A STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION WITH MULTIPLICATIVE NOISE

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ABSTRACT. We study a conservative stochastic nonlinear Schrödinger equation with a multiplicative noise. We show the global existence and uniqueness of square integrable solutions for subcritical nonlinearities, the critical exponent being the same, in dimension 1 or 2, as the critical exponent of the deterministic equation.

CONTENTS

1. INTRODUCTION

The nonlinear Schrödinger equation is one of the basic models for nonlinear waves. It arises in various areas of physics such as hydrodynamics, nonlinear optics or plasma physics. In some circumstances, randomness has to be taken into account and it often occurs through a random potential. When this potential depends only on the time variable, it can easily be removed by the use of a gauge transform, and has no effect on the dynamics, in the sense that the noise only affects the phase of the solutions. However, in some situations, the noise depends both on time and space variables. Such a model has been proposed in [1] (see also [2], [6], [10]) in the context of molecular aggregates with thermal fluctuations. In this case, the potential may be a white noise, and the equation has the form

$$i\,\frac{dz}{dt} - (\Delta z + |z|^{2\sigma} z) = \dot{\eta} z, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1.1}$$

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where z is a complex valued process defined on $\mathbb{R}^n \times \mathbb{R}^+$, $\sigma = 1$. Also $\dot\eta$ is a real valued gaussian process with correlation function

$$\mathbb{E}(\dot\eta(x,t)\dot\eta(y,s)) = c(x,y)\delta(t-s). \tag{1.2}$$

The case $c(x,y) = \delta(x-y)$ corresponds to a space-time white noise but it is difficult to treat mathematically, and smoother correlation functions will be considered here.

Other kinds of random potentials are studied in [8] in the context of fibers.

The product on the right hand side of (1.1) has to be interpreted correctly. Classically, two kinds of products are used for stochastic processes. Itô product is preferred in mathematics since it allows the use of many powerful probabilistic tools. Stratonovitch product is often more natural in physics. It arises naturally as the limit of random equations with non vanishing correlation length and allows to use classical calculus when differentiating functions. A Stratonovitch equation is always equivalent to a Itô equation in which the drift is modified through the addition of a corrective term.

The product arising in equation (1.1) is a Stratonovitch product, the only one which allows the equation to preserve the L^2– norm of the solution – this is easily seen from the fact that η is real valued and that classical calculus can be used. This conservation of the L^2– norm of z is motivated by physical arguments.

It is convenient at this stage to give a mathematical definition of η and to give the equivalent Itô form of (1.1). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\mathcal{F}_t)_{t\geq 0}$ be a filtration, and let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of independent brownian motions associated to this filtration. Given $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R}^n, \mathbb{R})$ (the space of real valued square integrable functions on $\mathbb{R}^n$), and a symmetric linear operator Φ on $L^2(\mathbb{R}^n,\mathbb{R})$, the process

$$W(t,x,\omega) = \sum_{k=0}^{\infty} \beta_k(t,\omega)\Phi e_k(x), \quad t \geq 0, \quad x \in \mathbb{R}^n, \quad \omega \in \Omega$$

is a Wiener process on $L^2(\mathbb{R}^n,\mathbb{R})$ with covariance operator Φ. Then we set

$$\dot\eta = \frac{dW}{dt}.$$

If Φ is defined through a real valued kernel k:

$$\Phi u(x) = \int_{\mathbb{R}^n} k(x,y)u(y)dy, \quad u \in L^2(\mathbb{R}^n,\mathbb{R}) \tag{1.3}$$

we have

$$c(x,y) = \int_{\mathbb{R}^n} k(x,z)k(y,z)dz.$$

Then, (1.1) is rewritten as

$$idz - (\Delta z + f(|z|^2)z)dt = z \circ dW \tag{1.4}$$

where $\circ$ stands for a Stratonovitch product, and $f(s) = s^\sigma$. The equivalent Itô equation is given by

$$idz - (\Delta z + f(|z|^2)z)dt = zdW - \frac{1}{2}izF_\Phi dt. \tag{1.5}$$

Where the function F_Φ in the Itô correction depends only on Φ and is more precisely given by

$$F_\Phi(x) = \sum_{k=0}^{\infty} (\Phi e_k(x))^2, \ x \in \mathbb{R}^n. \tag{1.6}$$

This quantity does not depend on the basis $(e_k)_{k\in\mathbb{N}}$. For instance, in the case of (1.3),

$$F_\Phi(x) = |k(x,\cdot)|^2_{L^2(\mathbb{R}^n)}$$

Equation (1.4) is supplemented with the initial condition

$$z(0) = z_0. \tag{1.7}$$

In this work, we are concerned with existence and uniqueness of solutions of (1.5), (1.7). In the deterministic theory, a fixed point argument is used on the integral form of the equation

$$z(t) = S(t)z_0 - i\int_0^t S(t-s)\big(f(|z(s)|^2)z(s)\big)ds$$

where $S(t) = e^{-it\Delta}$ is the evolution operator associated with the linear equation. The natural space to work with is given by the nonlinearity and by the Strichartz estimate. This space is[1] $\mathcal{C}([0,T];L^2(\mathbb{R}^n)) \cap L^\gamma([0,T];L^q(\mathbb{R}^n))$ with $q = 2\sigma+2$ and $\gamma = \frac{4(\sigma+1)}{n\sigma}$. Existence and uniqueness of solutions for initial data in $L^2(\mathbb{R}^n)$ can be proved under the condition $\sigma < \frac{2}{n}$ (see [5], [11]).

In the stochastic case, the integral form of (1.5) is

$$\begin{aligned} z(t) =& S(t)z_0 - i\int_0^t S(t-s)\big(f(|z(s)|^2)z(s)\big)ds \\ &- i\int_0^t S(t-s)\big(z(s)dW(s)\big) - \frac{1}{2}\int_0^t S(t-s)\big(z(s)F_\Phi\big)ds. \end{aligned} \tag{1.8}$$

[1] In all the article, unless explicitly mentionned, $L^p(\mathbb{R}^n)$ is the space of complex valued functions with integrable p^{th} power.

Our aim is to get existence and uniqueness of solutions for (1.8) by using a fixed point argument. Since the noise we consider is multiplicative, we cannot solve (1.8) pathwise. Instead, we have to work in $L^\rho(\Omega,\mathcal{E})$ for some $\rho\geq 1$, and some space-time function space $\mathcal{E}$.

We also have to use a truncation argument, because the nonlinear term is not lipschitz. As follows from the deterministic theory, the natural space is $\mathcal{E}=\mathcal{C}([0,T];L^2(\mathbb{R}^n))\cap L^\gamma([0,T];L^q(\mathbb{R}^n))$ and, as easily seen, the nonlinear term has to be truncated in the space $L^\gamma([0,T];L^q(\mathbb{R}^n))$. When estimating the stochastic integral or the correction term in this space , it can be seen that the smaller q is, the stronger assumption is required on Φ. Apparently, q is determined by σ, but using the flexibility of Strichartz estimates, a fixed point argument can be used in $\mathcal{C}([0,T];L^2(\mathbb{R}^n))\cap L^r(0,T;L^p(\mathbb{R}^n))$ for any admissible pair (r,p) such that $p\geq 2\sigma+2$. Hence, in order to assume less regularity on Φ, we work in such a space with p large enough. The truncation argument, however, has to be suited to the nonlinear term. Thus, we first consider the equation

$$
\begin{aligned}
z(t)=&S(t)z_0-i\int_0^t S(t-s)\big(\theta(|z|_{L^\gamma(0,s;L^q(\mathbb{R}^n))})f(|z(s)|^2)z(s)\big)ds\\
&-i\int_0^t S(t-s)\big(z(s)dW(s)\big)-\frac{1}{2}\int_0^t S(t-s)\big(z(s)F_\Phi\big)ds
\end{aligned}
\tag{1.9}
$$

where θ is a truncation function and where F_Φ is given by (1.6). We use a fixed point argument in

$$L^\rho(\Omega;\mathcal{C}([0,T];L^2(\mathbb{R}^n))\cap L^r(0,T;L^p(\mathbb{R}^n))).$$

This can be done under the assumption that Φ is a Hilbert-Schmidt operator in $L^2(\mathbb{R}^n;\mathbb{R})$, and a γ- radonifying operator from $L^2(\mathbb{R}^n)$ into $L^{2+\delta}(\mathbb{R}^n)$ (see [3] and Section 2 for a definition of γ- radonifying operators). Here δ depends on n and σ. In the case (1.3), this amounts to assume that

$$k\in L^2(\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R})\cap L^{2+\delta}(\mathbb{R}^n_x;L^2(\mathbb{R}^n_y;\mathbb{R})).$$

The fixed point argument works under the assumption that

$$\sigma<\min\big(\frac{2}{n},\frac{1}{n-1}\big)$$

which is the same as in the deterministic theory in dimension $n=1$ or 2, but more restrictive if $n\geq 3$. The study of equation (1.9) is the object of section 3.

In order to get a solution of the original equation (1.8), an estimate on the solution of (1.9) is needed in the space $L^1(\Omega;L^\gamma(0,T;L^q(\mathbb{R}^n)))$. This is obtained through an estimate in the smaller space $L^1(\Omega;L^\infty(0,T;L^2(\mathbb{R}^n)))\cap L^1(\Omega;L^r(0,T;L^p(\mathbb{R}^n)))$. Since the L^2- norm of the solution is conserved by equation (1.9), we have a trivial estimate in

$L^1(\Omega; L^\infty(0,T; L^2(\mathbb{R}^n)))$. However, it is much more difficult to get an estimate in the second space. This is part of the aim in Section 4, in which a global solution of (1.8) is obtained. Uniqueness is also proved in Section 4.

