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Families Index for Manifolds with Boundary, Superconnections, and Cones. I. Families of Manifolds with Boundary and Dirac Operators

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This is Part I of a work, in which we establish a formula for the Chern character of a family of Dirac operators of Atiyah–Patodi–Singer on even-dimensional manifolds with boundary. The key tools are the superconnections of Quillen, the cone method, and the Levi–Civita superconnection. In this Part I, we construct a family of Dirac operators on manifolds with boundary and we introduce the corresponding Levi–Civita superconnections.

CONTENTS

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0. Introduction

The purpose of this work is to establish a formula for the Chern character associated with a family of Dirac operators on manifolds with boundary and the corresponding global boundary conditions of Atiyah, Patodi, and Singer [APS1].

We first recall the results of [APS1]. Let Z be an even-dimensional compact oriented spin manifold with boundary ∂Z endowed with a metric g^Z which is a product near the boundary. Let D be the Dirac operator on Z, and let $D^{\partial Z}$ be the Dirac operator on ∂Z. Atiyah, Patodi, and Singer [APS1] introduced global pseudo-differential conditions on the boundary associated with the spectral decomposition of $D^{\partial Z}$ and so defined a Fredholm differential operator D_+. They gave a formula for the index $\operatorname{Ind} D_+$ of the form

$$\operatorname{Ind} D_+ = \int_Z \omega - \bar{\eta}(0). \tag{0.1}$$

In (0.1), ω is the local Atiyah–Singer characteristic polynomial [AS1, AS2, ABP], and $\bar{\eta}(0)$ is a spectral invariant of $D^{\partial Z}$, called the reduced êta invariant of $D^{\partial Z}$. Formula (0.1) was used in [APS1] to calculate the signature of the manifold Z.

In [C1, C2], Cheeger gave a different approach to the calculation of the signature of a manifold with boundary. In [C1, C2] a cone $C(\partial Z)$ is attached to the boundary ∂Z, so that $Z' = Z \cup_{\partial Z} C(\partial Z)$ is a manifold with conical singularity. If r is the radial coordinate on $C(\partial Z)$, if $g^{\partial Z}$ is the metric on ∂Z, the cone $C(\partial Z)$ is equipped with the metric

$$dr^2 + r^2 g^{\partial Z}. \tag{0.2}$$

A Dirac operator is then defined on the manifold Z' which is proved to be Fredholm. Its L_2 is equal to the index of Atiyah, Patodi, and Singer [APS1]. *More precisely, the kernel and cokernel of this new Dirac operator are shown to be canonically isomorphic to the kernel and cokernel of the Dirac operator of* [APS1], *while for nonzero eigenvalues, the eigenspaces are unrelated.* By using the classical heat equation method in index theory [ABP], together with the functional calculus on cones [C1, C2]—which involves the Bessel functions—Cheeger provides an alternative proof of the result of Atiyah, Patodi, and Singer [APS1] for the signature of a manifold with boundary. In particular, the contribution of the conical singularity to the index was shown to be equal to the reduced êta invariant of [APS1] by using the functional calculus. The results of [C1, C2] were extended by Chou [Ch] to general Dirac operators.

In [B1], Bismut gave a local heat equation version of the index theorem of Atiyah and Singer for families [AS2]. If $M \to_Z B$ is a submersion with fiber Z, a canonical closed differential form representing the Chern character of the considered family of Dirac operators was exhibited in [B1]. The proof of [B1] uses the superconnection formalism of Quillen [Q] in an infinite-dimensional situation. A key tool in the proof of the local families index is the Levi–Civita superconnection of the fibration $M \to_Z B$. The Levi–Civita superconnection should be thought of as being the Levi–Civita connection on TM for a singular metric which is infinite in the horizontal directions of M.

A remarkable feature of Quillen's superconnections [Q] as used in [B1] is that, for a given fibration $M \to_Z B$, the deformation process involved in the proof is largely independent of the spectral theory of the Dirac operators in the fibers Z. In particular, the natural local geometric object for the local families index theorem which is the Levi–Civita superconnection, ignores the spectral decomposition of the Dirac operators in the fibers.

One is led naturally to try extending the methods of [B1] to calculate the Chern character of a family of Dirac operators D_+ acting on the fibers Z of a Riemannian submersion $M \to_Z B$, where the fibers Z are now even-dimensional manifolds with boundary ∂Z, with the global boundary conditions of Atiyah, Patodi, and Singer [APS1].

Here it is necessary to take into account the spectral flow of the family of Dirac operators $D^{\partial Z}$ on the boundaries $D^{\partial Z}$ [APS3], which introduces jumps in $\bar{\eta}(0)$, and so jumps in formula (0.1) for $\operatorname{Ind} D_+$. In order for the index bundle to be defined, it is essential to assume that $\operatorname{Ker} D^{\partial Z} = \{0\}$ or, more generally, that $\operatorname{Ker} D^{\partial Z}$ is a vector bundle on B. In particular, the family of operators $D^{\partial Z}$ is trivial in $K^1(B)$.

One can then try to adapt the superconnection formalism to obtain a formula for the Chern character of the index bundle associated with the family of Dirac operators of [APS1]. This seems to be very difficult. In fact the boundary conditions of Atiyah, Patodi, and Singer [APS1] are of global nature and directly involve the spectrum of $D^{\partial Z}$. As we pointed out before, one of the strengths of the superconnection formalism is that one can disregard the way the spectrum of the considered family of operators varies from fiber to fiber.

Our idea is then to replace the family of manifolds with boundary Z by the family of manifolds with conical singularity $Z' = Z \cup_{\partial Z} C(\partial Z)$. In particular, for $\varepsilon > 0$, we endow $C(\partial Z)$ with the metric

$$\frac{dr^2}{\varepsilon} + r^2 g^{\partial Z} \tag{0.3}$$

and we construct the family of Dirac operators D^{ε}_+ of [C1, C2].

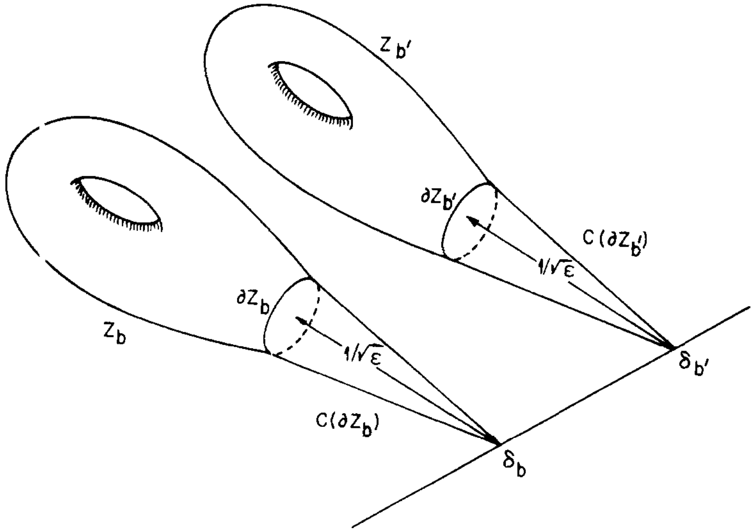

FIGURE 1

We then prove in Theorem 3.2 that, for $\varepsilon > 0$ small enough, the index bundle $\text{Ind}\, D^{\varepsilon}_{+}$ coincides with the Atiyah–Patodi–Singer index bundle. Also, because the Dirac operator D^{ε}_{+} on $Z \cup_{\partial Z} C(\partial Z)$ incorporates the spectral boundary connections of Atiyah, Patodi, and Singer [APS1] in a very implicit way, we prove in Sections 3–6 that the superconnection formalism of Quillen [Q] and also the Levi-Civita superconnection of [B1] can be used to study the family of Dirac operators D^{ε}_{+}. The price we pay is that instead of using the functional calculus on cones of [C1, C2], we must extend the classical theory of elliptic operators to manifolds with conical singularities in order to deal with deformations of such elliptic operators.

Thus, in Theorem 6.2, we obtain a formula for the Chern character of the Atiyah, Patodi, and Singer index bundle $\text{Ind}\, D_{+}$. Apparently, this formula still depends on the parameter $\varepsilon > 0$ defining the metric on the cones $C(\partial Z)$ and the boundary term is not yet identified explicitly.

Now the idea is to make $\varepsilon > 0$ tend to 0 in the formula for the Chern character. Equivalently we take the "adiabatic" limit of the Chern character formula, as the length of the cones $C(\partial Z)$ tends to infinity. The idea of taking the adiabatic limit of global quantities has already appeared in other contexts In [B1], it was used to give a cohomological proof of the families index theorem. Witten [Wi] suggested that the holonomy of determinant bundles was related to adiabatic limits of êta invariants. This was proved rigorously in [BF2] and in [C3]. Also adiabatic limits of êta invariants have been studied in a broader context in Bismut and Cheeger [BC1]. In

[MM] the adiabatic limit is related to the Leray spectral sequence of a fibration (see also [D]).

In our situation, we blow up the metric of the cones $C(\partial Z)$ in the one-dimensional radial direction, so that we can use the same techniques as in [BF2, C3]. We then obtain a formula for $\mathrm{ch}(\mathrm{Ind}\, D_+)$

$$\mathrm{ch}(\mathrm{Ind}\, D_+) = \int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \mathrm{Tr}\left[\exp\left(-\frac{L^\xi}{2i\pi}\right)\right] - \tilde{\eta}. \tag{0.4}$$

In (0.4), $\hat{A}(R^Z/2\pi)\,\mathrm{Tr}[\exp(-L^\xi/2i\pi)]$ is the differential form which was obtained in [B1] in the local families index theorem for manifolds without boundary. $\tilde{\eta}$ is a differential form associated with the family of Dirac operators $D^{\partial Z}$ on the fibers ∂Z. It satisfies

$$d\tilde{\eta} = \int_{\partial Z} \hat{A}\left(\frac{R^{\partial Z}}{2\pi}\right) \mathrm{Tr}\left[\exp - \left(\frac{L^\xi}{2i\pi}\right)\right]. \tag{0.5}$$

Formula (0.5) reflects the fact that the family $D^{\partial Z}$ is trivial in $K^1(B)$, and that the corresponding odd Chern forms of [BF2, Section 2] are exact.

The form $\tilde{\eta}$ already appeared in our previous work [BC1], where we calculated the adiabatic limit of the reduced êta invariant of Dirac operators on a manifold M' fibering over B', where the metric of M' is blown up in the horizontal directions.

The techniques of this paper combine those of [B1, BF2, C1, C2, C3]. In particular we develop a version of elliptic theory on cones which is more general than the functional calculus of [C1, C2]. This is done by a probabilistic technique which gives us a generalized form of Kato's comparison theorem for self-adjoint semi-groups. We also establish certain estimates on Bessel functions by probabilistic methods (see Proposition 1.15). As far as we can tell, these do not appear in the literature.

Our work is divided into two parts. This paper contains Part I and consists of three sections.

To make the reading of the paper easier, we devote Section 1 to establish the index theorem of Atiyah, Patodi, and Singer for one single manifold with boundary, by using the cone method of [C1, C2]. The essential new ingredient of Section 1 with respect to [C1, C2] is that we identify the êta invariant by taking an adiabatic limit rather than by using functional calculus on cones. Also, in Proposition 1.15, we establish an estimate on Bessel functions which plays a key role in Sections 1 and 6.

In Section 2, we construct the Levi–Civita superconnection on a family of manifolds with isolated conical singularities. In Section 3, we prove the existence of the index bundle associated with the family of

Atiyah–Patodi–Singer Dirac operators D_+ and also the existence of the index bundle associated with the family of Dirac operators D^{ε}_{+} of [C1, C2] constructed on our manifolds with conical singularities. Then we prove that these index bundles coincide in $K^0(B)$.

The next three Sections will appear as Part II in [BC2].

A companion paper to this work is our article [BC1] on adiabatic limits of êta invariants. In [BC3], we give the cohomological interpretation of the results contained in this paper. In particular, we prove that the index of Dirac operators on manifolds with boundary is asymptotically multiplicative We also extend our results to odd dimensional manifolds with boundary. We use notations of [Q, B1]. In particular, Tr_s will be our notation for supertraces. Also if K is a Z_2 graded algebra, we note $[A, B]$ the supercommutator,

$$[A, B] = AB - (1)^{\deg A \deg B} BA. \tag{0.6}$$

The results obtained in this paper were announced in [BC4].

I. Index Theorem for Manifolds with Boundary, Cones, and Adiabatic Limits

In this section, we establish the index theorem of Atiyah, Patodi, and Singer [APS1] for manifolds with boundary using the cone technique. This section is intended to be a simple introduction to the more complicated techniques and results of Sections 4 and 6. As opposed to what is done in [C1, C2, Ch], however, we identify the contribution from the singularity by passing to the adiabatic limit in the radial direction (compare also [BC1, Section 3]). Remarkably enough, the methods of Section 1 will be used with little modification for *families* of Dirac operators on manifolds with boundary, where the methods of [APS1] or [C1, C2, Ch] do not obviously apply.

This section is organized in the following way. In (a) we consider a manifold Z with boundary ∂Z, the cone $C(\partial Z)$, and construct the manifold $Z' = Z \cup_{\partial Z} C(\partial Z)$. In (b), we calculate the Levi–Civita connection on Z' for a metric $g^{Z',\varepsilon}$, which is given on $C(\partial Z)$ by

$$\frac{dr^2}{\varepsilon} + r^2 g^{\partial Z}.$$

In (c), we construct the Dirac operator of [C1, C2, Ch] on Z'. In (d), we briefly describe the main results on this Dirac operator which were established in [C1, C2, Ch]. In (e), we calculate the asymptotics of the heat equation formula for the index of the Dirac operator on Z'. In (f), we compute a similar asymptotics on an infinite cone. In (g), we obtain the formula of [C1, C2, Ch] for the index of the Dirac operator on Z' in terms of a local characteristic polynomial on Z and of the contribution of the conical singularity J^ε.

In (h), by passing to the adiabatic limit, i.e., by making $\varepsilon \to 0$, we prove that J^ε coincides with the reduced êta invariant of [APS1]. For this we need to establish certain key estimates which we will also use in Section 6 for families. We then obtain the index theorem of Atiyah, Patodi, and Singer [APS1] in the form given in [C1, C2, Ch].

(a) *A Manifold with Boundary Z and Its Associated Space Z′ with Conical Singularity*

Let Z denote a smooth connected compact manifold with smooth compact boundary ∂Z. We assume that Z has even dimension $n = 2l$, is oriented and spin. Let $C(\partial Z)$ be the cone constructed over ∂Z, i.e. $C(\partial Z)$ is the compact set

$$C(\partial Z) = (]0, 1] \times \partial Z) \cup \{\delta\}$$

If r denotes the radial coordinate in $]0, 1]$, then for any $y \in \partial Z$, as $r \downarrow 0$, $(r, y) \to \delta$.

Let $\mathscr{U}$ be a tubular neighborhood of ∂Z in Z, which we identify with $[1, 2[\times \partial Z$ so that ∂Z is identified with $\{1\} \times \partial Z$. If r still denotes the coordinate which varies in $[1, 2[$, we piece together Z and $C(\partial Z)$ along their common boundary ∂Z so that the coordinate r patches smoothly. We thus obtain a new manifold

$$Z' = Z \cup_{\partial Z} C(\partial Z).$$

Z' is a smooth manifold with a conical singularity at δ.

Set $f_1 = -\partial/\partial r$. If $(e_1, ..., e_{n-1})$ is an oriented base of $T\partial Z$, $(f_1, e_1, ..., e_{n-1})$ is an oriented base of TZ.

Let $g^{\partial Z}$ be a smooth Riemannian metric on ∂Z. Let ε be a positive parameter, which we fix for the moment. We equip the cone $C(\partial Z)$ with the metric considered in [C1, C2, Ch],

$$\frac{dr^2}{\varepsilon} + r^2 g^{\partial Z}. \tag{1.1}$$

Note here a slight difference of terminology with [C1, C2, C3], where given a metric $g^{\partial Z}$ on ∂Z, the cone $C(\partial Z)$ is defined to be $]0, +\infty[\times \partial Z$ equipped with the metric $dr^2 + r^2 g^{\partial Z}$, and $g^{\partial Z}$ is considered as the metric on the cross section $r=1$. In the formalism of [C1, C2, C3], if $r=\sqrt{\varepsilon}\, r'$, the metric (1.1) sould be rewritten in the form $dr'^2+r'^2\varepsilon g^{\partial Z}$.

Let $g^{Z,\varepsilon}$ be a Riemannian metric on Z which is such that on the tubular neighborhood $\mathscr{U}$, $g^{Z,\varepsilon}$ is also given by (1.1). Clearly the metric $g^{Z,\varepsilon}$ patches smoothly with the metric (1.1) on the cone $C(\partial Z)$. Therefore Z' is equipped with a smooth metric which we note $g^{Z',\varepsilon}$.

Let $F=F_+\oplus F_-$ be the Z_2 graded Hermitian vector bundle of spinors on Z associated with the metric $g^{Z,\varepsilon}$. F_+ and F_- are the bundles of positive and negative spinors on Z. TZ acts on $F=F_+\oplus F_-$ by Clifford multiplication. If $X\in TZ$, we also denote the corresponding Clifford multiplication operator by X.

In particular on ∂Z, as a Clifford multiplication operator, f_1 interchanges F_+ and F_-. Therefore we can identify $h\in F_+$ and $f_1 h\in F_-$.

Also ∂Z is an odd-dimensional manifold, which is oriented and spin. In order to be consistent with classical orientation conventions, we identify the Hermitian bundle of spinors on ∂Z with F_+.

On ∂Z for the metric $g^{Z,1}$, f_1 acts on $F=F_+\oplus F_-$ as the matrix

$$f_1=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

If $y\in\partial Z$ and $r\in[1,2[$, we will identify $F_{(r,y)}=F_{+,(r,y)}\oplus F_{-,(r,y)}$ with $F_y=F_{+,y}\oplus F_{-,y}$. In particular, for the metric $g^{Z,\varepsilon}$, the Clifford multiplication operator associated with $-\partial/\partial r$ is now $f_1/\sqrt{\varepsilon}$. Similarly, if $e_1,\ldots,e_{n-1}$ is an orthonormal base of $T_y(\partial Z)$ with respect to $g^{\partial Z}$, $e_1,\ldots,e_{n-1}$ act by Clifford multiplication on F_y. Identifying $F_{(r,y)}$ with F_y, for the metric $g^{Z,\varepsilon}$, the vectors $e_1,\ldots,e_{n-1}$ act by Clifford multiplication at (r,y) like $re_1,\ldots,re_{n-1}$.

We extend the vector bundle $F=F_+\oplus F_-$ on ∂Z to the cone $C(\partial Z)$ in a trivial way. The whole manifold Z' is now equipped with the smooth Hermitian vector bundle $F=F_+\oplus F_-$. By extending the Clifford multiplication operators from the tubular neighborhood $\mathscr{U}$ into $C(\partial Z)$ in the obvious way, $F=F_+\oplus F_-$ can now be considered as the Hermitian vector bundle of spinors on Z' associated with the metric $g^{Z',\varepsilon}$.

