TD 1 - EXERCISE 1 - Almost everywhere Vitali's theorem.

COURS : Lemme de Vitali

$(B_i)_{i\in I}$ une famille de boules fermées dans $\mathbb{R}^n$ (resp. toutes ouvertes)

On suppose : $\sup_{i\in I} r_i < +\infty$

ALORS, il existe $J\subset I$, J au plus dénombrable, tel que les boules B_j, $j\in J$ sont 2 à 2 disjoints et

$$\bigcup_{i\in I} B_i \subset \bigcup_{j\in J} 5B_j$$

ou n'importe quoi ≥ 3

$B = B(x,r)$
$\hookrightarrow 5B = B(x, 5r)$

Hyp : $\sup_{i\in I} r_i < +\infty$ sinon $\mathbb{R}^n \subset B(0,i)$ $\leadsto$ pb. 2 à 2 disjoints

Notation : $B(x,r)$ boule fermée

BUT : montrer une version "mesure" de ce lemme

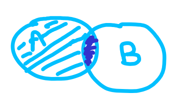
THM. $E\subset\mathbb{R}^n$ Borel set
$\mathcal{F}\subset\{\text{closed balls in } \mathbb{R}^n\}$ is a Vitali covering of E i.e :

$$\forall x\in E,\quad \inf\{\operatorname{diam} B : B\in\mathcal{F} \text{ and } x\in B\} = 0 \qquad (1)$$

THEN, there exists a countable family of 2 by 2 disjoint balls $\mathcal{G} = \{B_k\}_{k\in\mathbb{N}} \subset \mathcal{F}$ s.t.

$$E - \bigcup_{k\in\mathbb{N}} B_k \text{ is Lebesgue negligible}$$

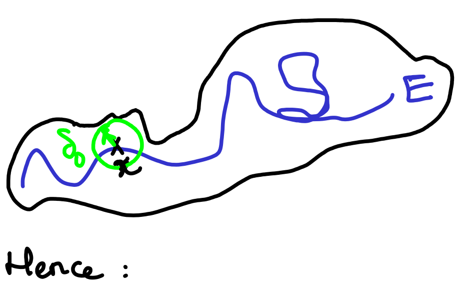
$$A - B := A - A\cap B = A\cap(\mathbb{R}^n - B)$$



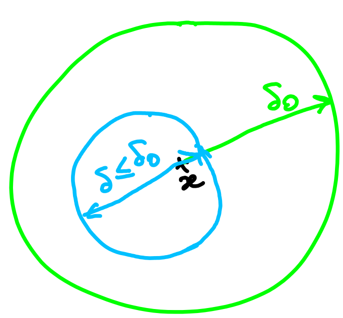
1 - Let $W\subset\mathbb{R}^n$ be an open set s.t. $E\subset W$. Check that

$$\forall x\in E,\quad \inf\{\operatorname{diam} B : B\in\mathcal{F} \text{ and } x\in B \text{ and } B\subset W\} = 0$$

Let $x\in E$. As $x\in W$ open : $\exists \delta_0 > 0$ s.t. $B(x,\delta_0)\subset W$



and any ball of diameter $< \delta_0$ containing x is included in $B(x,\delta_0)\subset W$



Hence :

$$\{B\in\mathcal{F} : x\in B \text{ and } \operatorname{diam} B < \delta_0\} \subset \{B\in\mathcal{F} : x\in B \text{ and } B\subset W\}$$

As a conclusion :

$$0 = \inf\{\operatorname{diam} B : B \in \mathcal{F} \text{ and } x \in B\}$$

$$= \inf\{\operatorname{diam} B : B \in \mathcal{F} \text{ and } x \in B \text{ and } \operatorname{diam} B \leq \delta_0\}$$

$$0 \geq \inf\{\operatorname{diam} B : B \in \mathcal{F} \text{ and } x \in B \text{ and } B \subset W\}$$

$A \subset B$

$\Rightarrow \inf_A \dots \geq \inf_B \dots$

2/ We assume $\mathcal{L}^n(E) < +\infty$. Let $1 - \frac{1}{2} \cdot \frac{1}{5^n} < \theta < 1$ be fixed.

