{\rho}^{\mathrm{ss}})$, $\ell(\tau) > i$ as subquotient. Set $D_0(\overline{\rho}^{\mathrm{ss}})_{\le i} \overset{\mathrm{def}}{=} \bigoplus_{j\le i} D_0(\overline{\rho}^{\mathrm{ss}})_j$. We obtain

$$\mathrm{JH}(D_0(\overline{\rho})_i) = \mathrm{JH}(D_0(\overline{\rho})) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_i), \quad \text{and} \tag{65}$$

$$\mathrm{JH}(D_0(\overline{\rho})_{\le i}) = \mathrm{JH}(D_0(\overline{\rho})) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_{\le i}). \tag{66}$$

Indeed, (65) implies (66). For (65), the inclusion $\subseteq$ is obvious, but both sides form a partition of $\mathrm{JH}(D_0(\overline{\rho}))$ as i varies, so equality holds.

Since $D_0(\overline{\rho})$ is multiplicity free and decomposes as $\bigoplus_{\sigma\in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})$, we see that $D_0(\overline{\rho})_{\le i}$ also decomposes as a direct sum

$$D_0(\overline{\rho})_{\le i} = \bigoplus_{\sigma\in W(\overline{\rho})} D_{0,\sigma}(\overline{\rho})_{\le i}, \tag{67}$$

where $D_{0,\sigma}(\overline{\rho})_{\le i} \overset{\mathrm{def}}{=} D_{0,\sigma}(\overline{\rho}) \cap D_0(\overline{\rho})_{\le i}$. (Note that by (66) we have $D_{0,\sigma}(\overline{\rho})_{\le i} \ne 0$ if and only if $\ell(\sigma) \le i$.) Similarly, $D_0(\overline{\rho})_i$ also decomposes as a direct sum $\bigoplus_{\tau\in W(\overline{\rho}^{\mathrm{ss}}),\ell(\tau)=i} D_{0,\tau}(\overline{\rho})_i$, where $D_{0,\tau}(\overline{\rho})_i \overset{\mathrm{def}}{=} D_0(\overline{\rho})_i \cap D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})$.

We remark that by (66) and Lemma 4.3.14 we have for any $\sigma \in W(\overline{\rho})$:

$$\frac{D_{0,\sigma}(\overline{\rho})_{\leq i}}{D_{0,\sigma}(\overline{\rho})_{\leq i-1}} = \bigoplus_{\tau \in W(\overline{\rho}^{\rm ss}), \ell(\tau)=i, J_\sigma = J_{\overline{\rho}} \cap J_\tau} D_{0,\tau}(\overline{\rho})_i. \tag{68}$$

Lemma 4.3.14. *Let $\tau \in W(\overline{\rho}^{\rm ss})$ and $\sigma \in W(\overline{\rho})$ be the element such that $J_\sigma = J_{\overline{\rho}} \cap J_\tau$. Then*

$$\mathrm{JH}(D_{0,\tau}(\overline{\rho}^{\rm ss})) \cap \mathrm{JH}(D_0(\overline{\rho})) \subseteq \mathrm{JH}(D_{0,\sigma}(\overline{\rho})).$$

Proof. This is a consequence of [BP12, Lemma 15.3]. □

Theorem 4.3.15. *Assume that $\overline{\rho}$ is 6-generic. Let π_1 be a subrepresentation of π. Then there exists a unique integer $i_0 = i_0(\pi_1)$ with $-1 \leq i_0 \leq f$ such that*

$$\pi_1^{K_1} = D_0(\overline{\rho})_{\leq i_0}.$$

Proof. If $\pi_1^{K_1} = 0$ (resp. $\pi_1^{K_1} = D_0(\overline{\rho})$) we are done, with $i_0 = -1$ (resp. $i_0 = f$). Otherwise, by (63) there exists a unique integer $-1 < i_0 < f$ such that $D_0(\overline{\rho})_{\leq i_0} \subseteq \pi_1^{K_1}$ and $D_0(\overline{\rho})_{\leq i_0+1} \nsubseteq \pi_1^{K_1}$. We need to prove that the (first) inclusion is an equality. Suppose this is not the case. Then we may find a Serre weight τ which embeds in $\pi_1^{K_1}/D_0(\overline{\rho})_{\leq i_0}$, hence also embeds in $D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0}$. This implies that $\tau \in W(\overline{\rho}^{\rm ss})$ with $\ell(\tau) > i_0$ by (63) and (64). Thus, there exists a Serre weight τ satisfying the condition

$$\tau \in W(\overline{\rho}^{\rm ss}) \cap \mathrm{JH}(\pi_1^{K_1}), \quad \ell(\tau) > i_0. \tag{69}$$

We choose τ satisfying (69) such that $\ell(\tau)$ is minimal.

Step 1. We prove that $\ell(\tau) = i_0+1$. First assume $\tau \in W(\overline{\rho}^{\rm ss}) \setminus W(\overline{\rho})$ and let $\sigma \in W(\overline{\rho})$ be the Serre weight with $J_\sigma = J_{\overline{\rho}} \cap J_\tau$. Note that $\tau \in \mathrm{JH}(D_0(\overline{\rho}))$ by Lemma 4.1.2, so $I(\sigma,\tau) \hookrightarrow D_0(\overline{\rho})$ by Lemma 4.1.3, and thus $I(\sigma,\tau) \subseteq \pi_1^{K_1}$. Since $\sigma \neq \tau$, we have $\mathrm{rad}_\Gamma(I(\sigma,\tau)) \neq 0$, and by using again Lemma 4.1.3 we have

$$\mathrm{JH}(\mathrm{rad}_\Gamma(I(\sigma,\tau))) \subseteq W(\overline{\rho}^{\rm ss}) \cap \mathrm{JH}(\pi_1^{K_1}).$$

By the choice of τ, we must have $\ell(\tau') \leq i_0$ for any $\tau' \in \mathrm{JH}(\mathrm{rad}_\Gamma(I(\sigma,\tau)))$. Then by the second sentence of Lemma 4.1.3 and remembering that $\ell(\tau') = |J_{\tau'}|$ for $\tau' \in W(\overline{\rho}^{\rm ss})$, this forces $\ell(\tau) \leq i_0+1$, hence $\ell(\tau) = i_0+1$ (as $\ell(\tau) > i_0$ by construction).

Next, we assume that $\tau \in W(\overline{\rho})$, i.e. τ occurs in the $\mathrm{GL}_2(\mathcal{O}_K)$-socle of π. This is equivalent to $J_\tau \subseteq J_{\overline{\rho}}$. Note that in this case we have $\tau \hookrightarrow \pi_1^{K_1}$. By assumption, $\ell(\tau) = i_0+1 > 0$. By Lemma 3.1.3 (resp. Lemma 4.1.1), using the observation $J_{\mu^{[s]}} = \delta(J_\mu)$ for $\mu \in \mathscr{D}^{\rm ss}$, the Serre weight $\tau^{[s]}$ occurs in $D_{0,\tau_1}(\overline{\rho}^{\rm ss})$ (resp. $D_{0,\sigma_1}(\overline{\rho})$), where $\tau_1 \in W(\overline{\rho}^{\rm ss})$ and $\sigma_1 \in W(\overline{\rho})$ are uniquely determined by

$$J_{\tau_1} = \delta(J_\tau), \quad J_{\sigma_1} = J_{\overline{\rho}} \cap \delta(J_\tau).$$

Moreover, the image of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\tau^s \to \pi_1$ is equal to $I(\sigma_1, \tau^{[s]})$, which contains τ_1 as a subquotient (by Lemma 4.3.10 applied to $\overline{\rho}^{\rm ss}$ and $\chi_\mu = \chi_{\tau^{[s]}}$), so we have $\tau_1 \in \mathrm{JH}(\pi_1^{K_1})$. We note that $\ell(\tau_1) = \ell(\tau) > i_0$, thus τ_1 also satisfies (69) and $\ell(\tau_1)$ is minimal subject to (69). If again $\tau_1 \in W(\overline{\rho})$, i.e. $J_{\tau_1} \subseteq J_{\overline{\rho}}$, we may continue this procedure to obtain τ_2 and σ_2. Since

$J_\tau \neq \emptyset$, $J_{\overline{\rho}} \neq \{0,\dots,f-1\}$, we finally arrive at some τ_n with $J_{\tau_n} = \delta^n(J_\tau) \nsubseteq J_{\overline{\rho}}$, equivalently $\tau_n \in W(\overline{\rho}^{\mathrm{ss}}) \setminus W(\overline{\rho})$, and we are reduced to the case in the previous paragraph. Thus $\ell(\tau) = \ell(\tau_n) = i_0 + 1$ as desired.