(b) *The Levi–Civita Connection on TZ'*

Let $\nabla^{Z,\varepsilon}$ (resp. $\nabla^{Z',\varepsilon}$) by the Levi–Civita connection on TZ (resp. TZ') associated with the metric $g^{Z,\varepsilon}$ (resp. $g^{Z',\varepsilon}$) and let $R^{Z,\varepsilon}$ (resp. $R^{Z',\varepsilon}$) be the corresponding curvature tensor.

We will briefly describe the connection $\nabla^{Z',\varepsilon}$ on the cone $C(\partial Z)$. Let $\nabla^{\partial Z}$ be the Levi–Civita connection on $T\partial Z$ with respect to the metric $g^{\partial Z}$.

DEFINITION 1.1. ∇ denotes the Euclidean connection on the restriction of TZ' to $C(\partial Z)\cup_{\partial Z}\mathscr{U}$ defined by the following relations: if X, Y are smooth vector fields on ∂Z, then

$$\begin{aligned}\nabla_X Y&=\nabla^{\partial Z}_X Y; &\qquad \nabla_X \frac{\partial}{\partial r}&=0\\ \nabla_{\partial/\partial r} X&=\frac{X}{r}; &\qquad \nabla_{\partial/\partial r}\frac{\partial}{\partial r}&=0.\end{aligned} \tag{1.2}$$

Let T be the torsion tensor of ∇. Let $\tilde{S}^{\varepsilon}$ be the tensor on $C(\partial Z)\cup_{\partial Z}\mathscr{U}$:

$$\tilde{S}^{(\varepsilon}=\nabla^{Z',\varepsilon}-\nabla. \tag{1.3}$$

PROPOSITION 1.2. *If X, $Y\in T(\partial Z)$, then*

$$\begin{aligned}&T(X,Y)=0\\ &T\left(\frac{\partial}{\partial r},X\right)=\frac{X}{r}\\ &\tilde{S}^{\varepsilon}\left(\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}=0; \qquad \tilde{S}^{\varepsilon}\left(\frac{\partial}{\partial r}\right)X=0\\ &\tilde{S}^{\varepsilon}(X)\frac{\partial}{\partial r}=\frac{X}{r}; \qquad \tilde{S}^{\varepsilon}(X)Y=-\varepsilon r\langle X,Y\rangle_{g^{\partial Z}}\frac{\partial}{\partial r}\\ &R^{Z',\varepsilon}\left(X,\frac{\partial}{\partial r}\right)=0.\end{aligned} \tag{1.4}$$

Proof. The first four lines of (1.4) follow from straightforward computations which are left to the reader. Also,

$$\begin{aligned}R^{Z',\varepsilon}\left(X,\frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}&=\nabla^{Z',\varepsilon}_X\nabla^{Z',\varepsilon}_{\partial/\partial r}\frac{\partial}{\partial r}-\nabla^{Z',\varepsilon}_{\partial/\partial r}\nabla^{Z',\varepsilon}_X\frac{\partial}{\partial r}\\ &=-\nabla^{Z',\varepsilon}_{\partial/\partial r}\tilde{S}^{\varepsilon}(X)\frac{\partial}{\partial r}\\ &=-\nabla^{Z',\varepsilon}_{\partial/\partial r}\frac{X}{r}=\frac{X}{r^2}-\frac{X}{r^2}=0.\end{aligned}$$

Similarly,

$$R^{Z',\varepsilon}\left(X,\frac{\partial}{\partial r}\right)Y=\nabla_X^{Z',\varepsilon}\nabla_{\partial/\partial r}^{Z',\varepsilon}Y-\nabla_{\partial/\partial r}^{Z',\varepsilon}\nabla_X^{Z',\varepsilon}Y$$

$$=\nabla_X^{Z',\varepsilon}\left(\frac{Y}{r}\right)-\nabla_{\partial/\partial r}^{Z',\varepsilon}\left(\nabla_X Y-\varepsilon r\langle X, Y\rangle_{g^{\partial Z}}\frac{\partial}{\partial r}\right)$$

$$=\frac{1}{r}\nabla_X Y-\varepsilon\langle X, Y\rangle_{g^{\partial Z}}\frac{\partial}{\partial r}-\frac{1}{r}\nabla_X Y+\varepsilon\langle X, Y\rangle_{g^{\partial Z}}\frac{\partial}{\partial r}$$

$$=0.$$

The proposition is proved. ∎

Let ξ be a smooth complex vector bundle on Z. If ρ: $\mathscr{U}\to\partial Z$ is defined by $\rho(r, y)=y$, we have $\xi=\rho^*\xi_{\partial Z}$.

We assume that ξ is equipped with a metric h^ξ and with a unitary connection ∇^ξ which both split as products in $\mathscr{U}$, i.e., h^ξ and ∇^ξ are the pull-back by ρ^* of the restrictions of h^ξ and ∇^ξ to ∂Z.

Let L^ξ be the curvature of ∇^ξ. Clearly on $\mathscr{U}$, if $X\in T^{\partial Z}$,

$$L\left(\frac{\partial}{\partial r}, X\right)=0. \tag{1.5}$$

We now extend ξ from ∂Z to $C(\partial Z)$ in the obvious way. On $C(\partial Z)$, ξ inherits the corresponding metric and connection, and (1.5) still holds. Therefore ξ is a smooth Hermitian vector bundle with unitary connection ∇^ξ on Z'.

(c) *The Dirac Operator on Z'*

The Levi-Civita connection $\nabla^{Z',\varepsilon}$ on TZ' lifts into a unitary connection on $F=F_+\oplus F_-$, which we still denote $\nabla^{Z',\varepsilon}$.

The Hermitian vector bundle $F\otimes\xi$ is equipped with a unitary connection $\nabla^{Z',\varepsilon}\otimes 1+1\otimes\nabla^\xi$, which we still denote $\nabla^{Z',\varepsilon}$.

Let $H^\infty=H_+^\infty\oplus H_-^\infty$ be the vector space of smooth sections of $F\otimes\xi=(F_+\otimes\xi)\oplus(F_-\otimes\xi)$ on Z' which vanish at δ together with their derivatives. D^ε denotes the Dirac operator acting on H^∞ associated with the metric $g^{Z',\varepsilon}$ and the connection ∇^ξ. If $e'_1, ..., e'_n$ is an orthonormal basis of TZ' for the metric $g^{Z',\varepsilon}$, then D^ε is given by

$$D^\varepsilon=\sum_1^n e'_i\nabla_{e'_i}^{Z',\varepsilon}. \tag{1.6}$$

The operator D^ε interchanges H_+^∞ and H_-^∞. Let $D_\pm^\varepsilon$ be the restriction of D^ε to $H_\pm^\infty$. In particular, D_+^ε maps H_+^∞ into H_-^∞.

The Levi–Civita connection $\nabla^{\partial Z}$ on $T\,\partial Z$ lifts to the restriction of F_+ to ∂Z. Therefore on ∂Z, the Hermitian vector bundle $F_+ \otimes \xi$ is endowed with the unitary connection $\nabla^{\partial Z} \otimes 1 + 1 \otimes \nabla^{\xi}$, which we still denote $\nabla^{\partial Z}$. Also if $X \in T\,\partial Z$, X acts by Clifford multiplication on $F_+ \otimes \xi$.

Let $D^{\partial Z}$ be the Dirac operator on ∂Z, which acts on the smooth sections of $F_+ \otimes \xi$ on ∂Z and is naturally associated with the metric $g^{\partial Z}$ and the connection ∇^{ξ}. If $e_1, ..., e_{n-1}$ is an orthonormal base of $T\,\partial Z$, then $D^{\partial Z}$ is given by

$$D^{\partial Z} = \sum_{i=1}^{n-1} e_i \nabla^{\partial Z}_{e_i}. \tag{1.7}$$

Of course, on $C(\partial Z) \cup_{\partial Z} \mathscr{U}$, we will use the description of $F = F_+ \oplus F_-$ which was given earlier. Recall that on $C(\partial Z) \cup_{\partial Z} \mathscr{U}$, F_+ and F_- are identified. If $X \in T\,\partial Z$, then X acts by Clifford multiplication on $F = F_+ \oplus F_-$ as

$$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}.$$

So, on $C(\partial Z) \cup_{\partial Z} \mathscr{U}$, $D^{\partial Z}$ acts on the set of smooth sections of $F \otimes \xi = (F_+ \otimes \xi) \oplus (F_- \otimes \xi)$ as

$$\begin{bmatrix} 0 & D^{\partial Z} \\ D^{\partial Z} & 0 \end{bmatrix}.$$

Proposition 1.3. *On $C(\partial Z) \cup_{\partial Z} \mathscr{U}$, D^{ε} is given by*

$$D^{\varepsilon} = \sqrt{\varepsilon}\, f_1 \left(\frac{-\partial}{\partial r} - \frac{n-1}{2r} \right) + \frac{D^{\partial Z}}{r}. \tag{1.8}$$

Equivalently, in matrix form, D^{ε} is given by

$$D^{\varepsilon} = \begin{bmatrix} 0 & -\sqrt{\varepsilon} \left(\frac{\partial}{\partial r} + \frac{n-1}{2r} \right) + \frac{D^{\partial Z}}{r} \\ \sqrt{\varepsilon} \left(\frac{\partial}{\partial r} + \frac{n-1}{2r} \right) + \frac{D^{\partial Z}}{r} & 0 \end{bmatrix}. \tag{1.9}$$

Proof. We know that

$$\nabla^{Z', \varepsilon} = \nabla + \tilde{S}^{\varepsilon}.$$

Clearly, the Euclidean connection ∇ on $T(C(\partial Z) \cup_{\partial Z} \mathscr{U})$ lifts into a unitary connection on $F = F_+ \oplus F_-$. If $e'_1, ..., e'_n$ is an orthonormal base of TZ' for $g^{Z', \varepsilon}$, by [BF2, Eq. (1.2)], we find that

$$\nabla^{Z', \varepsilon} = \nabla_{\cdot} + \tfrac{1}{4} \langle \tilde{S}^{\varepsilon}(\cdot)\, e'_i, e'_j \rangle\, e'_i e'_j.$$

By Proposition 1.2, we see that for $X \in T(\partial Z)$,

$$\nabla^{Z',\varepsilon}_{\partial/\partial r} = \nabla_{\partial/\partial r}$$
$$\nabla^{Z',\varepsilon}_{X} = \nabla_X + \frac{\sqrt{\varepsilon}}{2} X f_1 .$$

Let $e_1, ..., e_{n-1}$ be an orthonormal base of $T\partial Z$ for $g_{\partial Z}$. Using (1.6), we find that

$$D^\varepsilon = \sum_1^{n-1} \frac{e_i}{r} \nabla^{Z',\varepsilon}_{e_i} + \sqrt{\varepsilon} f_1 \nabla^{Z',\varepsilon}_{f_1}$$
$$= \sum_1^{n-1} \frac{e_i}{r} \left(\nabla^{\partial Z}_{e_i} + \frac{\sqrt{\varepsilon}}{2} e_i f_1 \right) + \sqrt{\varepsilon} f_1 \nabla_{f_1}. \tag{1.10}$$

The proposition is proved. ∎

We now calculate $(D^\varepsilon)^2$.

PROPOSITION 1.4. *On $C(\partial Z) \cup_{\partial Z} \mathscr{U}$, $(D^\varepsilon)^2$ is given by the formula*

$$(D^\varepsilon)^2 = -\varepsilon \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right) - \frac{\varepsilon}{4r^2} (n-1)(n-3) + \frac{\sqrt{\varepsilon} f_1 D^{\partial Z}}{r^2} + \frac{(D^{\partial Z})^2}{r^2}. \tag{1.11}$$

Equivalently, on $C(\partial Z) \cup_{\partial Z} \mathscr{U}$, $(D^\varepsilon)^2$ is given in matrix form by

$$(D^\varepsilon)^2 = \varepsilon \begin{bmatrix} -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{n-1}{2} + \frac{D^{\partial Z}}{\sqrt{\varepsilon}}\right)\left(\frac{n-3}{2} - \frac{D^{\partial Z}}{\sqrt{\varepsilon}}\right) & 0 \\ 0 & -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{n-1}{2} - \frac{D^{\partial Z}}{\sqrt{\varepsilon}}\right)\left(\frac{n-3}{2} + \frac{D^{\partial Z}}{\sqrt{\varepsilon}}\right) \end{bmatrix}. \tag{1.12}$$

Proof. (1.11) and (1.12) are obvious consequences of (1.8) and (1.9). ∎

(d) *The Dirac Operator on Z' and the Operator of Atiyah, Patodi, and Singer on Z*

To simplify the exposition, we will assume in the sequel that

$$\operatorname{Ker} D^{\partial Z} = \{0\}. \tag{1.13}$$

Assumption (1.13) can be easily lifted by the method of [C1, C2, Ch].

However in the context of the families situation of Section 3, the analogous point has a significance which is not purely technical.

Observe that by assumption (1.13), if ε is small enough, for any λ in the spectrum of $D^{\partial Z}$, we have $|\lambda|/\sqrt{\varepsilon} > \frac{1}{2}$.

We now recall the main results given in [C1, C2, Ch]. Observe that H^{∞} is naturally equipped with a L_2 Hermitian product.

Then by [Ch, Theorem 3.2], for $\varepsilon > 0$ small enough, D^{ε} is essentially self-adjoint on H^{∞}, and the operator D^{ε}_{+} is Fredholm. In particular, the index of D^{ε}_{+} is given by the formula

$$\text{Ind } D^{\varepsilon}_{+} = \dim \text{Ker } D^{\varepsilon}_{+} - \dim \text{Ker } D^{\varepsilon}_{-}. \tag{1.14}$$

Also for $\varepsilon > 0$ small enough, the operator $(D^{\varepsilon})^2$ is essentially self-adjoint of H^{∞}, the operator $\exp(-t(D^{\varepsilon})^2)$ is trace class and, moreover, for any $t > 0$,

$$\text{Ind } D^{\varepsilon}_{+} = \text{Tr}_s[\exp(-t(D^{\varepsilon})^2)]. \tag{1.15}$$

The properties of $\exp(-t(D^{\varepsilon})^2)$ were established in [C1, C2] using the functional calculus of cones. We will establish more general results on such operators in Section 4.

On the other hand, Atiyah, Patodi, and Singer [APS1] defined a Dirac operator D on Z with global boundary conditions.

Namely, let g^Z be a smooth metric on TZ which has the following two properties

- g^Z coincides with $g^{\partial Z}$ on ∂Z.
- g^Z is product on $\mathscr{U}$, i.e., on $\mathscr{U}$, g^Z is given

$$g^Z = dr^2 + g^{\partial Z}. \tag{1.16}$$

$F = F_{+} \oplus F_{-}$ still denotes the Hermitian vector bundle of spinors on Z for the metric g^Z and the given spin structure on Z. (ξ, ∇^{ξ}) is taken as before. In [APS1], a Dirac operator D is formally defined on Z as in (1.6). D still splits into

$$D = \begin{bmatrix} 0 & D_{-} \\ D_{+} & 0 \end{bmatrix}.$$

Also the Dirac operator $D^{\partial Z}$ on ∂Z has a discrete spectrum. Let P_{+} and P_{-} be the orthogonal projection operators (with respect to the natural L_2 Hermitian product) on the direct sum of eigenspaces of $D^{\partial Z}$ corresponding to nonnegative or negative eigenvalues of $D^{\partial Z}$.

Then Atiyah, Patodi, and Singer [APS1] restrict the operator D_+ to the C^∞ sections ϕ of $F_+ \otimes \xi$ such that

$$P_+ \phi = 0. \tag{1.17}$$

They construct a natural family of Sobolev spaces on Z (which incorporate the boundary condition (1.17)) such that the restriction of D_+ to such spaces is Fredholm.

Similarly, the operator D_- acts on the C^∞ sections ψ of $F_- \otimes \xi$ on Z such that

$$P_- \psi = 0. \tag{1.18}$$

Then

$$\operatorname{Ind} D_+ = \dim(\operatorname{Ker} D_+) - \dim(\operatorname{Ker} D_-). \tag{1.19}$$

Giving an a priori identification of Ind D^ε_+ and Ind D_+, first as integers and later as virtual bundles, is of the utmost importance in the sequel. Here we prove the simple result.

THEOREM 1.5. *For $\varepsilon > 0$, small enough,*

$$\operatorname{Ind} D^\varepsilon_+ = \operatorname{Ind} D_+. \tag{1.20}$$

Proof We give two proofs of Theorem 1.5 which are essentially equivalent.

Proof No. 1. Inspection of the arguments in [APS1] shows that we can define the Dirac operator $\tilde{D}^\varepsilon_+$ of [APS1] on Z associated with the metric $g^{Z,\varepsilon}$ (which is not a product on $\mathscr{U}$) with the boundary conditions (1.17), (1.18), that $\tilde{D}^\varepsilon_+$ is Fredholm and that Ind $\tilde{D}^\varepsilon_+ = \operatorname{Ind} D_+$.

We now will show that Ind $\tilde{D}^\varepsilon_+ = \operatorname{Ind} D^\varepsilon_+$. If h is a L_2 section of $F_+ \otimes \xi$ on Z' such that $D^\varepsilon_+ h = 0$, then on $C(\partial Z)$, we have

$$\sqrt{\varepsilon}\left(\frac{\partial}{\partial r} + \frac{n-1}{2r}\right) h + \frac{D^{\partial Z}}{r} h = 0. \tag{1.21}$$

Let

$$h(r) = \sum_{\lambda \in \operatorname{Sp}(D^{\partial Z})} h_\lambda(r)$$

be the decomposition of $h(r, \cdot)$ according to the eigenspaces of $D^{\partial Z}$. Clearly for any $\lambda \in \operatorname{Sp}(D^{\partial Z})$,

$$\sqrt{\varepsilon}\left(\frac{\partial}{\partial r} + \frac{n+1}{2r}\right) h_\lambda + \frac{\lambda h_\lambda}{r} = 0$$

and so we find that

$$h_\lambda(r) = r^{-(\lambda/\sqrt{\varepsilon} + (n-1)/2)} h_\lambda(1). \tag{1.22}$$

Also the contribution of the cone $C(\partial Z)$ to the L_2 norm of h is given by

$$\int_{[0,1]\times\partial Z} |h^2(r, y)|^2 \, r^{n-1} \, dr \, dy = \sum_\lambda \int_0^1 |h_\lambda(r)|^2 \, r^{n-1} \, dr. \tag{1.23}$$

From (1.22) and (1.23), we find that, since (1.23) is finite, if $h_\lambda(1) \neq 0$, then $\lambda/\sqrt{\varepsilon} < \frac{1}{2}$. Since $0 \notin \mathrm{Sp}(D^{\partial Z})$, we find that for $\varepsilon > 0$ small enough, if $D^\varepsilon_+ h = 0$, then $P_+ h = 0$. Thus we have shown that for $\varepsilon > 0$ small enough, if $h \in \mathrm{Ker}\, D^\varepsilon_+$, then the restriction h' of h to Z is in $\mathrm{Ker}\, \tilde{D}^\varepsilon_+$.