In the course of the proof of these results, we need estimates on the stochastic integral $\int_0^t S(t-s)\,(z(s)dW(s))$ in the space $L^\rho(\Omega; L^r(0,T; L^p(\mathbb{R}^n)))$. Since $L^p(\mathbb{R}^n)$ is not Hilbert, we use the theory of stochastic integral with values in Banach space developed for instance in [3], [4]. Of special interest are the Bürkholder inequalities for these integrals.

Next section is devoted to setting the notations, introducing a few mathematical tools, and the main result of the article is stated there. A slightly more general nonlinear term will be considered.

2. Notations and Main Result

In general, a norm in a vector space X will be denoted by $|\cdot|_X$ or sometimes $\|\cdot\|_X$. We use the classical Lebesgue space (of complex valued functions) $L^p(\mathbb{R}^n)$, and the inner product in $L^2(\mathbb{R}^n)$ (where $\mathbb{C}$ is identified with $\mathbb{R}^2$) is denoted by $(\cdot,\cdot)$, i.e.

$$(u,v) = Re \int_{\mathbb{R}^n} u(x)\bar{v}(x)dx$$

for $u, v \in L^2(\mathbb{R}^n)$. We define the usual space $H^s(\mathbb{R}^n)$ of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ whose Fourier transform $\hat{u}$ satisfies $(1+|\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$.

We also use the space $L^2(\mathbb{R}^n;\mathbb{R})$ of real valued square integrable functions.

If I is an interval of $\mathbb{R}$, X is a Banach space, and $1 \le r \le +\infty$, then $L^r(I;X)$ is the space of strongly Lebesgue measurable functions u from I into X such that the function $t \mapsto |u(t)|_X$ is in $L^r(I)$.

We will sometimes use the shorter notation $L^r(0,T;L^p_x)$ for $L^r([0,T];L^p(\mathbb{R}^n))$, or even $L^r_t(L^p_x)$ when the interval $[0,T]$ is specified and fixed.

We consider a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t\ge 0}$.

We then define similarly as above the spaces $L^r(\Omega;X)$ when X is a Banach space. We will also use notations like $L^\rho_\omega(L^r_t(L^p_x)))$,...

Given two separable Hilbert Spaces H and $\tilde{H}$, we denote by $L^0_2(H;\tilde{H})$ the space of Hilbert-Schmidt operators from H into $\tilde{H}$. The corresponding norm is then given by

$$\|\Phi\|^2_{L^0_2(H,\tilde{H})} = tr(\Phi^*\Phi) = \sum_{k\in\mathbb{N}} |\Phi e_k|^2_{\tilde{H}},$$

where $(e_k)_{k\in\mathbb{N}}$ is any orthonormal basis of H. When $H = \tilde{H} = L^2(\mathbb{R}^n;\mathbb{R})$ (space of real-valued functions), $L_2^0(L^2(\mathbb{R}^n;\mathbb{R}); L^2(\mathbb{R}^n;\mathbb{R}))$ is simply denoted by $L_2^0(L^2(\mathbb{R}^n;\mathbb{R}))$, or even L_2^0.

Given a Banach space B, we will also consider bounded linear operators from $L^2(\mathbb{R}^n)$ into B. Then, in order to replace the notion of Hilbert-Schmidt operators, we define, following [3], [4], a γ- radonifying operator as an operator $K : H \to B$ such that the image by K of the canonical gaussian distribution on H extends to a Borel probability measure on B. As shown in [4], proposition 3.1, this is equivalent to the convergence of the series $\sum_{k=1}^{\infty} \gamma_k K e_k$ in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, B)$, for any sequence $(\gamma_k)_{k\in\mathbb{N}}$ of independent normal real valued random variables on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H. Moreover, the space of γ- radonifying operators from H into B can be equipped with the norm

$$\|K\|_{R(H,B)} = \left(\tilde{\mathbb{E}} \left| \sum_{k=1}^{\infty} \gamma_k K e_k \right|_B^2 \right)^{1/2}$$

(which does not depend on $(\gamma_k)_{k\in\mathbb{N}}$, nor on $(e_k)_{k\in\mathbb{N}}$) and is a Banach space. We will use the following result.

Lemma 2.1. *Let H be a separable Hilbert space and E, F two separable Banach spaces. Given $K \in R(H;E)$ and $L \in \mathcal{L}(E;F)$ (the space of bounded linear operators from E into F), then $LK \in R(H;F)$ and*

$$\|LK\|_{R(H;F)} \leq \|L\|_{\mathcal{L}(E;F)} \|K\|_{R(H;E)}.$$

Proof. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space, $(\gamma_k)_{k\in\mathbb{N}}$ a sequence of independent normal real-valued random variables on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis of H. Then, for $n, p \in \mathbb{N}$,

$$\tilde{\mathbb{E}} \left| \sum_{k=n}^{n+p} \gamma_k L K e_k \right|_F^2 = \tilde{\mathbb{E}} \left| L \sum_{k=n}^{n+p} \gamma_k K e_k \right|_F^2 \leq \|L\|_{\mathcal{L}(E;F)}^2 \tilde{\mathbb{E}} \left| \sum_{k=n}^{n+p} \gamma_k K e_k \right|_E^2 .$$

It follows that the series $\sum_{k=1}^{\infty} \gamma_k L K e_k$ is convergent. Taking $n = 1$ and letting p go to $+\infty$, we conclude the proof. □

Clearly, if B is a Hilbert space, $R(H;B) = L_2^0(H;B)$. If $2 \leq q \leq \frac{2n}{n-s}$ $(s < n)$, by the sobolev embedding $H^s(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and Lemma 2.1, we deduce that a Hilbert-Schmidt operator from $L^2(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$ is also γ-radonifying from $L^2(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

We consider the stochastic nonlinear Schrödinger equation (1.5) written in Itô form, where F_Φ is given by (1.6), and we assume the following concerning the nonlinearity f and the Wiener process W:

(i) $f : \mathbb{R}^+ \to \mathbb{R}$ is locally lipschitz continuous and there are constants $C > 0$ and σ with $0 < \sigma < \frac{2}{n}$ if $n = 1$ or 2, $0 < \sigma < \frac{1}{n-1}$ if $n \geq 3$, such that

$$|f(|\zeta|^2)\zeta - f(|\zeta'|^2)\zeta'| \leq C(|\zeta|^{2\sigma} + |\zeta'|^{2\sigma})|\zeta - \zeta'|$$

for any $\zeta, \zeta' \in \mathbb{C}$.

A typical example of such nonlinear term is the function $f(s) = s^\sigma$.

Let $\tilde{W}$ be a cylindrical Wiener process on $L^2(\mathbb{R}^n; \mathbb{R})$ associated to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, so that for any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^n; \mathbb{R})$ we have

$$\tilde{W} = \sum_{k=0}^{\infty} \beta_k(t) e_k$$

for a sequence (β_k) of real independent brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We assume that the Wiener process W satisfies

(ii) there exists Φ in $L_2^0(L^2(\mathbb{R}^n; \mathbb{R}))$ and in $R(L^2(\mathbb{R}^n; \mathbb{R}); L^{2+\delta}(\mathbb{R}^n))$ for some $\delta > 2(n-1)$ such that

$$W = \Phi \tilde{W}.$$

Note that W is a real valued process.

We denote by $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ the laplacian operator on $\mathbb{R}^n$, and by $S(t) = e^{-it\Delta}$ the unitary group on $H^s(\mathbb{R}^n)$ generated by the linear equation

$$i\frac{\partial u}{\partial t} = \Delta u.$$

We will say that z is a solution of (1.5) on [0,T] with the initial condition (1.7) if it is a mild solution of the Itô form of the equation, i.e. it satisfies for each $t \in [0, T]$ and for almost each $\omega \in \Omega$

$$\begin{aligned} z(t) =& S(t)z_0 - i\int_0^t S(t-s)\big(f(|z(s)|^2)z(s)\big)ds \\ &- i\int_0^t S(t-s)\left(z(s)dW(s)\right) - \frac{1}{2}\int_0^t S(t-s)\left(z(s)F_\Phi\right)ds \end{aligned} \tag{2.1}$$

where $F_\Phi = \sum_{k=0}^\infty (\Phi e_k)^2$, for any orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^n, \mathbb{R})$. We will see that since $\Phi \in R(L^2(\mathbb{R}^n; \mathbb{R}), L^{2+\delta}(\mathbb{R}^n))$, F_Φ makes sense as an element of $L^{1+\frac{\delta}{2}}(\mathbb{R}^n)$. We may now state our result:

Theorem 2.1. *Assume that f and W satisfy (i) and (ii); let $p \geq \max(2\sigma+2, \frac{2(2+\delta)}{\delta})$ with the additional assumption that $p < \frac{2n}{n-1}$ if $n \geq 2$, and let r be such that $\frac{2}{r} = n(\frac{1}{2}-\frac{1}{p})$; then for any $\rho \geq \max(r, (2\sigma+2)(\frac{4\sigma}{2-n\sigma}+1))$, for any $z_0 \in L^\rho(\Omega; L^2(\mathbb{R}^n))$ $\mathcal{F}_0$ measurable and for any $T_0 > 0$, there is a unique solution z of (1.5), (1.7) such that $z \in L^\rho(\Omega; \mathcal{C}([0,T_0]; L^2(\mathbb{R}^n))) \cap L^1(\Omega; L^r(0,T_0; L^p(\mathbb{R}^n)))$. Moreover, for a.e. $\omega \in \Omega$, and each $t \in [0, T_0]$*

$$|z(t)|_{L^2(\mathbb{R}^n)} = |z_0|_{L^2(\mathbb{R}^n)}$$

Remark 2.1. Note that the assumptions on p in Theorem 2.1 are compatible, since we assumed $\sigma < \frac{1}{n-1}$ and $\delta > 2(n-1)$. Also, the assumption $p \geq \frac{2(2+\sigma)}{\delta}$ may be written as $\frac{nr}{2} \leq 2+\delta$ and $p < \frac{2n}{n-2}$ as $\frac{r}{2} > 2$.

