On relations between entropy and Hausdorff dimension of measures

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Abstract : *We characterize the measures for which the Hausdorff dimension can be calculated by an entropy formula. Several examples and counterexamples are proposed.*

1 Introduction.

Let D be an integer greater than 1 and m be a probability measure in $[0,1)^D$. Fix $\ell \geq 1$ and denote by $\mathcal{F}_n$ the family of ℓ-adic cubes of the n^{th} generation, that is

$$\mathcal{F}_n = \left\{ I = \prod_{i=1}^{D} [k_i/\ell^n, (k_i+1)/\ell^n) \ ; \ 0 \leq k_i < \ell^n \right\} .$$

For an arbitrary point x in $[0,1)^D$, let $I_n(x)$ be the unique cube $I \in \mathcal{F}_n$ such that $x \in I$.

We want to give estimates for the lower and upper dimensions of the measure m. These are respectively defined by

$$\begin{cases} \dim_*(m) = \inf(\dim(E) \ ; \ m(E) > 0) \\ \dim^*(m) = \inf(\dim(E) \ ; \ m(E) = 1) \end{cases} .$$

It is well known that there exist some relations between these quantities and the function τ which appears in the multifractal formalism. More precisely, if we let :

$$\tau_n(t) = \frac{1}{n \log \ell} \log \left(\sum_{I \in \mathcal{F}_n} m(I)^t \right) \quad \text{and} \quad \tau(t) = \limsup_{n \to +\infty} \tau_n(t) \ ,$$

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it is proved in [Heu98] that

$$-\tau_+'(1) \le \dim_*(m) \le h_*(m) \tag{1}$$

where $h_*(m)$ is the lower entropy (also called lower Rényi dimension) of the measure m, defined as

$$h_*(m) = \liminf_{n\to+\infty} h_n(m) \quad \text{where} \quad h_n(m) = -\tau_n'(1) = \frac{-1}{n\,\log\ell}\sum_{I\in\mathcal{F}_n} m(I)\,\log(m(I))\ .$$

In [Heu98], we also give sufficient conditions for the equality $-\tau_+'(1) = \dim_*(m)$ to hold.

In this paper we are interested in describing the measures m satisfying $\dim_*(m) = h_*(m)$. Theorem 2.1 states that $\dim_*(m) = h_*(m)$ holds if and only if there exists a subsequence n_k such that for dm-almost every $x \in [0,1)^D$,

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k\,\log\ell} = \dim_*(m)\ .$$

In particular, such measures are unidimensional (i.e. $\dim_*(m) = \dim^*(m)$). If we denote by $\dim(m)$ this common value, the measure m is supported by a set of dimension $\dim(m)$ but every set of dimension strictly less than $\dim(m)$ is negligible. Nevertheless, unidimensionality is not a sufficient condition to have $\dim_*(m) = h_*(m)$. An example of a measure of exact dimension (i.e. unidimensional) for which $\dim_*(m) < h_*(m)$ is proposed in proposition 4.1.

Similar results can be established, comparing the upper entropy

$$h^*(m) = \limsup_{n\to+\infty} h_n(m)$$

and the packing dimension of the measure m. Following [Heu98] or [Fal97], we can introduce

$$\begin{cases} \operatorname{Dim}_*(m) = \inf(\operatorname{Dim}(E)\ ;\ m(E) > 0) \\ \operatorname{Dim}^*(m) = \inf(\operatorname{Dim}(E)\ ;\ m(E) = 1) \end{cases},$$

where $\operatorname{Dim}(E)$ is the packing dimension of the set E. As proved in [Heu98],

$$h^*(m) \le \operatorname{Dim}^*(m) \le -\tau_-'(1) \tag{2}$$

and we characterize in theorem 2.2 of the present paper the measures for which the equality $h^*(m) = \operatorname{Dim}^*(m)$ holds.

Several examples of measures satisfying $\dim_*(m) = h_*(m)$ and $\operatorname{Dim}^*(m) = h^*(m)$ are also proposed. In particular, this is the case in an ergodic situation (example 2.3), for quasi-Bernoulli measures (example 2.5) and in a context where the strong law of large numbers can be applied (example 2.6).