On the other hand, if $h' \in \mathrm{Ker}\, \tilde{D}^\varepsilon_+$, then on ∂Z, h' has the expansion

$$h = \sum_{\substack{\lambda \in \mathrm{Sp}(D^{\partial Z}) \\ \lambda < 0}} h_\lambda \tag{1.24}$$

We extend h to $C(\partial Z)$ by the formula

$$h = \sum_{\substack{\lambda \in \mathrm{Sp}(D^{\partial Z}) \\ \lambda < 0}} r^{-(\lambda/\sqrt{\varepsilon} + (n-1)/2)} h_\lambda. \tag{1.25}$$

Clearly, for $\varepsilon > 0$ small enough, h is square-integrable on Z' and, moreover, $D^\varepsilon_+ h = 0$. Therefore we have proved that for $\varepsilon > 0$ small enough, $\mathrm{Ker}\, D^\varepsilon_+$ and $\mathrm{Ker}\, \tilde{D}^\varepsilon_+$ are isomorphic vector spaces. The same result can be proved for $\mathrm{Ker}\, D^\varepsilon_-$ and $\mathrm{Ker}\, \tilde{D}^\varepsilon_-$. The proposition is proved. ∎

Proof No. 2. By modifying the metric $g^{Z,\varepsilon}$ on $\mathscr{U}$, we can assume that $g^{Z,\varepsilon}$ is given on $\mathscr{U}$ by

$$\frac{dr^2}{\varepsilon} + f(r)\, g^{\partial Z}, \tag{1.26}$$

where $f(r)$ is a positive C^∞-function such that

$$f(r) = r^2 \quad \text{for} \quad 1 \leqq r \leqq \tfrac{3}{2}$$

$$f(r) = 1 \quad \text{for} \quad \tfrac{7}{4} \leqq r \leqq 2.$$

We now consider the new manifold with boundary $\tilde{Z}$ which is given by

$$\tilde{Z} = Z \backslash \{(r, y) \in \mathscr{U};\, |r| < \tfrac{7}{4}\} \tag{1.27}$$

whose boundary is $\{\frac{7}{4}\} \times \partial Z$. If $\tilde{C}(\partial\tilde{Z}) = (]0, \frac{7}{4}] \times \partial Z) \cup \{\delta\}$, then $Z' = \tilde{Z} \cup_{\partial\tilde{Z}} \tilde{C}(\partial\tilde{Z})$. The metric on $\tilde{Z}$ is now a product near the boundary $\partial\tilde{Z}$.

Also if $\tilde{D}$ is the Dirac operator of [APS1] on $\tilde{Z}$, one verifies immediately that Ind $\tilde{D}_+ = \text{Ind}\, D_+$. By proceeding as before, if is now easy to prove that Ind $\tilde{D}_+ = \text{Ind}\, D^\varepsilon_+$.

(e) *The Asymptotics of the Heat Kernel on Z′*

Let dx be the Riemannian orientation form on Z' for the metric $g^{Z',1}$. Given $\varepsilon > 0$ small enough, for $t > 0$, let $P^\varepsilon_t(x, x')$ be the smooth kernel associated with the operator $\exp(-t(D^\varepsilon)^2)$.

If $h \in H^\infty$, $x \in Z' \backslash \{\delta\}$, then

$$\exp(-t(D^\varepsilon)^2)\, h(x) = \int_{Z'} P^\varepsilon_t(x, x')\, h(x')\, dx'. \tag{1.28}$$

Let $\hat{A}$ be the Hirzebruch polynomial. $\hat{A}$ is an ad $O(n)$ invariant polynomial defined on (n, n) antisymmetric matrices. If B is a (n, n) antisymmetric matrix with diagonal entries $\left[\begin{smallmatrix} 0 & x_i \\ -x_i & 0 \end{smallmatrix}\right]$, then

$$\hat{A}(B) = \prod_1^l \frac{x_i/2}{\sinh(x_i/2)}. \tag{1.29}$$

THEOREM 1.6. *For $\varepsilon > 0$ small enough, for any $x \in Z' \backslash \{\delta\}$, then*

$$\lim_{t \to 0} \text{Tr}_s[P^\varepsilon_t(x, x)]\, dx = \left\{\hat{A}\left(\frac{R^{Z',\varepsilon}}{2\pi}\right) \text{Tr}\left[\exp\left(-\frac{L^\xi}{2i\pi}\right)\right]\right\}^{\max} \tag{1.30}$$

and the convergence is uniform over compact subsets of $Z' \backslash \{\delta\}$. In particular,

$$\lim_{t \to 0} \text{Tr}_s[P^\varepsilon_t(x, x)] = 0 \qquad \text{on} \quad C(\partial Z) \cup_{\partial Z} \mathscr{U}. \tag{1.31}$$

Proof. (1.22) follows form the local index formula for Dirac operators [ABP, BeV, B4, Ge, Gi, P]. By Proposition 1.2, $R^{Z',\varepsilon}(\partial/\partial r, \cdot)$ vanishes on $C(\partial Z) \cup_{\partial Z} \mathscr{U}$. Also by (1.5), $L(\partial/\partial r, \cdot)$ also vanishes. Therefore the right-hand side of (1.30) vanishes on $C(\partial Z) \cup_{\partial Z} \mathscr{U}$. ∎

(f) *The Heat Kernel on the Infinite Cone*

Let $C^\infty(\partial Z)$ be the infinite cone with vertex δ:

$$C^\infty(\partial Z) = (]0, +\infty[\times \partial Z) \cup \{\delta\}.$$

If $e_1, ..., e_{n-1}$ is an oriented base of $T\partial Z$, we orient $C^\infty(\partial Z)$ by the base $(-\partial/\partial r, e_1, ..., e_{n-1})$. For $\varepsilon > 0$, we still endow $C^\infty(\partial Z)$ with the metric $g^{\varepsilon,\infty}$ given by

$$g^{\varepsilon,\infty} = \frac{dr^2}{\varepsilon} + r^2 g^{\partial Z}. \tag{1.32}$$

We extend $F = F_+ \oplus F_-$ and ξ from ∂Z to $C^\infty(\partial Z)$ in the obvious way. Let $D^{\varepsilon,\infty}$ be the Dirac operator acting on the smooth sections of $F \otimes \xi$ on the cone $C^\infty(\partial Z)$. $D^{\varepsilon,\infty}$ is exactly given by the right-hand side of Eq. (1.8).

By [C1, C2, Ch], for $\varepsilon > 0$ small enough, $(D^\varepsilon)^2$ is essentially self-adjoint on the set of C^∞ sections of $F \otimes \xi$ on $C^\infty(\partial Z) \setminus \{\delta\}$ with compact support.

Let dy denote the Riemannian orientation form of ∂Z for the metric $g^{\partial Z}$ and let dx be the Riemannian orientation form of $C^\infty(\partial Z)$ for the metric $g^{1,\infty}$. Clearly,

$$dx = r^{n-1}\, dy\, dr.$$

Given $\varepsilon > 0$ sufficiently small, for $t > 0$, let $P_t^{\varepsilon,\infty}$ be the smooth kernel on $C^\infty(\partial Z)$ associated with the operator $\exp(-t(D^{\varepsilon,\infty})^2)$. $P_t^{\varepsilon,\infty}$ is, of course, calculated with respect to the volume form dx. Then by [C1, C2, Ch] for any $t > 0$, $M > 0$,

$$\int_{[0,M] \times \partial Z} \mathrm{Tr}[P_t^{\varepsilon,\infty}((r, y), (r, y))]\, dx < +\infty. \tag{1.33}$$

The following simple observation is the crucial first step of [C1, C2].

Proposition 1.7. *For any $t > 0$, $(r, y) \in\,]0, +\infty[\, \times \partial Z$, then*

$$P_t^{\varepsilon,\infty}((r, y), (r, y)) = \frac{1}{r^n} P_{t/r^2}^{\varepsilon,\infty}((1, y), (1, y)). \tag{1.34}$$

Also as $t \downarrow 0$,

$$\mathrm{Tr}_s[P_t^{\varepsilon,\infty}((1, y), (1, y))] = O(t) \tag{1.35}$$

and $O(t)$ is uniform on ∂Z.

Proof. For $s > 0$, let h_s be the dilation of the cone $C^\infty(\partial Z)$,

$$(r, y) \to h_s(r, y) = (sr, y). \tag{1.36}$$

One verifies that

$$h_{1/s}\, D^{\varepsilon,\infty} h_s = s D^{\varepsilon,\infty} \tag{1.37}$$

and so

$$h_{1/s}(D^{\varepsilon,\infty})^2\, h_s = s^2 (D^{\varepsilon,\infty})^2. \tag{1.38}$$

(1.34) immediately follows from (1.37). The proof of (1.35) is similar to the proof of (1.31) in Theorem 1.6. ∎

(g) *A Formula for* Ind D^{ε}_{+}

We here establish the formula of [C1, C2, Ch] for Ind D^{ε}_{+}.

DEFINITION 1.8. For $\varepsilon > 0$ small enough, set

$$J^{\varepsilon} = -\int_{0}^{+\infty} \frac{ds}{2s} \int_{\partial Z} \operatorname{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy. \tag{1.39}$$

The integral which defines J^{ε} converges. In fact,

- As $s \downarrow 0$, Proposition 1.7 takes care of the convergence.
- By (1.34), we know that

$$P_s^{\varepsilon,\infty}((1,y),(1,y)) = \frac{1}{s^{n/2}} P_1^{\varepsilon,\infty}\left(\left(\frac{1}{\sqrt{s}}, y\right), \left(\frac{1}{\sqrt{s}}, y\right)\right). \tag{1.40}$$

Due to (1.33), it is clear that the integral in (1.39) also converges as $s \uparrow +\infty$.

THEOREM 1.9. *For* $\varepsilon > 0$ *small enough, then*

$$\operatorname{Ind} D^{\varepsilon}_{+} = \int_{Z} \hat{A}\left(\frac{R^{Z,\varepsilon}}{2\pi}\right) \operatorname{Tr}\left[\exp\left(\frac{-L^{\xi}}{2i\pi}\right)\right] - J^{\varepsilon}. \tag{1.41}$$

Proof By (1.15), we know that for $\varepsilon > 0$ small enough, for any $t > 0$,

$$\operatorname{Ind} D^{\varepsilon}_{+} = \int_{Z'} \operatorname{Tr}_s[P_t^{\varepsilon}(x,x)]\,dx. \tag{1.42}$$

Take β such that $0 < \beta < 1$. Let $C^{\beta}(\partial Z)$ be the cone

$$C^{\beta}(\partial Z) = (]0,\beta] \times \partial Z) \cup \{\delta\}.$$

Set

$$Z'^{1/2} = Z' \backslash C^{1/2}(\partial Z).$$

By Theorem 1.6, we find that

$$\lim_{t \downarrow 0} \int_{Z'^{1/2}} \operatorname{Tr}_s[P_t^{\varepsilon}(x,x)]\,dx = \int_{Z'^{1/2}} \hat{A}\left(\frac{R^{Z',\varepsilon}}{2\pi}\right) \operatorname{Tr}\left[\exp\left(\frac{-L^{\xi}}{2i\pi}\right)\right]. \tag{1.43}$$

On the other hand, by (1.31),

$$\lim_{t\downarrow 0}\int_{C^{1/2}(\partial Z)\setminus C^{1/4}(\partial Z)}\mathrm{Tr}_s[P_t^{\varepsilon}(x,x)]\,dx=0. \tag{1.44}$$

By standard estimates on heat kernels, we know that there is $\alpha>0$ such that for any $x\in C^{1/4}(\partial Z)$,

$$|P_t^{\varepsilon}(x,x)-P_t^{\varepsilon,\infty}(x,x)|\leqq C\exp(-\alpha/t). \tag{1.45}$$

Therefore,

$$\left|\int_{C^{1/4}(\partial Z)}\mathrm{Tr}_s[P_t^{\varepsilon}(x,x)]\,dx-\int_{C^{1/4}(\partial Z)}\mathrm{Tr}_s[P_t^{\varepsilon,\infty}(x,x)]\,dx\right|$$
$$\leqq C\exp\left(\frac{-\alpha}{t}\right). \tag{1.46}$$

Using (1.34), we find that

$$\int_{r\leqq 1/4}\mathrm{Tr}_s[P_t^{\varepsilon,\infty}((r,y),(r,y))]\,r^{n-1}\,dr$$
$$=\int_{r\leqq 1/4}\mathrm{Tr}_s[P_{t/r^2}^{\varepsilon,\infty}((1,y),(1,y))]\,\frac{dr}{r}$$
$$=\int_{r\leqq 1/4\sqrt{t}}\mathrm{Tr}_s[P_{1/r^2}^{\varepsilon,\infty}((1,y),(1,y))]\,\frac{dr}{r}$$
$$=\frac{1}{2}\int_{s\geqq 16t}\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,\frac{ds}{s}. \tag{1.47}$$

By (1.47), we find that as $t\downarrow 0$, the integral in the r.h.s. of (1.47) converges to J^{ε}. Theorem 1.9 follows from (1.43), (1.44), (1.46), (1.47). ∎

Remark 1.10. For $a\in C$, $\mathrm{Re}\,(a)>0$, set

$$J^{\varepsilon}(a)=-\int_0^{+\infty}(2\varepsilon s)^a\,\frac{ds}{2s}\int_{\partial Z}\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy. \tag{1.48}$$

By [C1, C2, Ch], we know that for any $\beta>0$, for ε small enough, $P_1^{\varepsilon,\infty}((r,y),(r,y))$ decays faster than r^{β} as $r\downarrow 0$.

Take $M > n/2$. For $\varepsilon>0$ small enough, for any a such that $n/2 < \mathrm{Re}\,(a)\leqq M$, using (1.40), the integral defining $J^{\varepsilon}(a)$ converges. Of course, by (1.35) we know that this is also the case for $-1<\mathrm{Re}\,(a)\leqq M$.

For $\nu \geq 0$, let I_ν be the modified Bessel function of order ν. By [C1, C2, CaJ, PY: Eqs. (2.h), (2.i), So] we know that for any $b \in R$, the kernel on R_+ associated with the operator $\exp s\{d^2/dr^2 + (n-1)/r\, d/dr - b^2/r^2\}$ has a density $q_s^b(r, r')$ with respect to the measure $r'^{n-1}\, dr'$ which is given by

$$q_s^b(r, r') = \frac{1}{2s} (rr')^{-(n-2)/2} \exp\left\{ -\frac{r^2 + r'^2}{4s} \right\} I_{\{(n-2)/2)^2 + b^2\}^{1/2}} \left(\frac{rr'}{2s} \right). \tag{1.49}$$

Let $\{\lambda\}$ be the discrete family of real eigenvalues of $D^{\partial Z}$. Since $0 \notin \operatorname{Spec} D^{\partial Z}$, for $\varepsilon > 0$ small enough, for any λ, $|\lambda/\sqrt{\varepsilon}| > \frac{1}{2}$. Using (1.12) and (1.49), we find that for $\varepsilon > 0$ small enough,

$$\int_{\partial Z} \operatorname{Tr}_s[P_s^{\varepsilon, \infty}((1, y), (1, y))]\, dy$$
$$= \frac{1}{2\varepsilon s} \exp\left(\frac{-1}{2\varepsilon s} \right) \sum \left[I_{|\lambda/\sqrt{\varepsilon} + 1/2|} \left(\frac{1}{2\varepsilon s} \right) - I_{|\lambda/\sqrt{\varepsilon} - 1/2|} \left(\frac{1}{2\varepsilon s} \right) \right]. \tag{1.50}$$

Therefore, for $\operatorname{Re}(a)$ large enough,

$$J^\varepsilon(a) = \frac{1}{2} \sum \int_0^{+\infty} u^{-a} \exp(-u) [I_{|\lambda/\sqrt{\varepsilon} + 1/2|} - I_{|\lambda/\sqrt{\varepsilon} - 1/2|}](u)\, du. \tag{1.51}$$

Using the recursion relations on Bessel functions [Wa, p. 79]

$$I_\nu(u) = \frac{\nu + 1}{u} I_{\nu+1} + I'_{\nu+1}, \tag{1.52}$$

we find that for ν large enough,

$$-\frac{1}{2} \int_0^{+\infty} u^{-a} \exp(-u) (I_{\nu+1} - I_\nu)(u)\, du$$
$$= \frac{1}{2} (\nu + a + 1) \int_0^{+\infty} u^{-(a+1)} \exp(-u)\, I_{\nu+1}(u)\, du. \tag{1.53}$$

By a formula of Henkel and Gegenbauer [Wa, p. 384], we know that

$$\frac{1}{2} (\nu + a + 1) \int_0^{+\infty} u^{-(a+1)} \exp(-u)\, I_{\nu+1}(u)\, du$$
$$= 2^{a-1} \frac{\Gamma(a + 1/2)}{\Gamma(1/2)} \frac{\Gamma(\nu - a + 1)}{\Gamma(\nu + a + 1)}. \tag{1.54}$$

So we find that if $n/2 < \mathrm{Re}\,(a) \leqq M$, for $\varepsilon > 0$ small enough,

$$J^\varepsilon(a) = \frac{2^a}{2} \frac{\Gamma(a+1/2)}{\Gamma(1/2)} \sum (\mathrm{sgn}\,\lambda) \frac{\Gamma(|\lambda|/\sqrt{\varepsilon} + 1/2 - a)}{\Gamma(|\lambda|/\sqrt{\varepsilon} + 1/2 + a)}. \qquad (1.55)$$

Equation (1.35) then shows that $J^\varepsilon(a)$ extends into a meromorphic function of a for $\mathrm{Re}\,(a) \leqq M$, which is holomorphic at $a = 0$.

A closely related formula for $J^\varepsilon(a)$ was obtained in [C2, Eq. (6.10)], to show that $J^\varepsilon(0)$ coincides with $\bar{\eta}(0)$.

(h) *Bessel Functions, Adiabatic Limit of J^ε and the Atiyah–Patodi–Singer Index Theorem*

We now will make $\varepsilon \downarrow 0$ in Theorem 1.9 to reobtain the index theorem in the form given by Atiyah, Patodi, and Singer [APS1].

Let g^Z be a smooth metric on Z taken as in Section 1(d). In particular, on $\mathscr{U}$, g^Z is given by (1.16). Let R^Z be the curvature of the Levi–Civita connection ∇^Z on (TZ, g^Z).

PROPOSITION 1.11. *For any $\varepsilon > 0$, the following equality holds*:

$$\int_Z \hat{A}\left(\frac{R^{Z,\varepsilon}}{2\pi}\right) \mathrm{Tr}\left[\exp\left(\frac{-L}{2i\pi}\right)\right] = \int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \mathrm{Tr}\left[\exp\left(\frac{-L}{2i\pi}\right)\right]. \qquad (1.56)$$

Proof. We use the same argument as in [C2, Section 6]. The metric $dr^2/\varepsilon + r^2 g^{\partial Z}$ is conformally equivalent to the metric $dr^2/\varepsilon r^2 + g^{\partial Z}$. If $v = (\log r)/\sqrt{\varepsilon}$, the metric $dr^2/\varepsilon r^2 + g^{\partial Z}$ is exactly the metric $dv^2 + g^{\partial Z}$.