Step 2. Let $\lambda \in \mathscr{D}^{\mathrm{ss}}$ be the element corresponding to τ, and define the f-tuple $\mu = (\mu_j(x_j))$ as in (59), i.e. $\mu_j(x_j) = p - 1 - x_j$ if $j \in J_1$ and $\mu_j(x_j) = \lambda_j(x_j)$ otherwise, where

$$J_1 \stackrel{\mathrm{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p - 3 - x_j\} \quad (\text{and } J_2 = \emptyset).$$

It is direct to check that $\mu \in \mathscr{P}$ and $\chi_\lambda = \chi_\mu \prod_{j\in J_1} \alpha_j^{-1}$. We also note that $J_{\overline{\rho}} \cap J_\lambda \subseteq J_\mu \subseteq J_\lambda$, i.e. $J_{\overline{\rho}} \cap J_\lambda = J_\mu$. Let

$$\widetilde{J}_1 \stackrel{\mathrm{def}}{=} \{j : \lambda_j(x_j) \in \{x_j + 1, p - 2 - x_j\}\} = \{j : \mu_j(x_j) \in \{x_j + 1, p - 2 - x_j\}\}.$$

Then $J_1 \cap \widetilde{J}_1 = \emptyset$ and $J \stackrel{\mathrm{def}}{=} J_1 \sqcup \widetilde{J}_1 \subseteq Y(\mu)$, where $Y(\mu)$ is defined in (56). By Lemma 4.3.9, there is a unique (up to scalar) I-equivariant embedding $\iota : W(\chi_\mu, \chi'') \hookrightarrow \pi|_I$, where

$$\chi'' \stackrel{\mathrm{def}}{=} \chi_\mu^{J,\emptyset} = \chi_\mu \prod_{j\in J} \alpha_j^{-1}.$$

Note that $W(\chi_\mu, \chi_\lambda) \hookrightarrow W(\chi_\mu, \chi'')$ by construction (Lemma 4.3.1).

Step 3. We prove that $\mathrm{im}(\iota)$ is contained in π_1. It is equivalent to prove that V is contained in π_1, where V denotes the image of the $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant morphism (induced by Frobenius reciprocity)

$$\widetilde{\iota} : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu, \chi'') \to \pi|_{\mathrm{GL}_2(\mathcal{O}_K)}.$$

Note that V is contained in $\pi^{K_1} \cong \bigoplus_{\sigma' \in W(\overline{\rho})} D_{0,\sigma'}(\overline{\rho})$, since $W(\chi_\mu, \chi'')$ is fixed by K_1 by Lemma 4.3.1. By Lemma 4.3.9, V is contained in $D_{0,\sigma}(\overline{\rho})$, where $\sigma \in W(\overline{\rho})$ is as in Step 1. For $J' \subseteq J$, let $\tau^{J'}$ be the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu \prod_{j\in J'} \alpha_j^{-1}$, i.e. the unique Serre weight with $(\tau^{J'})^{I_1} = \chi_\mu \prod_{j\in J'} \alpha_j^{-1}$. Note that $\tau^{J_1} = \tau$. It follows from Proposition 4.3.6(i) that

$$\mathrm{cosoc}_\Gamma(V) \cong \bigoplus_{J' \subseteq J,\ \tau^{J'} \in \mathrm{JH}(V)} \tau^{J'}. \tag{70}$$

By multiplicity freeness of π^{K_1} it suffices to show that $\tau^{J'}$ occurs in $\pi_1^{K_1}$ for each $J' \subseteq J$ satisfying $\tau^{J'} \in \mathrm{JH}(V)$. If $J' = J_1$, this is true by assumption, so we may assume $J' \neq J_1$ in the following.

We have $I(\sigma, \tau) \hookrightarrow V$ by Lemma 4.3.11, and $\mathrm{JH}(I(\sigma,\tau)) \subseteq W(\overline{\rho}^{\mathrm{ss}})$ by Lemma 4.1.3. Moreover, we have

$$\mathrm{JH}(V/I(\sigma,\tau)) \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset. \tag{71}$$

This follows from Lemma 4.3.11 by noting that if $\chi' \in \mathrm{JH}(W(\chi_\mu, \chi'')) \setminus \mathrm{JH}(W(\chi_\mu, \chi_\lambda))$, then $\chi' \notin \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})^{I_1})$ by the explicit description of $\mathscr{P}^{\mathrm{ss}}$ and so $\mathrm{JH}(\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi') \cap W(\overline{\rho}^{\mathrm{ss}}) = \emptyset$ by [Bre14, Prop. 4.2].

Now fix $J' \subseteq J$ satisfying $J' \neq J_1$ and $\tau^{J'} \in \mathrm{JH}(V)$. In particular $\tau^{J'} \neq \tau$. As V is K_1-invariant, $I(\sigma, \tau^{J'}) \subseteq V$. If $I(\sigma, \tau^{J'}) \nsubseteq D_0(\overline{\rho})_{\leq i_0}$, equivalently the morphism $I(\sigma, \tau^{J'}) \to D_0(\overline{\rho})/D_0(\overline{\rho})_{\leq i_0}$ is nonzero, then $\mathrm{JH}(I(\sigma, \tau^{J'}))$ would contain some element $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ with

$\ell(\tau') \geq i_0+1$, by (64). As τ' must contribute to $I(\sigma,\tau)$ by (71), by Lemma 4.1.3 we deduce $\tau'=\tau$ (as otherwise $\ell(\tau')<\ell(\tau)=i_0+1$) and hence $\tau\in \mathrm{JH}(I(\sigma,\tau^{J'}))$. But τ is a quotient of V by (70) and hence of $I(\sigma,\tau^{J'})$, so $\tau^{J'}=\tau$, contradiction. Hence $\tau^{J'}$ occurs in $D_0(\overline{\rho})_{\leq i_0}\subseteq \pi_1^{K_1}$, as desired.

Step 4. Our goal is to prove that $D_0(\overline{\rho})_{\leq i_0+1}\subseteq \pi_1^{K_1}$, which will contradict our choice of i_0. By the multiplicity freeness of π^{K_1}, it suffices to prove

$$\mathrm{JH}(D_0(\overline{\rho})_{i_0+1})\subseteq \mathrm{JH}(\pi_1^{K_1}),$$

or equivalently (by (65)),

$$\mathrm{JH}(D_{0,\tau'}(\overline{\rho}^{\mathrm{ss}}))\cap \mathrm{JH}(D_0(\overline{\rho}))\subseteq \mathrm{JH}(\pi_1^{K_1}) \tag{72}$$

for any $\tau'\in W(\overline{\rho}^{\mathrm{ss}})$ satisfying $\ell(\tau')=i_0+1$. In this step we prove that (72) holds under the additional hypothesis that $\tau'\in \mathrm{JH}(\pi_1^{K_1})$.

We may assume that $\tau'=\tau$ and let again $\lambda\in\mathscr{D}^{\mathrm{ss}}$ correspond to τ. Since π_1 carries an action of $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$, we deduce an injective morphism $\kappa: W(\chi_\mu^s,\chi''^s)\hookrightarrow \pi_1|_I$ from Step 3, hence a $\mathrm{GL}_2(\mathcal{O}_K)$-equivariant morphism (induced by Frobenius reciprocity)

$$\widetilde{\kappa}:\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu^s,\chi''^s)\to \pi_1|_{\mathrm{GL}_2(\mathcal{O}_K)}.$$

Let $\sigma_1\in W(\overline{\rho})$ be the Serre weight such that χ_μ^s contributes to $D_{0,\sigma_1}(\overline{\rho})^{I_1}$. Then σ_1 occurs in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_\mu^s$ and is parametrized by $X(\mu^{[s]})$ (recall from (50) that $\mu^{[s]}\in\mathscr{P}$ is the f-tuple corresponding to χ_μ^s). Similarly, let $\tau_1=\delta(\tau)\in W(\overline{\rho}^{\mathrm{ss}})$ be such that χ_λ^s contributes to $D_{0,\tau_1}(\overline{\rho}^{\mathrm{ss}})^{I_1}$, then τ_1 occurs in $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)}\chi_\lambda^s$ and is parametrized by $X^{\mathrm{ss}}(\lambda^{[s]})$. By Lemma 4.3.9 (applied with $(\mu^{[s]},\emptyset,J)$ for (μ,J_1,J_2) there) we see that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathrm{im}(\widetilde{\kappa}))=\sigma_1$. By Lemma 4.3.11 (applied with $\lambda^{[s]}$, resp. $\mu^{[s]}$, instead of λ, resp. μ, and noting that $W(\chi_\mu^s,\chi_\lambda^s)\hookrightarrow W(\chi_\mu^s,\chi''^s)$) we deduce that $I(\sigma_1,\tau_1)\subseteq \mathrm{im}(\widetilde{\kappa})^{K_1}\subseteq\pi_1^{K_1}$. In particular, $\tau_1\in\mathrm{JH}(\pi_1^{K_1})$. We also note that $J_{\sigma_1}=J_{\overline{\rho}}\cap J_{\tau_1}$. (Using Lemmas 4.1.1 and 3.1.3 we have $J_{\sigma_1}=J_{\overline{\rho}}\cap J_{\mu^{[s]}}$, $J_{\tau_1}=J_{\lambda^{[s]}}$, and recall from Step 2 that $\lambda_j=\mu_j$, hence $\lambda_j^{[s]}=\mu_j^{[s]}$, for all $j\in J_1^c\supseteq J_{\overline{\rho}}$.)