a/ Show that there exists a countable (at most) family $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$, 2 by 2 disjoint and such that

$$\mathcal{L}^n\left(E - \bigsqcup_{i \in \mathbb{N}} B_i\right) \leq \left(1 - \frac{1}{2} \cdot \frac{1}{5^n}\right) \mathcal{L}^n(E) \text{ and } \forall i,\ B_i \subset W$$

We show that we gain a small factor in the measure by withdrawing a countable family of balls $\rightsquigarrow$ we want that $\mathcal{L}^n(\bigcup_{i \in \mathbb{N}} B_i) \geq \text{cte}\ \mathcal{L}^n(E)$.

By exterior regularity of Lebesgue measure: let U be an open set of $\mathbb{R}^n$ s.t.

$$E \subset U \text{ and } \mathcal{L}^n(U) \leq \mathcal{L}^n(E) + \underbrace{\text{some } \varepsilon > 0}_{\frac{1}{2 \cdot 5^n}\mathcal{L}^n(E)}$$

to be chosen conveniently.

$U \to U \cap W$ open

As $E \subset U \cap W$ and $\mathcal{L}^n(U \cap W) \leq \mathcal{L}^n(U)$ we assume $U \subset W$

Thanks to question 1., the family of closed balls

$$\tilde{\mathcal{F}} = \mathcal{F} \cap \{B : B \subset U \text{ and } \operatorname{diam} B \leq 1\}$$ is a Vitali cover of E.

We apply Vitali's covering lemma to $E \subset \bigcup_{B \in \tilde{\mathcal{F}}} B$: let $\mathcal{G} = \{B_i\}_{i \in \mathbb{N}}$ be a countable family of 2 by 2 disjoint balls of $\tilde{\mathcal{F}}$ s.t.

$$E \subset \bigcup_{B \in \tilde{\mathcal{F}}} B \subset \bigcup_{i \in \mathbb{N}} 5B_i \Rightarrow \mathcal{L}^n(E) \leq \mathcal{L}^n\left(\bigcup_{i \in \mathbb{N}} 5B_i\right)$$

$$\leq \sum_{i \in \mathbb{N}} \underbrace{\mathcal{L}^n(5B_i)}_{= 5^n \mathcal{L}^n(B_i)}$$

$$\leq 5^n \underbrace{\sum_{i \in \mathbb{N}} \mathcal{L}^n(B_i)}_{= \mathcal{L}^n(\bigsqcup_{i \in \mathbb{N}} B_i)}$$

disjoint countable

"the union of balls B_i covers a portion of E in measure"

whence $\frac{1}{5^n} \mathcal{L}^n(E) \leq \mathcal{L}^n\left(\bigsqcup_i B_i\right)$.

As $E \subset U$ then $E - \bigsqcup_i B_i \subset U - \bigcup_i B_i$

$$\Rightarrow \mathcal{L}^n\left(E - \bigsqcup_i B_i\right) \leq \mathcal{L}^n\left(U - \bigcup_i B_i\right)$$

$$= \mathcal{L}^n(U) - \mathcal{L}^n\left(\bigcup_i B_i\right)$$

$$\leq \mathcal{L}^n(U) - \frac{1}{5^n}\mathcal{L}^n(E) \leq \left(1 - \frac{1}{5^n}\right)\mathcal{L}^n(E) + \underbrace{\varepsilon}_{= \frac{1}{2}\frac{1}{5^n}\mathcal{L}^n(E)}$$

"point where we "need" U

$A, B, C \subset \mathbb{R}^n$

$A \subset B \Rightarrow A - C \subset B - C$

$A \cap (\mathbb{R}^n - C) \subset B \cap (\mathbb{R}^n - C)$

2-b/- Infer that there exists a finite family of balls $\{B_i\}_{i=1,\dots,N} \subset \mathcal{F}$ 2 by 2 disjoint and such that

$$\mathcal{L}^n\Big(E - \bigsqcup_{i=1}^{N} B_i\Big) \le \theta\, \mathcal{L}^n(E) \qquad \text{and} \qquad \forall i,\ B_i \subset W$$