It follows that the Riemannian manifold with boundary $(Z, g^{Z,\varepsilon})$ is conformally equivalent to the manifold Z endowed with a metric g'^Z which coincides with $dv^2 + g^{\partial Z}$ on a tubular neighborhood of ∂Z.

On the other hand, we know that the Pontryagin forms of TZ are conformally invariant. (1.56) follows. ∎

Remark 1.12. By Theorem 1.5, we know that for $\varepsilon > 0$ small enough, $\mathrm{Ind}\, D^\varepsilon_+$ does not depend on ε. It thus follows from Theorem 1.9 and Proposition 1.11 that for $\varepsilon > 0$ small enough, J^ε is independent of ε.

Let $\bar{\eta}(0)$ be the reduced êta invariant of Atiyah, Patodi, and Singer [APS1] associated with the self-adjoint operator $D^{\partial Z}$. Recall that by [APS2, p. 84; BF2, Theorem 2.4] as $s \downarrow 0$ then

$$\mathrm{Tr}[D^{\partial Z} \exp(-s(D^{\partial Z})^2)] = O(\sqrt{s}). \qquad (1.57)$$

Using (1.57), we find the classical expression for $\bar{\eta}(0)$,

$$\bar{\eta}(0) = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} \mathrm{Tr}[D^{\partial Z} \exp(-s(D^{\partial Z})^2)] \frac{ds}{\sqrt{s}}. \qquad (1.58)$$

We now will identify J^ε and $\bar{\eta}(0)$ by an adiabatic limit procedure. This technique is quite different from the one which is used in [C1, C2, Ch] and will be extended in Section 5 to the families index theorem.

THEOREM 1.13. *The following identity holds*:

$$\lim_{\varepsilon \downarrow 0} J^\varepsilon = \bar{\eta}(0). \tag{1.59}$$

Proof. The proof of Theorem 1.13 is divided into two main steps. By (1.57), $/\sqrt{s}\ \mathrm{Tr}[D^{\partial Z} \exp -s(D^{\partial Z})^2]$ can be extended by continuity at $s=0$.

The first step in the proof of Theorem 1.13 is the following.

PROPOSITION 1.14. *For any* $s \geqq 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{s} \int_{\partial Z} \mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1, y), (1, y))]\, dy$$

$$= -\frac{1}{\sqrt{\pi s}} \mathrm{Tr}[D^{\partial Z} \exp(-s(D^{\partial Z})^2)] \tag{1.60}$$

and the convergence is uniform over the compact subsets of R_+.

Proof. Let $Q_s(y, y')$ be the C^∞ kernel associated with the operator $D^{\partial Z} \exp(-s(D^{\partial Z})^2)$. We will prove that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{s} \mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1, y), (1, y))] = \frac{-1}{\sqrt{\pi s}} \mathrm{Tr}[Q_s(y, y)] \tag{1.61}$$

with the required uniformity. By Proposition 1.3, we know that

$$D^\varepsilon = \sqrt{\varepsilon} f_1 \left(\frac{-\partial}{\partial r} - \frac{n-1}{2r}\right) + \frac{1}{r} D^{\partial Z}$$

and so we get

$$(D^\varepsilon)^2 = -\varepsilon \left(\frac{\partial}{\partial r} + \frac{n-1}{2r}\right)^2 + \frac{(D^{\partial Z})^2}{r^2} + \frac{\sqrt{\varepsilon} f_1}{r^2} D^{\partial Z}. \tag{1.62}$$

It should now be clear that the proof of (1.61) is closely related to the second proof of the families index theorem given in [B1, Theorems 5.3] and to [BF2, Theorem 3.12; C3], where a similar adiabatic limit problem was considered (compare especially [BC1, Section 3]).

There are, however, two differences. The first is substantial. While in [BF2, C3] the variable r was assumed to vary in S_1 and no singular terms like $1/r$ or $1/r^2$ appeared, here, because we work on a cone, such singular terms are present. Still, we claim that, to study the limit of $\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1, y), (1, y))]$ as $\varepsilon \downarrow 0$, we can localize the problem in the region $]\frac{1}{2}, \frac{3}{2}[\times \partial Z$. So we can replace the cone $C^\infty(\partial Z)$ by $]-\infty, +\infty[\times \partial Z$ and assume that for $|r|$ large enough, $(D^\varepsilon)^2$ is simply the operator

$$-\frac{d^2}{dr^2} + (D^{\partial Z})^2.$$

This is essentially because on the diagonal, away from the tip $\{\delta\}$, $P_s^{\varepsilon,\infty}$ behaves like a standard heat kernel on a nonsingular space.

Once localization is proved, we must now interest ourselves in the algebra which determines the existence of the limit.

With respect to [BF2, Theorem 3.12], a minor difference is that while in [BF2], the dimension of the total manifold was odd, the dimension of $C^\infty(\partial Z)$ is even. However, this is compensated by the fact that in [BF2], only traces where considered, while here we consider supertraces. As explained at length in [BF2, Sections 1(b) and 2(f); BC1, Section 3], as far as adiabatic limits are concerned, these two situations are essentially equivalent.

In particular remember that if $e_1, ..., e_{n-1}$ is an oriented orthonormal base of $T\partial Z$, if $f_1 e_1 \cdots e_{n-1} \in c^{\text{even}}(TZ)$ is considered as acting on $F = F_+ \oplus F_-$, and $e_1 \cdots e_{n-1} \in c^{\text{odd}}(T\partial Z)$ is considered as acting on F_+, then by [BF2, Eqs. (1.6), (1.7)]:

$$\begin{aligned} \mathrm{Tr}_s[f_1 e_1 \cdots e_{n-1}] &= (-2i)^l \\ \mathrm{Tr}[e_1 \cdots e_{n-1}] &= (-i)^l 2^{l-1}. \end{aligned} \tag{1.63}$$

Finally observe the critical fact that in formula (1.62) for $(D^\varepsilon)^2$, the Clifford variable f_1 appears with the weight $\sqrt{\varepsilon}$.

It is then possible to reproduce exactly the steps of the proof of [BF2, Theorems 3.12] in a much simpler situation and to obtain (1.61). To prove uniformity as $s \to 0$ in (1.61), we can also exactly proceed as in [BF2, Theorem 3.12; see also BC1, Section 4].

In view of (1.39), (1.58), and (1.60), to prove Theorem 1.13, we need to show that the dominated convergence theorem can be used in the integral (1.39) defining J^ε. We thus establish an estimate on ratios of modified Bessel functions, which will also be used in Section 6.

Recall that by [Wa, p. 54],

$$I_{1/2}(z) = \sqrt{2/\pi z}\,\sinh(z). \tag{1.64}$$

PROPOSITION 1.15. *For any $v \geqq 10$, and z such that $1 \leqq z \leqq 3v^2/32$,*

$$\frac{I_v(z)}{I_{1/2}(z)} \leqq 7 \exp\left(\frac{-v}{2(6z)^{1/2}}\right). \tag{1.65}$$

Proof. Let r be the generic element in $\mathscr{C}(R_+; R_+^*)$. Let Q be the probability law on $\mathscr{C}(R_+; R_+^*)$ of the Bes($\frac{1}{2}$) bridge such that $r_0 = 1$, $r_{1/z} = 1$. Bessel bridges are extensively discussed in Pitman and Yor [PY, Section 2].

If $s \to w_s$ is a standard Brownian motion in R^3 such that $|w_0| = 1$, which is conditioned to be on the unit where at time $1/z$, the probability law of $|w_\cdot|$ is exactly Q.

Let E^Q be the expectation operator with respect to Q. By [PY, Eq. (2.i)], we know that

$$\frac{I_v}{I_{1/2}}(z) = E^Q \exp\left\{-\frac{1}{2}\left(v^2 - \frac{1}{4}\right)\int_0^{1/z} \frac{dt}{r_t^2}\right\}. \tag{1.66}$$

Since $v \geqq 2$, we find that

$$\frac{I_v}{I_{1/2}}(z) \leqq E^Q\left[\exp\left\{-\frac{v^2}{4z \sup_{0 \leqq t \leqq 1/z} |r_t|^2}\right\}\right] \tag{1.67}$$

Take n, n' in R^3. Let $P_{n,n'}$ be the probability law of the Brownian bridge $\tilde{w}$ in R^3 such that $\tilde{w}_0 = n$, $\tilde{w}_{1/z} = n'$. Then under $P_{n,n'}$, the probability law of the process $\tilde{w}_s - (1 - sz)n - szn'$ is exactly $P_{0,0}$ [Si, p. 40].

If $X \in R^3$ is such that its Euclidean norm $|X|$ is larger than l, at least one component of X has an absolute value which is larger than $l/\sqrt{3}$. By [IMK, p. 27], we find that for any $l > 0$,

$$P_{0,0}[\sup_{0 \leqq s \leqq 1/z} |\tilde{w}_s| \geqq l] \leqq 6 \exp(-\tfrac{3}{2} l^2 z). \tag{1.68}$$

Therefore, if n, n' are unit vectors in R^3, we find that for $l > 1$,

$$P_{n,n'}[\sup_{0 \leqq s \leqq 1/z} |\tilde{w}_s| \geqq l] \leqq 6 \exp(-\tfrac{2}{3}(l-1)^2 z)$$

and so, if $l \geqq 2$,

$$P_{n,n'}[\sup_{0 \leqq s \leqq 1/z} |\tilde{w}_s| \geqq l] \leqq 6 \exp\left(-\frac{l^2}{6} z\right). \tag{1.69}$$

Let $d\sigma$ be the uniform probability measure on the sphere S_2. If $n=(1,0,0)$, under the probability law on $\mathscr{C}(R_+; R^3)$,

$$\int_{S_2} P_{n,n'} \exp\left\{-\frac{|n'-n|^2}{2}z\right\} d\sigma(n') \Big/ \int_{S_2} \exp\left\{-\frac{|n'-n|^2}{2}z\right\} d\sigma(n'), \tag{1.70}$$

the probability law of the process $|\tilde{w}|$ is exactly Q. Therefore from (1.69), we find that for any $l \geqq 2$,

$$Q[\sup_{0 \leqq s \leqq 1/z} r_s \geqq l] \leqq 6 \exp\left(-\frac{z}{6}l^2\right). \tag{1.71}$$

From (1.67) and (1.71), we find that for $l \geqq 2$,

$$\frac{I_\nu}{I_{1/2}}(z) \leqq \exp\left(-\frac{\nu^2}{4zl^2}\right) + 6\exp\left(-\frac{zl^2}{6}\right). \tag{1.72}$$

If $z \leqq 3\nu^2/32$ we can choose $l=[3\nu^2/2z]^{1/4}$ in (1.72). We obtain

$$\frac{I_\nu}{I_{1/2}}(z) \leqq \exp\left(-\frac{\nu}{2(6z)^{1/2}}\right) + 6\exp\left(-\frac{z\nu}{2(6z)^{1/2}}\right). \tag{1.73}$$

If $1 \leqq z \leqq 3\nu^2/32$, from (1.73), we obtain (1.65). ∎

We now complete the proof of Theorem 1.13. We need to dominate the function

$$\frac{1}{s}\left|\int_{\partial Z} \mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy\right| \tag{1.74}$$

as $s \uparrow +\infty$ by an integrable function which does not depend on ε. Note that by an initial scaling of the metric $g^{\partial Z}$, we may and we will assume that the eigenvalues λ of $D^{\partial Z}$ are such that $|\lambda| \geqq 4$. By (1.50), we know that

$$\int_{\partial Z} \mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy$$
$$=\frac{1}{2\varepsilon s}\exp\left(-\frac{1}{2\varepsilon s}\right)\sum_\lambda \left(I_{|\lambda/\sqrt{\varepsilon}+1/2|} - I_{|\lambda/\sqrt{\varepsilon}-1/2|}\right)\left(\frac{1}{2\varepsilon s}\right). \tag{1.75}$$

Clearly for $\lambda > 0$, and ε small enough, since I_ν decreases as $\nu \geqq 0$ increases, we get

$$0 \leqq I_{\lambda/\sqrt{\varepsilon}-1/2} - I_{\lambda/\sqrt{\varepsilon}+1/2} \leqq I_{\lambda/\sqrt{\varepsilon}-1} - I_{\lambda/\sqrt{\varepsilon}+1}. \tag{1.76}$$

Also by the recursion formula on Bessel functions [Wa, p. 79], we know that

$$(I_{\lambda/\sqrt{\varepsilon}-1}-I_{\lambda/\sqrt{\varepsilon}+1})(z)=\frac{2\lambda}{\sqrt{\varepsilon}\, z}\, I_{\lambda/\sqrt{\varepsilon}}(z). \tag{1.77}$$

Using (1.75)–(1.77), we find that for $\varepsilon>0$ small enough, for any $s>0$,

$$\left|\int_{\partial Z}\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy\right| \leqq \frac{2\exp(-1/2\varepsilon s)}{\sqrt{\varepsilon}}\sum_\lambda |\lambda|\, I_{|\lambda/\sqrt{\varepsilon}|}\left(\frac{1}{2\varepsilon s}\right). \tag{1.78}$$

In the sequel, we assume that $s\geqq 1$.

- f $2\varepsilon s\leqq 1$, using formula (1.64) for $I_{1/2}$, we write (1.78) in the form

$$\left|\int_{\partial Z}\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy\right| \leqq \frac{1}{\sqrt{\pi}}\sqrt{s}\sinh\left(\frac{1}{2\varepsilon s}\right)\exp\left(-\frac{1}{2\varepsilon s}\right)\sum_\lambda |\lambda|\,\frac{I_{\lambda/\sqrt{\varepsilon}}}{I_{1/2}}\left(\frac{1}{2\varepsilon s}\right). \tag{1.79}$$

Note at this stage that the potentially diverging term $1/\sqrt{\varepsilon}$ has disappeared, at least formally.

For $\varepsilon>0$ small enough, any λ in the spectrum of $D^{\partial Z}$ satisfies $|\lambda|/\sqrt{\varepsilon}\geqq 10$. Moreover, $1/2\varepsilon s\geqq 1$. Also, since $s\geqq 1$ and $|\lambda|\geqq 4$, we find that $1\leqq 1/2\varepsilon s\leqq 1/2\varepsilon\leqq 3\lambda^2/32\varepsilon$. So we can use Proposition 1.15 with $z=1/2\varepsilon s$, $v=|\lambda|/\sqrt{\varepsilon}$, and we obtain from (1.79),

$$\left|\int_{\partial Z}\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy\right| \leqq C\sqrt{s}\sum_\lambda |\lambda|\exp\left(-\frac{|\lambda|\sqrt{s}}{2\sqrt{3}}\right). \tag{1.80}$$

So we find that for $\varepsilon>0$ small enough, if $2\varepsilon s\leqq 1$, then

$$\frac{1}{s}\left|\int_{\partial Z}\mathrm{Tr}_s[P_s^{\varepsilon,\infty}((1,y),(1,y))]\,dy\right| \leqq \frac{C}{\sqrt{s}}\mathrm{Tr}\left[|D|^{\partial Z}\exp\left(-\frac{\sqrt{s}}{2\sqrt{3}}|D|^{\partial Z}\right)\right]. \tag{1.81}$$

Clearly the r.h.s. of (1.81) is integrable as $s\uparrow+\infty$.

• We now consider the case where $2\varepsilon s \geqq 1$. By Poisson's integral formula [Wa, p. 79], we know that for $\nu > 0$,

$$I_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \int_{-1}^{+1} \operatorname{ch}(zs)(1-s^2)^{\nu - 1/2}\, ds. \tag{1.82}$$

Clearly $\Gamma(\nu + \frac{1}{2}) \geqq \Gamma(\nu) = \Gamma(\nu+1)/\nu$, and $\nu/2^\nu$ is uniformly bounded. From (1.82), we find that there is $C > 0$ such that for any z with $0 \leqq z \leqq 1$ and any $\nu > 0$,

$$I_\nu(z) \leqq C \frac{z^\nu}{\Gamma(\nu+1)}. \tag{1.83}$$

So using (1.78), (1.82), (1.83), we find that if $2\varepsilon s \geqq 1$, then

$$\left| \int_{\partial Z} \operatorname{Tr}_s[P_s^{\varepsilon,\infty}((1, y), (1, y))]\, dy \right| \leqq C\sqrt{s} \sum_\lambda |\lambda| \frac{(2\varepsilon s)^{-|\lambda|/\sqrt{\varepsilon}}}{\Gamma(|\lambda|/\sqrt{\varepsilon} + 1)}. \tag{1.84}$$

If $2\varepsilon s \geqq 1$, we have

$$(2\varepsilon s)^{-|\lambda|/\sqrt{\varepsilon}} \leqq (2\varepsilon s)^{-|\lambda|/2\sqrt{\varepsilon}}. \tag{1.85}$$

Also by Stirling's formula, since λ is bounded away from 0, we know that for any $\varepsilon > 0$,

$$\Gamma\left(\frac{|\lambda|}{\sqrt{\varepsilon}} + 1\right) \geqq \left(\frac{|\lambda|}{\sqrt{\varepsilon}\, e}\right)^{|\lambda|/\sqrt{\varepsilon}}. \tag{1.86}$$

From (1.84)–(1.86), we get

$$\left| \int_{\partial Z} \operatorname{Tr}_s[P_s^{\varepsilon,\infty}((1, y), (1, y))]\, dy \right| \leqq C\sqrt{s} \sum |\lambda| \left(\frac{e}{\lambda\sqrt{2s}}\right)^{|\lambda|/\sqrt{\varepsilon}}. \tag{1.87}$$

If $\varepsilon > 0$ is small enough, for any λ in the spectrum of $D^{\partial Z}$, $|\lambda|/\sqrt{\varepsilon} \geqq n+4$. Since $D^{\partial Z}$ is an elliptic operator of order 1 and ∂Z has dimension $n-1$, we know that

$$\sum \frac{1}{|\lambda|^{n+3}} < +\infty.$$

Therefore, for $\varepsilon>0$ small enough and s such that $2\varepsilon s \geqq 1$ and also $e/\sqrt{2s} \leqq 1$, then

$$\frac{1}{s}\left|\int_{\partial Z} \operatorname{Tr}_s[P_s^{\varepsilon,\infty}((1, y), (1, y))]\, dy\right| \leqq \frac{C'}{s^{(n+5)/2}} \tag{1.88}$$

The r.h.s. of (1.88) is integrable as $s \uparrow +\infty$.

From (1.81) and (1.88) we find that we can use the dominated convergence theorem in the integral defining J^ε and we obtain Theorem 1.13. ▮

Remark 1.16. Let us make at this stage several observations. The first observation is that the final argument in the proof of Proposition 1.14 also shows that as $s \downarrow 0$

$$\operatorname{Tr}[Q_s(y, y)] = O(\sqrt{s}), \tag{1.89}$$

from which the fact that $\bar{\eta}(a)$ is holomorphic at $a=0$ follows. In other words, local index theory on the even-dimensional cone $C^\infty(\partial Z)$ implies that $\bar{\eta}(a)$ is holomorphic at $a=0$. This argument is also discussed in [BC1, Section 3]. Note that in [BF2, Remark 2.5], the local holomorphy of $\bar{\eta}(a)$ at $a=0$ was shown to be a consequence of the local families index theorem [B1] for the family $\varepsilon \to \varepsilon D^{\partial Z}$.