Remark 2.2. If $n = 1$, then the assumption on σ is $\sigma < 2$, which is the same as for global existence of L^2 solutions for the deterministic equation. The assumption on δ reduce in this case to $\delta > 0$. If $n = 2$, the assumption on σ reduces again to the same as in the deterministic equation : $\sigma < 1$, but we need $\delta > 2$. For $n \geq 3$, we require $\sigma < \frac{1}{n-1}$ which is less than the limiting case of the deterministic theory.

In order to prove Theorem 2.1, we will first consider, in Section 3, an equation in which the nonlinear term has been multiplied by a truncating function, and we will prove an existence and uniqueness result for this truncated equation. Then in Section 4, we will show that provided that the support of the truncating function is taken large enough (depending on the initial data z_0), the solutions of the truncated equation are solutions of (2.1). We will also prove in Section 4 the uniqueness part of Theorem 2.1.

All along the paper, C will denote various constants, which may change form one line to another.

3. A Truncated Equation

Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$ with supp $\theta \subset (-2,2)$, $\theta(x) = 1$ for $x \in [-1,1]$ and $0 \leq \theta(x) \leq 1$ for $x \in \mathbb{R}$. Let $R > 0$ and $\theta_R(x) = \theta(x/R)$.

Let $q = 2\sigma+2$ and $\gamma = \frac{4(\sigma+1)}{n\sigma}$ so that $\frac{2}{\gamma} = n(\frac{1}{2}-\frac{1}{q})$.

We then consider the following equation

$$\begin{aligned} z(t) =& S(t)z_0 - i\int_0^t S(t-s)\big(\theta_R(|z|_{L^\gamma(0,s;L_x^q)})f(|z(s)|^2)z(s)\big)ds \\ &- i\int_0^t S(t-s)\,(z(s)dW(s)) - \frac{1}{2}\int_0^t S(t-s)\,(z(s)F_\Phi)\,ds \end{aligned} \tag{3.1}$$

which is the mild form for the Itô equation

$$idz - \left(\Delta z + \theta_R(|z|_{L^\gamma(0,t;L^q_x)}) f(|z|^2)z\right) dt = zdW - \frac{1}{2} izF_\Phi dt. \tag{3.2}$$

This section is devoted to the proof of the next result.

Proposition 3.1. *Let p, r, σ be as in Theorem 2.1, let $\rho \geq r$, $z_0 \in L^\rho(\Omega; L^2(\mathbb{R}^n))$ be $\mathcal{F}_0-$measurable and let $T_0 > 0$; then (3.1) has a unique solution $z \in L^\rho(\Omega; \mathcal{C}([0, T_0]; L^2(\mathbb{R}^n))) \cap L^\rho(\Omega; L^r(0, T_0; L^p(\mathbb{R}^n)))$.*

In order to prove Proposition 3.1, we use a fixed point argument in the Banach space $L^\rho(\Omega; \mathcal{C}([0, T]; L^2(\mathbb{R}^n))) \cap L^\rho(\Omega; L^r(0, T; L^p(\mathbb{R}^n)))$ for some sufficiently small T depending only on R. To do this, we will need estimates on the stochastic integral

$$Jz(t) = \int_0^t S(t-s)\left(z(s)dW(s)\right).$$

We state, for later use, a slightly more general estimate in the next lemma, from which an estimate on $Jz(t)$ in $L^\rho(\Omega; L^r(0, T; L^p(\mathbb{R}^n)))$ is deduced as a corollary.

Lemma 3.1. *Let W satisfy assumption (ii) of Section 2, and let r and p be as in Theorem 2.1. For each $\rho \geq 2$, $T > 0$ and each adapted process $z \in L^\rho(\Omega; L^\infty(0, T; L^2_x))$, if Iz is defined for $t_0, t \in [0, T]$ by*

$$Iz(t_0, t) = \int_0^{t_0} S(t-s)\left(z(s)dW(s)\right),$$

then for any $t \in [0, T]$,

$$\begin{aligned} &\mathbb{E}\left(\sup_{0 \leq t_0 \leq T} |Iz(t_0, t)|^\rho_{L^p(\mathbb{R}^n)}\right) \\ &\leq C\|\Phi\|^\rho_{R(L^2_x; L^{nr/2}_x)} T^{\rho(\frac{1}{2}-\frac{1}{r})} \mathbb{E}\left(|z|^\rho_{L^\infty(0,T;L^2_x)}\right). \end{aligned}$$

Remark 3.1. Using Hölder inequality, it is not difficult to see that, since $2 \leq \frac{nr}{2} \leq 2 + \delta$, we have $L^0_2(L^2_x) \cap R(L^2_x; L^{2+\delta}_x) \subset R(L^2_x; L^{nr/2}_x)$ and

$$\|\Phi\|_{R(L^2_x, L^{nr/2}_x)} \leq C(\|\Phi\|_{L^0_2(L^2_x)} + \|\Phi\|_{R(L^2_x; L^{2+\delta}_x)})$$

so that the right hand side in Lemma 3.1 is finite.

Proof of Lemma 3.1. We write

$$\mathbb{E}\left(\sup_{0\le t_0\le T} \quad |Iz(t_0,t)|^{\rho}_{L^p(\mathbb{R}^n)}\right)$$
$$= \mathbb{E}\left(\sup_{0\le t_0\le T} \quad \left|\int_0^{t_0} S(t-s)\,(z(s)dW(s))\right|^{\rho}_{L^p(\mathbb{R}^n)}\right).$$

Next, we use Bürkholder inequality in the Banach space $L^p(\mathbb{R}^n)$ (see [3] and [4]; note that $p\ge 2$, so that $L^p(\mathbb{R}^n)$ is a UMD and type 2 Banach space by Proposition 3.3 in [4]). This inequality allows us to bound the preceding expression above by

$$C\mathbb{E}\left(\int_0^T \|S(t-s)(z(s)\Phi)\|^2_{R(L^2_x;L^p_x)}ds\right)^{\rho/2}.$$

By Lemma 2.1, applied with $K=\Phi$ and $L: u\mapsto S(t-s)(z(s)u)$, we have

$$\|S(t-s)(z(s)\Phi)\|_{R(L^2_x;L^p_x)} \le \|L\|_{\mathcal{L}(L_x^{nr/2};L^p_x)}\|\Phi\|_{R(L^2_x;L_x^{nr/2})}.$$

For any $u\in L^{nr/2}(\mathbb{R}^n)$, the well known decay estimate on the linear group $S(t)$ (see for example [5]) gives for p' such that $\frac{1}{p}+\frac{1}{p'}=1$:

$$\begin{aligned}|Lu|_{L^p(\mathbb{R}^n)} &= |S(t-s)z(s)u|_{L^p(\mathbb{R}^n)}\\ &\le C|t-s|^{-n(\frac{1}{2}-\frac{1}{p})} \quad |z(s)u|_{L^{p'}(\mathbb{R}^n)}\\ &\le C|t-s|^{-\frac{2}{r}}|z(s)|_{L^2(\mathbb{R}^n)}|u|_{L^{nr/2}(\mathbb{R}^n)}\end{aligned}$$

we deduce that

$$\|S(t-s)(z(s)\Phi)\|_{R(L^2_x;L^p_x)}$$
$$\le C|t-s|^{-2/r}|z(s)|_{L^2(\mathbb{R}^n)}\|\Phi\|_{R(L^2_x;L_x^{nr/2})}$$

and

$$\mathbb{E}\left(\sup_{0\le t_0\le T}|Iz(t_0,t)|^{\rho}_{L^p(\mathbb{R}^n)}\right)$$
$$\le C\|\Phi\|^{\rho}_{R(L^2_x;L_x^{nr/2})}\mathbb{E}\left(\int_0^T |t-s|^{-4/r}|z(s)|^2_{L^2(\mathbb{R}^n)}ds\right)^{\rho/2}$$
$$\le C\|\Phi\|^{\rho}_{R(L^2_x,L_x^{nr/2})}T^{\rho(\frac{1}{2}-\frac{2}{r})}\mathbb{E}\left(|z|^{\rho}_{L^\infty(0,T;L^2_x)}\right)$$

which achieves the proof of Lemma 3.1. $\square$

Remark 3.2. We have seen in the proof of Lemma 3.1 that we need $\frac{4}{r} < 1$, which leads to the assumption that $p < \frac{2n}{n-1}$ and requires a restriction on σ compared with the deterministic equation, for $n > 2$. This is actually the only point in all the proof where this restriction is necessary.