This is an easy consequence of 2a/- and $\begin{cases} \Big(E - \bigsqcup_{i=1}^{k} B_i\Big)_{k\in\mathbb{N}^*} \text{ is } \searrow \\ \mathcal{L}^n(E) < +\infty \end{cases}$

$\Big(E-\bigsqcup_{i=1}^{k} B_i\Big) = \bigcap_{i=1}^{k} (E - B_i) \searrow$

decreases w.r.t. $k \in \mathbb{N}^*$

$$\Downarrow$$

$$\mathcal{L}^n\Big(E - \bigsqcup_{i=1}^{k} B_i\Big) \xrightarrow[k\to+\infty]{} \mathcal{L}^n\Big(E - \bigsqcup_{i=1}^{\infty} B_i\Big) \underset{2a/-}{\le} \underbrace{\Big(1 - \frac{1}{2\cdot 5^n}\Big)}_{<\theta} \mathcal{L}^n(E)$$

so that for N large enough,

$$\mathcal{L}^n\Big(E - \bigsqcup_{i=1}^{N} B_i\Big) \le \theta\, \mathcal{L}^n(E)$$

2c/- Construct a countable family of balls $\{D_i\}_{i\in\mathbb{N}^*} \subset \mathcal{F}$ 2 by 2 disjoint together with an increasing sequence $(N_k)_k$

s.t.

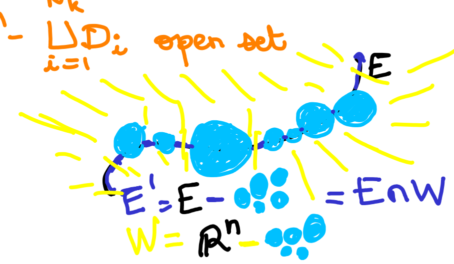
$$\mathcal{L}^n\Big(E - \bigsqcup_{i=1}^{N_k} D_i\Big) \le \theta^k\, \mathcal{L}^n(E) \qquad (2)$$

Construct the family by induction on $k \in \mathbb{N}^*$.

- $k = 1$: $N_1 = N$ and $\{D_i\}_{i=1}^{N_1}$ are given by question 2b/- ($W = \mathbb{R}^n$)
- Let $k \in \mathbb{N}^*$ be fixed and assumed that we have $N_k \in \mathbb{N}^*$ and a family $\{D_i\}_{i=1}^{N_k} \subset \mathcal{F}$ of 2 by 2 disjoint balls s.t. (2) is satisfied

We iterate 2a/- and 2b/- replacing $E \leftrightarrow E' = E - \bigsqcup_{i=1}^{N_k} D_i$: $\mathcal{L}^n(E') < +\infty$.

to ensure that the balls we add are disjoint from the N_k previous ones (!) taking $W = \mathbb{R}^n - \bigsqcup_{i=1}^{N_k} D_i$ open set



$\mathcal{F}$ is a Vitali cover of E'

By 2b/ $\Rightarrow$ there exists a finite family of 2 by 2 disjoint balls $\{B_i\}_{i=1}^{N}$, s.t.

$$\underbrace{\mathcal{L}^n\Big(E' - \bigsqcup_{i=1}^{N} D_i\Big)}_{= \mathcal{L}^n\big(E - \bigcup_{i=1}^{N_{k+1}} D_i\big)} \le \theta\, \underbrace{\mathcal{L}^n(E')}_{\underset{(HR)}{\le} \theta^k \mathcal{L}^n(E)} \le \theta^{k+1} \mathcal{L}^n(E)$$

$\begin{cases} N_{k+1} = N_k + N \\ D_{N_k+i} = B_i \quad (i = 1,\dots,N) \end{cases}$

$\{D_i\}_{i=1}^{N_{k+1}}$ are 2 by 2 disjoint $\Leftarrow$ $\begin{cases} \text{As } D_{N_k+1},\dots,D_{N_k+N} \subset W = \mathbb{R}^n - \bigsqcup_{i=1}^{N_k} D_i \text{, the "new" balls are disjoint from the "old" ones} \\ \oplus\ \{D_i\}_{i=1}^{N_k} \text{ were 2 by 2 disjoint by (HR)} \end{cases}$

2d/ Conclude the case $\mathcal{L}^n(E) < +\infty$.