These two arguments are very directly connected. They express a version of the multiplicativity of the local index.

Another important observation concerns the fact that dominated convergence can be used in the proof of Theorem 1.13. The situation is essentially different from [BF2, Theorem 3.14] since $(1/r^2)\,(D^{\partial Z})^2$ is no longer *uniformly* elliptic because r can take arbitrarily large values. Still note that the estimate in (1.78) explicitly takes into account the fact that we calculate the difference of two traces. What the proof after Eq. (1.78) does is to make explicit an argument of Getzler [Ge] which is used in [BF2, Theorem 3.14].

This rescaling argument will be used explicitly in our proof of the families index theorem for manifolds with boundary.

From Theorem 1.5, Theorem 1.9, Proposition 1.11, and Theorem 1.13, we deduce the index theorem of Atiyah, Patodi, and Singer [APS1] in the form obtained by Cheeger [C1, C2] and Chou [Ch].

THEOREM 1.17. *For $\varepsilon>0$ small enough,*

$$J^\varepsilon = \bar{\eta}(0)$$

$$\operatorname{Ind} D_+ = \operatorname{Ind} D_+^\varepsilon = \int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \operatorname{Tr}\left[\exp\left(-\frac{L^\xi}{2i\pi}\right)\right] - \bar{\eta}(0). \tag{1.90}$$

II. The Geometry of Families of Manifolds with Conical Singularities

In this section, we consider a family of manifolds with boundary. As in Section 1, we replace each manifold with boundary by a manifold with a conical singularity. We then extend the constructions of [B1] to this new family of manifolds with conical singularities. In particular, by extending [B1, Section 3], we construct the Levi–Civita superconnection of the family.

This section is organized in the following way. In (a), we consider a family of manifolds with boundary Z and we construct the associated family of manifolds with conical singularities Z'. In (b), we construct connections on the vector bundles TZ' and $T\partial Z$. In (c) we define a family of Dirac operators on the fibers Z'. In (d), we construct the Levi–Civita superconnections of [B1] associated with the families of manifolds Z' and ∂Z.

(a) *A Family of Manifolds with Boundary and the Associated Spaces with Conical Singularities*

Let B denote a compact connected manifold of dimension m. Let X be a compact connected manifold with smooth boundary ∂X. Set $n = \dim X$. We assume that X is orientable and has spin.

Let M be a compact connected manifold with smooth boundary ∂M. Assume that the dimension of M is $n+m$.

Let $\pi\colon M \to B$ be a submersion of M on B, which defines a fibration, whose fibers are diffeomorphic to X. Namely we assume that there is an open covering $\mathscr{V}$ of B such that if $V \in \mathscr{V}$, there is a smooth diffeomorphism $\phi_V\colon \pi^{-1}(V) \to V \times X$, and moreover if $V, V' \in \mathscr{V}$ are such that $V \cap V' \neq \phi$, $\phi_V \circ \phi_{V'}^{-1}$: $V \cap V' \times X \to V \cap V' \times X$ is given by a map $(b, x) \to (b, f_{V,V'}(b, x))$, where $f_{V,V'}(b, .)$ is a smooth diffeomorphism of the manifold with boundary X which depends smoothly on $b \in V \cap V'$.

For $b \in B$, set

$$Z_b = \pi^{-1}\{b\}. \tag{2.1}$$

Z_b is a manifold with boundary ∂Z_b, which is such that

$$Z_b \subset M, \qquad \partial Z_b \subset \partial M_b, \tag{2.2}$$

both inclusions in (2.2) being embeddings.

In particular $\pi\colon \partial M \to B$ is a fibration of the compact manifold ∂M on B with compact fiber ∂Z. Note that in general, ∂M and ∂Z may be non-connected.

Let TZ (resp. $T\partial Z$) be the subbundle of TM (resp. $T\partial M$) whose fiber at $x \in M$ (resp. $x \in \partial M$) is the tangent space at x to the fiber Z (resp. ∂Z). We assume that TZ is an oriented spin vector bundle on M.

Let $\mathbf{n}$ be a nonzero vector field defined on ∂M with values in TZ which points inward to M. By equipping TZ with a metric and by exponentiating the vector field $\mathbf{n}$ by geodesics in the fibers Z, we may and we will assume that there is a tubular neighborhood $\mathscr{U}$ of ∂M in M, which has the following two properties:

- The set $\mathscr{U}$ is diffeomorphic to $[1, 2[\times \partial M$ and ∂M is identified with $\{1\} \times \partial M$.
- Under the previous identifications, for any $b \in B$, $U \cap Z_b$ is identified with $[1, 2[\times \partial Z_b$.

r will denote the coordinate varying in $[1, 2]$. Clearly, $\partial/\partial r \in TZ$. Then $T\partial Z$ is an oriented spin vector bundle on ∂M. Set

$$f_1 = -\partial/\partial r.$$

If $e_1, ..., e_{n-1}$ is an oriented base of $T\partial Z$, TZ is oriented by $(f_1, e_1, ..., e_{n-1})$.

Assume that $g^{\partial M}$ is a metric on $T^{\partial M}$, that g^M is a metric on TM which is of the form $dr^2 + g^{\partial M}$ on $\mathscr{U}$ and define $T^H M$ to be the orthogonal complement of TZ in TM. Then $T^H M$ has the following three properties:

(a) For any $x \in M$,

$$T_x M = T_x Z \oplus T_x^H M. \tag{2.3}$$

(b) For any $x \in \partial M$, $T_x^H M \subset T_x \partial M$ and so

$$T_x \partial M = T_x \partial Z \oplus T_x^H M. \tag{2.4}$$

(c) If $(r, y) \in [1, 2[\times \partial M$, then $T^H_{(r,y)} M = T_y^H M$.

For every $x \in M$, the linear map $\pi_*: T_x M \to T_{\pi(x)} B$ induces a linear isomorphism from $T_x^H M$ into $T_{\pi(x)} B$.

We now use the cone construction of Section 1 for each individual fiber Z. Namely, if $b \in B$, let $C(\partial Z_b)$ be the cone

$$C(\partial Z_b) = (]0, 1] \times \partial Z_b) \cup \{\delta_b\},$$

where δ_b compactifies $C(\partial Z_b)$ as $r \in]0, 1]$ tends to 0. Set

$$Z'_b = Z_b \bigcup_{\partial Z_b} C(\partial Z_b).$$

$$M' = \bigcup_{b \in B} Z'_b \backslash \{\delta_b\}.$$

M' clearly fibers on B and the fibration map is still denoted π.

Remark 2.1. Note that $\bigcup_b C(\partial Z_b)$ is not identified with $C(\partial M)$, since the tips δ_b have not been identified. Of course, since it is the metric on a cone which determines its topology, the distinction is somewhat irrelevant.

If $(r, y) \in C(\partial Z)$, $r > 0$, set

$$T^H_{(r,y)} M' = T^H_y M.$$

Clearly $T^H M'$ on $C(\partial Z)$ patches smoothly with $T^H M$ on $\mathscr{U}$. So the total manifold M' is now equipped with a vector bundle $T^H M'$ such that

$$TM' = TZ' \oplus T^H M'. \tag{2.5}$$

Let $g^{\partial Z}$ be any metric on $T\partial Z$. Let ε denote a positive real number which we fix for the moment.

We equip $TC(\partial Z)$ with the metric

$$\frac{dr^2}{\varepsilon} + r^2 g^{\partial Z}. \tag{2.6}$$

Let $g^{Z,\varepsilon}$ be any metric on TZ which coincides with (2.6) on the tubular neighborhood $\mathscr{U}$. Again the metric on TZ patches smoothly with the metric (2.6) on $TC(\partial Z)$. We note $g^{Z',\varepsilon}$ this metric on TZ'.

Let g^B be a smooth metric on TB. g^B lifts naturally into a smooth metric on $T^H M'$. Let $g^{M',\varepsilon} = g^B \oplus g^{Z',\varepsilon}$ be the metric on M' which coincides with g^B on $T^H M'$, with $g^{Z',\varepsilon}$ on TZ' and is such that $T^H M'$ and TZ' are orthogonal. Let $\langle\,,\rangle$ be the corresponding scalar product.

As in [B1], it turns out that the objects in which we are ultimately interested will not depend on g^B. Note that in [BC1, Section 4a)] only the splitting $TM = T^H M \oplus TZ$ is used and not any metric on the base B.

(b) *Euclidean Connections on TZ' and $T\partial Z$*

By [B1, Theorem 1.9], to the triples $(TZ', g^{Z',\varepsilon}, T^H M')$ and $(T\partial Z, g^{\partial Z}, T^H M)$ we can associate Euclidean connections $\nabla^{Z',\varepsilon}$ on TZ' and $\nabla^{\partial Z}$ on $T\partial Z$. These connections generalize the Levi–Civita connection on individual fibers Z' or ∂Z. Let us briefly recall the construction of [B1, Section 1].

Let $\nabla^{M',\varepsilon}$ be the Levi–Civita connection or TM' for the metric $g^{M',\varepsilon}$. Let $P^{Z'}$ be the orthogonal projection operator $TM' \to TZ'$.

By [B1, Theorem 1.9], we know that

$$\nabla^{Z',\varepsilon} = P^{Z'} \nabla^{M',\varepsilon}. \tag{2.7}$$

Let $\nabla^{Z,\varepsilon}$ be the restriction of $\nabla^{Z',\varepsilon}$ to TZ. Let $R^{Z',\varepsilon}$, $R^{Z,\varepsilon}$ be the curvatures of the connections $\nabla^{Z',\varepsilon}$, $\nabla^{Z,\varepsilon}$.

The construction of the connection $\nabla^{\partial Z}$ on the vector bundle $T\partial Z$ is very similar. Namely, if the manifold ∂M is endowed with the metric $g^B \oplus g^{\partial Z}$, let $\nabla^{\partial M}$ be the Levi–Civita connection on $T\partial M$. Then by [B1, Theorem 1.9], if $\tilde{P}^{\partial Z}$ is the projection $T\partial M = T^H M \oplus T\partial Z \to T\partial Z$, we have the identity

$$\nabla^{\partial Z} = \tilde{P}^{\partial Z} \nabla^{\partial M}. \tag{2.8}$$

Let $R^{\partial Z}$ be the curvature of the connection $\nabla^{\partial Z}$. Let ∇^B be the Levi–Civita connection on TB. ∇^B lifts into an Euclidean connection on $T^H M$ which we still note ∇^B. Let ∇^ε be the Euclidean connection on $TM' = TZ' \oplus T^H M'$ given by $\nabla^\varepsilon = \nabla^{Z',\varepsilon} \oplus \nabla^B$.

Let T^ε be the torsion tensor of ∇^ε, and let S^ε be the tensor $S^\varepsilon = \nabla^{M,\varepsilon} - \nabla^\varepsilon$. Let us briefly recall the main properties of S^ε and T^ε listed in [B1, Theorem 1.9; BF2, Section 1 d)]:

- T^ε takes values in TZ'.
- If $U, V \in TZ'$, $T^\varepsilon(U, V) = 0$.
- T^ε and the $(3, 0)$ tensor $\langle S^\varepsilon(.).,.\rangle$ do not depend on g^B.
- For any U in TM, $S^\varepsilon(U)$ maps TZ' into $T^H M'$.
- For any $U, V \in T^H M'$, $S^\varepsilon(U) V \in TZ'$.
- If $U \in T^H M$, $S^\varepsilon(U) U = 0$.

Similarly let $T^{\partial Z}$ and $S^{\partial Z}$ be the corresponding objects on ∂M canonically associated with the triple $(T\partial Z, g^{\partial Z}, T^H M)$. Let $P^{\partial Z}$ be the orthogonal projection operator from TZ onto $T\partial Z$.

THEOREM 2.2. *On the manifold ∂M, we have the equality of connections on the vector bundle $T\partial Z$*

$$\nabla^{\partial Z} = P^{\partial Z} \nabla^{Z',\varepsilon}. \tag{2.9}$$

On $\mathscr{U} \bigcup_{\partial M} M' \setminus M$, T^ε takes values in ∂Z. On ∂M, when restricted to vectors in $T\partial M$, T^ε and $\langle S^\varepsilon(.).,.\rangle_{g^{M',\varepsilon}}$ coincide with $T^{\partial Z}$ and with $\langle S^{\partial Z}(.).,.\rangle_{g^{\partial M}}$.

On $\mathscr{U} \bigcup_{\partial M} M' \setminus M$, if $Y \in T^H M'$, $U \in T\partial Z$,

$$\begin{gathered}
\nabla_Y^{Z',\varepsilon} \frac{\partial}{\partial r} = 0; \qquad \nabla_Y^{Z',\varepsilon} U = \nabla_Y^{\partial Z} U \\
T^\varepsilon\left(., \frac{\partial}{\partial r}\right) = 0; \qquad S^\varepsilon\left(\frac{\partial}{\partial r}\right) = 0; \qquad S^\varepsilon(.)\frac{\partial}{\partial r} = 0.
\end{gathered} \tag{2.10}$$

Finally on $\mathscr{U} \bigcup_{\partial M} M' \backslash M$, *if* $Y, Y' \in T^H M'$, $U \in T\partial Z$,

$$R^{Z',\varepsilon}\left(\frac{\partial}{\partial r}, .\right) = 0$$

$$R^{Z',\varepsilon}(Y, Y')\frac{\partial}{\partial r} - \nabla^{Z',\varepsilon}_{T^{\varepsilon}(Y, Y')}\frac{\partial}{\partial r} = 0 \tag{2.11}$$

$$R^{Z',\varepsilon}_{(y,r)}(Y, Y')U = R^{\partial Z}_{y}(Y, Y')U.$$

Proof. Let $P^{\partial M}$ be the orthogonal projection operator $TM \to T\partial M$. The connection $P^{\partial M}\nabla^{M',\varepsilon}$ on $T\partial M$ is exactly the Levi–Civita connection $\nabla^{\partial M}$ of the manifold ∂M. It follows from (2.7) that the connection $P^{\partial Z}\nabla^{Z',\varepsilon}$ coincides with $\nabla^{\partial Z}$.

We now define a new connection ∇' on TZ restricted to the tubular neighborhood $\mathscr{U}$. Namely if U' is a smooth section of $T\partial Z$, and Y a smooth section of $T^H M$, set

$$\begin{aligned} \nabla'_Y U' &= \nabla^{\partial Z}_Y U' \\ \nabla'_Y \frac{\partial}{\partial r} &= 0. \end{aligned} \tag{2.12}$$

Similarly if $Y \in TZ$, we assume that ∇'_Y is the Levi–Civita covariant differentiation operator of the fiber $Z \cap \mathscr{U}$.

Remember that on $\mathscr{U}$, $T^H M \subset T\partial M$, and that $T^H M$ is the pull back of its restriction to ∂M. This implies in particular that the coordinate r is preserved along integral curves of $T^H M$. It immediately follows that the connection ∇' preserves the metric of TZ.

Let T'^{ε} be the torsion of the Euclidean connection $\nabla' \oplus \nabla^B$ on TM. We claim that

(a) T'^{ε} vanishes on $TZ \times TZ$. This is clear since ∇' restricts on a fiber Z to the Levi–Civita connection of Z.

(b) We have the equality

$$T'^{\varepsilon}\left(\frac{\partial}{\partial r}, .\right) = 0. \tag{2.13}$$

To prove (2.13), we only need to check that if $V' \in TB$, if V^H denotes the horizontal lift of V in $T^H M$, then

$$T'^{\varepsilon}\left(\frac{\partial}{\partial r}, V^H\right) = 0. \tag{2.14}$$

Now observe that

$$\nabla^B_{\partial/\partial r} V^H = 0; \qquad \nabla'_{V^H} \frac{\partial}{\partial r} = 0. \tag{2.15}$$

Also since on $\mathscr{U}$, $T^H_{(r,y)}M = T^H_y M$, we find that

$$\left[\frac{\partial}{\partial r}, V^H\right] = 0. \tag{2.16}$$

(2.14) immediately follows from (2.15), (2.16).

(c) T'^{ε} takes its values in $T\,\partial Z$. In fact, if V, V' are smooth sections of TB, $P^Z[V^H, V'^H] \in T\,\partial Z$, since $T^H_{(r,y)}M = T^H_y M$ and $T^H_y M \subset T_y\,\partial M$.

Since ∇^B is torsion-free,

$$T'^{\varepsilon}(V^H, V'^H) = -P^{Z'}[V^H, V'^H] \tag{2.17}$$

and so $T'^{\varepsilon}(V^H, V'^H) \in T\,\partial Z$. Also if U is a smooth section of $T\,\partial Z$,

$$T'^{\varepsilon}(V^H, U) = \nabla'^{\partial Z}_{V^H} U - [V^H, U]. \tag{2.18}$$

By the same properties on $T^H M$ as before, $[V^H, U] \in T\,\partial Z$ and so $T'^{\varepsilon}(V^H, U) \in T\,\partial Z$. Using (b), it is now clear that T'^{ε} takes its values in $T\,\partial Z$.

We claim that if $U, U' \in TZ$, $Y \in T^H M$, then

$$\langle T'^{\varepsilon}(Y, U), U'\rangle - \langle T'^{\varepsilon}(Y, U'), U\rangle = 0. \tag{2.19}$$

If U or U' are equal to $\partial/\partial r$, (2.19) is a consequence of properties (b) and (c) which we proved before. To prove (2.19), we assume that U and U' lie in $T\,\partial Z$. Then $T'^{\varepsilon}(Y, U) = T^{\partial Z}(Y, U)$, $T'^{\varepsilon}(Y, U') = T^{\partial Z}(Y, U')$.

By [B1, Eq. (1.28)], we know that, since $T^{\partial Z}(U, U') = 0$, then

$$2\langle S^{\partial Z}(Y)U, U'\rangle + \langle T^{\partial Z}(Y, U), U'\rangle + \langle T^{\partial Z}(U', Y), U\rangle = 0. \tag{2.20}$$

On the other hand, the properties of S^{ε} listed after (2.8) also hold for $S^{\partial Z}$ In particular, we find that since U, $U' \in T\,\partial Z$, then $\langle S^{\partial Z}(Y)U, U'\rangle = 0$. It is now clear that (2.19) holds.

Therefore, the connection ∇' on TZ is Euclidean, T'^{ε} has properties (a), (c), and also is such that (2.19) holds. It is then elementary to verify that $\nabla' = P^{Z'} \nabla^{M',\varepsilon}$ and so $\nabla^{Z',\varepsilon} = \nabla'$. Therefore, we have proved the first two equalities in (2.10).