In the last section, we construct a measure m of exact dimension for which

$$\dim(m) = h_*(m) \quad \text{but} \quad \operatorname{Dim}(m) > h^*(m)\ .$$

2 Main results and examples.

The main result of this paper is the following.

Theorem 2.1 *Let m be a probability measure in $[0,1)^D$. Then*

$$\dim_*(m) \leq h_*(m) \ .$$

Moreover, the following properties are equivalent :

(i) $\dim_*(m) = h_*(m)$
(ii) $\dim_*(m) = \dim^*(m) = h_*(m)$
(iii) *There exists a subsequence $(n_k)_{k\geq 1}$ such that for dm-almost every $x \in [0,1)^D$,*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \dim_*(m) \ .$$

A similar result can also be established, comparing the upper entropy $h^*(m)$ of a measure m with its packing dimension $\mathrm{Dim}^*(m)$.

Theorem 2.2 *We also have*

$$h^*(m) \leq \mathrm{Dim}^*(m),$$

and the following properties are equivalent :

(i) $\mathrm{Dim}^*(m) = h^*(m)$
(ii) $\mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = h^*(m)$
(iii) *There exists a subsequence $(n_k)_{k\geq 1}$ such that for dm-almost every $x \in [0,1)^D$,*

$$\lim_{k\to+\infty} \frac{\log m(I_{n_k}(x))}{-n_k \log \ell} = \mathrm{Dim}^*(m) \ .$$

Remark. As was pointed out in the introduction, unidimensionality is not sufficient to establish the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ (see propositions 4.1 and 5.1). In fact, the statements of theorems 2.1 and 2.2 stem from a deep homogeneity property.

Let us now give some useful examples of measures m for which the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ hold.

Example 2.3 Suppose that the sequence

$$\frac{\log m(I_n(x))}{-n \log \ell} \tag{3}$$

is almost surely converging to a constant d. Then, the equivalent properties of theorems 2.1 and 2.2 are satisfied and we have

$$d = \dim(m) = \mathrm{Dim}(m) \ . \tag{4}$$

In particular, this is the case in an ergodic context. Let us denote the elements of $\mathcal{F}_n$ by $I_{\varepsilon_1,\ldots,\varepsilon_n}$ with $\varepsilon_i \in \{0,\ldots,\ell^D-1\}$ and

$$I_{\varepsilon_1,\ldots,\varepsilon_{n+1}} \subset I_{\varepsilon_1,\ldots,\varepsilon_n} \ .$$

Define $J(x)$ to be the unique element $(\varepsilon_i)_{i\geq 1}$ of the Cantor set $\{0,\ldots,\ell^D-1\}^{\mathbb{N}^*}$ such that $\{x\} = \bigcap_n I_{\varepsilon_1,\ldots,\varepsilon_n}$, and consider the image $\tilde{m}$ of m with respect to the application J. If we suppose that the mesure $\tilde{m}$ is invariant and ergodic with respect to the shift operator, Shannon-McMillan's theorem ensures that (3) admits an almost sure constant limit d and that (4) is satisfied (see [Heu98] for more details and [Zin97] for basic facts on ergodic theory). •

Example 2.4 The following situation was described by S.M. Ngai in [Nga97]. Suppose that the function τ of the multifractal formalism admits a derivative $\tau'(1)$. Then, using inequalities (1) and (2), we conclude that the measure m is unidimensional and satisfies

$$\dim(m) = \mathrm{Dim}(m) = h(m) = -\tau'(1)$$

where $h(m) = h_*(m) = h^*(m)$ is the genuine limit of the sequence $h_n(m)$. •

Example 2.5 The case of quasi-Bernoulli measures is related to examples 2.3 and 2.4. Suppose that there exists a constant $C > 0$ such that for every $\varepsilon_1,\ldots,\varepsilon_{n+p}$ we have :

$$\frac{1}{C}\, m\left(I_{\varepsilon_1,\ldots,\varepsilon_n}\right) m\left(I_{\varepsilon_{n+1},\ldots,\varepsilon_{n+p}}\right) \leq m\left(I_{\varepsilon_1,\ldots,\varepsilon_{n+p}}\right) \leq C\, m\left(I_{\varepsilon_1,\ldots,\varepsilon_n}\right) m\left(I_{\varepsilon_{n+1},\ldots,\varepsilon_{n+p}}\right) \ .$$