Again $\left.\begin{array}{l} \bullet \left(E - \bigcup_{i=1}^{N_k} D_i\right)_{k\in\mathbb{N}^*} \text{ is } \searrow \\ \bullet\ \mathcal{L}^n(E) < +\infty \end{array}\right\} \Rightarrow \lim_{k\infty} \mathcal{L}^n\left(E - \bigcup_{i=1}^{N_k} D_i\right) = \mathcal{L}^n\left(E - \bigcup_{i=1}^{+\infty} D_i\right)$

while $\lim_{k\infty} \underbrace{\theta^k}_{\to 0} \underbrace{\mathcal{L}^n(E)}_{<+\infty} = 0$. therefore $\boxed{\mathcal{L}^n\left(E - \bigcup_{i\in\mathbb{N}^*} D_i\right) = 0}$.

and by construction the balls $D_i \in \mathcal{F}$ are 2 by 2 disjoint.

3/ Prove the theorem (without assuming $\mathcal{L}^n(E) < +\infty$).

We partition $\mathbb{R}^n$ into a union of open rings + spheres
each of Lebesgue meas. 0.

Let $j \in \mathbb{N}^*$ and $R_j = \{x \in \mathbb{R}^n : j-1 < |x| < j\}$ then $\mathcal{L}^n\left(\mathbb{R}^n - \bigsqcup_{j\in\mathbb{N}^*} R_j\right) = 0$

open set.

countable union of spheres.

Let $E_j = E \cap R_j$. $\mathcal{L}^n(E_j) \leq \mathcal{L}^n(B(0,j)) < +\infty$

and by question 2/ with $W = R_j$, there exists a countable family $\mathcal{G}_j \subset \mathcal{F}$ of 2 by 2 disjoint balls
s.t. and $\forall B \in \mathcal{G}_j,\ B \subset R_j$.

$\mathcal{L}^n\left(E_j - \bigcup_{B\in\mathcal{G}_j} B\right) = 0$

In particular, for all $B \in \mathcal{G}_{j_1}$, $D \in \mathcal{G}_{j_2}$ with $j_1 \neq j_2$

$B \cap D = \phi$: B and D are disjoint

$R_{j_1} \cap R_{j_2} = \phi$

Take $\mathcal{G} = \bigcup_{j\in\mathbb{N}^*} \mathcal{G}_j \subset \mathcal{F}$ countable, balls in $\mathcal{G}$ are 2 by 2 disjoint

and $\mathcal{L}^n\left(E - \bigcup_{B\in\mathcal{G}} B\right) \leq \sum_{j\in\mathbb{N}^*} \underbrace{\mathcal{L}^n\left(E_j - \bigcup_{B\in\mathcal{G}} B\right)}_{= \mathcal{L}^n\left(E_j - \bigcup_{B\in\mathcal{G}_j} B\right) = 0} + \underbrace{\sum_{j\in\mathbb{N}^*} \mathcal{L}^n(S_j - \ldots)}_{=0}$

$\leq \mathcal{L}^n(S_j) = 0$

$E = \bigsqcup_{j\in\mathbb{N}} S_j \sqcup \bigsqcup_{j\in\mathbb{N}} R_j$

$S_j = \{x \in \mathbb{R}^n : |x| = j\}$
sphere of radius j

$= 0$

4/ Check that we could replace $\mathcal{L}^n$ with a Radon measure μ assuming that μ is <u>doubling</u> i.e. $\exists C \geq 1$ s.t. $\forall x \in \mathbb{R}^n$ and $r > 0$,

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \quad (3)$$

Note that assuming (3) for closed balls or open balls is equivalent here.
Indeed,

- if (3) is known for closed balls $B(x,r)$, $x\in\mathbb{R}^n$, $r>0$.