Corollary 3.1. *Let W, r and p be as in Lemma 3.1; let also $T > 0$ and $\rho \geq r$, and set for any adapted process $z \in L^\rho(\Omega; L^\infty(0,T; L^2(\mathbb{R}^n)))$, $Jz(t) = Iz(t,t)$. Then $Jz \in L^\rho(\Omega; L^r(0,T; L^p(\mathbb{R}^n)))$ and*

$$\begin{aligned}&\mathbb{E}\left(|Jz(\cdot)|^\rho_{L^r_t(0,T;L^p_x)}\right)\\ &\leq CT^{\rho(\frac{1}{2}-\frac{1}{r})}(\|\Phi\|_{L^0_2} + \|\Phi\|_{R(L^2_x;L^{2+\delta}_x)})^\rho \mathbb{E}\left(|z|^\rho_{L^\infty(0,T;L^2_x)}\right)\end{aligned}$$

Proof. We write, using Hölder inequality and Lemma 3.1

$$\begin{aligned}\mathbb{E}\left(|Jz(\cdot)|^\rho_{L^r_t(0,T;L^p_x)}\right) &= \mathbb{E}\left(\int_0^T |Iz(t,t)|^r_{L^p_x} dt\right)^{\rho/r}\\ &\leq T^{\rho/r-1}\mathbb{E}\left(\int_0^T |Iz(t,t)|^\rho_{L^p_x} dt\right)\\ &\leq CT^{\rho/r-1}\int_0^T \mathbb{E}\left(\sup_{0\leq t_0\leq T} |I_z(t_0,t)|^\rho_{L^p_x}\right) dt\\ &\leq C\|\Phi\|^\rho_{R(L^2_x;L^{nr/2}_x)} T^{\rho(\frac{1}{2}-\frac{1}{r})}\mathbb{E}\left(|z|^\rho_{L^\infty(0,T;L^2_x)}\right)\end{aligned}$$

and, the conclusion follows from Remark 3.1. ☐

We also need an estimate on Jz in the space $L^\rho_\omega(L^\infty_t(L^2_x))$; it is given by the next Lemma.

Lemma 3.2. *Let W, r, p be as in Theorem 2.1 and let $T > 0$, $\rho \geq 2$ and z be an adapted process in $L^\rho(\Omega; L^r(0,T; L^p(\mathbb{R}^n)))$. If Jz is defined as in Corollary 3.1, then $Jz \in L^\rho(\Omega; \mathcal{C}([0,T]; L^2(\mathbb{R}^n)))$ and*

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,T]} |Jz(t)|^\rho_{L^2(\mathbb{R}^n)}\right)\\ &\leq C\left(\|\Phi\|_{L^0_2} + \|\Phi\|_{R(L^2_x;L^{2+\delta}_x)}\right)^\rho T^{\rho(\frac{1}{2}-\frac{1}{r})}\mathbb{E}\left(|z|^\rho_{L^r(0,T;L^p_x)}\right).\end{aligned}$$

Proof. $S(t)$ being a unitary group in $L^2(\mathbb{R}^n)$, Theorem 6.10 in [7] (with 2 replaced by ρ, but the proof is the same) tells us that, provided that $z\Phi \in L^\rho(\Omega; L^2(0,T; L^0_2(L^2(\mathbb{R}^n))))$, Jz has a continuous modification with values in $L^2(\mathbb{R}^n)$, and

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Jz(t)|^\rho_{L^2(\mathbb{R}^n)}\right) \leq C\mathbb{E}\left(\int_0^T |z(s)\Phi|^2_{L^0_2} ds\right)^{\rho/2}.$$

Then, using $\frac{1}{2}=\frac{1}{p}+\frac{2}{nr}$ and Lemma 2.1,

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T]}|Jz(t)|^{\rho}_{L^2(\mathbb{R}^n)}\right)\\
&\le C\|\Phi\|^{\rho}_{R(L^2_x;L^{nr/2}_x)}\mathbb{E}\left(\int_0^T|z(s)|^2_{L^p(\mathbb{R}^n)}ds\right)^{\rho/2}\\
&\le C\|\Phi\|^{\rho}_{R(L^2_x;L^{nr/2}_x)}T^{\rho(\frac{1}{2}-\frac{1}{r})}\mathbb{E}\left(|z|^{\rho}_{L^r(0,T;L^p_x)}\right).
\end{aligned}$$

We conclude thanks to Remark 3.1 again. □

Proof of Proposition 3.1. We set

$$\begin{aligned}
\mathcal{T}z(t)=&S(t)z_0-i\int_0^t S(t-s)\big(\theta_R(|z|_{L^\gamma(0,s;L^q_x)})f(|z(s)|^2)z(s)\big)ds\\
&-i\int_0^t S(t-s)\,(z(s)dW(s))-\frac{1}{2}\int_0^t S(t-s)\,(z(s)F_\Phi)\,ds
\end{aligned}$$

where we recall that $q=2\sigma+2$ and $\gamma=\frac{4(\sigma+1)}{n\sigma}$. We will show that $\mathcal{T}$ is a contraction mapping in $L^\rho(\Omega;\mathcal{C}([0,T];L^2(\mathbb{R}^n))\cap L^r(0,T;L^p(\mathbb{R}^n)))=X_T$ provided that T is sufficiently small, depending only on R. With this aim in view, let $z_1,z_2\in X_T$ being adapted processes; making use of Strichartz inequalities for the operator Λ defined by $\Lambda g(t)=\int_0^t S(t-s)g(s)ds$, namely

$$|\Lambda g|_{L^r(0,T;L^p(\mathbb{R}^n))}\le C|g|_{L^{\alpha'}(0,T,L^{\beta'}(\mathbb{R}^n))} \tag{3.3}$$

for any α',β' such that $\frac{1}{\alpha'}+\frac{1}{\alpha}=\frac{1}{\beta'}+\frac{1}{\beta}=1$, and (α,β) is an admissible pair, i.e. $2\le\beta<\frac{2n}{n-2}$, ($2\le\beta\le+\infty$ if $n=1$) and $\frac{2}{\alpha}=n(\frac{1}{2}-\frac{1}{\beta})$ (see for example [5], [9]), we get for a.e. ω

$$\begin{aligned}
&|\mathcal{T}z_1-\mathcal{T}z_2|_{L^r(0,T;L^p_x)}\\
&\le C\big|\theta_R(|z_1|_{L^\gamma(0,t,L^q_x)})f(|z_1|^2)z_1-\theta_R(|z_2|_{L^\gamma(0,t,L^q_x)})f(|z_2|^2)z_2\big|_{L^{\gamma'}(0,T,L^{q'}_x)}\\
&\quad+\left|\int_0^t S(t-s)\big((z_1(s)-z_2(s))dW(s)\big)\right|_{L^r(0,T;L^p_x)}\\
&\quad+C\,|(z_1-z_2)F_\Phi|_{L^{r'}(0,T;L^{p'}_x)}\\
&\le I+II+III
\end{aligned} \tag{3.4}$$

where r',p',γ',q', are respectively conjugates of r,p,γ,q. Note that (r,p) and (γ,q) are admissible pairs. In order to estimate I, we set for $i=1,2$,

$$t_i^R=\sup\left\{t\le T,|z_i|_{L^\gamma(0,t;L^q_x)}\le 2R\right\},$$

and assume that $t_1^R \le t_2^R$ (here we keep R fixed). Then, writing $[0,T] = [0,t_1^R] \cup [t_1^R, t_2^R] \cup [t_2^R, T]$, we get

$$\begin{aligned} I \ &\le C \left| \left(\theta_R(|z_1|_{L^\gamma(0,t;L^q_x)}) - \theta_R(|z_2|_{L^\gamma(0,t;L^q_x)})\right) f\left(|z_1|^2\right) z_1 \right|_{L^{\gamma'}(0,t_1^R;L^{q'}_x)} \\ &\quad + C \left| \theta_R(|z_2|_{L^\gamma(0,t;L^q_x)}) \left(f(|z_1|^2) z_1 - f(|z_2|^2) z_2\right) \right|_{L^{\gamma'}(0,t_1^R;L^{q'}_x)} \\ &\quad + C \left| \theta_R(|z_2|_{L^\gamma(0,t;L^q_x)}) f(|z_2|^2) z_2 \right|_{L^{\gamma'}(t_1^R,t_2^R;L^{q'}_x)} \\ &\le I_1 + I_2 + I_3. \end{aligned}$$

we make use of the following lemma to estimate I_1.