By Lemma 4.3.13 applied to $\widetilde{R}_{\widetilde{J_1}}(\chi_\lambda)$, and noting that we have a surjection

$$\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu^s,\chi''^s)\twoheadrightarrow \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\lambda^s,\chi''^s)=\widetilde{R}_{\widetilde{J_1}}(\chi_\lambda),$$

we see that the unique quotient Q_{τ_1} of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu^s,\chi''^s)$ with socle $\delta(\tau)=\tau_1$ contains $D_{0,\tau_1}(\overline{\rho}^{\mathrm{ss}})$. As τ_1 occurs in $Q_{\sigma_1}=\mathrm{im}(\widetilde{\kappa})$ (the unique quotient with socle σ_1), we see that Q_{σ_1} surjects onto Q_{τ_1} and hence contains $D_{0,\tau_1}(\overline{\rho}^{\mathrm{ss}})$ as subquotient. By Lemma 4.3.14, we have

$$\mathrm{JH}(D_{0,\tau_1}(\overline{\rho}^{\mathrm{ss}}))\cap\mathrm{JH}(D_0(\overline{\rho}))\subseteq \mathrm{JH}(D_{0,\tau_1}(\overline{\rho}^{\mathrm{ss}}))\cap\mathrm{JH}(D_{0,\sigma_1}(\overline{\rho}))\subseteq \mathrm{JH}(Q_{\sigma_1})\cap\mathrm{JH}(D_{0,\sigma_1}(\overline{\rho}))\subseteq \mathrm{JH}(Q_{\sigma_1}^{K_1}),$$

where the last inclusion results from Proposition 4.3.8 (applied with $\sigma=\sigma_1$ and varying τ). As $Q_{\sigma_1}=\mathrm{im}(\widetilde{\kappa})\subseteq\pi_1$, (72) holds for $\tau'=\tau_1$. Repeating the same argument with $\tau'=\tau_1\in W(\overline{\rho}^{\mathrm{ss}})\cap\mathrm{JH}(\pi_1^{K_1})$, which still has length i_0+1, we see that (72) holds for all $\delta^n(\tau)$, in particular for τ itself as $\delta(\cdot)$ is periodic. Thus, we deduce that (72) holds for all $\tau'\in W(\overline{\rho}^{\mathrm{ss}})$ such that $\ell(\tau')=i_0+1$ and $\tau'\in\mathrm{JH}(\pi_1^{K_1})$.

Step 5. We modify the proof of [BP12, Thm. 15.4] to show that (72) holds for any $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ with $\ell(\tau') = i_0 + 1$. We may assume that $i_0 + 1 < f$. As in the previous step we start with $\tau \in W(\overline{\rho}^{\mathrm{ss}}) \cap \mathrm{JH}(\pi_1^{K_1})$ and recall that $\ell(\tau) = i_0 + 1$. Write $J_\tau = \mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_r$ with $\mathcal{S}_i = \{a_i, a_i + 1, \ldots, b_i = a_i + \ell_i - 1\}$ (thought of inside $\mathbb{Z}/f\mathbb{Z}$), $0 \le a_1 < a_2 < \cdots < a_r < f$, and $b_i + 1 \notin J_\tau$ for each $1 \le i \le r$. In particular, $\ell(\tau) = \sum_{i=1}^r \ell_i$. Fix $1 \le i \le r$. Define an f-tuple λ as follows (note that λ has a different meaning than in the previous steps):

$$\lambda_j(x_j) = \begin{cases} p - 3 - x_j & \text{if } j \in J_\tau \setminus \{b_i\}, \\ x_j + 1 & \text{if } j = b_i, \\ p - 2 - x_j & \text{if } j = b_i + 1, \\ p - 1 - x_j & \text{otherwise.} \end{cases}$$

Then it is direct to check that $\lambda \in \mathscr{P}^{\mathrm{ss}}$ and $|J_\lambda| = i_0 + 1$. Moreover, letting $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ be the element such that χ_λ^s contributes to $D_{0,\tau'}(\overline{\rho}^{\mathrm{ss}})$, by Lemma 3.1.3 applied to χ_λ^s we have

$$J_{\tau'} = (J_\tau \setminus \{b_i\}) \sqcup \{b_i + 1\}, \tag{73}$$

so in particular, $\ell(\tau') = \ell(\tau) = i_0 + 1$. Below we will prove that $\tau' \in \mathrm{JH}(\pi_1^{K_1})$, so that (72) holds for τ' by Step 4. By repeating this procedure, it is easy to see using (73) that (72) holds for any $\tau' \in W(\overline{\rho}^{\mathrm{ss}})$ of length $i_0 + 1$.

We define $\mu \in \mathscr{P}$ as in Step 2, with $J_1 \overset{\mathrm{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p - 3 - x_j\}$ (and $J_2 = \emptyset$). Then $W(\chi_\mu, \chi_\lambda) \hookrightarrow \pi|_{I_1}$ by Lemma 4.3.9. We claim that the image is contained in π_1, or equivalently that the image V of the induced map $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu, \chi_\lambda) \to \pi^{K_1}$ is contained in $\pi_1^{K_1}$. As in Step 3, letting $\tau^{J'}$ denote the cosocle of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} \chi_\mu \prod_{j \in J'} \alpha_j^{-1}$, where $J' \subseteq J_1$, it suffices to show that $\tau^{J'} \in \mathrm{JH}(V)$ implies $\tau^{J'} \in \mathrm{JH}(\pi_1^{K_1})$ for any $J' \subseteq J_1$. Assume $\tau^{J'} \in \mathrm{JH}(V)$ for some $J' \subseteq J_1$ and define an f-tuple μ' by $\mu'_j(x_j) = \mu_j(x_j) - 2 = \lambda_j(x_j)$ if $j \in J'$, $\mu'_j(x_j) = \mu_j(x_j)$ otherwise, so that $\chi_{\mu'} = \chi_\mu \prod_{j \in J'} \alpha_j^{-1}$. Then $\mu' \in \mathscr{P}^{\mathrm{ss}}$ and $|J_{\mu'}| \le |J_\lambda| = i_0 + 1$, with equality holding if and only if $\mu' = \lambda$ (i.e. $J' = J_1$). If $J' = J_1$, then $\tau^{J'} \in \mathrm{JH}(D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})) \cap \mathrm{JH}(D_0(\overline{\rho}))$ (by Lemma 3.1.3) and so $\tau^{J'} \in \mathrm{JH}(\pi_1^{K_1})$ by (72) for $\tau' = \tau$ in Step 4. If $J' \subsetneq J_1$, then $\tau^{J'} \in \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_{\le i_0}) \cap \mathrm{JH}(D_0(\overline{\rho})) = \mathrm{JH}(D_0(\overline{\rho})_{\le i_0}) \subseteq \mathrm{JH}(\pi_1^{K_1})$, by assumption. This proves the claim. By Lemma 4.3.11, τ' is contained in the K_1-invariants of the image of $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu^s, \chi_\lambda^s)$ in π_1, hence $\tau' \in \mathrm{JH}(\pi_1^{K_1})$ as desired. □

Corollary 4.3.16. *Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in Theorem 4.3.15. Then*

$$\mathrm{JH}(\pi_1^{I_1}) = \{\chi_\lambda : \lambda \in \mathscr{P} \text{ such that } |J_\lambda| \le i_0\}. \tag{74}$$

Proof. By Lemma 3.1.3, χ is contained in the right-hand side of (74) if and only if $\chi \in \mathrm{JH}(D_0(\overline{\rho})^{I_1}) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})^{I_1}_{\le i_0})$. As $\mathrm{JH}(\pi_1^{I_1}) \subseteq \mathrm{JH}(D_0(\overline{\rho})^{I_1}) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})^{I_1}_{\le i_0})$ by (66) and Theorem 4.3.15, we deduce that "$\subseteq$" holds in (74). Conversely, if $\chi \in \mathrm{JH}(D_0(\overline{\rho})^{I_1}) \cap \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})^{I_1}_{\le i_0})$, then χ contributes to $D_0(\overline{\rho})^{I_1}_i$ for some i, hence to $D_0(\overline{\rho}^{\mathrm{ss}})^{I_1}_i$, which implies $i \le i_0$. In particular, χ does not contribute to $D_0(\overline{\rho})^{I_1}_i$ for any $i > i_0$, so χ contributes to $D_0(\overline{\rho})^{I_1}_{\le i_0} = \pi_1^{I_1}$ by Theorem 4.3.15. □

Corollary 4.3.17. *Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in Theorem 4.3.15. Then*

$$\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1) \cong \bigoplus_{\sigma \in W(\overline{\rho}), \ell(\sigma) \le i_0} \sigma.$$

Proof. Note that $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)$ is multiplicity free by Corollary 4.3.16. If $\sigma \subseteq \pi_1|_{\mathrm{GL}_2(\mathcal{O}_K)}$ is an irreducible subrepresentation, then $\sigma \in W(\overline{\rho})$, $\ell(\sigma) \le i_0$ by Theorem 4.3.15 and (64). Conversely, suppose that $\sigma \in W(\overline{\rho})$ with $\ell(\sigma) \le i_0$. Then $\sigma \hookrightarrow D_0(\overline{\rho})_{\le i_0}$ by the sentence after (64), hence $\sigma \subseteq \pi_1|_{\mathrm{GL}_2(\mathcal{O}_K)}$ by Theorem 4.3.15. □

Remark 4.3.18. In particular, a subrepresentation π_1 is not determined by $\pi_1^{I_1}$. For example, if $J_{\overline{\rho}} = \emptyset$, then it follows from the definitions that $|J_\lambda| \le f/2$ for all $\lambda \in \mathscr{P}$. Likewise, π_1 is not determined by $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}(\pi_1)$. (On the other hand, π_1 is determined by $\pi_1^{K_1}$ by Theorem 4.4.8.)

We conclude with a result on higher Iwahori invariants.