Let $x\in\mathbb{R}^n$, $r_0>0$ and $r_k=(1-\frac{1}{k})r_0 \longrightarrow r_0$ for $k\to+\infty$.

note that $\boxed{\bigcup_{k\in\mathbb{N}^*} B(x,r_k) = U(x,r_0)}$ open ball of radius $r_0>0$, center x.

$$\bigcup_{k\in\mathbb{N}^*} B(x,2r_k) = U(x,2r_0)$$

$$\mu(B(x,2r_k))\leq C\mu(B(x,r_k))$$

$$\Big\downarrow{\scriptstyle k\to\infty}$$

$$\mu(B(x,2r_0))\leq C\mu(U(x,r_0))$$

We use inner closed balls to approximate the open ball and we can similarly approximate open balls with larger closed ones:

- $\boxed{B(x,r_0) = \bigcap_{k\in\mathbb{N}^*} U(x,r_k)}$ with $r_k = r_0+\frac{1}{k} \searrow r_0$ for $k\to+\infty$

which gives the converse implication

Let us review the "measure" ingredients in the proof: $\mu(B)<+\infty$ for any ball B.

- Homogeneity of $\mathcal{L}^n$: $\mathcal{L}^n(5B) = 5^n\mathcal{L}^n(B)$

actually we only used $\leq$

For μ: let $B=B(x,r)$ a closed ball, using the doubling property,

$$\mu(5B)=\mu(B(x,\underset{\frac{5}{2}\cdot 2}{\underset{\shortparallel}{5}}r)) \leq C\,\mu(B(x,\underset{\underbrace{\frac{5}{8}}_{<1}\cdot 2\cdot 2\cdot r}{\underset{\shortparallel}{\tfrac{5}{2}r}}))$$

$$\leq C^3\,\mu(B(x,\tfrac{5}{2^3}))$$

$$\mu(5B)\leq C^3\mu(B) \quad \leadsto \text{change } \theta \text{ for } 1-\frac{1}{2}\frac{1}{C^3}<\theta<1$$

- Exterior regularity: still true for Radon measure.
- Sphere have zero Lebesgue measure: NOT true with all spheres when it comes to μ
[e.g. $\mu=\delta_1$ in $\mathbb{R}$ and $S(0,1)=\{-1\}\cup\{1\}$]

However,

$$\boxed{\{r>0 : \mu(S(0,r))>0\} \text{ is at most countable.}}$$

Indeed, $\begin{array}{ccc}]0,+\infty[& \longrightarrow & \mathbb{R} \\ r & \longmapsto & \mu(B(0,r)) \end{array}$ is increasing and thus it has at most a countable number of discontinuity points $X\subset]0,+\infty[$.

For $r_0 \notin X$, $\mu(B(0,r)) \xrightarrow[r \to r_0]{} \mu(B(0,r_0))$

in particular, for $r < r_0$, $\mu(B(0,r)) \leq \mu(B(0,r) \cup S(0,r_0)) \leq \mu(B(0,r_0))$

$$\mu(B(0,r) \cup S(0,r_0)) = \mu(B(0,r)) + \mu(S(0,r_0))$$

and $r \to r_0$ gives $\mu(S(0,r_0)) = 0$ $\square$

Take a sequence $(r_j)_{j \in \mathbb{N}^*}$ in $]0,+\infty[\setminus X$ s.t. (r_j) is stricly $\nearrow$ and $\xrightarrow[j \to +\infty]{} +\infty$

and replace $R_j = \{x : j-1 < |x| < j\}$ $j \in \mathbb{N}^*$

with $R_j = \{x : r_{j-1} < |x| < r_j\}$ and with $R_1 = \{x : |x| < r_1\}$

TD1 - EXERCISE 2 - A weak version of Sard's lemma.

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ differentiable

$Z = \{x \in \mathbb{R}^n : Df(x) \text{ is not invertible } \}$

AIM : $f(Z)$ is Lebesgue negligible

1/- For $x \in Z$ fixed, show that there exists C_x st. for all $0 < \delta \leq 1$,

$\exists \rho_{x,\delta} > 0, \ \forall r < \rho_{x,\delta}$, $f(B(x,r))$ can be covered by less than $C_x \delta^{1-n}$ balls of radius δr.