Lemma 3.3. *Let $z_1, z_2 \in L^\gamma(0,T;L^q(\mathbb{R}^n))$ and $z_3 \in L^{\gamma'}(0,T;L^{q'}(\mathbb{R}^n))$, then we have the estimate*

$$\begin{aligned} A \ &\equiv \left| \left(\theta_R(|z_1|_{L^\gamma(0,t;L^q_x)}) - \theta_R(|z_2|_{L^\gamma(0,t;L^q_x)})\right) z_3 \right|_{L^{\gamma'}(0,T;L^{q'}_x)} \\ &\le C_R |z_1 - z_2|_{L^\gamma(0,T;L^q_x)} |z_3|_{L^{\gamma'}(0,T;L^{q'}_x)}. \end{aligned}$$

Proof. It suffices to write

$$\begin{aligned} A^{\gamma'} &\le \int_0^T \left| \left(\theta_R(|z_1|_{L^\gamma(0,t;L^q_x)}) - \theta_R(|z_2|_{L^\gamma(0,t;L^q_x)})\right) z_3(t) \right|_{L^{q'}_x}^{\gamma'} dt \\ &\le \frac{|\theta'|_{L^\infty(\mathbb{R})}}{R} \int_0^T \left| |z_1|_{L^\gamma(0,t;L^q_x)} - |z_2|_{L^\gamma(0,t;L^q_x)} \right|^{\gamma'} |z_3(t)|_{L^{q'}_x}^{\gamma'} dt \\ &\le C_R \int_0^T |z_1 - z_2|_{L^\gamma(0,t;L^q_x)}^{\gamma'} |z_3(t)|_{L^{q'}_x}^{\gamma'} dt \\ &\le C_R |z_1 - z_2|_{L^\gamma(0,T;L^q_x)}^{\gamma'} |z_3|_{L^{\gamma'}(0,T;L^{q'}_x)}^{\gamma'} \end{aligned}$$

which proves the lemma. □

As an application of Lemma 3.3 (with $T = t_1^R$), we have

$$I_1 \le C_R |z_1 - z_2|_{L^\gamma(0,T;L^q_x)} |f(|z_1|^2) z_1|_{L^{\gamma'}(0,t_1^R;L^{q'}_x)};$$

by assumption (i) and the use of Hölder inequality together with the definition of t_1^R, this term is bounded above by

$$\begin{aligned} &C |z_1 - z_2|_{L^\gamma(0,T;L^q_x)} \left| |z_1|^{2\sigma+1} \right|_{L^{\gamma'}(0,t_1^R;L^{q'}_x)} \\ &\le C T^{1-n\sigma/2} |z_1 - z_2|_{L^\gamma(0,T;L^q_x)} |z_1|_{L^\gamma(0,t_1^R;L^q_x)}^{2\sigma+1} \end{aligned}$$

hence,

$$I_1 \le C R^{2\sigma+1} T^{1-n\sigma/2} |z_1 - z_2|_{L^\gamma(0,T;L^q_x)}$$

We estimate I_2 in the same way, using the fact that θ_R is bounded:

$$\begin{aligned} I_2 &\leq C\Big|\left(|z_1|^{2\sigma}+|z_2|^{2\sigma}\right)|z_1-z_2|\Big|_{L^{\gamma'}(0,t_1^R;L_x^{q'})} \\ &\leq CT^{1-n\sigma/2}\left(|z_1|^{2\sigma}_{L^\gamma(0,t_1^R;L_x^q)}+|z_2|^{2\sigma}_{L^\gamma(0,t_1^R;L_x^q)}\right)|z_1-z_2|_{L^\gamma(0,t_1^R;L_x^q)} \\ &\leq 2CR^{2\sigma}T^{1-n\sigma/2}|z_1-z_2|_{L^\gamma(0,T;L_x^q)}. \end{aligned}$$

Lastly, since $\theta_R(|z_1|_{L^\gamma(0,t;L_x^q)})=0$ if $t\in(t_1^R,t_2^R)$, we may use Lemma 3.3 again to estimate

$$\begin{aligned} I_3 &= C\Big|\left(\theta_R(|z_1|_{L^\gamma(0,t;L_x^q)})-\theta_R(|z_2|_{L^\gamma(0,t;L_x^q)})\right)f(|z_2|^2)z_2\Big|_{L^{\gamma'}(t_1^R,t_2^R;L_x^{q'})} \\ &\leq C_R|z_1-z_2|_{L^\gamma(0,T;L_x^q)}\Big||z_2|^{2\sigma+1}\Big|_{L^{\gamma'}(t_1^R,t_2^R;L_x^{q'})} \\ &\leq C_RT^{1-n\sigma/2}|z_1-z_2|_{L^\gamma(0,T;L_x^q)}|z_2|^{2\sigma+1}_{L^\gamma(t_1^R,t_2^R;L_x^q)} \\ &\leq C_RT^{1-n\sigma/2}R^{2\sigma+1}|z_1-z_2|_{L^\gamma(0,T;L_x^q)}. \end{aligned}$$

Thus, we have proved that

$$I \leq C_RT^{1-n\sigma/2}|z_1-z_2|_{L^\gamma(0,T;L_x^q)}.$$

The expectation of the second term in (3.4) will be estimated by using Corollary 3.1 ; to estimate the third term, we first use Hölder's inequality:

$$\begin{aligned} III &\leq C|z_1-z_2|_{L^{r'}(0,T;L_x^p)}|F_\Phi|_{L_x^{nr/4}} \\ &\leq CT^{1-\frac{2}{r}}|z_1-z_2|_{L^r(0,T;L_x^p)}|F_\Phi|_{L_x^{nr/4}}. \end{aligned}$$

Then, we notice that since Φ is Hilbert-Schmidt in $L^2(\mathbb{R}^n,\mathbb{R})$, it is given by a kernel $k(x,y)\in L^2(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$:

$$\Phi u(x)=\int_{\mathbb{R}^n}k(x,y)u(y)dy,\quad \forall u\in L^2(\mathbb{R}^n).$$

By Remark 3.2 in [4], we have

$$|k(x,y)|_{L_x^{nr/2}(L_y^2)}\leq C\|\Phi\|_{R(L_x^2;L_x^{nr/2})}$$

and since, by Plancherel identity

$$\begin{aligned} |k(x,\cdot)|^2_{L_y^2} &= \sum_{\ell=0}^\infty\left(\int_{\mathbb{R}^n}k(x,y)e_\ell(y)dy\right)^2 \\ &= \sum_{\ell=0}^\infty(\Phi e_\ell)^2(x)=F_\Phi(x) \end{aligned}$$

we obtain

$$\begin{aligned} |F_\Phi|_{L_x^{nr/4}} &= |k(x,y)|^2_{L_x^{nr/2}(L_y^2)} \\ &\le C\|\Phi\|^2_{R(L_x^2;L_x^{nr/2})} \\ &\le C(\|\Phi\|^2_{L_2^0(L_x^2)} + \|\Phi\|^2_{R(L_x^2;L_x^{2+\delta})}) \end{aligned}$$

by Remark 3.1. It follows that

$$III \le CT^{1-\frac{2}{r}} \left(\|\Phi\|^2_{L_2^0(L_x^2)} + \|\Phi\|^2_{R(L_x^2;L_x^{2+\delta})} \right) |z_1 - z_2|_{L^r(0,T;L_x^p)}.$$

Collecting all these estimates, and using Corollary 3.1, we get for any $\rho \ge r$

$$\begin{aligned} &\big|\mathcal{T} z_1 - \mathcal{T} z_2\big|_{L_\omega^\rho(L^r(0,T;L_x^p))} \\ &\le C_R T^{1-n\sigma/2} \big|z_1 - z_2\big|_{L_\omega^\rho(L^\gamma(0,T;L_x^q))} \\ &\quad + C\left(\|\Phi\|_{L_2^0} + \|\Phi\|_{R(L_x^2;L_x^{2+\delta})} \right) T^{1/2-1/r} \big|z_1 - z_2\big|_{L_\omega^\rho(L^\infty(0,T;L_x^2))} \\ &\quad + C(\|\Phi\|^2_{L_2^0} + \|\Phi\|^2_{R(L_x^2;L_x^{2+\delta})}) T^{1-2/r} \big|z_1 - z_2\big|_{L_\omega^\rho(L^r(0,T;L_x^p))}. \end{aligned}$$

Interpolating $L^\gamma(0,T;L^q(\mathbb{R}^n))$ between $L^\infty(0,T;L^2(\mathbb{R}^n))$ and $L^r(0,T;L^p(\mathbb{R}^n))$, and assuming that $T \le T_0$, we finally obtain

$$\begin{aligned} &|\mathcal{T} z_1 - \mathcal{T} z_2|_{L_\omega^\rho(L^r(0,T;L^p(\mathbb{R}^n)))} \\ &\le C(R, T_0, \|\Phi\|_{L_2^0}, \|\Phi\|_{R(L_x^2;L_x^{2+\delta})}) T^\nu |z_1 - z_2|_{X_T} \end{aligned} \tag{3.5}$$

where $\nu = \inf(1 - \frac{n\sigma}{2}, \frac{1}{2} - \frac{1}{r}) > 0$. We now need an estimate on $|\mathcal{T} z_1 - \mathcal{T} z_2|_{L_\omega^\rho(L_t^\infty(L_x^2))}$. Let z_1, z_2 be adapted and in X_T; making use of Strichartz inequality (3.3) again, but with $L^r(0,T;L_x^p)$ replaced by $L^\infty(0,T;L_x^2)$, we may write for a.e. $\omega \in \Omega$

$$\begin{aligned} &|\mathcal{T} z_1 - \mathcal{T} z_2|_{L^\infty(0,T;L_x^2)} \\ &\le C\big|\theta_R(|z_1|_{L^\gamma(0,t;L_x^q)}) f(|z_1|^2) z_1 - \theta_R(|z_2|_{L^\gamma(0,t;L_x^q)}) f(|z_2|^2) z_2\big|_{L^{\gamma'}(0,T;L_x^{q'})} \\ &\quad + \Big| \int_0^t S(t-s)\big((z_1(s) - z_2(s)) dW(s)\big)\Big|_{L^\infty(0,T;L_x^2)} \\ &\quad + C|(z_1 - z_2) F_\Phi|_{L^{r'}(0,T;L_x^{p'})} \\ &\le I + II' + III \end{aligned}$$

where I and III are the same as in (3.4), and II' is estimated by using Lemma 3.2, so that we get for any $\rho \ge r$,

$$\begin{aligned} &|\mathcal{T} z_1 - \mathcal{T} z_2|_{L_\omega^\rho(L^\infty(0,T;L_x^2))} \\ &\le C(R, T_0, \|\Phi\|_{L_2^0}, \|\Phi\|_{R(L_x^2;L_x^{2+\delta})}) T^\nu |z_1 - z_2|_{X_T} \end{aligned} \tag{3.6}$$

assuming again that $T \le T_0$. Now (3.5) and (3.6) show that provided we choose for example

$$T \le \inf \left(T_0, \frac{1}{\Big(2C(R, T_0, \|\Phi\|_{L_2^0}, \|\Phi\|_{R(L_x^2, L_x^{2+\delta})})\Big)^{1/\nu}} \right), \tag{3.7}$$