Also using [B1, Eq. (1.28)] again, we find that if U, $V \in TM$, then

$$2\left\langle S^{\varepsilon}\left(\frac{\partial}{\partial r}\right) U, V\right\rangle + \left\langle T^{\varepsilon}\left(\frac{\partial}{\partial r}, U\right), V\right\rangle + \left\langle T^{\varepsilon}\left(V, \frac{\partial}{\partial r}\right), U\right\rangle$$
$$- \left\langle T^{\varepsilon}(U, V), \frac{\partial}{\partial r}\right\rangle = 0. \tag{2.21}$$

The second and the third term in the l.h.s. of (2.21) clearly vanish. Also since $T^{\varepsilon}(U, V) \in T\,\partial Z$, the fourth term also vanishes. Therefore we find from (2.21) that $S^{\varepsilon}(\partial/\partial r) = 0$. The proof that $S^{\varepsilon}(.)(\partial/\partial r) = 0$ is strictly similar.

To complete the proof of the theorem, we only need to prove (2.11). Remember that $\nabla^{Z',\varepsilon}$ restricted to one given fiber Z' is the Levi–Civita connection of this fiber. It follows from Proposition 1.2 that if $U \in TZ$, then

$$R^{Z',\varepsilon}\left(\frac{\partial}{\partial r}, U\right) = 0. \tag{2.22}$$

Let now V, U' be smooth sections of TB and TZ, respectively, and let V^H be the horizontal lift of V in $T^H M$. Then using (2.16), we get

$$R^{Z',\varepsilon}\left(\frac{\partial}{\partial r}, V^H\right) U' = \nabla^{Z',\varepsilon}_{\partial/\partial r} \nabla^{Z',\varepsilon}_{V^H} U' - \nabla^{Z',\varepsilon}_{V^H} \nabla^{Z',\varepsilon}_{\partial/\partial r} U'. \tag{2.23}$$

By Definition 1.1 and Proposition 1.2, we know that $\nabla^{Z',\varepsilon}_{\partial/\partial r} \dfrac{\partial}{\partial r} = 0$. Also by (2.12), we know that $\nabla^{Z',\varepsilon}_{V^H}(\partial/\partial r) = 0$. So we obtain

$$R^{Z',\varepsilon}\left(\frac{\partial}{\partial r}, V^H\right) \frac{\partial}{\partial r} = 0. \tag{2.24}$$

We now assume that U' is a smooth section of $T\,\partial Z$ on ∂M, so that U' does not depend on the variable r. We then know that $\nabla^{Z',\varepsilon}_{V^H} U' \in T\,\partial Z$, and more precisely that

$$\nabla^{Z',\varepsilon}_{V^H} U' = \nabla^{\partial Z}_{V^H} U'. \tag{2.25}$$

From (2.25), we find that $\nabla'^{Z,\varepsilon}_{V^H} U'$ does not depend either on the variable r. Using Definition 1.1 and Proposition 1.2, we find that

$$\nabla^{Z',\varepsilon}_{\partial/\partial r} \nabla^{Z',\varepsilon}_{V^H} U' = \frac{1}{r} \nabla^{Z',\varepsilon}_{V^H} U. \tag{2.26}$$

Also we know that

$$\nabla^{Z',\varepsilon}_{\partial/\partial r} U' = \frac{U'}{r}.$$

Since $T^{H}M$ does not vary with r, it follows that

$$\nabla^{Z',\varepsilon}_{V^H}\nabla^{Z',\varepsilon}_{\partial/\partial r}U' = \frac{1}{r}\nabla^{Z',\varepsilon}_{V^H}U'. \tag{2.27}$$

From (2.26) and (2.27), we get

$$R^{Z',\varepsilon}\left(\frac{\partial}{\partial r}, V^H\right)U' = 0. \tag{2.28}$$

The first equality in (2.11) follows from (2.22), (2.24), and (2.28).

If $Y, Y' \in T^HM$, by (2.17), $T^\varepsilon(Y, Y') = -P^{Z'}[Y, Y']$. Using (2.10), we obtain

$$R^{Z',\varepsilon}(Y, Y')\frac{\partial}{\partial r} = -\nabla^{Z',\varepsilon}_{[Y,Y']}\frac{\partial}{\partial r} = -\nabla^{Z',\varepsilon}_{P^{Z'}[Y,Y']}\frac{\partial}{\partial r}. \tag{2.29}$$

The second line in (2.11) is proved. The third line in (2.11) is a consequence of the equality $\nabla^{Z',\varepsilon} = \nabla'$. The proof of the theorem is completed. ∎

Remark 2.3. The boundary ∂Z is not totally geodesic in Z for the metric $g^{Z,\varepsilon}$. However, $\nabla^{Z,\varepsilon}$ preserve $T\,\partial Z$ for horizontal displacements. This simple fact will enable us to define the Levi–Civita superconnections on Z and on ∂Z in a compatible way. This will be of utmost importance in our proof of the families index theorem.

(c) *The Case where* dim Z *Is Even*: *A Family of Dirac Operators*

Recall that TZ is an oriented spin vector bundle, which is equipped with a metric $g^{Z,\varepsilon}$. We now assume that the dimension n of the fibers Z is even, so that $n = 2l \geqslant 2$. We fix once for all a spin structure on TZ.

Let $F = F_+ \oplus F_-$ be the Z_2 graded Hermitian vector bundle of spinors over TZ'. F_+ and F_- are, of course, the bundles of positive and negative spinors over TZ' for the metric $g^{Z',\varepsilon}$. The Euclidean connection $\nabla^{Z',\varepsilon}$ immediately extends into a unitary connection on F_+ and F_-.

For every fiber Z'_b, we can use the results of Section 1. In particular, the restriction of F_+ to ∂M is the bundle of spinors over $T\,\partial Z$. The restriction of F_+ to ∂M is therefore equipped with the unitary connection $\nabla^{\partial Z}$. Moreover, if $(r, y) \in \,]0, 1[\, \times \partial M$, we identify $F_{(r,y)} = F_{+(r,y)} \oplus F_{-(r,y)}$ with $F_y = F_{+,y} \oplus F_{-,y}$.

Let ξ be a complex vector bundle on M. If ρ is the map $(r, y) \in \mathscr{U} \to \rho(r, y) = y \in \partial M$, we have the identification $\xi = \rho^*\xi_{\partial M}$. So we may and we will assume that ξ is equipped with a Hermitian metric h^ξ and a unitary connection ∇^ξ which are product on $\mathscr{U}$, i.e., are the pull-back by ρ of a metric and a connection on $\xi_{\partial M}$. Let L^ξ be the curvature of ∇^ξ.

Clearly,

$$L^{\xi}\left(\frac{\partial}{\partial r},\cdot\right)=0 \qquad \text{on} \quad \mathscr{U}. \tag{2.30}$$

In the same way as in Section 1, we extend ξ and ∇^{ξ} to the whole manifold M'. Of course, (2.30) still holds on $M' \setminus M$.

On M', the Hermitian vector bundles $F_{\pm} \otimes \xi$ are equipped with the connection $\nabla^{Z',\varepsilon} \otimes 1 + 1 \otimes \nabla^{\xi}$, which we still denote $\nabla^{Z',\varepsilon}$. Similarly, on ∂M, the Hermitian vector bundle $F_{+} \otimes \xi$ is equipped with the unitary connection $\nabla^{\partial Z} \otimes 1 + 1 \otimes \nabla^{\xi}$, which we denote by $\nabla^{\partial Z}$.

Remember that $f_1 = -\partial/\partial r$. We identify f_1 with the corresponding Clifford multiplication operator.

DEFINITION 2.4. For $b \in B$, let $H_b^{\infty} = H_{+,b}^{\infty} \oplus H_{-,b}^{\infty}$ be the vector space of smooth sections of $F \otimes \xi = (F_{+} \otimes \xi) \oplus (F_{-} \otimes \xi)$ over the fiber Z'_b which vanish at δ_b together with their derivatives.

D_b^{ε} denotes the Dirac operator acting on H_b^{∞} associated with the metric $g^{Z',\varepsilon}$ and the connection ∇^{ξ}. D_b^{ε} interchanges $H_{+,b}^{\infty}$ and $H_{-,b}^{\infty}$. We write D_b^{ε} in the form

$$D_b^{\varepsilon} = \begin{bmatrix} 0 & D_{-,b}^{\varepsilon} \\ D_{+,b}^{\varepsilon} & 0 \end{bmatrix}. \tag{2.31}$$

For $b \in B$, let $H_b'^{\infty}$ be vector space of smooth sections of $F_{+} \otimes \xi$ over the fiber ∂Z_b. Let $D_b^{\partial Z}$ be the Dirac operator acting on H_b^{∞}, naturally associated with the metric $g^{\partial Z}$ and the connection ∇^{ξ}.

Of course, for every $b \in B$, formulas (1.6)–(1.9) for D_b^{ε} and $D_b^{\partial Z}$ remain valid.

Let dx be the volume element in the fiber Z' with respect to the metric $g^{Z',1}$. Let dy be the volume element in the fiber ∂Z with respect to the metric $g^{\partial Z}$. If $h_1, h_2 \in H_b^{\infty}$ (resp. if $h'_1, h'_2 \in H_b'^{\infty}$) set

$$\langle h_1, h_2 \rangle = \int_{Z'_b} \langle h_1, h_2 \rangle (x)\, dx \tag{2.32}$$

(resp.

$$\langle h'_1, h'_2 \rangle = \int_{\partial Z_b} \langle h'_1, h'_2 \rangle (y)\, dy). \tag{2.33}$$

Remark 2.5. It is here appropriate to observe that $\operatorname{Ker} D_b^{\varepsilon}$ is not included in H_b^{∞}, since, as we saw in (1.22), for $\varepsilon > 0$ small enough, elements of Ker D_b^ε vanish at $\{\delta_b\}$ with arbitrary large, but still finite, order. H_b^∞ will only here play the role of a minimal domain for D_b^ε. Still, we will have to be careful when constructing connections on Ker D_b^ε—in the case where it is a vector bundle—from a connection on H_b^∞, since Ker D_b^ε is not included in H_b^∞.

(d) *The Levi–Civita Superconnections*

We now will define the Levi–Civita superconnection associated with our two families of operators D^ε and $D^{\partial Z}$. At this stage our discussion does not involve any sophisticated analytic arguments. It is closely related to [B1, Sections 2 and 3].

If $V \in TB$, let V^H be the horizontal lift of V in T^HM' so that $V^H \in T^HM'$, $\pi_* V^H = V$.

DEFINITION 2.6. $\tilde{\nabla}$ (resp. $\tilde{\nabla}'$) denotes the connection on the vector bundle H^∞ (resp. H'^∞) over B which is such that if h (resp. h') is a smooth section of H^∞ (resp. H'^∞) then

$$\tilde{\nabla}_Y h = \nabla^{Z',\varepsilon}_{Y^H} h \tag{2.34}$$

(resp.

$$\tilde{\nabla}'_Y h' = \nabla^{\partial Z}_{Y^H} h'). \tag{2.35}$$

As proved in [B1, Proposition 1.11] the curvature tensors of $\tilde{\nabla}$ and $\tilde{\nabla}'$ take their values in the set of first-order differential operators (which act fiberwise on Z' or ∂Z).

Following [BF1, BF2], we briefly show how to construct unitary connections on H^∞ and on H'^∞ with respect to the Hermitian products (2.32) and (2.33).

DEFINITION 2.7. Let $e'_1, ..., e'_n$ be an orthonormal base of TZ'. Let k^ε denote the vector in T^HM',

$$k^\varepsilon = -\tfrac{1}{2} S^\varepsilon(e'_i)e'_i. \tag{2.36}$$

If $e_1, ..., e_{n-1}$ is an orthonormal base of ∂Z, since by Theorem 2.2, $S^\varepsilon(\partial/\partial r) = 0$, we find that

$$k^\varepsilon = -\tfrac{1}{2} S^\varepsilon(e_i)e_i \qquad \text{on} \quad \partial M. \tag{2.37}$$

We now define the connections $\tilde{\nabla}^u$ and $\tilde{\nabla}'^u$ on H^∞ and H'^∞ as in [BF1, Definition 1.3].

DEFINITION 2.8. $\tilde{\nabla}^u$ (resp. $\tilde{\nabla}'^u$) denotes the connection on the vector bundle H^∞ (resp. H'^∞) such that if $V \in TB$,

$$\tilde{\nabla}^u = \tilde{\nabla} + \langle k, V^H \rangle \tag{2.38}$$

(resp.

$$\tilde{\nabla}^u = \tilde{\nabla}' + \langle k, V^H \rangle). \tag{2.39}$$

Then by [BF1, Proposition 1.4; BF2, Proposition 1.4] $\tilde{\nabla}^u$ (resp. $\tilde{\nabla}'^u$) is a unitary connection with respect to the Hermitian product (2.32) on H^∞ (resp. the Hermitian product (2.33) on H'^∞).

A final ingredient in the definition of the Levi–Civita superconnections is the torsion tensor T^ε. In fact, the Clifford algebra $c(TZ)$ acts by Clifford multiplication on $F \otimes \xi = (F_+ \otimes \xi) \oplus (F_- \otimes \xi)$. Similarly, the Clifford algebra $c(T\partial Z)$ acts by Clifford multiplication on the restriction of $F_+ \otimes \xi$ to ∂M.

We now use Quillen's superconnection formalism [Q] as in [B1]. Namely, on M', our computations will be done in the Z_2 graded algebra $(\Lambda_b(T^*B) \hat{\otimes} c_x(TZ')) \otimes \mathrm{End}_x \xi$ (when x varies in the fiber Z'_b). Similarly, on ∂Z, our computations are done in the Z_2 graded algebra $(\Lambda_b(T^*B) \hat{\otimes} c_y(T\partial Z)) \otimes \mathrm{End}_y \xi$ (where $y \in \partial Z_b$).

The vector bundle $H^\infty = H^\infty_+ \oplus H^\infty_-$ is Z_2 graded. Let τ be the involution defining the grading i.e. $\tau = \pm 1$ or $H^\infty_\pm$. End H^∞ is a Z_2 graded algebra, the even (resp. odd) elements commuting (resp. anticommuting) with τ.

As explained in [B1, Section 2b)], our rules of computation on Z' require that we work in the graded tensor product $\Lambda(T^*B) \hat{\otimes} \mathrm{End}\, H^\infty$.

Our computations on ∂Z are slightly subtler. As explained in [BF2, Section 2f)], our conventions are compatible with the conventions of Quillen [Q, Section 5].

Let $f_1, ..., f_m$ be a base of TB, $db^1, ..., db^m$ the corresponding dual base in T^*B. We identify $f_1, ..., f_m$ with their horizontal lifts $f_1^H, ..., f_m^H$. As in Section 1, we identify $X \in TZ'$ with the corresponding element in the Clifford algebra $c(TZ')$.

DEFINITION 2.9. For $x \in M$, $c_x(T^\varepsilon)$ denotes the odd element of $\Lambda_{\pi(x)}(T^*B) \hat{\otimes} c_x(TZ)$

$$c_x(T^\varepsilon) = \sum_{\alpha<\beta} db^\alpha\, db^\beta\, T^\varepsilon_x(f_\alpha, f_\beta). \tag{2.40}$$

For $y \in \partial M$, $c_y(T^{\partial Z})$ denotes the odd element in $\Lambda_{\pi(x)}(T^*B) \hat{\otimes} c_y(T\partial Z)$

$$c_y(T^{\partial Z}) = \sum_{\alpha<\beta} db^\alpha\, db^\beta T^{\partial Z}_y(f_\alpha, f_\beta). \tag{2.41}$$

Note that by Theorem 2.2, on $\mathscr{U}$, T^ε takes its values in $T(\partial Z)$. In particular, on ∂M, $c(T^\varepsilon)$ restricts to the corresponding element $c(T^{\partial Z})$ in $\Lambda(T^*B) \hat{\otimes} c(T\partial Z)$.

In the sense of Quillen [Q], for any $t>0$, $\tilde{\nabla}^u + \sqrt{t}\, D^\varepsilon$ is a superconnection on the Z_2 graded vector bundle H^∞, and $\tilde{\nabla}'^u + \sqrt{t}\, D^{\partial Z}$ is a superconnection on the vector bundle H'^∞. Following [B1, Section 3], [BF2, Proposition 1.18] we now define the Levi–Civita superconnection of parameter $t>0$.

Definition 2.10. For $t>0$, A_t^ε denotes the superconnection on the Z_2 graded vector bundle H^∞,

$$A_t^\varepsilon = \tilde{\nabla}^u + \sqrt{t}\, D^\varepsilon - \frac{c(T^\varepsilon)}{4\sqrt{t}}. \tag{2.42}$$

Similarly, A'_t denotes the superconnection on the vector bundle H'^∞,

$$A'_t = \tilde{\nabla}^u + \sqrt{t}\, D^{\partial Z} - \frac{c(T^{\partial Z})}{4\sqrt{t}}. \tag{2.43}$$

Note that A'_t is exactly the superconnection considered in [BF2, Section 2f)]. Also A_t^ε is the obvious extension of the superconnection constructed in [B1, Section 3] to the manifold with conical singularities M'. It is a remarkable fact that, in the same way as D^ε "restricts" on ∂Z to $D^{\partial Z}$, the Levi–Civita superconnection A_t^ε "restricts" on ∂M to A'_t.

We now briefly describe the superconnection A_t^ε on the manifold $M' \backslash M$. Remember that for $(r, y) \in]0, 1] \times \partial M$, we identified $(F \otimes \xi)_{(r,y)}$ with $(F \otimes \xi)_y$

If h is a smooth section of H_+^∞ on B, one verifies easily that if $V \in TB$, for $r \in]0, 1]$

$$\begin{aligned} (\tilde{\nabla}_V h)(r, y) &= \tilde{\nabla}'_V h(r, .)(y) \\ (\tilde{\nabla}^u_V h)(r, y) &= \tilde{\nabla}'^u_V h(r, .)(y). \end{aligned} \tag{2.44}$$

The meaning of (2.44) is that the value of the coordinate r is irrelevant in the computation of $(\tilde{\nabla}_V h)(r, y)$ or $\tilde{\nabla}^u_V h(r, y)$, i.e., that the calculation can be done at $r=1$.

If $\tilde{e}_1, ..., \tilde{e}_{n-1}$ is an orthonormal base of $T\partial Z$ at (r, y) for the metric $g^{Z',\varepsilon}$, then

$$c_{(r,y)}(T^\varepsilon) = \sum_{\substack{\alpha<\beta \\ i \in [1, n-1]}} \langle T^\varepsilon_{(r,y)}(f_\alpha, f_\beta), \tilde{e}_i \rangle \, db^\alpha \, db^\beta \tilde{e}_i.$$

Let $e_1, ..., e_{n-1}$ be an orthonormal base of $T\partial Z$ for the metric $g^{\partial Z}$. The restriction of $g'^{Z,\varepsilon}$ to $T\partial Z$ at (r, y) is $r^2 g^{\partial Z}$.