We show that $f(B(x,r))$ is included in a box constrained by

- $f(B(x,r)) \subset$ ball of radius proportional to r ← just because f is differentiable at x

- $f(B(x,r)) \subset$ band of width δr

↳ combine and cover the box

- f is differentiable at x :

$$f(x+h) = f(x) + Df(x).h + |h| \varepsilon(h) \quad \text{où} \quad \varepsilon(h) \xrightarrow[h \to 0]{} 0$$

let $\delta > 0$, there exists $\rho_{x,\delta} > 0$ such that for all $h \in \mathbb{R}^n$, $|h| \leq \rho_{x,\delta}$

$$|\varepsilon(h)| \leq \delta$$

In particular, for all $|h| \leq r < \rho_{x,\delta}$

$$|f(x+h) - f(x)| \leq \|Df(x)\| \, |h| + \delta |h| \leq (\|Df(x)\| + 1)|h|$$

so that for all $y \in B(x,r)$ we have $f(y) \in B(f(x), C_x r)$ with $\boxed{C_x = \|Df(x)\| + 1}$

$$\Rightarrow \boxed{f(B(x,r)) \subset B(f(x), C_x r)}$$

- Differentiability tells more here: for such $y \in B(x,r)$:

$$|f(y) - [\underbrace{f(x) + Df(x).h}_{\in \text{ affine space } f(x) + Df(x)(\mathbb{R}^n)}]| \leq \delta |h|$$

T key point: $\dim Df(x)(\mathbb{R}^n) \leq n-1$ since $Df(x)$ NOT INVERTIBLE

this implies that $\text{dist}(f(y), T) \leq \delta \underbrace{|y-x|}_{\leq r}$ for T affine sub-space $\dim T \leq n-1$

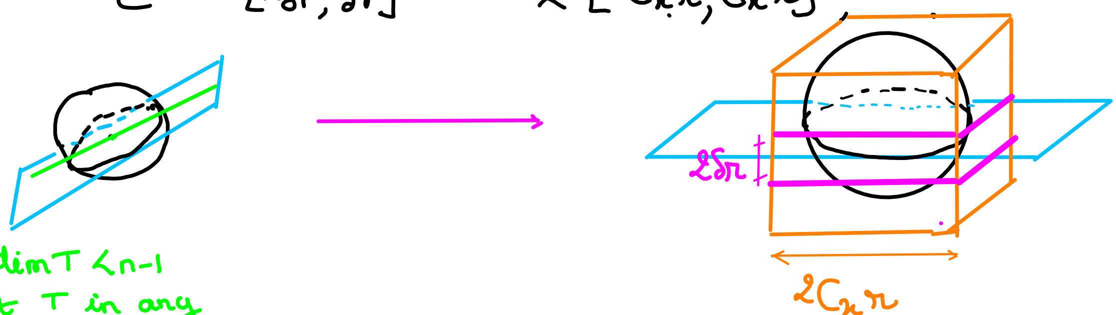
Therefore, $\boxed{f(B(x,r)) \text{ is contained in an } (\delta r)\text{- neighborhood of } T}$

- Combining both inclusions, we can put $f(B(x,r))$, up to translation and rotation

$$\text{in } B(f(x), C_x r) \cap [-\delta r, \delta r]^{n-\dim T} \times \mathbb{R}^{\dim T}$$

and including the ball into a "box" we get:

$$\subset [-\delta r, \delta r]^{n-\dim T} \times [-C_x r, C_x r]^{\dim T},$$



if $\dim T < n-1$
put T in any
sub-space of dim $n-1$

$$\subset [-\delta r, \delta r]^1 \times [-C_x r, C_x r]^{n-1}$$

2/ Let $\eta > 0$ and $\Omega \subset \mathbb{R}^n$ bounded open set. Let $x \in Z \cap \Omega$.

We have $\mathcal{L}^n(B(x,r)) = \omega_n r^n$; $\omega_n = \mathcal{L}^n(B(0,1))$

$$\mathcal{L}^n(f(B(x,r))) \le 2\delta r \times (2 C_x r)^{n-1} \quad \text{for } r < \rho_{x,\delta}$$

$$\le 2^n r^n C_x^{n-1} \delta$$

$$\le \underbrace{\frac{2^n}{\omega_n}(\|Df(x)\|+1)^{n-1}\delta}_{? \le \eta} \; \underbrace{\omega_n r^n}_{= \mathcal{L}^n(B(x,r))}$$

$0 < \delta \le 1$ was free up to this stage.