Proposition 4.3.19. *Assume that $\overline{\rho}$ is $\max\{6, 2f+1\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in Theorem 4.3.15. Then for any $\lambda \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$ such that $|J_\lambda| = i_0 + 1$, the character χ_λ does not occur in $\pi_1[\mathfrak{m}^{f+1}]$.*

Proof. Define disjoint subsets J_1, J_2 of $\{0, 1, \ldots, f-1\}$ and $\mu \in \mathscr{P}$ as in (58) and (59).

We let again $\sigma \in W(\overline{\rho})$ be the Serre weight determined by $J_\sigma = J_{\overline{\rho}} \cap J_\mu$ and $\tau \in W(\overline{\rho}^{\mathrm{ss}})$ be the Serre weight determined by $J_\tau = J_\lambda$. We also recall that there is a unique embedding $\iota : W(\chi_\mu, \chi_\lambda) \hookrightarrow \pi|_I$ and let V be the image of the induced morphism $\widetilde{\iota} : \mathrm{Ind}_I^{\mathrm{GL}_2(\mathcal{O}_K)} W(\chi_\mu, \chi_\lambda) \to \pi$. By Lemma 4.3.11, $I(\sigma, \tau) \subseteq V^{K_1}$.

Note that χ_λ contributes to $D_{0,\tau}(\overline{\rho}^{\mathrm{ss}})^{I_1}$ by Lemma 3.1.3, so $\ell(\tau) = |J_\lambda| = i_0 + 1$.

Suppose by contradiction that $\chi_\lambda \in \mathrm{JH}(\pi_1[\mathfrak{m}^{f+1}])$. As $|J_1| + |J_2| \le f$ we see by Lemma 4.3.1 and multiplicity freeness of $\pi[\mathfrak{m}^{f+1}]$ (which holds by Corollary 2.4.3(ii), applied with $n = f+1$ and $r = 1$) that $\mathrm{im}(\iota) \subseteq \pi_1$ and hence $V \subseteq \pi_1$. Since $I(\sigma, \tau) \subseteq V^{K_1}$, we deduce that $\tau \in \mathrm{JH}(\pi_1^{K_1})$. By Theorem 4.3.15, $\mathrm{JH}(\pi_1^{K_1}) \subseteq \mathrm{JH}(D_0(\overline{\rho}^{\mathrm{ss}})_{\le i_0})$, contradicting $\ell(\tau) = i_0 + 1$. □

4.4 Finite length

We prove that (the duals of) subrepresentations and quotients of π are Cohen–Macaulay Λ-modules of grade $2f$. We deduce many results on the structure of π as a $\mathrm{GL}_2(K)$-representation, including that it is of finite length.

In the proofs we will use the functor $D_\xi^\vee$ (see the paragraph preceding Proposition 3.2.2). We first state a theorem of Yitong Wang [Wan, Thm. 1.2] that will be essential for our proof.

Theorem 4.4.1 (Y. Wang)**.** *Assume that $2f < r_j < p - 2 - 2f$ for all $0 \le j \le f-1$. Let π_1 be a subrepresentation of π. Then we have*

$$\dim_{\mathbb{F}((X))} D_\xi^\vee(\pi_1) = |\,\mathrm{JH}(\pi_1^{K_1}) \cap W(\overline{\rho}^{\mathrm{ss}})|.$$

By equation (64) we deduce the following corollary.

Corollary 4.4.2. *Assume that $\overline{\rho}$ is $\max\{6, 2f+1\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \le i_0 \le f$ be as in Theorem 4.3.15. Then*

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1) = \sum_{i \le i_0} \binom{f}{i}.$$

We denote by N the graded module defined in § 2.3, namely

$$N \stackrel{\text{def}}{=} \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{R}{\mathfrak{a}(\lambda)}.$$

If $\overline{\rho}$ is moreover 9-generic we have $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \cong N$ by Theorem 2.1.2. From now on, we thus assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic (in addition to assumptions (i)–(iv)).

Proposition 4.4.3. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. Let $0 \subsetneq \pi_1 \subsetneq \pi$ be a subrepresentation of π and let $\pi_2 \stackrel{\text{def}}{=} \pi/\pi_1$. Then both $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ and $\mathrm{gr}_F(\pi_2^\vee)$ are Cohen–Macaulay $\mathrm{gr}(\Lambda)$-modules of grade $2f$, where F denotes the filtration induced from $\pi^\vee$. In particular, $\pi_1^\vee$ and $\pi_2^\vee$ are Cohen–Macaulay Λ-modules of grade $2f$.*

Remark 4.4.4. It is easy to see that F does not coincide with the $\mathfrak{m}$-adic filtration in general (when $\overline{\rho}$ is nonsplit), for example because $\pi_2^{I_1}$ is bigger than $\pi^{I_1}/\pi_1^{I_1}$ already when $f = 1$ and π_1 is a principal series representation. We will determine $\mathrm{gr}_{\mathfrak{m}}(\pi_2^\vee)$ in [BHH+c].

Recall the ideals $I(J_1, J_2, d)$ and $I(J_1, J_2, d, \underline{t}) = I(J_1, J_2, d) + (\underline{t})$ of $\overline{R}$ from Definition 4.2.4, where J_1, J_2 are disjoint subsets of $\{0, \ldots, f-1\}$, $d \in \mathbb{Z}$, and $t_j \in \{y_j, z_j, y_j z_j\}$ for all $0 \le j \le f-1$. If $d \ge 1$, the ideal $I(J_1, J_2, d)$ is generated by all $\prod_{j \in J_1'} y_j \prod_{j \in J_2'} z_j$ with $J_1' \subseteq J_1$, $J_2' \subseteq J_2$, $|J_1'| + |J_2'| = d$ (plus all t_j for $I(J_1, J_2, d, \underline{t})$). If $d \le 0$ these ideals equal $\overline{R}$.

For $\lambda \in \mathscr{P}$ define the ideal of $\overline{R}$,

$$\mathfrak{a}_1^{i_0}(\lambda) \stackrel{\text{def}}{=} I(J_1, J_2, i_0 + 1 - |J_\lambda|) + \mathfrak{a}(\lambda), \tag{75}$$

where $J_1 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p - 1 - x_j\}$ and $J_2 \stackrel{\text{def}}{=} \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = x_j\}$. In other words, $\mathfrak{a}_1^{i_0}(\lambda) = I(J_1, J_2, i_0 + 1 - |J_\lambda|, \underline{t})$, where $t_j \in \{y_j, z_j, y_j z_j\}$ is defined in (12) in terms of λ. (Note that $t_j = y_j z_j$ for all $j \in J_1 \sqcup J_2$.) By definition, $\mathfrak{a}_1^{i_0}(\lambda) = \overline{R}$ if $i_0 < |J_\lambda|$ and $\mathfrak{a}_1^{i_0}(\lambda) = \mathfrak{a}(\lambda)$ if $|J_1| + |J_2| < i_0 + 1 - |J_\lambda|$.

Proof of Proposition 4.4.3. For most of the proof we allow the extreme cases $\pi_1 = 0$ and $\pi_1 = \pi$.

Step 1. We show that for $\lambda \in \mathscr{P}$ the ideal $\mathfrak{a}_1^{i_0}(\lambda)$ kills the χ_λ^{-1}-eigenspace of $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)_0 \cong (\pi_1^{I_1})^\vee$ inside $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$.

By Corollary 4.3.16 we may assume that $|J_\lambda| \le i_0$. We already know that the χ_λ^{-1}-eigenspace is killed by $\mathfrak{a}(\lambda)$ ([BHH+a, Thm. 3.3.2.1], [HW22, Cor. 8.12]), so let us take a monomial $\prod_{j \in J_1'} y_j \prod_{j \in J_2'} z_j$ with $J_1' \subseteq J_1$, $J_2' \subseteq J_2$, $|J_1'| + |J_2'| = i_0 + 1 - |J_\lambda|$ (in particular of degree > 0).

Define $\lambda' \in \mathscr{P}^{\mathrm{ss}}$ by letting $\lambda'_j(x_j) \stackrel{\text{def}}{=} \lambda_j(x_j) - 2$ if $j \in J'_1$, $\lambda'_j(x_j) \stackrel{\text{def}}{=} \lambda_j(x_j) + 2$ if $j \in J'_2$, and $\lambda'_j(x_j) \stackrel{\text{def}}{=} \lambda_j(x_j)$ otherwise. Then $\lambda' \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$ using the definition of $\mathscr{P}^{\mathrm{ss}}$ and (9). Moreover, $|J_{\lambda'}| = |J_\lambda| + (i_0 + 1 - |J_\lambda|) = i_0 + 1$. By Proposition 4.3.19 we deduce (on the dual side) that the monomial $\prod_{j\in J'_1} y_j \prod_{j \in J'_2} z_j$ kills the χ_λ^{-1}-eigenspace of $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)_0$ inside $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$.