It is well known (see [Car85]) that such a measure is equivalent to an invariant and ergodic measure. Moreover, it is proved in [Heu98] that $\tau'(1)$ exists in this case. Let us also remember that the multifractal formalism is available for such measures (see [BMP92]). •

Example 2.6 Let us write

$$X_n(x) = \log\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right)$$

and suppose that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i)) = 0 \quad dm \ a.s.$$

where the expectation is relative to the probability m. If we remark that

$$h_n(m) = \frac{-1}{n\log\ell}\sum_{i=1}^{n}\mathbb{E}(X_i),$$

we can easily conclude that the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ are satisfied in this situation. In particular, using the strong law of large numbers, this is the case when the random variables X_i are bounded (according to [Wu98], the measure m is called ℓ-adic doubling) and uncorrelated.

A classical case (called Bernoulli product) of such measures is often described in the literature (see for example [BK90], [Bis95] or [Heu98]). Fix a sequence $(p_i)_{i\geq 1}$ of real numbers with $0 < p_i < 1$ and consider a sequence $(Y_i)_{i\geq 1}$ of independant random variables such that

$$\mathbb{P}(Y_i = 0) = p_i \quad \text{and} \quad \mathbb{P}(Y_i = 1) = 1 - p_i \ .$$

Then, the law m of the random variable

$$\sum_{i=1}^{+\infty} 2^{-i} Y_i$$

satisfies

$$\begin{cases} \dim(m) = h_*(m) = \liminf\limits_{n\to+\infty} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} s(p_i) \\ \mathrm{Dim}(m) = h^*(m) = \limsup\limits_{n\to+\infty} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} s(p_i) \end{cases} \tag{5}$$

where the function $s(t)$ is defined by

$$s(t) = -[t\log_2 t + (1-t)\log_2(1-t)] \quad \text{for} \quad t \in [0,1]$$

and $\log_2$ is the logarithm in base 2. Particular cases of this example will be used in section 4. $\bullet$

Example 2.7 Example 2.6 can be seen as a particular case of a more general situation which is described in [Heu98], Corollaire 4.2. If we suppose that the sequence of functions $(\tau'_n(t))_{n\geq 1}$ is equicontinous at $t = 1$, then

$$\dim_*(m) = h_*(m) = -\tau'_+(1) \quad \text{and} \quad \mathrm{Dim}^*(m) = h^*(m) = -\tau'_-(1) \ .$$

$\bullet$

3 Proof of theorems 2.1 and 2.2.

We prove theorem 2.1. Since the proof of theorem 2.2 is similar, we will only sketch it at the end of the section.

The inequality $\dim_*(m) \leq h_*(m)$ is well known (see for example [Heu98] or [You82]). Nevertheless, let us give an elementary proof in order to make easier the study of the equality case. If $x \in [0,1)^D$, put :

$$\alpha_n(x) = \frac{\log m(I_n(x))}{-n \log \ell} \quad \text{and} \quad \underline{\alpha}(x) = \liminf_{n \to +\infty} \alpha_n(x) \; .$$

It is well known ([Fan94], [Fal97] or [Heu98]) that :

$$\dim_*(m) = \sup(\{\alpha \geq 0 \; ; \; \underline{\alpha} \geq \alpha \quad dm \; a.s.\}) = \inf \operatorname{ess}(\underline{\alpha}) \; . \tag{6}$$

Using Fatou's lemma and the fact that m is a probability measure, we get

$$\dim_*(m) \leq \int \underline{\alpha}(x) \, dm(x) \leq h_*(m) \; . \tag{7}$$

Proof of (iii) $\Rightarrow$ (i). Suppose that there exists a subsequence $(n_k)_{k \geq 1}$ such that for dm-almost every $x \in [0,1)^D$,

$$\lim_{k \to +\infty} \alpha_{n_k}(x) = \dim_*(m) \; .$$

Using the dominated convergence theorem (see lemma 3.1 below), we obtain

$$\begin{aligned} \dim_*(m) &= \int \lim_{k \to +\infty} \alpha_{n_k}(x) \, dm(x) \\ &= \lim_{k \to +\infty} \int \alpha_{n_k}(x) \, dm(x) \\ &= \lim_{k \to +\infty} h_{n_k}(m) \\ &\geq h_*(m) \end{aligned}$$

and we are done. Observe that due to the following lemma, we may apply Lebesgue's dominated convergence theorem.