$\mathcal{T}$ is a contraction mapping in X_T. Moreover, the unitarity of $S(t)$ in $L^2(\mathbb{R}^n)$, together with the fact that $S(\cdot)$ maps $L^2(\mathbb{R}^n)$ into $\mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^n))$, Strichartz inequality

$$|S(\cdot)u_0|_{L^r(\mathbb{R}, L^p(\mathbb{R}^n))} \le C|u_0|_{L^2(\mathbb{R}^n)},$$

(see [5], [9]), and estimates on the stochastic and nonlinear parts of $\mathcal{T}$ following (3.5) and (3.6) show that $\mathcal{T}$ maps X_T into X_T for each $T > 0$. Hence $\mathcal{T}$ has a unique fixed point in X_T, which is the unique solution of (3.1). Moreover, since the choice of T only depends on R, the solution may be extended to the whole interval $[0, T_0]$, achieving to prove Proposition 3.1. □

4. Proof of Theorem 2.1.

Let z_0 be $\mathcal{F}_0$ measurable and in $L^\rho(\Omega; L^2(\mathbb{R}^n))$, with $\rho \ge \max\left(r, (2\sigma+2)(\frac{4\sigma}{2-n\sigma}+1)\right)$. We denote for $m \in \mathbb{N}$ by z_m the unique solution of (3.1) with $R = m$ given by Proposition 3.1. We define

$$\tau_m(\omega) = \sup \left\{ t \in [0, T_0],\ |z_m|_{L^\gamma(0,t;L_x^q)} \le m \right\}$$

where we recall that $q = 2\sigma + 2$ and $\gamma = \frac{4(\sigma+1)}{n\sigma}$. We first show the following lemma.

Lemma 4.1. *$z_m(t) = z_{m+1}(t)$ for each $t \in [0, \min(\tau_m, \tau_{m+1})]$, for a.e. $\omega \in \Omega$.*

Proof. Fix $m \in \mathbb{N}$ and $T > 0$; let $\tau = \min(\tau_m, \tau_{m+1})$ and let $t \in [0, \tau]$; then

$$\begin{aligned} & z_{m+1}(t) - z_m(t) \\ & = -i \int_0^t S(t-s)\big(f(|z_{m+1}(s)|^2)z_{m+1}(s) - f(|z_m(s)|^2)z_m(s)\big)ds \\ & \quad - i \int_0^t S(t-s)\big((z_{m+1}(s) - z_m(s))dW(s)\big) \\ & \quad - \frac{1}{2}\int_0^t S(t-s)\big((z_{m+1}(s) - z_m(s))F_\Phi\big)ds, \end{aligned}$$

since for $0 \le s \le t \le \tau$, we have

$$\theta_m(|z_m|_{L^\gamma(0,s;L_x^q)}) = \theta_{m+1}(|z_{m+1}|_{L^\gamma(0,s;L_x^q)}) = 1.$$

In the case where $\tau < T_0$, we define y_m as the solution on $[\tau, T_0]$ of the equation

$$i\,dy - \Delta y\,dt = y\,dW - \frac{i}{2} y F_\Phi dt$$

with $y_m(\tau) = z_m(\tau)$, that is y_m satisfies for $t \in [\tau, T_0]$:

$$\begin{aligned} y_m(t) = & S(t-\tau) z_m(\tau) - i \int_\tau^t S(t-s)(y_m(s) dW(s)) \\ & - \frac{1}{2} \int_\tau^t S(t-s)(y_m(s) F_\Phi) ds. \end{aligned} \tag{4.1}$$

The existence and uniqueness of $y_m \in L^\rho_\omega(\mathcal{C}([\tau, T_0]; L^2_x)) \cap L^\rho_\omega(L^r(\tau, T; L^p_x))$ satisfying (4.1) follows easily by standard arguments from Strichartz inequalities (see the proof of Proposition 3.1), since (4.1) is a linear equation.

Now, we set

$$\tilde{z}_m(t) = \begin{cases} z_m(t) & \text{if } t \in [0, \tau] \\ y_m(t) & \text{if } t \in [\tau, T_0] \end{cases}$$

and we define y_{m+1} and $\tilde{z}_{m+1}$ in the same way, replacing m by $m+1$; we will show that $\tilde{z}_m = \tilde{z}_{m+1}$ on $[0, T]$ provided that T is sufficiently small, and the lemma will follow by a reiteration procedure. We have for $t \in [0, T]$,

$$\begin{aligned} & \tilde{z}_{m+1}(t) - \tilde{z}_m(t) \\ & = -i \int_0^{t \wedge \tau} S(t-s) \big(f(|\tilde{z}_{m+1}(s)|^2) \tilde{z}_{m+1}(s) - f(|\tilde{z}_m(s)|^2) \tilde{z}_m(s) \big) ds \\ & \quad - i \int_0^t S(t-s) \big((\tilde{z}_{m+1}(s) - \tilde{z}_m(s)) dW(s) \big) \\ & \quad - \frac{1}{2} \int_0^t S(t-s) \big((\tilde{z}_{m+1}(s) - \tilde{z}_m(s)) F_\Phi \big) ds \\ & = I_1(\omega, t, x) + I_2(\omega, t, x) + I_3(\omega, t, x). \end{aligned}$$

For a fixed $\omega \in \Omega$, we estimate the $L^r(0,T;L^p_x)$– norm of I_1 as follows, by using Strichartz inequality (3.3)

$$\begin{aligned}
&|I_1|_{L^r(0,T;L^p_x)} \\
&= \Big| \int_0^t S(t-s)\chi_{\{s\le\tau\}}\big(f(|\tilde z_{m+1}(s)|^2)\tilde z_{m+1}(s) - f(|\tilde z_m(s)|^2)\tilde z_m(s)\big)\,ds \Big|_{L^r(0,T;L^p_x)} \\
&\le C\big|\chi_{[0,\tau]}\big(f(|\tilde z_{m+1}|^2)\tilde z_{m+1} - f(|\tilde z_m|^2)\tilde z_m\big)\big|_{L^{\gamma'}(0,T;L^{q'}_x)} \\
&\le C\big|f(|\tilde z_{m+1}|^2)\tilde z_{m+1} - f(|\tilde z_m|^2)\tilde z_m\big|_{L^{\gamma'}(0,\tau\wedge T;L^{q'}_x)} \\
&\le CT^{1-n\sigma/2}(|\tilde z_{m+1}|^{2\sigma}_{L^\gamma(0,\tau;L^q_x)} + |\tilde z_m|^{2\sigma}_{L^\gamma(0,\tau;L^q_x)})|\tilde z_{m+1} - \tilde z_m|_{L^\gamma(0,T;L^q_x)}
\end{aligned}$$

and since, from the definition of τ, we have

$$|\tilde z_j|_{L^\gamma(0,\tau;L^q_x)} \le j \text{ for } \; j = m,\, m+1,$$

we finally get

$$\begin{aligned}
&\Big(\mathbb{E}|I_1|^\rho_{L^r(0,T;L^p_x)}\Big)^{1/\rho} \\
&\le C(m)T^{1-n\sigma/2}\Big(\mathbb{E}|\tilde z_{m+1} - \tilde z_m|^\rho_{L^\gamma(0,T;L^q_x)}\Big)^{1/\rho}.
\end{aligned} \tag{4.2}$$

We estimate I_2 by using Corollary 3.1 and Lemma 3.2, and I_3 as we estimated III in Section 3, to conclude, as in (3.5) and (3.6) that provided $T \le T_0$ and $\nu = \inf(1-\frac{n\sigma}{2}, \frac12 - \frac1r)$, we have

$$\begin{aligned}
&\Big\{\mathbb{E}\big(|\tilde z_{m+1} - \tilde z_m|^\rho_{L^r(0,T;L^p_x)} + |\tilde z_{m+1} - \tilde z_m|^\rho_{L^\infty(0,T;L^2_x)}\big)\Big\}^{1/\rho} \\
&\le T^\nu C(T_0, m, \|\Phi\|_{L_2^0}, \|\Phi\|_{R(L^2_x;L^{2+\delta}_x)})\Big\{\mathbb{E}\big(|\tilde z_{m+1} - \tilde z_m|^\rho_{L^r(0,T;L^p_x)} + |\tilde z_{m+1} - \tilde z_m|^\rho_{L^\infty(0,T;L^2_x)}\big)\Big\}^{1/\rho}.
\end{aligned}$$

Thus, we have $\tilde z_{m+1} = \tilde z_m$ on $[0,T]$ for a.e. $\omega \in \Omega$, provided that T is chosen sufficiently small, but depending only on m and Φ, so that we can conclude that $z_{m+1} = z_m$ on $[0,\tau]$ for a.e. $\omega \in \Omega$. $\square$

As a consequence of Lemma 4.1, τ_m is nondecreasing with m, and we can define

$$\tau^* = \lim_{m\to+\infty} \tau_m. \tag{4.3}$$

Our aim is now to show that

$$\mathbb{P}\,\{\tau^* = T_0\} = 1$$

and for that purpose, we need an estimate independent of m on z_m in $L^1(\Omega; L^\gamma(0,T;L^q_x))$.