It remains to choose $\delta \le \dfrac{\omega_n}{2^n C_x^{n-1}} \eta$ — the non uniformity w.r.t. x is "absorbed" by the choice of $\delta = \delta_x$ while $\eta > 0$ is FREE.

We can take any $0 < r_x < \rho_{x,\delta}$ and such that $B(x, r_x) \subset \Omega$

(≤ 1 no bound in order to apply Vitali's covering lemma)

3/ Take $R>0$ and $\Omega = \{x \in \mathbb{R}^n : |x| < R\}$

Apply Vitali's covering to the family of balls

$$\mathcal{F} = \{ B(x, \tfrac{r_x}{5}) : x \in \Omega \cap Z \}$$ where $r_x > 0$ given by quest° 2 (depend on $\eta > 0$).

By quest° 2/ for any $x \in Z \cap \Omega$, $B(x, r_x) \subset \Omega \Rightarrow B(x, \frac{r_x}{5}) \subset \Omega$

and $\mathcal{L}^n(B(x, 5 \cdot \frac{r_x}{5})) \leq \eta \mathcal{L}^n(B(x, 5 \frac{r_x}{5}))$

And by definition of $\mathcal{F}$: $Z \cap \Omega \subset \bigcup_{B \in \mathcal{F}} B$

By Vitali's covering lemma, $\exists \mathcal{G} = \{B_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$
$\{B_i\}_i$ 2 by 2 disjoint s.t.

$$Z \cap \Omega \subset \bigcup_{B \in \mathcal{F}} B \subset \bigcup_{i \in \mathbb{N}} 5B_i$$

$$\Rightarrow f(Z \cap \Omega) \subset f(\bigcup_{i \in \mathbb{N}} 5B_i) = \bigcup_{i \in \mathbb{N}} f(5B_i)$$

Note: measurability of $f(Z \cap \Omega)$ is not clear while $\underbrace{f(5B_i)}_{\text{compact: Borel}}$ OK!

and $\bigcup_{i \in \mathbb{N}}$ countable union of Borel set

$$\mathcal{L}^n(\bigcup_{i \in \mathbb{N}} f(5B_i)) \leq \sum_{i \in \mathbb{N}} \mathcal{L}^n(f(5B_i))$$

$$\leq \sum_{i \in \mathbb{N}} \eta \mathcal{L}^n(5B_i)$$

$$\leq \eta 5^n \sum_{i \in \mathbb{N}} \mathcal{L}^n(B_i) \overset{\text{disjoint}}{=} \mathcal{L}^n(\bigsqcup_{i \in \mathbb{N}} B_i) \times 5^n \eta$$

$$\Rightarrow \underbrace{\mathcal{L}^n(\bigcup_{i \in \mathbb{N}} f(5B_i))}_{= 0} \leq \eta 5^n \underbrace{\mathcal{L}^n(\Omega)}_{< +\infty} \xrightarrow[\eta \to 0]{} 0$$ and $\Omega = \{|x| < R\}$

Finally, $f(Z \cap \{|x| < N\})$ is negligible for all $N \in \mathbb{N}^*$

$\Rightarrow f(Z) \subset \bigcup_{N \in \mathbb{N}^*} f(Z \cap \{|x|<N\})$ is negligible.

41 - The proof does not require covering lemma if we asume f of class C^1

instead: we can use partition by cubes

- NOT POSSIBLE before because the size $r_x > 0$ of balls we can use cannot be chosen uniformly: strongly depends on x
  ↳ in a grid of cubes: all cubes have same size Problem

- for f of class C^1, we show that $r_x > 0$ can be chosen uniformly w.r.t. x in a (big) compact set $K = B(0, R+1)$ e.g.

KEY POINTS:
- Df continuous $\to$ $C_x = \|Df(x)\| + 1 \leq C =: \sup_{x \in K} \|Df(x)\| + 1$
- Df UNIFORMLY CONTINUOUS $\rightsquigarrow$ $\rho_{x,\delta}$ can be chosen uniform / x !

Let $K = B(0, R+1)$, K compact $\Rightarrow$
- Df is bounded in K: $\forall x \in K$, $C_x = \|Df(x)\| + 1 \leq C$
- Df is uniformly continuous in K.