Step 2. Define $N_1^{i_0} \stackrel{\text{def}}{=} \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \overline{R}/\mathfrak{a}_1^{i_0}(\lambda)$ and let $N_2^{i_0}$ be the kernel of the natural map $N \twoheadrightarrow N_1^{i_0}$. Consider the induced short exact sequence

$$0 \to \mathrm{gr}_F(\pi_2^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \to \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \to 0,$$

where F is the filtration on $\pi_2^\vee$ induced from the $\mathfrak{m}$-adic filtration on $\pi^\vee$. By Step 1 the morphism $N \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ factors through $N_1^{i_0}$, hence we get an induced commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{gr}_F(\pi_2^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi^\vee) \arrow[r] & \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \arrow[r] & 0 \\ 0 \arrow[r] & N_2^{i_0} \arrow[r] \arrow[u, hook] & N \arrow[r] \arrow[u, "\cong"] & N_1^{i_0} \arrow[r] \arrow[u, two heads] & 0 \end{tikzcd}$$

with injective (resp. surjective) vertical map on the left (resp. right). Thus

$$\mathcal{Z}(N_1^{i_0}) \ge \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)), \quad \mathcal{Z}(N_2^{i_0}) \le \mathcal{Z}(\mathrm{gr}_F(\pi_2^\vee)). \tag{76}$$

Step 3. We show that $N_1^{i_0}$ and $N_2^{i_0}$ are Cohen–Macaulay of grade $2f$, or zero.

First note that $j_{\mathrm{gr}(\Lambda)}(N_1^{i_0}) \ge j_{\mathrm{gr}(\Lambda)}(N) = 2f$. By Corollary 4.2.5 and [BHH$^+$a, Lemma 3.3.1.9], $N_1^{i_0}$ is a Cohen–Macaulay $\mathrm{gr}(\Lambda)$-module of grade $2f$, or zero. (We may omit the terms in the direct sum with $|J_\lambda| > i_0$, as they vanish.) As $N_2^{i_0} = \ker(N \twoheadrightarrow N_1^{i_0})$ and both N and $N_1^{i_0}$ are Cohen–Macaulay of grade $2f$, or zero, so is $N_2^{i_0}$.

Step 4. We show that $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ and $\mathrm{gr}_F(\pi_2^\vee)$ are Cohen–Macaulay of grade $2f$.

By assumption (iii) we have $\mathrm{E}_\Lambda^{2f}(\pi^\vee) \cong \pi^\vee \otimes \det(\overline{\rho})\omega^{-1}$ as $\mathrm{GL}_2(K)$-representations. As in the proof of [BHH$^+$a, Prop. 3.3.5.3(iii)] we may construct a subrepresentation $\widetilde{\pi}_2 \subseteq \pi$ such that $\mathcal{Z}(\mathrm{gr}(\widetilde{\pi}_2^\vee)) = \mathcal{Z}(\mathrm{gr}(\pi_2^\vee))$ (with respect to any good filtrations). By [BHH$^+$a, Prop. 3.3.5.3(i)] and the exactness of $D_\xi^\vee$ we have

$$\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\widetilde{\pi}_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_2) = \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) - \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1).$$

By Corollary 4.4.2 we deduce that $i_0(\widetilde{\pi}_2) = f - 1 - i_0(\pi_1)$.

In particular, noting that $N_1^{i_0}$ only depends on $i_0 = i_0(\pi_1)$, we deduce by (76) applied to the subrepresentation $\widetilde{\pi}_2$ that $\mathcal{Z}(N_1^{f-1-i_0}) \ge \mathcal{Z}(\mathrm{gr}_{\mathfrak{m}}(\widetilde{\pi}_2^\vee))$. Hence

$$\mathcal{Z}(N_1^{f-1-i_0}) \ge \mathcal{Z}(\mathrm{gr}(\widetilde{\pi}_2^\vee)) = \mathcal{Z}(\mathrm{gr}(\pi_2^\vee)) \ge \mathcal{Z}(N_2^{i_0}) = \mathcal{Z}(N) - \mathcal{Z}(N_1^{i_0}). \tag{77}$$

We claim that equality holds, and it suffices to show that $m(N_1^{i_0}) + m(N_1^{f-1-i_0}) = m(N)$.

As $N_1^{i_0} = \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \overline{R}/\mathfrak{a}_1^{i_0}(\lambda)$ and the involution $\lambda \mapsto \lambda^*$ preserves (i.e. induces a bijection on) $\mathscr{P}$ by [BHH$^+$a, Lemma 3.3.1.7(i)], it suffices to show that

$$m(\overline{R}/\mathfrak{a}_1^{i_0}(\lambda)) + m(\overline{R}/\mathfrak{a}_1^{f-1-i_0}(\lambda^*)) = m(\overline{R}/\mathfrak{a}(\lambda)) \;\text{ for each } \lambda \in \mathscr{P}. \tag{78}$$

Fix now $\lambda \in \mathscr{P}$. Recall that $\mathfrak{a}_1^{i_0}(\lambda) = I(J_1, J_2, i_0 + 1 - |J_\lambda|, \underline{t})$, where $J_1 = \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = p-1-x_j\}$, $J_2 = \{j \in J_{\overline{\rho}}^c : \lambda_j(x_j) = x_j\}$, and $t_j \in \{y_j, z_j, y_j z_j\}$ is defined in (12). Let $J \overset{\text{def}}{=} J_1 \sqcup J_2$. By Lemma 4.1.4 we have $|J_\lambda| + |J_{\lambda^*}| + |J| = f$ (and J is unchanged when λ is replaced by λ^*). By Lemma 4.2.6 we have

$$m(\overline{R}/\mathfrak{a}_1^{i_0}(\lambda)) = 2^{|\{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\}|} \left(\sum_{i < i_0 + 1 - |J_\lambda|} \binom{|J|}{i} \right), \tag{79}$$

noting that $\{j \in J^c : t_j = y_j z_j\} = \{j : \lambda_j(x_j) \in \{x_j + 1, p-2-x_j\}\}$. In particular, taking $i_0 = f$ and noting that $|J_\lambda| + |J| \leq f$ by Lemma 4.1.4 (or by arguing directly) we have,

$$m(\overline{R}/\mathfrak{a}(\lambda)) = 2^{|\{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\}|} \cdot 2^{|J|}. \tag{80}$$

From (79) and the definition of $\lambda \mapsto \lambda^*$ in [BHH+a, Def. 3.3.1.6] we obtain

$$m(\overline{R}/\mathfrak{a}_1^{f-1-i_0}(\lambda^*)) = 2^{|\{j : \lambda_j(x_j) \in \{x_j+1, p-2-x_j\}\}|} \left(\sum_{i < f - i_0 - |J_{\lambda^*}|} \binom{|J|}{i} \right),$$

By Lemma 4.1.4,

$$\sum_{i < f - i_0 - |J_{\lambda^*}|} \binom{|J|}{i} = \sum_{i < |J| + |J_\lambda| - i_0} \binom{|J|}{i} = \sum_{i > i_0 - |J_\lambda|} \binom{|J|}{i},$$

and we deduce (78) and hence equality in (77) and (76).

Since $N_1^{i_0}$ is Cohen–Macaulay, hence pure (by combining Prop. 3.5(v)(a) and Prop. 3.9(i) in [Ven02]), or since $N_1^{i_0} = 0$, any nonzero submodule of $N_1^{i_0}$ has a nonzero cycle. Hence the surjection $N_1^{i_0} \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee)$ must be an isomorphism and consequently $\mathrm{gr}_F(\pi_2^\vee) \cong N_2^{i_0}$ by Step 2. We finally assume that $\pi_1 \neq 0$ and $\pi_1 \neq \pi$. Then the isomorphisms we just established show that $N_1^{i_0} \neq 0$ and $N_2^{i_0} \neq 0$, so both $N_1^{i_0}$ and $N_2^{i_0}$ are Cohen–Macaulay by Step 3. Hence $\pi_1^\vee$ and $\pi_2^\vee$ are Cohen–Macaulay, because if a finitely generated Λ-module M admits a good filtration such that the associated graded module is Cohen–Macaulay, then M itself is Cohen–Macaulay by [LvO96, Prop. III.2.2.4]. $\square$

Corollary 4.4.5. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. Let $i_0 = i_0(\pi_1)$ with $-1 \leq i_0 \leq f$ be as in Theorem 4.3.15. Then*

$$\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\overline{R}}{\mathfrak{a}_1^{i_0}(\lambda)}$$

and

$$\mathrm{gr}_F(\pi_2^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}(\lambda)},$$

where F denotes the filtration induced from $\pi^\vee$.

Corollary 4.4.6. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. Suppose that $\pi' = \pi_1'/\pi_1$ is any nonzero subquotient of π, where $\pi_1 \subsetneq \pi_1' \subseteq \pi$. Let $i_0 \overset{\text{def}}{=} i_0(\pi_1)$, $i_0' \overset{\text{def}}{=} i_0(\pi_1')$, so $-1 \leq i_0 < i_0' \leq f$.*

Let F denote the subquotient filtration on $\pi'^\vee$ induced from the $\mathfrak{m}$-adic filtration on $\pi^\vee$. Then

$$\mathrm{gr}_F(\pi'^\vee) \cong \bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0}(\lambda)}{\mathfrak{a}_1^{i_0'}(\lambda)}. \tag{81}$$

Moreover, $\mathrm{gr}_F(\pi'^\vee)$ (resp. $\pi'^\vee$) is Cohen–Macaulay of grade $2f$.