Proposition 4.1. *Let $\rho \geq \max(r, (2\sigma+2)(\frac{4\sigma}{2-n\sigma}+1))$, $z_0 \in L^\rho(\Omega; L^2(\mathbb{R}^n))$ $\mathcal{F}_0$ measurable, and z_m be the solution of (3.1) with $R = m$ given by Proposition 3.1. Then, for a.e. $\omega \in \Omega$, and for each $t \in [0, T_0]$,*

$$|z_m(t)|_{L^2(\mathbb{R}^n)} = |z_0|_{L^2(\mathbb{R}^n)}$$

(in particular, $|z_m|_{L^\infty(0,T_0;L^2(\mathbb{R}^n))} = |z_0|_{L^2(\mathbb{R}^n)}$ for a.e. $\omega \in \Omega$) and there is a constant $M > 0$, depending on T_0 and $|z_0|_{L^\rho_\omega(L^2_x)}$, but independent of m, such that

$$\mathbb{E}\big(|z_m|_{L^r(0,T_0;L^p_x)}\big) \leq M.$$

By interpolating $L^1(\Omega; L^\gamma(0, T_0; L^q_x))$ between $L^1(\Omega; L^\infty(0, T_0; L^2_x))$ and $L^1(\Omega; L^r(0, T_0; L^p_x))$, we have

Corollary 4.1. *Under the same assumptions as in Proposition 4.1, there exists a constant $M' > 0$, depending on T_0 and $|z_0|_{L^\rho_\omega(L^2_x)}$, but independent of m, such that*

$$\mathbb{E}\big(|z_m|_{L^\gamma(0,T_0;L^q_x)}\big) \leq M'.$$

Proof of Proposition 4.1. To show the conservation of L^2- norm, we recall that z_m satisfies the integral equation (3.1), which is the mild form of the Itô equation

$$idz - \big(\Delta z + \theta(|z|_{L^\gamma(0,t;L^q_x)})f(|z|^2)z\big)dt = zdW - \frac{i}{2}zF_\Phi dt \tag{4.4}$$

with $z_m(0) = z_0$.

Applying Itô formula to the functional $F(t) = |z_m(t)|^2_{L^2(\mathbb{R}^n)}$, we get, using that f and Φ are real valued,

$$\begin{aligned}
|z_m(t)|^2_{L^2_x} =& |z_0|^2_{L^2_x} + 2Im \int_0^t \big(\theta_m(|z_m|_{L^\gamma(0,s;(L^q_x))})f(|z_m|^2)z_m, z_m\big)ds \\
&+ 2Im \sum_{k=0}^\infty \int_0^t (z_m \Phi e_k, z_m)d\beta_k - Re \int_0^t (z_m \sum_{k=0}^\infty (\Phi e_k)^2, z_m)ds \\
&+ \int_0^t \sum_{k=0}^\infty |z_m \Phi e_k|^2_{L^2_x} ds \\
=& |z_0|^2_{L^2_x}.
\end{aligned}$$

This computation may easily be justified by using a regularization procedure, and it proves the first part of the lemma.

Next, applying Strichartz inequality to the integral equation (3.1), we obtain as in (3.4), for a.e. $\omega \in \Omega$ and for $0 \leq T \leq T_0$,

$$\begin{aligned}
|z_m|_{L^r(0,T;L^p_x)} \leq& C|z_0|_{L^2_x} + C|\theta_m(|z_m|_{L^\gamma(0,t;L^q_x)})f(|z_m|^2)z_m|_{L^{\gamma'}(0,T;L^{q'}_x)} \\
&+ |Jz_m|_{L^r(0,T;L^p_x)} + C|z_m F_\Phi|_{L^{r'}(0,T;L^{p'}_x)}
\end{aligned}$$

where Jz is defined as in Corollary 3.1.

Proceeding as we estimated I and III in (3.4), we then get

$$\begin{aligned}|z_m|_{L^r(0,T;L^p_x)} \leq C_1|z_0|_{L^2_x} &+ C_2T^{1-n\sigma/2}|z_m|^{2\sigma+1}_{L^\gamma(0,T;L^q_x)}\\ &+ C_3T^{1-2/r}(\|\Phi\|^2_{L^0_2} + \|\Phi\|^2_{R(L^2_x;L^{2+\delta}_x)})|z_m|_{L^r(0,T;L^p_x)}\\ &+ |Jz_m|_{L^r(0,T;L^p_x)}.\end{aligned}$$

Now, we assume that $T \leq T_1$ with

$$C_3T_1^{1-2/r}(\|\Phi\|^2_{L^0_2} + \|\Phi\|^2_{R(L^2_x;L^{2+\delta}_x)}) = \frac{1}{2}, \tag{4.5}$$

so that the preceding estimate leads to

$$\begin{aligned}|z_m|_{L^r(0,T;L^p_x)} \leq &2C_1|z_0|_{L^2_x} + 2C_2T^{1-n\sigma/2}|z_m|^{2\sigma+1}_{L^\gamma(0,T;L^q_x)}\\ &+ 2|Jz_m|_{L^r(0,T_0;L^p_x)}.\end{aligned}$$

Let θ be such that $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p}$ (hence $0 < \theta < 1$), then interpolating $L^\gamma(0,T;L^q_x)$ between $L^\infty(0,T;L^2_x)$ and $L^r(0,T;L^p_x)$, and using the fact that $|z_m|_{L^\infty(0,T;L^2_x)} = |z_0|_{L^2_x}$, we see that if $T \leq \inf(T_0,T_1)$

$$\begin{aligned}|z_m|_{L^r(0,T;L^p_x)} \leq\ & 2C_1|z_0|_{L^2_x} + 2|Jz_m|_{L^r(0,T_0;L^p_x)}\\ &+2C_2T^{1-n\sigma/2}|z_0|^{\theta(2\sigma+1)}_{L^2_x}|z_m|^{(1-\theta)(2\sigma+1)}_{L^r(0,T;L^p_x)}\\ \leq\ & C_4(T_0)|z_0|_{L^2_x}(1+|z_0|^{2\sigma+1}_{L^2_x})\\ &+2(|Jz_m|_{L^r(0,T_0;L^p_x)} + \sup\nolimits_{0\leq t_0\leq T_0}|Iz_m(t_0,\cdot)|_{L^r(0,T_0;L^p_x)})\\ &+T^{1-n\sigma/2}|z_m|^{2\sigma+1}_{L^r(0,T;L^p_x)}\end{aligned}$$

where we have used Young's inequality in the last estimate, and where the second term in the right hand side has been modified for later purpose, $Iz(t_0,t)$ being defined as in Lemma 3.1. Hence, setting

$$\begin{aligned}K_m(\omega) =&C_4(T_0)|z_0|_{L^2_x}(1+|z_0|^{2\sigma+1}_{L^2_x})\\ &+ 2\big(|Jz_m|_{L^r(0,T_0;L^p_x)} + \sup_{0\leq t_0\leq T_0}|Iz_m(t_0,\cdot)|_{L^r(0,T_0;L^p_x)}\big)\end{aligned} \tag{4.6}$$

we have for all $T \leq \inf(T_0,T_1)$

$$|z_m|_{L^r(0,T;L^p_x)} \leq K_m(\omega) + T^{1-n\sigma/2}|z_m|^{2\sigma+1}_{L^r(0,T;L^p_x)}.$$

It follows that for T small enough,

$$|z_m|_{L^r(0,T;L^p_x)} \leq 2K_m(\omega). \tag{4.7}$$

More precisely, (4.7) is satisfied if we choose for instance $T = \inf(T_0, T_1, T^m(\omega))$ with

$$T^m(\omega) = (2^{\sigma+1} K_m^\sigma)^{-\frac{4}{2-n\sigma}}. \tag{4.8}$$

Note that by Lemma 3.1 and Corollary 3.1, we have $T^m(\omega) > 0$ for a.e. $\omega \in \Omega$.

We may now reiterate the previous process on each interval $[jT, (j+1)T] \subset [0, T_0]$ (keeping m fixed and varying j); using the integral equation

$$\begin{aligned} z_m(t) =& S(t - jT)z_m(jT) - i \int_{jT}^t S(t-s)\big(\theta_m(|z_m|_{L^\gamma(0,s;L_x^q)}) f(|z_m|^2) z_m\big) ds \\ & - i \int_{jT}^t S(t-s)\big(z_m(s) dW(s)\big) - \frac{1}{2} \int_{jT}^t S(t-s)\big(z_m(s) F_\Phi\big) ds, \end{aligned}$$

we easily get by the same computations as before,

$$\begin{aligned} |z_m|_{L^r(jT,(j+1)T;L_x^p)} \le & \; 2C_1 |z_0|_{L_x^2} \\ & + 2 \sup_{0 \le t_0 \le T_0} \left| \int_{t_0}^t S(t-s)\big(z_m(s) dW(s)\big) \right|_{L^r(t_0,T_0;L_x^p)} \\ & + 2C_2 T^{1-n\sigma/2} |z_0|_{L_x^2}^{\theta(2\sigma+1)} |z_m|_{L^r(jT,(j+1)T;L_x^p)}^{(1-\theta)(2\sigma+1)}. \end{aligned}$$

Now, using the trivial fact that

$$\begin{aligned} \int_{t_0}^t S(t-s)(z_m(s) dW(s)) = & \int_0^t S(t-s)(z_m(s) dW(s)) \\ & - \int_0^{t_0} S(t-s)(z_m(s) dW(s)), \end{aligned}$$

we have

$$\begin{aligned} & \sup_{0 \le t_0 \le T_0} \left| \int_{t_0}^t S(t-s)\big(z_m(s) dW(s)\big) \right|_{L^r(t_0,T_0;L_x^p)} \\ & \le |J z_m|_{L^r(0,T_0;L_x^p)} + \sup_{0 \le t_0 \le T_0} |I z_m(t_0, \cdot)|_{L^r(0,T_0;L_x^p)} \end{aligned}$$

so that with T still defined as $T = \inf(T_0, T_1, T^m(\omega))$ (see (4.8)), we see that

$$|z_m|_{L^r(jT,(j+1)T;L_x^p)} \le 2K_m(\omega)$$

for each j such that $[jT, (j+1)T] \subset [0, T_0]$, and where $K_m(\omega)$ is defined in (4.6).

