TD 2 : Interpolation, a theorem of M. Riesz

We consider measurable 2π–periodic functions with values in $\mathbb{C}$, which we identify with measurable functions on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.
Recall that for $1 \leq p < \infty$, such a function $f : \mathbb{R} \to \mathbb{C} \in \mathrm{L}^p$ $(i.e.\ \mathrm{L}^p(\mathbb{T}))$ if

$$\|f\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\, dt\right)^{\frac{1}{p}} < \infty.$$

We denote by $\mathcal{T}_n = \mathrm{vect}(e_k, |k| \leq n, k \in \mathbb{Z})$ with $e_k : \mathbb{R} \to \mathbb{C}$, $t \mapsto e^{ikt}$ and $\mathcal{T} = \mathrm{vect}(e_k, k \in \mathbb{Z})$. For $f \in \mathrm{L}^1$, we define

$$c_k(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\, dt \quad \text{and} \quad S_n(f) = \sum_{k=-n}^{n} c_k(f)e_k = f * \sum_{k=-n}^{n} e_k = f * D_n.$$

The aim of the present exercises is to prove the following theorem:

Thorem (M. Riesz, 1927)**.** *Let $1 < p < \infty$ and $f \in \mathrm{L}^p(\mathbb{T})$, then $S_n f$ converges to f in L^p.*

We recall that:

- we know that the theorem is true for $p = 2$ (Plancherel),
- we know that $S_n : \mathrm{L}^p \to \mathrm{L}^p$ is continuous since

$$\|S_n(f)\|_p = \|f * D_n\|_p \leq \|D_n\|_1 \|f\|_p.$$

- we know that $\mathcal{T}$ is dense in L^p $(1 \leq p < \infty)$, using the fact that Fejér's kernel is an approximate identity.[1]

Exercise 1.— *Approximation and interpolation*
Given $1 < p < +\infty$, we consider the following property :

$$\exists C_p > 0,\ \forall f \in \mathcal{T},\ \forall n \in \mathbb{N},\ \|S_n(f)\|_p \leq C_p \|f\|_p. \qquad (\mathcal{P}_p)$$

1. Show that $(\mathcal{P}_p)$ is equivalent to

$$\forall f \in \mathrm{L}^p,\ \|S_n(f) - f\|_p \xrightarrow[n \to +\infty]{} 0\,.$$

2. Show that if $(\mathcal{P}_p)$ holds for a given $1 < p < \infty$ then $(\mathcal{P}_q)$ also holds for q conjugated to p, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$.
Notice that $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \overline{f} S_n g = \frac{1}{2\pi}\int_0^{2\pi} g\overline{S_n(f)}\,.$

3. Assume that there exists an increasing sequence $(p_m)_{m\in\mathbb{N}}$ satisfying $p_m \xrightarrow[m\to\infty]{} \infty$, $1 < p_m < \infty$ and such that property $(\mathcal{P}_{p_m})$ holds for all $m \in \mathbb{N}$. The existence of such a sequence is then the topic of Exercise 2. Prove the theorem.

[1] Fejér's kernel is defined as $\sigma_n = \frac{1}{n}\sum_{k=0}^{n-1} D_k$ and for $f \in \mathrm{L}^p$, $\|f - \sigma_n f\|_p \xrightarrow[n\to+\infty]{} 0$.

Exercise 2.— *Hilbert Transform on $\mathbb{T}$.*
For $f \in \mathcal{T}$ we define its Hilbert Transform :

$$(Hf)(x) = -i \sum_{k \in \mathbb{Z}} \operatorname{sgn}(k) c_k(f) e^{ikx} \quad \text{with} \quad \operatorname{sgn}(k) = \begin{cases} 1 & \text{si} \quad k > 0 \\ -1 & \text{si} \quad k < 0 \\ 0 & \text{si} \quad k = 0 \end{cases}$$

and

$$(H_+ f)(x) = \sum_{k>0} c_k(f) e^{ikx} \quad \text{and} \quad (H_- f)(x) = \sum_{k<0} c_k(f) e^{ikx} .$$

1. Let $1 < p < \infty$. Show that if there exists $K_p > 0$ such that

$$\forall f \in \mathcal{T},\ \|Hf\|_p \leq K_p \|f\|_p,$$

   then $(\mathcal{P}_p)$ holds.
   We can start with connecting $S_n(f)$ to $H_\pm(e_{\pm n} f)$.

2. Show that it is possible to reduce the question to the case where $f \in \mathcal{T}$ is real-valued and has zero integral (on a period).

3. We now assume that $f \in \mathcal{T}$ is real-valued and has zero integral (on a period). Let $m \in \mathbb{N}^*$.

   (a) Show that $\displaystyle\int_0^{2\pi} (f + iHf)^{2m} = 0$ and Hf is real-valued.

   (b) Let $0 < \epsilon < 1$, show that for $k \in \mathbb{N}$ such that $1 \leq k \leq m$ and for all $a, b \geq 0$,

$$a^k b^{m-k} \leq \frac{a^m}{\epsilon^m} + \epsilon b^m,$$

   in the sense $a^m \leq \epsilon^{-m} a^m$ if $b = 0$ and $k = m$.

   (c) Show that there exists $C_m > 0$ such that $\forall f \in \mathcal{T},\ \|Hf\|_{2m} \leq C_m \|f\|_{2m}$.

4. Conclude.