TD 8 : A Lusin Type Theorem.

We fix an open set $\Omega \subset \mathbb{R}^n$ of finite measure and a Borel function $f : \Omega \to \mathbb{R}^n$.

Exercise 1.— *The aim is to show that a Borel function from an open set of finite measure Ω to $\mathbb{R}^n$ coincides with the gradient of a function C^1 outside an open set of measure arbitrarily small.*

Theorem 1. *For every $\epsilon > 0$, there exists an open set $W \subset \mathbb{R}^n$ and a function $u \in \mathrm{C}^1_0(\Omega)$ such that*

$$\begin{cases} f(x) = \nabla u(x) \text{ for all } x \in \Omega \setminus W \\ |W| \le \epsilon |\Omega|, \\ \|\nabla u\|_p \le C\epsilon^{\frac{1}{p}-1}\|f\|_p,\ \forall p \in [1,\infty] \end{cases}$$

where $C > 0$ is a constant depending only on the dimension n.

1. Let $h : \Omega \to \mathbb{R}^n$ be continuous and let η, $\epsilon > 0$. We consider cartesian meshes of $\mathbb{R}^n$ constituted of cubes of side δ and centred on $(\delta\mathbb{Z})^n$. We denote by $\mathcal{K}_\delta$ the countable family of all those cubes for a given $\delta > 0$.

   (a) Prove that there exist $\delta > 0$ and a finite family of cubes $\{T_i\}_{i\in I} \subset \mathcal{K}_\delta$ included in Ω and a function h_δ that is constant inside each T_i satisfying

   $$|\Omega \setminus \cup_i T_i| \le \frac{\epsilon}{2}|\Omega| \quad \text{and} \quad \forall x \in \cup_i T_i,\ |h(x) - h_\delta(x)| \le \eta.$$

   (b) Prove that there exist a compact set $K \subset \Omega$ and $u \in \mathrm{C}^1_c(\Omega)$ such that $\operatorname{supp} u \subset K$ and

   $$\begin{cases} |f(x) - \nabla u(x)| \le \eta \text{ for all } x \in K, \\ |\Omega \setminus K| \le \epsilon|\Omega|, \\ \|\nabla u\|_p \le C'\epsilon^{\frac{1}{p}-1}\|f\|_p,\ \forall p \in [1,\infty] \end{cases}$$

   Hint: if $h \equiv a$ were constant on some subdomain then $u(x) =< a, x > +cte$ would suit on this part.

2. **Case 1:** Assume in addition that f is continuous and bounded. Let $\epsilon > 0$ and for all $k \in \mathbb{N}^*$, $\epsilon_k = 2^{-k}\epsilon > 0$ and $\eta_k > 0$ tending to 0 to be conveniently adjusted.

   (a) Define $f_0 = f$ and u_1, K_1 given by question 1 applied with $h = f_0$, η_1 and ϵ_1), then define $f_1 = f_0 - \nabla u_1$ (on K_1 ...) then iterate to obtain (u_k, K_k, f_k) satisfying for all $k \in \mathbb{N}^*$

   $$\begin{cases} |\Omega \setminus K_k| \le \epsilon_k|\Omega|, \\ |f_{k-1}(x) - \nabla u_k(x)| \le \|f_k\|_\infty \le \eta_k, \quad \forall x \in K_k, \\ \|\nabla u_k\|_p \le C'(\epsilon_k)^{\frac{1}{p}-1}\|f_{k-1}\|_p, \quad \forall p \in [1,+\infty]\,. \end{cases}$$

   (b) For all $p \in [1, +\infty]$, show that

   $$\sum_{k=1}^{\infty} \|\nabla u_k\|_p \le C''\epsilon^{\frac{1}{p}-1}\|f\|_p < +\infty\,.$$

   (You obtain a bound on η_k.)

(c) Infer that $u = \sum_{k=1}^{\infty}$ is well-defined and of class C^1. (You might refine the bound on η_k to ensure that $\sum_{k=1}^{\infty} \|u_k\|_\infty < \infty$.)

(d) Prove that u and $W = \Omega \setminus \cap_k K_k$ satisfy the conclusion of Theorem 1.

3. **Case 2:** Conclude in the case where f is Borel: apply Lusin theorem and truncate.

Exercise 2.— *The aim of this exercise is to prove a similar result in the case where f is L^1 and u is required to be* BV.

Theorem 2. *Let $f \in \mathrm{L}^1(\Omega, \mathbb{R}^n)$. There exist $u \in \mathrm{BV}(\mathbb{R}^n)$ and $g : \Omega \to \mathbb{R}^n$ Borel functions such that*

$$Du = f\mathcal{L}^n + g\mathcal{H}^{n-1} \quad \textit{and} \quad \int |g|\, d\mathcal{H}^{n-1} \leq C\|f\|_1,$$

where $C > 0$ is a constant depending on n only.

1. By a construction similar to the one of Exercise 1, prove that given $f \in \mathrm{L}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\eta > 0$, there exists $u \in \mathrm{BV}(\mathbb{R}^n)$ and $g^a, g^s : \mathbb{R}^n \to \mathbb{R}^n$ two Borel functions such that

$$Du = g^a\mathcal{L}^n + g^s\mathcal{H}^{n-1} \quad \text{and} \quad \begin{cases} \|u\|_1 \leq \|f\|_1 \\ \|f - g^a\|_1 \leq \eta \\ \int |g^s|\, d\mathcal{H}^{n-1} \leq C'\|f\|_1. \end{cases}$$

2. Infer the theorem.

This is a result of G. Alberti (*A Lusin Type Theorem,* 1991).