As a consequence, we have, by using (4.8)

$$\begin{aligned} |z_m|_{L^r(0,T_0;L_x^p)} & \le \sum_j |z_m|_{L^r(jT,(j+1)T;L_x^p)} \\ & \le 2\Big(\frac{T_0}{T} + 1\Big) K_m(\omega) \\ & \le C_5(T_0) K_m + C_6(T_0) K_m^{\frac{4\sigma}{2-n\sigma}+1}. \end{aligned}$$

Hence, if $\rho \geq (2\sigma+2)(\frac{4\sigma}{2-n\sigma}+1)$, and using (4.6) together with Hölder inequality, we get

$$\begin{aligned}
&\mathbb{E}\big(|z_m|_{L^r(0,T_0;L^p_x)}\big)\\
&\leq C_7(T_0)\left\{\mathbb{E}|z_0|^\rho_{L^2_x}+\mathbb{E}\big(\big|Jz_m\big|^\rho_{L^r(0,T_0;L^p_x)}\big)\right.\\
&\qquad\left.+\mathbb{E}\left(\sup_{0\leq t_0\leq T_0}|Iz_m(t_0,\cdot)|^\rho_{L^r(0,T_0;L^p_x)}\right)\right\}.
\end{aligned}$$

By Hölder inequality, Lemma 3.1 and Remark 3.1,

$$\begin{aligned}
&\mathbb{E}\big(\sup_{0\leq t_0\leq T_0}|Iz_m(t_0,\cdot)|^\rho_{L^r(0,T_0;L^r_x)}\big)\\
&\leq T_0^{\rho/r-1}\mathbb{E}\left(\int_0^{T_0}\sup_{0\leq t_0\leq T_0}|Iz_m(t_0,t)|^\rho_{L^p_x}dt\right)\\
&\leq C_8T_0^{\rho/r}(\|\Phi\|_{L_2^0}+\|\Phi\|_{R(L^2_x,L^{2+\delta}_x)})^\rho T_0^{\rho(\frac12-\frac1r)}\mathbb{E}\big(|z_m|^\rho_{L^\infty(0,T_0;L^2_x)}\big)
\end{aligned}$$

and the same bound is true for $\mathbb{E}\big(|Jz_m|^\rho_{L^r(0,T_0;L^p_x)}\big)$ by Corollary 3.1, so that the proof of Proposition 4.1 is ended by the fact that $|z_m|_{L^\infty(0,T_0;L^2_x)}=|z_0|_{L^2_x}$ for a.e. $\omega\in\Omega$; M is then given by

$$M=C_9(T_0)\mathbb{E}\big(|z_0|^\rho_{L^2_x}\big)(1+\|\Phi\|_{L^0_2}+\|\Phi\|_{R(L^2_x,L^{2+\delta}_x)})^\rho.$$

□

We now show that

$$\mathbb{P}\left\{\tau^*=T_0\right\}=1$$

where τ^* is defined by (4.3). By Corollary 4.1,

$$\mathbb{E}\big(|z_m|_{L^\gamma(0,T_0;L^q_x)}\big)\leq M'$$

with M' depending only on T_0, Φ and $\mathbb{E}\big(|z_0|^\rho_{L^2_x}\big)$. Hence,

$$\mathbb{P}\left\{\tau_m=T_0\right\}=\mathbb{P}\left\{|z_m|_{L^\gamma(0,T_0;L^q_x)}\leq m\right\}\geq 1-\frac{M'}{m}$$

and since $\tau_m\nearrow\tau^*$ for a.e. $\omega\in\Omega$, we conclude that

$$\mathbb{P}\left\{\tau^*=T_0\right\}=1.$$

We then define z on $[0,T_0]\subset\cup_{m\geq1}[0,\tau_m]$ by $z=z_m$ on $[0,\tau_m]$. Then, obviously, z is a solution of (2.1). The fact that $|z_m(t)|_{L^2_x}=|z_0(t)|_{L^2_x}$ implies easily that $|z(t)|_{L^2_x}=|z_0|_{L^2_x}$ for a.e. $\omega\in\Omega$ and each $t\in[0,T_0]$. In particular, $z\in L^\rho(\Omega;L^\infty(0,T_0;L^2_x))$ (and $z\in\mathcal{C}([0,T_0];L^2_x)$ for a.e. $\omega\in\Omega$). Moreover, the uniform bound on $|z_m|_{L^1(\Omega;L^r(0,T_0;L^p_x))}$ given by Proposition 4.1, and the definition of z imply by Fatou's lemma that $z\in L^1(\Omega;L^r(0,T_0;L^p_x))$.

It remains to prove the uniqueness part of Theorem 2.1. Let z_1, z_2 be two solutions of (2.1) in $L^\rho(\Omega; \mathcal{C}([0,T_0]; L^2_x)) \cap L^1(\Omega; L^r(0,T_0; L^p_x))$. Denoting, for a fixed positive R,

$$\tau_R = \sup\left\{t \in [0,T_0], \quad |z_1|_{L^\gamma(0,t;L^q_x)} \le R, \text{and} \quad |z_2|_{L^\gamma(0,t;L^q_x)} \le R\right\},$$

then z_1 and z_2 are for a.e. $\omega \in \Omega$, solutions of (3.1) on $[0,\tau_R]$, and by the uniqueness part of Proposition 3.1, $z_1 = z_2$ on $[0,\tau_R]$ for a.e. $\omega \in \Omega$. Taking then $R = k \in \mathbb{N}$, and using the continuity in time of z_1, z_2, we have

$$z_1 = z_2 \text{ on } [0, \overline{\lim_{k\to\infty}} \tau_k] \text{ a.e. } \omega \in \Omega. \tag{4.9}$$

Next, we remark that

$$\begin{aligned}\mathbb{P}\{\tau_k = T_0\} &= \mathbb{P}\left\{|z_1|_{L^\gamma(0,T_0;L^q_x)} \le k,\ |z_2|_{L^\gamma(0,T_0;L^q_x)} \le k\right\}\\ &\le \frac{1}{k}\mathbb{E}\left(|z_1|_{L^\gamma(0,T_0;L^q_x)} + |z_2|_{L^\gamma(0,T_0;L^q_x)}\right)\end{aligned}$$

hence, $\mathbb{P}\{\tau_k = T_0\} \to 1$ as $k \to +\infty$, i.e. $\mathbb{P}\left\{\overline{\lim}_{k\to\infty} \tau_k = T_0\right\} = 1$, and from this and (4.9), we deduce that $z_1 = z_2$ on $[0,T_0]$ for a.e. $\omega \in \Omega$. This concludes the proof of Theorem 2.1. □

References

[1] Bang, O., Christiansen, P.L., If, F., Rasmussen, K.O., *Temperature effects in a nonlinear model of monolayer Scheibe aggregates*, Phys. Rev. E **49**, 4627–4636 (1994).

[2] Bang, O., Christiansen, P.L., If,F., Rasmussen, K.O., *White Noise in the Two-dimensional Nonlinear Schrödinger Equation*, Appl. Anal. **57**, 3–15 (1995).

[3] Brzezniak, Z., *On stochastic convolution in Banach spaces, and applications*, Stochastics Stochastics Rep. **61**, 245–295 (1997).

[4] Brzezniak, Z., Peszat, S., *Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process*, preprint, 1997.

[5] Cazenave, T., *An Introduction to Nonlinear Schrödinger Equations*, Textos de Métodos Matématicos **26**, Instituto de Matématica-UFRJ Rio de Janeiro, Brazil, 1993.

[6] Christiansen, P.L., Rasmussen, K.O., Bang, O., Gaididei, Yu.B., *The temperature-dependent collapse regime in a nonlinear dynamical model of Scheibe aggregates*, Phys. D **87**, 321–324 (1995).

[7] Da Prato, G., Zabczyk, J., *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Application, Cambridge University Press, 1992.

[8] Fibich, G., Papanicolaou, G., *Self-focusing in the perturbed and unperturbed nonlinear Schrödinger equation in critical dimension*, Preprint, 1997.

[9] Ginibre, J., Velo, G., *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. Henri Poincaré, Analyse Non Linéaire **2**, 309–327 (1985).

[10] Rasmussen, K.O., Gaididei, Yu.B., Bang, O., Chrisiansen, P.L., *The influence of noise on critical collapse in the nonlinear Schrödinger equation*, Phys. Letters A **204**, 121–127 (1995).

[11] Tsutsumi, Y., *L^2-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funk. Ekva. **30**, 115–125 (1987).