Identifying $(F\otimes\xi)_{(r,y)}$ with $(F\otimes\xi)_y$, we thus find that

$$c(T^\varepsilon)_{(r,y)} = r \sum_{\substack{\alpha<\beta \\ i\in[i,n-1]}} \langle T_y^{\partial Z}(f_\alpha, f_\beta), e_i\rangle_{g^{\partial Z}}\, db^\alpha\, db^\beta\, e_i \tag{2.45}$$

or, equivalently,

$$c(T^\varepsilon)_{r,y)} = rc(T^{\partial Z})_y. \tag{2.46}$$

Using (1.8), (2.41), (2.46), we find that on $M'\backslash M$, A_t^ε is given by the formula

$$A_t^\varepsilon = \tilde{\nabla}^u + \sqrt{t}\left[\sqrt{\varepsilon}f_1\left(-\frac{\partial}{\partial r} - \frac{n-1}{2r}\right) + \frac{D^{\partial Z}}{r}\right] - \frac{rc(T^{\partial Z})}{4\sqrt{t}}. \tag{2.47}$$

Note that $D^{\partial Z}$ and $c(T)$ scale by the factors $1/r$ and r, respectively. The fact that these two scales differ will play a key role in Section 6.

III. Fredholm Properties, Existence, and Equality of the Index Bundles

In this section, we will establish that under natural assumptions, for $\varepsilon > 0$ small enough, the family of operators D_+^ε defines an index bundle $\operatorname{Ker} D_+^\varepsilon - \operatorname{Ker} D_-^\varepsilon$ in $K^0(B)$, which coincides with the corresponding index bundle of Atiyah, Patodi, and Singer [APS1].

Our assumptions and notations are the same as in Section 2(c).

Let g^Z be a smooth metric on the vector bundle TZ which has the following two properties:

- g^Z restricts to $g^{\partial Z}$ on $T\partial Z$.
- g^Z is product near the boundary ∂M, i.e., on the tubular neighborhood $\mathcal{U}$, g^Z is given by

$$g^Z = dr^2 + g^{\partial Z}. \tag{3.1}$$

For simplicity, we still denote by F the Hermitian vector bundle of TZ spinors for the metric g^Z (for the fixed spin structure of TZ).

For every $b \in B$, we can define the Dirac operator D_b of Atiyah, Patodi, and Singer [APS1], acting on the smooth section of $F\otimes\xi$ over the fiber Z_b, associated with the metric g^Z on Z_b and the connection ∇^ξ on ξ_{Z_b}. Let

$P_{+,b}$, $P_{-,b}$ be the orthogonal projection operators on the direct sum of eigenspaces of $D_b^{\partial Z}$ corresponding to nonnegative and negative eigenvalues.

By [APS1, Section 2], $P_{+,b}$ and $P_{-,b}$ are pseudo-differential operators of order 0, and so act on the various Sobolev spaces.

D_b splits into $\begin{bmatrix} 0 & D_{-,b} \\ D_{+,b} & 0 \end{bmatrix}$, where $D_{+,b}$ acts on the smooth sections ϕ of $F_+ \otimes \xi$ on Z_b such that $P_{+,b}\phi = 0$, and $D_{-,b}$ acts on the smooth sections ψ of $F_- \otimes \xi$ on Z_b such that $P_{-,b}\psi = 0$. As shown in [APS1], $D_{+,b}$ is a Fredholm operator, and its index $\operatorname{Ind} D_{+,b}$ is given by

$$\operatorname{Ind} D_{+,b} = \dim \operatorname{Ker} D_{+,b} - \dim \operatorname{Ker} D_{-,b}. \tag{3.2}$$

$\operatorname{Ind} D_{+,b}$ is given by the Atiyah-Patodi-Singer formula of Theorem 1.17.

Let $\bar{\eta}_b(0)$ be the reduced êta invariant for the operator $D_b^{\partial Z}$. In general, $b \to \bar{\eta}_b(0)$ is not continuous and has integer jumps. $\bar{\eta}_b(0)$ has a jump of $+1$ if a negative eigenvalue of $D_b^{\partial Z}$ reaches 0, and a jump of -1 if a 0 eigenvalue becomes negative. Since the index, $\operatorname{Ind} D_{+,b}$, can jump, $D_{+,b}$ does not define a continuous family of Fredholm operators and there is no well-defined index bundle $\operatorname{Ker} D_+ - \operatorname{Ker} D_-$ in the sense of Atiyah and Singer [AS2]. Therefore, a necessary condition for the existence of an index bundle $\operatorname{Ker} D_+ - \operatorname{Ker} D_-$ in $K^0(B)$ is that $\operatorname{Ker} D_b^{\partial Z}$ is itself a vector bundle on B.

On the other hand, $b \to D_b^{\partial Z}$ is a family of Fredholm self-adjoint operators. By [APS3, Section 3], it defines an element of $K^1(B)$. If $\operatorname{Ker} D^{\partial Z}$ is a vector bundle, the map $b \to D_b^{\partial Z}$ is homotopically trivial, i.e., the corresponding element in $K^1(B)$ is trivial.

In the sequel we make the fundamental assumption H1:

(H1) For any $b \in B$, $\operatorname{Ker} D_b^{\partial Z} = \{0\}$.

We now precisely describe the families of Dirac operators which we will consider. Recall that we identify $F_+ \otimes \xi$ and $F_- \otimes \xi$ on $(M' \backslash M) \cup \partial M$.

DEFINITION 3.1. For $l \geq 0$, $b \in B$, $\tilde{H}^l_{\pm,b}$ denote the lth Sobolev space of sections of $F_\pm \otimes \xi$ on the manifold Z_b, H'^l_b is the lth Sobolev space of sections of $F_+ \otimes \xi$ on ∂Z_b.

For $l \geq 1$, $\tilde{H}^l_{\pm,b}(P)$ is the subspace of sections of $F_\pm \otimes \xi$ in $\tilde{H}^l_{\pm,b}$ such that if $jh \in H'^{l-1/2}_b$ is the restriction of h to ∂Z_b, then $P_\pm(jh) = 0$.

For $l \geq 0$, $b \in B$, $H^l_{\pm,b}$ denotes the lth Sobolev space of section of $F_\pm \otimes \xi$ on Z'_b.

It is of utmost importance to observe that while $\tilde{H}^l_{\pm,b}$ does not depend on the metric $g^{Z,\varepsilon}$, $H^l_{\pm,b}$ depends in an essential way on the metric (2.6) on $C(\partial Z)$, because $Z'_b \backslash \{\delta_b\}$ is an open manifold. Still $H^l_{\pm,b}$ does not depend on $g^{Z,\varepsilon}$ or on $g^{\partial Z}$.

We now prove the essential result of this section, part of which is already in Atiyah, Patodi, and Singer [APS1, Section 2].

THEOREM 3.2. *For any* $l \geqq 0$ (*resp.* $l \geqq 1$), $\tilde{H}'^{l}_{\pm}$ (*resp.* $\tilde{H}'^{l}_{\pm}(P)$) *is a continuous Hilbert bundle on B. Moreover, for any* $l \geqq 1$, D_+ *is a continuous family of Fredholm operators in* $\mathrm{Hom}(\tilde{H}'^{l}_{+}(P), \tilde{H}'^{l-1}_{-})$. *The corresponding index bundle* $\mathrm{Ker}\, D_+ - \mathrm{Ker}\, D_- \in K^0(B)$ *does not depend on* $l \geqq 1$.

For any $l \geqq 0$, $H'^{l}_{\pm}$ *is continuous Hilbert bundle on B. For* $\varepsilon > 0$ *small enough,* D^{ε}_{+} *is a continuous family of Fredholm operators in* $\mathrm{Hom}(H^1_+, H^0_-)$, *and the corresponding index bundle* $\mathrm{Ker}\, D^{\varepsilon}_{+} - \mathrm{Ker}\, D^{\varepsilon}_{-} \in K^0(B)$ *does not depend on* $\varepsilon > 0$. *More precisely, for* $\varepsilon > 0$ *small enough*

$$\mathrm{Ker}\, D^{\varepsilon}_{+} - \mathrm{Ker}\, D^{\varepsilon}_{-} = \mathrm{Ker}\, D_+ - \mathrm{Ker}\, D_- \qquad \text{in } K^0(B). \tag{3.3}$$

Proof. The proof of Theorem 3.2 is divided into several steps.

(a) $\tilde{H}'^{l}_{\pm}$, $\tilde{H}'^{l}_{\pm}(P)$ *are continuous Hilbert bundles on B.* Let $b_0 \in B$ and let V be a small open ball in B centered at b_0 such that $\pi^{-1}(V)$ is diffeomorphic to $V \times Z_{b_0}$. If ρ is the projection $V \times Z_{b_0} \to Z_{b_0}$, using parallel transport along the horizontal lines in $V \times Z_{b_0}$ which lift the radial lines starting at b_0, we can smoothly identify $(F \otimes \xi)_{V \times Z_b}$ with $\rho^*(F \otimes \xi)_{Z_{b_0}}$. For $b \in V$, we can thus identify $\tilde{H}'^{l}_{\pm,b}$ with $\tilde{H}'^{l}_{\pm,b_0}$; i.e., we have found a trivialization of the vector bundle $\tilde{H}'^{l}_{\pm}$ on V. One then easily verifies that the transition maps associated with two such trivializations are continuous (and in general are not smooth). We have thus proved that $\tilde{H}'^{l}_{\pm}$ is a continuous Hilbert bundle on B. The same argument shows that H''^{l} is a continuous Hilbert bundle on B.

Since $\mathrm{Ker}\, D^{\partial Z} = \{0\}$, we find that

$$P_+ = \tfrac{1}{2}(D^{\partial Z})^{-1}(D^{\partial Z} + |D^{\partial Z}|). \tag{3.4}$$

Using the results of Seeley [Se], we know that P_+ is a continuous family of zero order pseudodifferential operators on the fibers ∂Z. Then H''^{l} splits into

$$H''^{l} = H''^{l}_{+} \oplus H''^{l}_{-},$$

where $H''^{l}_{+} = P_+[H'_+]$, $H''^{l}_{-} = (I - P_+)(H''^{l})$.

We claim that $H''^{l}_{\pm}$ is a continuous Hilbert subbundle of H''^{l}. In fact, $D^{\partial Z}$ is a continuous section of $\mathrm{Hom}(H''^{l}, H''^{l-1})$ (this last bundle being endowed with the norm topology). By Seeley [Se], $b \to |D^{\partial Z}|$ has the same property. Since $D^{\partial Z}$ is invertible, $b \to P_{+,b}$ is a continuous section of $\mathrm{Hom}(H''^{l}, H''^{l})$.

Take b_0, V as before, so that if $b \in V$, H''^{l}_{b} is identified with $H''^{l}_{b_0}$. For $b \in V$, $P_{+,b}$ now acts on $H''^{l}_{b_0}$, and so $H''^{l}_{+,b}$ is a subspace of $H''^{l}_{b_0}$. Since $P_{+,b}$

is the identity on $H'^{l}_{+,b}$, we then find that for b close to b_0, $P_{+,b_0}P_{+,b}$ (resp. $P_{+,b}P_{+,b_0}$) is invertible in $\mathrm{End}(H'_{+,b_0})$ (resp. $\mathrm{End}(H'_{+,b})$). Therefore for b close to b_0, $P_{+,b}$ is a one to one map from H'^{l}_{+,b_0} into $H'^{l}_{+,b}$. So $h \in H'^{l}_{+,b_0} \to P_{+,b}h \in H'^{l}_{+,b}$ is an explicit continuous trivialization of H'^{l}_{+} on a small neighborhood of b_0 in B. It is now clear that H'^{l}_{+} and H'^{l}_{-} are continuous Hilbert subbundles of H'^{l}.

Let $b_0 \in B$, and let V be an open ball in B centered at b_0 which is small enough so that the vector bundles $\tilde{H}^{l}_{\pm}$, H'^{l}, $H'^{l}_{\pm}$ are identified with $\tilde{H}^{l}_{\pm,b_0}$, $H'^{l}_{b_0}$, $H^{l}_{\pm,b_0}$.

For $l \geqq 1$, recall that the restriction map j: $\tilde{H}^{l} \to H'^{l-1/2}$ is surjective. Let Q_{b_0} be the orthogonal projection operator from $\tilde{H}^{l}_{+,b_0}$ on $\tilde{H}^{l}_{+,b_0}(P)$. For $b \in V$, let E_b be the linear map

$$E_b\colon \phi \in \tilde{H}^{l}_{+,b_0} \to (P_{+,b}j\phi, Q_{b_0}\phi) \in H'^{l-1/2}_{+,b_0} \oplus \tilde{H}^{l}_{+,b_0}(P). \tag{3.5}$$

Now E_{b_0} is clearly a one-to-one map. Therefore, for b close enough to b_0, E_b is one to one. Since $\tilde{H}^{l}_{+,b}(P) = \mathrm{Ker}(P_{+,b}j)$, we find that $\phi \in \tilde{H}^{l}_{+,b_0}(P) \to E_b^{-1}(\phi)$ provides an explicit trivialization of $\tilde{H}^{l}_{+}(P)$ on a small neighborhood of b_0 in B. It is now clear that $\tilde{H}^{l}_{+}(P)$ and $\tilde{H}^{l}_{-}(P)$ are continuous Hilbert bundles on B.

As in the proof of Theorem 1.5, for every $b \in B$, we consider the Dirac operator $\tilde{D}^{\varepsilon}_{b} = \left[\begin{smallmatrix} 0 & \tilde{D}^{\varepsilon}_{-,b} \\ \tilde{D}^{\varepsilon}_{+,b} & 0 \end{smallmatrix}\right]$ of Atiyah, Patodi, and Singer [APS1] associated with the metric $g^{Z,\varepsilon}$ on TZ_b, the connection ∇^{ξ} and also the boundary conditions of [APS1]. The only difference with [APS1] is that $g^{Z,\varepsilon}$ is not a product near the boundary ∂Z. However, inspection of [APS1, Sections 2 and 3] shows that the results of [APS1] are still valid in this case. Note that as in the proof of Theorem 1.5, we can give another proof which does not necessitate the introduction of the family $\tilde{D}^{\varepsilon}_{+}$.

(b) *For any* $\varepsilon > 0$, $l \geqq 1$, D_+ *and* $\tilde{D}^{\varepsilon}_{+}$ *are continuous families of Fredholm operators in* $\mathrm{Hom}(\tilde{H}^{l}_{+}(P), \tilde{H}^{l}_{-})$ *which define the same index bundle*. By [APS1, Section 3], for $l \geqq 1$, for every $b \in B$, $D_{+,b}$ and $\tilde{D}^{\varepsilon}_{+,b}$ are Fredholm operators in $\mathrm{Hom}(\tilde{H}^{l}_{+}(P), \tilde{H}^{l-1}_{-})$. Since $D_{+,b}$ and $\tilde{D}^{\varepsilon}_{+,b}$ have smooth coefficients, it is clear that D_+ and $\tilde{D}^{\varepsilon}_{+}$ are continuous sections of the bundle $\mathrm{Hom}(\tilde{H}^{l}_{+}(P), \tilde{H}^{l-1}_{-})$.

By [A, p. 158; AS2], the families D_+ and $\tilde{D}^{\varepsilon}_{+}$ define virtual index bundles $\mathrm{Ker}\, D_+ - \mathrm{Ker}\, D_-$ and $\mathrm{Ker}\, \tilde{D}^{\varepsilon}_{+} - \mathrm{Ker}\, \tilde{D}^{\varepsilon}_{-}$ in $K^0(B)$, which do not depend on l.

For $0 \leqq s \leqq 1$, if $g^{Z,\varepsilon}_{s} = (1-s)g^{Z} + sg^{Z,\varepsilon}$, we find that the corresponding family of Fredholm operators $\tilde{D}^{\varepsilon,s}_{+}$ depends continuously on s and has the same properties as D_+ and $\tilde{D}^{\varepsilon}_{+}$. Therefore, the families D_+ and $\tilde{D}^{\varepsilon}_{+}$ define the same index bundle in $K^0(B)$.

(c) *For* $\varepsilon > 0$ *small enough, the family* D^{ε}_{+} *is a continuous family of Fredholm operators in* $\mathrm{Hom}(H^{1}_{+}, H^{0}_{-})$. It follows from Chou [Ch] that for $\varepsilon > 0$ small enough, for any $b \in B$, D^{ε}_{b} is a self-adjoint operator defined on its natural domain, which is not explicitly determined in terms of the Sobolev spaces we are considering. We here need a more precise statement, in order to prove that the family D^{ε}_{+} is a continuous family of Fredholm operators in $\mathrm{Hom}(H^{1}_{+}, H^{0}_{-})$. To do this, we will establish several estimates in which the constants C may vary from line to line.

We first prove that for any $b \in B$, $D^{\varepsilon}_{+,b}$ map $H^{1}_{+,b}$ into $H^{0}_{-,b}$. In fact, from formula (1.8) for $D^{\varepsilon}_{+,b}$ on the cone $C(\partial Z_b)$, we find that if $h \in H^{-1}_{+,b}$, then

$$\int_{Z'_b} |D^{\varepsilon}_{+,b} h|^2 (x)\, dx \leqq C \left[\|h\|^2_{H^1_{+,b}} + \int_{C(\partial Z_b)} \frac{|f|^2}{r^2}\, dx \right]. \tag{3.6}$$

Since $D^{\partial Z}_{b}$ is invertible, we get

$$\int_{\partial Z_b} |f|^2 (r, y)\, dy \leqq C' \int_{\partial Z_b} |D^{\partial Z}_{b} f|^2 (r, y)\, dy \tag{3.7}$$

and so

$$\begin{aligned} \int_{C(\partial Z_b)} \frac{|f|^2}{r^2} (r, y)\, dx &\leqq C \int_{C(\partial Z_b)} \frac{|D^{\partial Z}_{b} f|^2}{r^2} (r, y)\, dx \\ &\leqq C\, \|h\|^2_{H^1_{+,b}}. \end{aligned} \tag{3.8}$$

From (3.6), (3.8), we find that

$$|D^{\varepsilon}_{+,b} h|^2 \leqq C\, \|h\|^2_{H^1_{+,b}} \tag{3.9}$$

and the constant C in (3.9) can be chosen independently of b (which varies in the compact manifold B).

To prove that $D^{\varepsilon}_{+,b}$ is Fredholm from $H^{1}_{+,b}$ into $H^{0}_{-,b}$, we will construct a parametrix for $D^{\varepsilon}_{+,b}$. To do this, we will patch a parametrix for $D^{\varepsilon}_{+,b}$ near δ_b with a "classical" parametrix for $D^{\varepsilon}_{+,b}$ far from δ_b. Let $C^{\infty}(\partial Z_b)$ be the infinite cone

$$C^{\infty}(\partial Z_b) = (]0, +\infty[\times \partial Z_b) \cup \{\delta_b\}$$

which we still endow with the metric

$$\frac{dr^2}{\varepsilon} + r^2 g^{\partial Z}.$$

Recall that F_{+} and F_{-} are identified on ∂Z_b and so F_{+} and F_{-} are identified on $C^{\infty}(\partial Z_b)$. Let $H^{l,\infty}_{+,b}$ denote the lth Sobolev space of sections of $F_+ \otimes \xi$ over $C^\infty(\partial Z)$ and let $H^{l,\mathrm{loc},\infty}_{+,b}$ be the set of sections h of $F_+ \otimes \xi$ on $C^\infty(\partial Z_b)$ such that if $f(r)$ is a C^∞ function of r which vanishes for r large enough, then $fh \in H^{l,\infty}_{+,b}$.