Proof. The exact sequence $0 \to \pi'^\vee \to \pi_1'^\vee \to \pi_1^\vee \to 0$ of Λ-modules gives rise to an exact sequence

$$0 \to \mathrm{gr}_F(\pi'^\vee) \to \mathrm{gr}(\pi_1'^\vee) \to \mathrm{gr}(\pi_1^\vee) \to 0.$$

The second map is identified with the natural map $\bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1}\otimes\overline{R}/\mathfrak{a}_1^{i_0'}(\lambda) \to \bigoplus_{\lambda\in\mathscr{P}} \chi_\lambda^{-1}\otimes\overline{R}/\mathfrak{a}_1^{i_0}(\lambda)$ by Corollary 4.4.5 (cf. Step 2 of the proof of Proposition 4.4.3). Formula (81) follows. As $\mathrm{gr}(\pi_1'^\vee)$, $\mathrm{gr}(\pi_1^\vee)$ (resp. $\pi_1'^\vee$, $\pi_1^\vee$) are Cohen–Macaulay of grade $2f$ by Proposition 4.4.3, so is $\mathrm{gr}_F(\pi'^\vee)$ (resp. $\pi'^\vee$). (If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of Λ-modules (resp. $\mathrm{gr}(\Lambda)$-modules) and M and M'' are Cohen–Macaulay of the same grade j, then M' is zero or Cohen–Macaulay of grade j by [LvO96, Cor. III.2.1.6].) □

For $0 \le j \le f$, let $\mathscr{P}_j^{\mathrm{ss}}$ (resp. $\mathscr{P}_j$) denote the subset of $\lambda \in \mathscr{P}^{\mathrm{ss}}$ (resp. $\lambda \in \mathscr{P}$) with $|J_\lambda| = j$.

Corollary 4.4.7. *Keep the assumptions and notation in Corollary 4.4.6. There is an H-equivariant isomorphism*

$$\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_F(\pi'^\vee) \cong \bigoplus_{\lambda} \chi_\lambda^{-1},$$

where λ runs through all $\lambda \in \mathscr{P}_{i_0+1}^{\mathrm{ss}} \cup \big(\bigcup_{i_0+2\le j\le i_0'} \mathscr{P}_j\big)$.

Proof. We first look at $X_{i_0,i_0'}(\lambda) \overset{\mathrm{def}}{=} \mathbb{F} \otimes_{\overline{R}} \mathfrak{a}_1^{i_0}(\lambda)/\mathfrak{a}_1^{i_0'}(\lambda)$ for $\lambda \in \mathscr{P}$. If $|J_\lambda| > i_0'$, then $\mathfrak{a}_1^{i_0}(\lambda) = \mathfrak{a}_1^{i_0'}(\lambda) = \overline{R}$, so $X_{i_0,i_0'}(\lambda) = 0$; if $i_0 < |J_\lambda| \le i_0'$, then $\mathfrak{a}_1^{i_0}(\lambda) = \overline{R}$ while $\mathfrak{a}_1^{i_0'}(\lambda) \subseteq \mathfrak{m}_{\overline{R}}$ (the unique maximal graded ideal in $\overline{R}$), so $X_{i_0,i_0'}(\lambda) \cong \mathbb{F}$. Finally suppose $|J_\lambda| \le i_0$, and recall $\mathfrak{a}_1^i(\lambda) = I(J_1, J_2, i+1-|J_\lambda|) + \mathfrak{a}(\lambda)$, where J_1, J_2 are as in (75). Hence $I(J_1, J_2, i_0'+1-|J_\lambda|) \subseteq \mathfrak{m}_{\overline{R}} I(J_1, J_2, i_0+1-|J_\lambda|)$ and so

$$X_{i_0,i_0'}(\lambda) \cong \mathbb{F} \otimes_{\overline{R}} I(J_1, J_2, i_0+1-|J_\lambda|) \cong \bigoplus_{(J_1',J_2')} \mathbb{F}\Big(\overline{\prod_{j\in J_1'} y_j \prod_{j\in J_2'} z_j}\Big),$$

where (J_1', J_2') runs through all pairs with $J_1' \subseteq J_1$, $J_2' \subseteq J_2$, $|J_1'| + |J_2'| = i_0+1-|J_\lambda|$. Step 1 of the proof of Proposition 4.4.3 shows that to each pair (J_1', J_2') as above, one can associate an element $\lambda' \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$ with $|J_{\lambda'}| = i_0+1$, such that $\chi_\lambda^{-1} \prod_{j\in J_1'} \alpha_j \prod_{j\in J_2'} \alpha_j^{-1} = \chi_{\lambda'}^{-1}$. Conversely, by the construction in (59), any element $\lambda' \in \mathscr{P}^{\mathrm{ss}} \setminus \mathscr{P}$ with $|J_{\lambda'}| = i_0+1$ arises in this way and λ' uniquely determines $\lambda \in \mathscr{P}$ and J_1', J_2'. The result follows from this combined with Corollary 4.4.6. □

Theorem 4.4.8. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic.*

(i) *Any subquotient of π is generated by its K_1-invariants.*

(ii) *The representation π is uniserial of length at most $f+1$. More precisely, suppose that π_1, π_1' are any subrepresentations of π. Then the following are equivalent:*

(a) $\pi_1 \subseteq \pi_1'$;

(b) $\pi_1^{K_1} \subseteq \pi_1'^{K_1}$;

(c) $i_0(\pi_1) \le i_0(\pi_1')$;

(d) $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1) \le \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1')$.

(iii) *If π' is any nonzero subquotient of π, then $D_\xi^\vee(\pi')$ is nonzero.*

Proof. (i) The quotient of any $\mathrm{GL}_2(K)$-representation generated by its K_1-invariants is generated by its K_1-invariants, hence it suffices to consider the case of a subrepresentation $\pi_1 \subseteq \pi$. Let $\pi_1' \stackrel{\text{def}}{=} \langle \mathrm{GL}_2(K) \cdot \pi_1^{K_1} \rangle$ be the subrepresentation of π_1 generated by $\pi_1^{K_1}$, so $\pi_1'^{K_1} = \pi_1^{K_1}$. By Theorem 4.3.15 we have $i_0(\pi_1') = i_0(\pi_1)$. By the proof of Proposition 4.4.3 the natural map $\mathrm{gr}_{\mathfrak{m}}(\pi_1^\vee) \twoheadrightarrow \mathrm{gr}_{\mathfrak{m}}(\pi_1'^\vee)$ is an isomorphism (consider the diagram in Step 2), so $\pi_1' = \pi_1$.

(ii) To show the equivalence, we note that (a)$\Rightarrow$(b) and the converse holds by part (i), (b)$\Leftrightarrow$(c) by Theorem 4.3.15, and (c)$\Leftrightarrow$(d) by Corollary 4.4.2. Finally, condition (c) implies that π is uniserial of length at most $f+1$.

(iii) Write $\pi' = \pi_1'/\pi_1$ for some subrepresentations $\pi_1 \subsetneq \pi_1' \subseteq \pi$. By part (ii) we deduce that $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1) < \dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi_1')$. We conclude by the exactness of $D_\xi^\vee$. □

Remark 4.4.9. The statement of Theorem 4.4.8(i) fails if we replace K_1 by I_1, already when $K = \mathbb{Q}_p$, by [BP12, Thm. 20.3(i)] (see also [Mor17, Thm. 1.1] for a different proof).

Corollary 4.4.10. *Assume that $\overline{\rho}$ is $\max\{9, 2f+1\}$-generic. The $\mathrm{GL}_2(K)$-representation π is multiplicity free (of length $\le f+1$).*

Proof. Let π' be any nonzero subquotient of π and F be the subquotient filtration on $\pi'^\vee$ induced from the $\mathfrak{m}$-adic filtration on $\pi^\vee$. As in the proof of Proposition 2.4.9, by replacing $\mathrm{gr}_{\mathfrak{m}}(\pi^\vee)$ by $\mathrm{gr}_F(\pi'^\vee)$ we obtain a spectral sequence $E_i^r \Longrightarrow \mathrm{Tor}_i^\Lambda(\mathbb{F}, \pi'^\vee)$ with $E_i^1 = \mathrm{Tor}_i^{\mathrm{gr}(\Lambda)}(\mathbb{F}, \mathrm{gr}_F(\pi'^\vee))$ for $i \ge 0$. In particular, we get a surjective graded morphism compatible with H-action

$$E_0^1 = \mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_F(\pi'^\vee) \twoheadrightarrow \mathrm{gr}(\mathbb{F} \otimes_\Lambda \pi'^\vee) = E_0^\infty,$$

hence an inclusion

$$\mathrm{JH}(\mathbb{F} \otimes_\Lambda \pi'^\vee) = \mathrm{JH}(\mathrm{gr}(\mathbb{F} \otimes_\Lambda \pi'^\vee)) \subseteq \mathrm{JH}(\mathbb{F} \otimes_{\mathrm{gr}(\Lambda)} \mathrm{gr}_F(\pi'^\vee)). \tag{82}$$

By Theorem 4.4.8(ii) there exists a unique composition series $0 = \pi_0 \subsetneq \pi_1 \subsetneq \cdots \subsetneq \pi_\ell = \pi$ of the $\mathrm{GL}_2(K)$-representation π, and moreover $-1 = i_0(\pi_0) < i_0(\pi_1) < \cdots < i_0(\pi_\ell) = f$. Corollary 4.4.6 implies that

$$\mathrm{gr}_F((\pi_j/\pi_{j-1})^\vee) \cong \bigoplus_{\lambda \in \mathscr{P}} \chi_\lambda^{-1} \otimes \frac{\mathfrak{a}_1^{i_0(\pi_{j-1})}(\lambda)}{\mathfrak{a}_1^{i_0(\pi_j)}(\lambda)}.$$

As $\mathbb{F} \otimes_\Lambda (\pi_j/\pi_{j-1})^\vee$ is dual to $(\pi_j/\pi_{j-1})^{I_1}$, we deduce from Corollary 4.4.7 and (82) that the sets $\mathrm{JH}((\pi_j/\pi_{j-1})^{I_1})$ (of H-representations) are disjoint for $1 \le j \le \ell$, which proves the multiplicity freeness of π. □

A Appendix: canonical filtrations on Tor and Ext groups

We prove useful lemmas on the canonical filtration on Tor and Ext groups of filtered modules.