So that (i) $f(B(x,r)) \subset B(f(x), Cr)$

(ii) $\exists \rho_\delta > 0$ INDEP. OF x ! s.t. $\forall r \leq \rho_\delta$:

$f(B(x,r)) \subset (\delta r)$-neighborhood of a $(n-1)$-affine subspace.

Indeed: by uniform continuity of Df, let $\rho_\delta > 0$ s.t.

- $\forall y, z \in K$, $|y - z| \leq \rho_\delta \Rightarrow \|Df(y) - Df(z)\| < \delta$

Let $y \in B(x,r)$ for $r \leq \rho_\delta$ and denote $h = y - x \Rightarrow y = x + h$

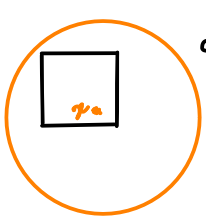
$$f(x+h) - f(x) = \int_0^1 Df(x+th).h \, dt \quad \text{since } f \text{ is } C^1 \text{ now.}$$

$$\Rightarrow |f(x+h) - f(x) - Df(x).h| \leq \int_0^1 |Df(x+th).h - Df(x).h| \, dt$$

$$\underset{\substack{|x+th-x| = t|h| \\ \leq |h| \\ \leq \rho_\delta}}{\leq} \delta |h| \leq \delta r$$

As C and ρ_δ are now uniform w.r.t $x \in K$:

- take cubes that can be included in balls of radius $\rho_\delta > 0$ and centered at any point of the cube
  ↳ cube of side-length L has diagonal $L\sqrt{n}$ and can be included in a ball of radius $L\sqrt{n}$



$$L\sqrt{n} < \rho_\delta \iff L < \frac{\rho_\delta}{\sqrt{n}}$$

• Cover $Z \cap B(0,R)$ with such cubes (of side-length $L = \frac{r}{\sqrt{n}}$ on a grid, disjoint up to faces) and cover each cube with a ball

$$B_i = B(x_i, r) : \begin{cases} x_i \in Z \cap B(0,R) \\ r = \rho_\delta \end{cases}, \quad i \in I_\delta$$

• volume of the image of such a cube $\lesssim r^n . \delta$

• $f(Z \cap B(0,R)) \subset \bigcup_i f(B_i)$ and $f(B_i) \subset$ set of volume

$$2\delta r \times (2Cr)^{n-1} = 2^n r^n C^{n-1} \delta = \frac{2^n}{\omega_n} C^{n-1} \mathcal{L}^n(B_i)\,\delta$$

• number of cubes in the grid $\sim \frac{1}{r^n}$

• We use at most N_δ cubes $\quad N L^n \leqslant R^n$

$L = \frac{r}{\sqrt{n}}$

$$\text{ie } N \leqslant \frac{R^n}{L^n} = (\sqrt{n})^n \frac{R^n}{r^n} = \frac{(\sqrt{n}R)^n \cdot \omega_n}{\mathcal{L}^n(B(0,r))}$$

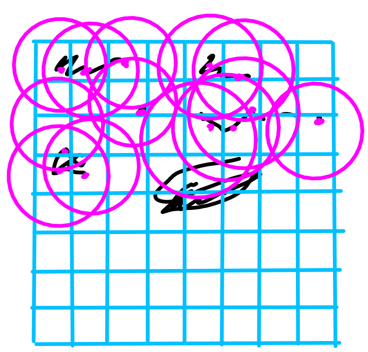
• summing up leaves δ

• Combine to get $\mathcal{L}^n\left(\bigcup_i f(B_i)\right) \leqslant N \cdot \frac{2^n C^{n-1}}{\omega_n} \mathcal{L}^n(B(0,r))\,\delta$

$$\leqslant (\sqrt{n}R)^n \cdot 2^n C^{n-1} \delta$$

Key point: cubes are disjoint allow to estimate the number

$$\Rightarrow f(Z \cap B(0,R)) \leqslant cte_n R^n \delta \xrightarrow[\delta \to 0]{} 0 \quad !!$$



... etc. number of balls $\lesssim$ number of cubes.