We now use a technique closely related to Atiyah *et al.* [APS1, Section 2]. For $r, r' > 0$, let $K^\varepsilon_b(r, r')$ be the operator

$$K^\varepsilon_b(r, r') = \frac{1}{\sqrt{\varepsilon}} \left(\frac{r'}{r}\right)^{(n-1)/2 + D^{\partial Z}_b/\sqrt{\varepsilon}} \times (P_{+,b} 1_{r' < r} - P_{-,b} 1_{r' > r}). \tag{3.10}$$

Since $D^{\partial Z}_b$ is elliptic, for $r' \neq r$, $K^\varepsilon_b(r, r')$ is given by a C^∞ kernel $K^\varepsilon_b((r, y), (r', y'))$ on ∂Z_b, which depends smoothly on r, r' (when $r \neq r'$). Also for any $\gamma > 0$, $m \in N$, for $\varepsilon > 0$ small enough, when r (resp. r') stays away from 0, the kernel $K^\varepsilon_b((r, y), (r', y'))$ and its derivatives in r, y (resp. its derivatives of order $\leq m$) tend to 0 as $r' \downarrow 0$ (resp. $r \downarrow 0$) faster than r'^γ (resp. r^γ)

Let $C^\infty_{+,b}$ (resp. $C^{\infty,\mathrm{comp}}_{+,b}$) be the set of C^∞ sections of $F_+ \otimes \xi$ on $]0, +\infty[\times \partial Z_b$ (resp. which have compact support). Let K^ε_b be the operator,

$$K^\varepsilon_b : h \in C^{\infty,\mathrm{comp}}_{+,b} \to \int_0^{+\infty} K^\varepsilon_b(r, r')\, h(r', y)\, dr'\, dy \in C^\infty_{+,b}. \tag{3.11}$$

We claim that K^ε_b extends into a continuous operator from $H^{0,\infty}_{+,b}$ into $H^{1,\mathrm{loc},\infty}_{+,b}$. Set $v = \log r$. When $0 < r < +\infty$, then $-\infty < v < +\infty$. Let $\mathscr{D}^{\varepsilon,\infty}_{+,b}$ be the operator,

$$\mathscr{D}^{\varepsilon,\infty}_{+,b} = \frac{\partial}{\partial v} + \frac{D^{\partial Z}_b}{\sqrt{\varepsilon}} + \frac{1}{2}. \tag{3.12}$$

If we use the new coordinate v, $D^{\varepsilon,\infty}_{+,b}$ is given by

$$D^{\varepsilon,\infty}_{+,b} = \sqrt{\varepsilon}\, e^{-v} \left(\frac{\partial}{\partial v} + \frac{D^{\partial Z}_b}{\sqrt{\varepsilon}} + \frac{n-1}{2}\right). \tag{3.13}$$

We find that

$$D^{\varepsilon,\infty}_{+,b} = \sqrt{\varepsilon}\, e^{-(n/2)v}\, \mathscr{D}^{\varepsilon,\infty}_{+,b} e^{((n-2)/2)v}. \tag{3.14}$$

Let $L^\varepsilon_t(v)$ be the operator

$$L^\varepsilon_b(v) = e^{-v\left(\frac{D^{\partial Z}_b}{\sqrt{\varepsilon}} + \frac{1}{2}\right)} (P_{+,b} 1_{v > 0} - P_{-,b} 1_{v < 0}). \tag{3.15}$$

Note that $L_b^\varepsilon(v)$ appears in [APS1] in a related context. For $l \geqq 0$, let $H_{+,b}^{l,v}$ be the lth Sobolev space of sections of $F_+ \otimes \xi$ on $]-\infty, +\infty[\times \partial Z_b$ (which is now endowed with the metric $dv^2 + g^{\partial Z}$). We claim that for $\varepsilon > 0$ small enough, for any $b \in B$, $l \geqq 0$,

$$h \to L_b^\varepsilon h\colon L_b^\varepsilon h(v, \cdot) = \int_{-\infty}^{+\infty} L_b^\varepsilon(v - v')\, h(v')\, dv' \tag{3.16}$$

is a well-defined continuous map from $H_{+,b}^{l,v}$ into $H_{+,b}^{l+1,v}$.

If $\hat{v}$ is a variable dual to v, by taking the Fourier transforms $\hat{h}$ and $\widehat{L_b^\varepsilon h}$ of h and $L_b^\varepsilon h$ in the variable v, we find that

$$\widehat{L_b^\varepsilon h} = \left(i\hat{v} + \frac{D_b^{\partial Z}}{\sqrt{\varepsilon}} + \frac{1}{2}\right)^{-1} \hat{h}.$$

Since $\operatorname{Ker} D_b^{\partial Z} = \{0\}$, we find that for $\varepsilon > 0$ small enough, L_b^ε has the required continuity property. Observe that

$$K_b^\varepsilon = \frac{1}{\sqrt{\varepsilon}} e^{-((n-2)/2)v} L_b^\varepsilon e^{(n/2)v}. \tag{3.17}$$

The map $h \in H_{+,b}^{0,\infty} \to e^{nv/2} h \in H_{+,b}^{0,v}$ is an isometry of Hilbert spaces. We claim that $h \to e^{-((n-2)/2)v} h$ defines a continuous map from $H_{+,b}^{1,v}$ into $H_{+,b}^{1,\mathrm{loc},\infty}$. In fact if $h \in H_{+,b}^{1,v}$, for any $A > 0$,

$$\begin{aligned} &\int_{]-\infty, A[\times \partial Z_b} \left[e^{-2v} \left(\left| \left(\frac{\partial}{\partial v} - \frac{n-2}{2} \right) h \right|^2 + |\nabla h|^2 \right) + |h|^2 \right] \\ &\qquad \times e^{-(n-2)v} e^{nv}\, dv\, dy \\ &\quad \leqq C \left[\int_{R \times \partial Z_b} \left[\left| \frac{\partial}{\partial v} h \right|^2 + |\nabla h|^2 \right] dv\, dy \right. \\ &\qquad \left. + e^{2A} \int_{R \times \partial Z_b} |h|^2\, dy\, dv \right]. \end{aligned} \tag{3.18}$$

From (3.17), (3.18), we find that K_b^ε is indeed continuous from $H_{+,b}^{0,\infty}$ into $H_{+,b}^{1,\mathrm{loc},\infty}$. Finally, using obvious notations, we find that

$$D_{+,b}^{\varepsilon,\infty} K_b^\varepsilon = K_b^\varepsilon D_{+,b}^{\varepsilon,\infty} = \text{Identity}. \tag{3.19}$$

We now will use K_b^ε as a parametrix for $D_{+,b}^\varepsilon$ near δ_b.

For $0 \leqq a < b < 1$, let $\rho(a, b)$ denote a function of r which is C^∞, non-negative, increasing, equal to 0 for $r \leqq a$ and to 1 for $r \geqq b$. $\rho(a, b)$ is well defined on $C(\partial Z_b)$. We extend $\rho(a, b)$ to Z_b by assuming that $\rho(a, b) = 1$ on Z_b. $\rho(a, b)$ is now defined on Z'_b.

Let $C^{\beta}(\partial Z_b)$ be the cone $(]0,\beta[\times\partial Z_b)\cup\{\delta_b\}$. Let A^{ε}_b be any parametrix for $D^{\varepsilon,b}_{+}$ on the open manifold $Z'_b\backslash C^{1/8}(\partial Z_b)$. We proceed as in [APS1, Section 3]. Set

$$Q^{\varepsilon}_{-,b}=(1-\rho(\tfrac{4}{5},1))\,K^{\varepsilon}_b(1-\rho(\tfrac{1}{2},\tfrac{3}{4}))+\rho(\tfrac{1}{4},\tfrac{2}{5})\,A^{\varepsilon}_b\rho(\tfrac{1}{2},\tfrac{3}{4}). \qquad (3.20)$$

Clearly for $\varepsilon>0$ small enough, $Q^{\varepsilon}_{-,b}$ maps continuously $H^0_{-,b}$ into $H^1_{+,b}$. We claim that for $\varepsilon>0$ small enough, $D^{\varepsilon}_{+,b}Q^{\varepsilon}_{-,b}-\mathrm{Id}_{H^0_{-,b}}$ is a compact operator from $H^0_{-,b}$ into itself.

Here, we essentially need to prove that for $\varepsilon>0$ small enough, the operator

$$\rho'(\tfrac{4}{5},1)K^{\varepsilon}_b(1-\rho(\tfrac{1}{2},\tfrac{3}{4})) \qquad (3.21)$$

is compact from $H^0_{-,b}$ into itself. If $\rho'(\tfrac{4}{5},1)(r)\neq 0$, then $r\geqq\tfrac{4}{5}$, and if $1-\rho(\tfrac{1}{2},\tfrac{3}{4})(r')\neq 0$, then $r'\leqq\tfrac{3}{4}$. Also we know that given $\gamma>0$, $\varepsilon>0$ small enough, and $r\geqq\tfrac{4}{5}$, the kernel $K^{\varepsilon}_b((r,y),(r',y'))$ and its derivatives in (r,y) tend to 0 as r' tends to 0 faster than r'^{γ}. Using Schwarz's inequality and an equicontinuity argument, it is now clear that the operator in (3.21) is compact in $\mathrm{End}(H^0_{-,b})$.

Similarly, set

$$\tilde{Q}^{\varepsilon}_{-,b}=(1-\rho(\tfrac{1}{2},\tfrac{3}{4}))\,K^{\varepsilon}_b(1-\rho(\tfrac{4}{5},1))+\rho(\tfrac{1}{2},\tfrac{3}{4})\,A^{\varepsilon}_b\rho(\tfrac{1}{4},\tfrac{2}{5}). \qquad (3.22)$$

Again, for $\varepsilon>0$ small enough, $\tilde{Q}^{\varepsilon}_{-,b}$ maps continuously $H^0_{-,b}$ into $H^1_{+,b}$. Recall that if $r'\geqq\tfrac{4}{5}$, for $\varepsilon>0$ small enough, $K^b_{\varepsilon}((r,y),(r',y'))$ and its derivatives of order $\leqq 3$ decay as $r\to 0$ faster than r^4. The same arguments as before shows that $\tilde{Q}^{\varepsilon}_{-,b}D^{\varepsilon}_{+,b}-\mathrm{Id}_{H^1_{+,b}}$ is a compact operator. We thus find that $\tilde{Q}^{\varepsilon}_{-,b}-Q^{\varepsilon}_{-,b}$ is a compact operator from $H^0_{-,b}$ into $H^1_{+,b}$, and so $Q^{\varepsilon}_{-,b}D^{\varepsilon}_{+,b}-\mathrm{Id}_{H^1_{+,b}}$ is also compact.

We have thus proved that for $\varepsilon>0$ small enough, for any $h\in B$, $D^{\varepsilon}_{+,b}$ is a Fredholm operator from $H^1_{+,b}$ into $H^0_{-,b}$.

It is also obvious that $b\in B\to D^{\varepsilon}_{+,b}$ is a continuous section of $\mathrm{Hom}(H^1_+,H^0_-)$. The family of operators $D^{\varepsilon}_{-,b}$ has similar properties. The existence of a parametrix for $D^{\varepsilon}_{-,b}$ also shows that for $\varepsilon>0$ small enough, for any $b\in B$, D^{ε}_b is a self-adjoint operator whose domain is H^1_b. In particular, $\mathrm{Coker}\,D^{\varepsilon}_{+,b}$ is isomorphic to $\mathrm{Ker}\,D^{\varepsilon}_{-,b}$.

(d) *Equality of the index bundles for D^{ε}_+ and $\tilde{D}^{\varepsilon}_+$*: We assume that $\varepsilon>0$ is small enough so that for any $b\in B$, if λ is in the spectrum of $D^{\partial Z}_b$, then $|\lambda/\sqrt{\varepsilon}|>\tfrac{1}{2}$.

Suppose first that $\mathrm{Ker}\,\tilde{D}^{\varepsilon}_-=\{0\}$. Then $\mathrm{Ker}\,\tilde{D}^{\varepsilon}_+$ is a continuous vector bundle on B. Now by the proof of Theorem 1.5, we find that for any $b\in B$, $\mathrm{Ker}\,D^{\varepsilon}_{-,b}=\{0\}$. Moreover, the proof of Theorem 1.5 shows that if $h\in\mathrm{Ker}\,D^{\varepsilon}_{+,b}$ and if U_bh denotes the restriction of h to Z, then $U_bh\in$

$\operatorname{Ker} \tilde{D}^{\varepsilon}_{+,b}$. It is also elementary to verify that U_b depends continuously on $b \in B$. Therefore, $\operatorname{Ker} D^{\varepsilon}_{+}$ and $\operatorname{Ker} \tilde{D}^{\varepsilon}_{+}$ are equal in $K^0(B)$.

Assume now that $\operatorname{Ker} \tilde{D}^{\varepsilon}_{-}$ is not $\{0\}$. Using the elliptic regularity of the index problem of [APS1] and by proceeding as in [AS2, Proposition 2.2], we find that there exist C^{∞} sections, $s_1, ..., s_q$, of $F_- \otimes \xi$ over M, which have the property:

If $\bar{\tilde{D}}^{\varepsilon}_{+,b}$ denotes the operator

$$(h, \lambda) \in \tilde{H}^1_{+,b}(p) \oplus C^q \to \bar{\tilde{D}}^{\varepsilon}_{+,b}(h, \lambda) = \tilde{D}^{\varepsilon}_{+,b} h + \sum_1^q \lambda^i s_i \in \tilde{H}^0_{-,b} \tag{3.23}$$

and if $\bar{\tilde{D}}^{\varepsilon}_{-,b}$ denotes its formal adjoint (when C^q is endowed with its canonical Hermitian inner product)

$$h' \in \tilde{H}^1_{-,b}(P) \to \bar{\tilde{D}}^{\varepsilon}_{-,b} h' = (\tilde{D}^{\varepsilon}_{-,b} h', \langle h', s_1 \rangle \cdots \langle h', s_q \rangle) \in \tilde{H}^0_{+,b} \oplus C^q$$

then for any $b \in B$, $\operatorname{Ker} \bar{\tilde{D}}^{\varepsilon}_{-,b} = \{0\}$.

Now $\bar{\tilde{D}}^{\varepsilon}_{+}$ is a family of Fredholm operators and $\operatorname{Ker} \bar{\tilde{D}}^{\varepsilon}_{+}$ is a vector bundle on B. We claim that we can take $s_1, ..., s_q$ so that $s_1, ..., s_q$ vanish on an open neighborhood of ∂M in M. In fact $s_1, ..., s_q$ can be approximated uniformly in all the Hilbert spaces $\tilde{H}^0_{-,b}$ by smooth sections of $F_- \otimes \xi$ which vanish on a neighborhood of ∂M. Since the condition $\operatorname{Ker} \bar{\tilde{D}}^{\varepsilon}_{-} = \{0\}$ is an open condition, we find that $s_1, ..., s_q$ can be assumed to vanish on a neighborhood of ∂M. By definition [AS2] the index bundle $\operatorname{Ker} \tilde{D}^{\varepsilon}_{+} - \operatorname{Ker} \tilde{D}^{\varepsilon}_{-}$ is represented in $K^0(B)$ by $\operatorname{Ker} \bar{\tilde{D}}^{\varepsilon}_{+} - C^q$.

We extend $s_1 \cdots s_q$ to $M' \backslash M$ by assuming they vanish on $M' \backslash M$. Then $s_1 \cdots s_q$ are smooth on $M' \backslash M$. We now define the operators $\bar{D}^{\varepsilon}_{+,b}$, $\bar{D}^{\varepsilon}_{+,b}$ by the formulas

$$(h, \lambda) \in H^{\infty}_{+,b} \oplus C^q \to \bar{D}^{\varepsilon}_{+,b}(h, \lambda) = D^{\varepsilon}_{+,b} h + \sum_1^q \lambda^i s_i \in H^{\infty}_{-,b}$$

$$h' \in H^{\infty}_{-,b} \to \bar{D}^{\varepsilon}_{-,b}(h', \lambda) = (D^{\varepsilon}_{-,b} h', \langle h', s_1 \rangle, ..., \langle h', s_q \rangle) \in H^{\infty}_{+,b} \oplus C^q. \tag{3.24}$$

For $\varepsilon > 0$ small enough, one verifies that $\bar{D}^{\varepsilon}_{+}$ is still a family of Fredholm operators. Also, by proceeding as in the proof of Theorem 1.5, we find that the restriction maps from $H^{\infty}_{+} \oplus C^q$ into $\tilde{H}^{\infty}_{+} \oplus C^q$ or from H'^{∞}_{-} into $\tilde{H}'^{\infty}_{-}$ permit us to identify $\operatorname{Ker} \bar{D}^{\varepsilon}_{+,b}$ with $\operatorname{Ker} \bar{\tilde{D}}^{\varepsilon}_{+,b}$ for every $b \in B$. Also for every $b \in B$, $\operatorname{Ker} \bar{D}^{\varepsilon}_{-,b} = \{0\}$. Thus $\operatorname{Ker} \bar{D}^{\varepsilon}_{+}$ is a vector bundle.

Moreover, one verifies easily that the identification of $\operatorname{Ker} \bar{D}^{\varepsilon}_{+}$ with $\operatorname{Ker} \bar{\bar{D}}^{\varepsilon}_{+}$ is a continuous identification of vector bundles.

Since the index bundle $\operatorname{Ker} D^{\varepsilon}_{+} - \operatorname{Ker} D^{\varepsilon}_{-}$ is equal in $K^0(B)$ to $\operatorname{Ker} \bar{D}^{\varepsilon}_{+} - C^q$, we find that for $\varepsilon > 0$ small enough,

$$\operatorname{Ker} D^{\varepsilon}_{+} - \operatorname{Ker} D^{\varepsilon}_{-} = \operatorname{Ker} \tilde{D}^{\varepsilon}_{+} - \operatorname{Ker} \tilde{D}^{\varepsilon}_{-} \qquad \text{in } K^0(B). \tag{3.25}$$

(e) *End of the proof of Theorem* 3.2: By (a), we know that $\operatorname{Ker} \tilde{D}^{\varepsilon}_{+} - \operatorname{Ker} \tilde{D}^{\varepsilon}_{-} = \operatorname{Ker} D_{+} - \operatorname{Ker} D_{-}$ in $K^0(B)$. Using (3.25) we obtain (3.3). Observe that as in the proof no. 2 of Theorem 1.5, by shrinking the fiber Z into the fibers $\tilde{Z}$ defined in (1.27) (or equivalently by "bending" the cones $C(\partial Z)$ near ∂M), we may avoid the introduction of the family $\tilde{D}^{\varepsilon}_{+}$. ∎

Remark 3.3. Inspection of part (c) in the proof of Theorem 3.2 shows that if $|\lambda|/\sqrt{\varepsilon} > \frac{1}{2}$ for every λ in $\operatorname{Sp}(D^{\partial Z})$, then D^{ε}_{+} is a Fredholm operator from H^1_{-} into H^0_{-}.

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