Let R be a filtered ring (not necessarily the ring R of § 1.3!), and let $\widetilde{R}$ be its Rees ring (see [LvO96, Def. I.4.3.5] or [BE90, § 4.1]). Then $\widetilde{R}$ is a graded ring, and we have a functor $N \mapsto \widetilde{N}$ from the category of filtered R-modules to the category of graded $\widetilde{R}$-modules (see [LvO96, § I.4.3]).

Letting $X \stackrel{\text{def}}{=} 1 \in \widetilde{R}_1$ be the canonical homogeneous element of degree 1 we have $\widetilde{R} = \bigoplus_{n\in\mathbb{Z}}(F_nR)X^n$ ([LvO96, Def. I.4.3.6(b)]). We thus define the *dehomogenization functor* $\mathcal{E}$ from the category of graded $\widetilde{R}$-modules to the category of filtered R-modules as follows: for a graded $\widetilde{R}$-module $W = \bigoplus_{n\in\mathbb{Z}} W_n$ we set $\mathcal{E}(W) \stackrel{\text{def}}{=} W/(1-X)W$, with filtration defined by

$$F_n(\mathcal{E}(W)) \stackrel{\text{def}}{=} (W_n + (1-X)W)/(1-X)W$$

for any $n \in \mathbb{Z}$. By [LvO96, Prop. I.4.3.7(5)] the functor $\mathcal{E}$ is exact, and by [LvO96, Prop. I.4.3.7(2), (3)] it induces an equivalence when restricted to the full subcategory of X-torsion-free graded $\widetilde{R}$-modules, with quasi-inverse $N \mapsto \widetilde{N}$. In particular, $\mathcal{E}(\widetilde{N}) \cong N$ for any filtered R-module N.

Lemma A.1. *Suppose that R is a filtered ring and that $N_1 \to N_2 \to N_3$ is an exact sequence of graded $\widetilde{R}$-modules. If N_3 is X-torsion-free, then the sequence $\mathcal{E}(N_1) \to \mathcal{E}(N_2) \to \mathcal{E}(N_3)$ of filtered R-modules is exact and the first morphism is strict. In particular, taking $N_3 = 0$: if $N_1 \to N_2$ is surjective, then $\mathcal{E}(N_1) \to \mathcal{E}(N_2)$ is a strict surjection.*

Proof. As recorded above (cf. [BE90, Prop. 5.3]), the Rees module $\widetilde{\mathcal{E}(N)}$ is identified with the largest X-torsion-free quotient of N. As N_3 is X-torsion-free, a diagram chase shows that the sequence $\widetilde{\mathcal{E}(N_1)} \to \widetilde{\mathcal{E}(N_2)} \to \widetilde{\mathcal{E}(N_3)}$ is exact. The result follows from [LvO96, Prop. I.4.3.8(2)]. $\square$

Suppose now that R, S are filtered rings such that the Rees ring $\widetilde{S}$ is noetherian, and let N be any filtered (S,R)-bimodule, i.e. equipped with a filtration F_nN ($n \in \mathbb{Z}$) such that with this filtration N is both a filtered left S-module and a filtered right R-module (cf. [LvO96, Def. I.2.2]). Then the notions in the previous paragraphs extend to filtered and graded bimodules, and we have a dehomogenization functor $\mathcal{E}$ from graded $(\widetilde{S},\widetilde{R})$-bimodules to filtered (S,R)-bimodules (in particular, $\mathcal{E}(\widetilde{N}) \cong N$ as filtered (S,R)-bimodules).

Following [BE90, § 5] in the case of $\mathrm{Ext}^i_R(-,R)$, we now explain that $\mathrm{Tor}^R_i(N,R)$ is canonically and functorially a filtered S-module. We also establish some basic properties of this canonical filtration.

If W is any graded $\widetilde{R}$-module, then

$$\mathcal{E}(\widetilde{N} \otimes_{\widetilde{R}} W) \cong S \otimes_{\widetilde{S}} \widetilde{N} \otimes_{\widetilde{R}} W \cong N \otimes_{\widetilde{R}} W \cong N \otimes_R \mathcal{E}(W), \tag{83}$$

where we used that $X = 1 \in \widetilde{S}_1$ (resp. $\widetilde{R}_1$) acts the same on the left and right of $\widetilde{N}$. Here, $\widetilde{N} \otimes_{\widetilde{R}} W$ is a graded $\widetilde{S}$-module (cf. the discussion at the end of § 2.2), $N \otimes_R \mathcal{E}(W)$ is a filtered S-module (cf. [LvO96, § I.6]) and (83) is easily checked to be an isomorphism of filtered S-modules.

Forgetting filtrations for a moment, as $\mathcal{E}$ is exact, we have a natural isomorphism

$$\mathcal{E}(\operatorname{Tor}_i^{\widetilde{R}}(\widetilde{N}, W)) \cong \operatorname{Tor}_i^R(N, \mathcal{E}(W)) \tag{84}$$

as S-modules for all $i \geq 0$. As $\operatorname{Tor}_i^{\widetilde{R}}(\widetilde{N}, W)$ is a graded $\widetilde{S}$-module, the isomorphism induces a canonical and functorial filtration on $\operatorname{Tor}_i^R(N, \mathcal{E}(W))$. In particular, if $W = \widetilde{M}$ for a filtered R-module M we obtain a canonical and functorial filtration on the S-module $\operatorname{Tor}_i^R(N, M)$.

Lemma A.2. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a strict exact sequence of filtered R-modules, then the long exact sequence*

$$\cdots \to \operatorname{Tor}_1^R(N, M_2) \to \operatorname{Tor}_1^R(N, M_3) \to N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \to 0$$

of S-modules respects filtrations.

The reason is that by strictness the induced sequence $0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3} \to 0$ is still exact.

Lemma A.3. *Suppose that $\widetilde{R}$ is noetherian, and suppose that N has the property that as a filtered S-module its filtration is good. Then for any filtered R-module M equipped with a good filtration, the canonical filtration on each $\operatorname{Tor}_i^R(N, M)$ is good.*

Note that the condition on N is equivalent to $\widetilde{N}$ being a finitely generated $\widetilde{S}$-module [LvO96, Prop. I.5.4(1)].

Proof. From the isomorphism (84) with $W = \widetilde{M}$ and [LvO96, Prop. I.4.3.7(2), (3)] it follows that the Rees module of $\operatorname{Tor}_i^R(N, M)$ is the largest X-torsion-free quotient of $\operatorname{Tor}_i^{\widetilde{R}}(\widetilde{N}, \widetilde{M})$. Hence by [LvO96, Prop. I.5.4(1)] it suffices to show that $\operatorname{Tor}_i^{\widetilde{R}}(\widetilde{N}, \widetilde{M})$ is a finitely generated $\widetilde{S}$-module for all i. By picking a gr-free resolution of $\widetilde{M}$ whose terms are moreover finitely generated (using $\widetilde{R}$ noetherian) and since $\widetilde{S}$ is noetherian, we reduce to the case $i = 0$ and $\widetilde{M}$ gr-free, in which case the claim follows from the assumption on N. $\square$

Lemma A.4. *Suppose that $\cdots \to F_1 \to F_0 \to M \to 0$ is a strict exact sequence with F_i filt-free for all i (see the beginning of § 2.2 for filt-free). Then the canonical filtration on $\operatorname{Tor}_i^R(N, M)$ coincides with the subquotient filtration on the i-th homology of the complex of filtered S-modules $N \otimes_R F_\bullet$ (each carrying the tensor product filtration).*

Proof. By strictness, the sequence $\cdots \to \widetilde{F_1} \to \widetilde{F_0} \to \widetilde{M} \to 0$ of graded $\widetilde{R}$-modules is exact. Hence $\operatorname{Tor}_i^{\widetilde{R}}(\widetilde{N}, \widetilde{M})$ is isomorphic to the i-th homology of the complex $\widetilde{N} \otimes_{\widetilde{R}} \widetilde{F}_\bullet$ of graded $\widetilde{S}$-modules. Let $C_i \stackrel{\text{def}}{=} \widetilde{N} \otimes_{\widetilde{R}} \widetilde{F}_i$, so $\mathcal{E}(C_i) \cong N \otimes_R F_i$ with the tensor product filtration. Let Z_i (resp. B_{i-1}) denote the kernel (resp. the image) of $C_i \to C_{i-1}$, and let $H_i \stackrel{\text{def}}{=} Z_i / B_i$. By exactness of $\mathcal{E}$ we have $\mathcal{E}(H_i) \cong \mathcal{E}(Z_i)/\mathcal{E}(B_i)$ as S-modules and we need to show that it carries the subquotient topology inside $\mathcal{E}(C_i)$, i.e. that the maps $\mathcal{E}(Z_i) \hookrightarrow \mathcal{E}(C_i)$ and $\mathcal{E}(Z_i) \twoheadrightarrow \mathcal{E}(H_i)$ are both strict. As $\widetilde{F_i}$ is gr-free, it follows that C_i is X-torsion-free, and hence so are B_i and Z_i. From Lemma A.1 we deduce that the sequences $0 \to \mathcal{E}(Z_i) \to \mathcal{E}(C_i) \to \mathcal{E}(B_{i-1}) \to 0$ and $\mathcal{E}(Z_i) \to \mathcal{E}(H_i) \to 0$ are strict exact. $\square$

Similarly, if N is a filtered (R, S)-bimodule, and M is an R-module with a *good* filtration, then the right S-module $\mathrm{Ext}^i_R(M, N)$ is canonically and functorially a filtered S-module. The reason is that for any finitely generated graded $\widetilde{R}$-module W we have a natural isomorphism of filtered right S-modules

$$\mathcal{E}(\mathrm{Ext}^i_{\widetilde{R}}(W, \widetilde{N})) \cong \mathrm{Ext}^i_R(\mathcal{E}(W), N)$$

and that $\mathrm{Hom}_R(W, -)$ is naturally graded [LvO96, Lemma I.4.1.1] and $\mathrm{Hom}_R(\mathcal{E}(W), N)$ is naturally filtered [LvO96, Prop. I.6.6], as W is finitely generated. The analogues of Lemmas A.2, A.3, and A.4 hold, with the analogous proofs, provided in the first lemma all M_i carry good filtrations and in the last lemma all F_i are filt-free of finite rank.

