On the mod *p* Langlands programme for GL_2

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Langlands programme

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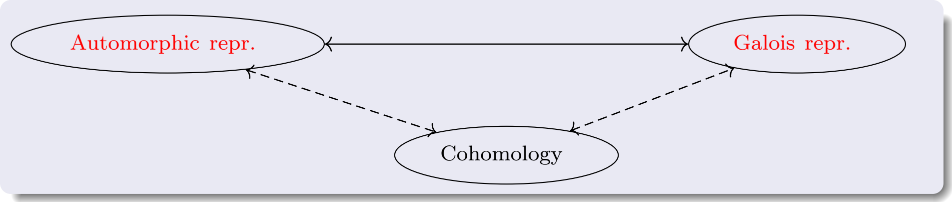

Automorphic repr.

Galois repr.

Cohomology

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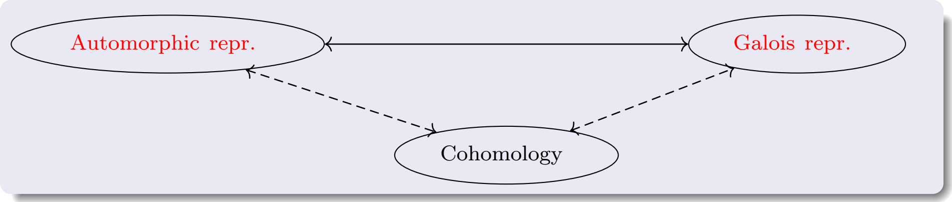

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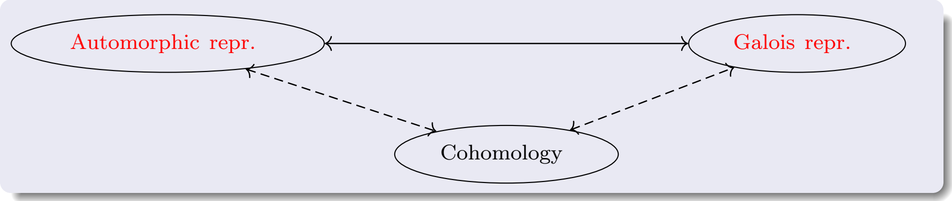

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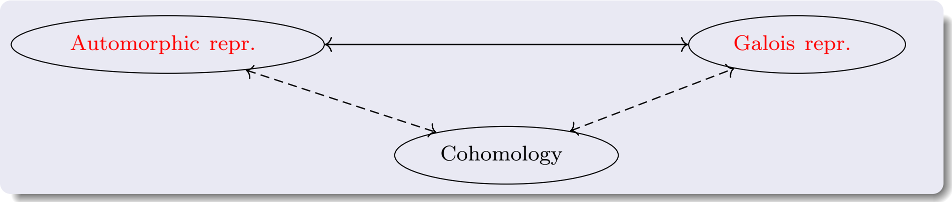

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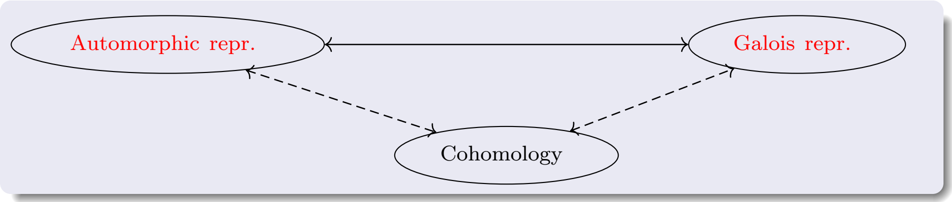

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Best way to define (some of) these varieties: use rings of *finite adèles*

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= finite disjoint union of quotients of Poincaré's upper half plane

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When $F = \mathbb{Q}$, modular curves = special case $D = \mathrm{M}_2$

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- $\mathrm{GL}_2(F_v)$ acts on $\varinjlim_{U_v} \mathrm{H}^i(Y_{U^vU_v}(\mathbb{C}),\mathbb{Z})$ but not $\mathrm{Gal}(\overline{F}/F)$!

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- $d_{U^v}(f) =$ finite multiplicity (depends on U^v and f)

Decomposition of cohomology

Theorem (Langlands, Deligne, Piatetski-Shapiro, Carayol, T. Saito

$$H^1(U^v) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \cong \bigoplus_f \left(\rho(f) \otimes_{\overline{\mathbb{Q}_\ell}} \pi_v(f)^{\oplus d_{U^v}(f)}\right) \bigoplus (*)$$

- f = new cuspidal Hilbert eigenforms of parallel weight $(2, \ldots, 2)$ and prime to v level U^v
- $\rho(f) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ irreducible representation of $\mathrm{Gal}(\overline{F}/F)$
- $\pi_v(f)$ = smooth irreducible (infinite dimensional) representation of $\mathrm{GL}_2(F_v)$ (smooth: $\pi_v(f) = \bigcup_n \pi_v(f)^{1+p^n\mathrm{M}_2(\mathcal{O}_{F_v})}$)
- $d_{U^v}(f)$ = finite multiplicity (depends on U^v and f)
- $(*)$ = non-cuspidal part = not interesting here

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For $\ell \neq p$ the isomorphism class of the representation $\pi_v(f)$ determines and only depends on the isomorphism class of the representation $\rho(f)|_{\mathrm{Gal}(\overline{F}_v/F_v)}$

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Remark: There is a version also when $\ell = p$

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Remark: Theorem should hold as soon as $F_v = \mathbb{Q}_p$ (not written so far)

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- Case $\rho|_{\mathrm{Gal}(\overline{F}_v/F_v)} \cong \chi_1 \oplus \chi_2$: the previous two principal series still occur in $\mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\rho, H^1(U^v) \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}_p})$ (assumed $\neq 0$) but there are also supersingular representations there as soon as $F_v \neq \mathbb{Q}_p$!

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Remark: p unramified in $F \Longrightarrow F_v/\mathbb{Q}_p$ = unramified extension

Recent results for $F_v \neq \mathbb{Q}_p$ III

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+ lots of long and clever *local* computations both on the $\mathrm{Gal}(\overline{F_v}/F_v)$-side (e.g. potentially crystalline deformation rings) and on the $\mathrm{GL}_2(F_v)$-side

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$$\overline{\mathbb{F}}_p[y_0, z_0, \dots, y_{f-1}, z_{f-1}]/(y_0 z_0, \dots, y_{f-1} z_{f-1})$$

where $\overline{\mathbb{F}}_p[y_i] = \mathrm{gr}(\overline{\mathbb{F}}_p[[\left(\begin{smallmatrix} 1 & \mathcal{O}_{F_v} \\ 0 & 1 \end{smallmatrix}\right)]])$ and $\overline{\mathbb{F}}_p[z_i] = \mathrm{gr}(\overline{\mathbb{F}}_p[[\left(\begin{smallmatrix} 1 & 0 \\ p\mathcal{O}_{F_v} & 1 \end{smallmatrix}\right)]])$

THANK YOU ☺