We finally specialize to the case where $R = S = \Lambda$ and M is a finitely generated (left) Λ-module equipped with a good filtration. In particular the right Λ-module $\mathrm{E}^i_\Lambda(M) = \mathrm{Ext}^i_\Lambda(M, \Lambda)$ carries a canonical and functorial filtration.

Lemma A.5. *Suppose that* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is a strict exact sequence of finitely generated filtered* Λ*-modules. Suppose that the filtration on* M_2 *(and hence on* M_1*,* M_3*) is good and that* $j \stackrel{\text{def}}{=} j_\Lambda(M_2)$*. Then the induced morphism* $0 \to \mathrm{E}^j_\Lambda(M_3) \to \mathrm{E}^j_\Lambda(M_2)$ *is strict.*

Proof. By strictness we get $0 \to \widetilde{M_1} \to \widetilde{M_2} \to \widetilde{M_3} \to 0$ of graded $\widetilde{\Lambda}$-modules, with $j_{\widetilde{\Lambda}}(\widetilde{M_2}) = j$ by [LvO96, § III.2.5]. Hence we obtain the exact sequence

$$0 \to \mathrm{E}^j_{\widetilde{\Lambda}}(\widetilde{M_3}) \to \mathrm{E}^j_{\widetilde{\Lambda}}(\widetilde{M_2}) \to \mathrm{E}^j_{\widetilde{\Lambda}}(\widetilde{M_1})$$

of graded right $\widetilde{\Lambda}$-modules. Each $\mathrm{E}^j_{\widetilde{\Lambda}}(\widetilde{M_i})$ is X-torsion-free by [BE90, Lemma 5.11], The result follows from Lemma A.1. □

References

[AJL83] Henning Haahr Andersen, Jens Jörgensen, and Peter Landrock, *The projective indecomposable modules of* $\mathrm{SL}(2, p^n)$, Proc. London Math. Soc. (3) **46** (1983), no. 1, 38–52. MR 684821

[BD14] Christophe Breuil and Fred Diamond, *Formes modulaires de Hilbert modulo p et valeurs d'extensions entre caractères galoisiens*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 5, 905–974. MR 3294620

[BDJ10] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, *On Serre's conjecture for mod ℓ Galois representations over totally real fields*, Duke Math. J. **155** (2010), no. 1, 105–161. MR 2730374

[BE90] J.-E. Björk and E. K. Ekström, *Filtered Auslander-Gorenstein rings*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math., vol. 92, Birkhäuser Boston, Boston, MA, 1990, pp. 425–448. MR 1103599

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in

Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956

[BHH$^+$a] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, *Conjectures and results on modular representations of* $\mathrm{GL}_n(K)$ *for a p-adic field* K, Mem. Amer. Math. Soc., to appear.

[BHH$^+$b] ______, *Multivariable* $(\varphi, \mathcal{O}_K^\times)$*-modules and local-global compatibility*, `https://arxiv.org/pdf/2211.00438`, preprint (2022).

[BHH$^+$c] ______, *On the constituents of the mod p cohomology of Shimura curves*, in preparation.

[BHH$^+$23] ______, *Gelfand-Kirillov dimension and* $\mathrm{mod}\, p$ *cohomology for* GL_2, Invent. Math. **234** (2023), no. 1, 1–128. MR 4635831

[Bjö89] Jan-Erik Björk, *The Auslander condition on Noetherian rings*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., vol. 1404, Springer, Berlin, 1989, pp. 137–173. MR 1035224

[BP12] Christophe Breuil and Vytautas Paškūnas, *Towards a modulo p Langlands correspondence for* GL_2, Mem. Amer. Math. Soc. **216** (2012), no. 1016, vi+114. MR 2931521

[Bre14] Christophe Breuil, *Sur un problème de compatibilité local-global modulo p pour* GL_2, J. Reine Angew. Math. **692** (2014), 1–76. MR 3274546

[Bru66] Armand Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. Algebra **4** (1966), 442–470. MR 202790

[CDP14] Pierre Colmez, Gabriel Dospinescu, and Vytautas Paškūnas, *The p-adic local Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$, Camb. J. Math. **2** (2014), no. 1, 1–47. MR 3272011

[CEG$^+$16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Camb. J. Math. **4** (2016), no. 2, 197–287. MR 3529394

[CEG$^+$18] ______, *Patching and the p-adic Langlands program for* $\mathrm{GL}_2(\mathbb{Q}_p)$, Compos. Math. **154** (2018), no. 3, 503–548. MR 3732208

[EGS15] Matthew Emerton, Toby Gee, and David Savitt, *Lattices in the cohomology of Shimura curves*, Invent. Math. **200** (2015), no. 1, 1–96. MR 3323575

[Eme] Matthew Emerton, *Local-global compatibility in the p-adic Langlands program for* $\mathrm{GL}_{2/\mathbb{Q}}$, `http://www.math.uchicago.edu/~emerton/pdffiles/lg.pdf`, preprint (2011).

[Eme10] ______, *Ordinary parts of admissible representations of p-adic reductive groups II. Derived functors*, Astérisque (2010), no. 331, 403–459. MR 2667883

[GW04] K. R. Goodearl and R. B. Warfield, Jr., *An introduction to noncommutative Noetherian rings*, second ed., London Mathematical Society Student Texts, vol. 61, Cambridge University Press, Cambridge, 2004. MR 2080008

[Hu10] Yongquan Hu, *Sur quelques représentations supersingulières de* $\mathrm{GL}_2(\mathbb{Q}_{p^f})$, J. Algebra **324** (2010), no. 7, 1577–1615. MR 2673752

[Hu16] ———, *Valeurs spéciales de paramètres de diagrammes de Diamond*, Bull. Soc. Math. France **144** (2016), no. 1, 77–115. MR 3481262

[HW18] Yongquan Hu and Haoran Wang, *Multiplicity one for the mod p cohomology of Shimura curves: the tame case*, Math. Res. Lett. **25** (2018), no. 3, 843–873. MR 3847337

[HW22] ———, *On the mod p cohomology for* GL_2*: the non-semisimple case*, Camb. J. Math. **10** (2022), no. 2, 261–431. MR 4461834

[Koh17] Jan Kohlhaase, *Smooth duality in natural characteristic*, Adv. Math. **317** (2017), 1–49. MR 3682662

[Le19] Daniel Le, *Multiplicity one for wildly ramified representations*, Algebra Number Theory **13** (2019), no. 8, 1807–1827. MR 4017535

[Lev92] Thierry Levasseur, *Some properties of noncommutative regular graded rings*, Glasgow Math. J. **34** (1992), no. 3, 277–300. MR 1181768

[LMS22] Daniel Le, Stefano Morra, and Benjamin Schraen, *Multiplicity one at full congruence level*, 2022, pp. 637–658. MR 4386824

[LvO96] Huishi Li and Freddy van Oystaeyen, *Zariskian filtrations*, K-Monographs in Mathematics, vol. 2, Kluwer Academic Publishers, Dordrecht, 1996. MR 1420862

[Mor17] Stefano Morra, *Sur les atomes automorphes de longueur 2 de* $\mathbf{GL}_2(\mathbf{Q}_p)$, Doc. Math. **22** (2017), 777–823. MR 3650223

[Sho16] Jack Shotton, *Local deformation rings for* GL_2 *and a Breuil-Mézard conjecture when* $\ell \neq p$, Algebra Number Theory **10** (2016), no. 7, 1437–1475. MR 3554238

[Ven02] Otmar Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc. (JEMS) **4** (2002), no. 3, 271–311. MR 1924402

[Wan] Yitong Wang, *On the rank of the multivariable* $(\varphi, \mathcal{O}_K^\times)$*-modules associated to mod p representations of* $\mathrm{GL}_2(k)$, `https://arxiv.org/pdf/2404.00389.pdf`, preprint (2024).

[Wan23] Yitong Wang, *On the* $\bmod p$ *cohomology for* GL_2, J. Algebra **636** (2023), 20–41. MR 4637601

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Camb. Stud. Adv. Math., vol. 38, Cambridge: Cambridge University Press, 1994. MR 1269324