Lemma 3.1 *Let* $\phi = \sup_{n \geq 1} \alpha_n$. *Then* $\phi \in L^1(m)$.

Proof. It is sufficient to prove that the real variable function

$$t \longmapsto m(\{x \; ; \; \phi(x) > t\})$$

is integrable in a neighbourhood of $+\infty$ with respect to the Lebesgue's measure. But we know that $\alpha_n(x) > t$ if and only if $m(I_n(x)) < \ell^{-nt}$. According to the fact that the partition $\mathcal{F}_n$ contains ℓ^{nD} elements,

$$m(\{x \ ; \ \alpha_n(x) > t\}) \leq \ell^{nD}\ell^{-nt} \ .$$

If $t > D$, we obtain

$$\begin{aligned} m(\{x \ ; \ \phi(x) > t\}) &\leq \sum_{n=1}^{+\infty} m(\{x \ ; \ \alpha_n(x) > t\}) \\ &\leq \sum_{n=1}^{+\infty} \ell^{n(D-t)} \\ &= \frac{\ell^{D-t}}{1-\ell^{D-t}} \end{aligned}$$

which proves the integrability of ϕ.

Proof of (i) $\Rightarrow$ (ii). Suppose that $\dim_*(m) = h_*(m)$. Using (6) and (7), we remark that for dm-almost every point x, $\underline{\alpha}(x) = \dim_*(m)$. Then, we also have

$$\dim_*(m) = \inf(\{\alpha \geq 0 \ ; \ \underline{\alpha} \leq \alpha \quad dm \ a.s.\}) = \sup \operatorname{ess}(\underline{\alpha})$$

and hence $\dim_*(m) = \dim^*(m)$ (see [Heu98] or [Fal97]). The measure m is thus unidimensional.

Proof of (ii) $\Rightarrow$ (iii). Let $d = \dim_*(m) = h_*(m)$. We will make use of the following lemma.

Lemma 3.2 *Let $\eta \in (0,1)$ and $n_0 \geq 1$. We can choose an integer $n_1 \geq n_0$ such that :*

$$m(\{x \ ; \ \alpha_{n_1} > d + \eta\}) \leq (2+d)\,\eta \ .$$

Proof. As was remarked before, the equality $\underline{\alpha} = d$ holds dm-almost surely. Using Egorov's argument, we can find $n_0' \geq n_0$ such that

$$m\left(\bigcap_{n \geq n_0'} \{x \ ; \ \alpha_n(x) > d - \eta^2\}\right) > 1 - \eta^2 \ .$$

On the other hand, by hypothesis (ii), we can choose an integer $n_1 \geq n_0'$ such that

$$h_{n_1}(m) = \int \alpha_{n_1}(x)\, dm(x) < d + \eta^2 \ .$$

This integer n_1 will be the right one. Let us denote

$$A = \{x \; ; \; \alpha_{n_1}(x) > d - \eta^2\}$$

and try to estimate the mass of the set

$$B = \{x \; ; \; \alpha_{n_1}(x) > d + \eta\} \; .$$

Obviously, $B \subset A$. We can write

$$\begin{aligned} d + \eta^2 &\geq \int \alpha_{n_1}(x)\, dm(x) \\ &\geq \int_{A \setminus B} \alpha_{n_1}(x)\, dm(x) + \int_B \alpha_{n_1}(x)\, dm(x) \\ &\geq (d - \eta^2)\,(m(A) - m(B)) + (d + \eta)\, m(B) \\ &\geq (d - \eta^2)\,(1 - \eta^2) + (\eta + \eta^2)\, m(B) \; . \end{aligned}$$

We then get

$$m(B) \leq \frac{2\eta^2 + d\eta^2}{\eta + \eta^2} \leq (2 + d)\, \eta$$

which is the conclusion of the lemma.

By using lemma 3.2 with $\eta = 2^{-k}$, we can construct a subsequence $(n_k)_{k \geq 1}$ such that for all $k \geq 1$,

$$m(\{x \; ; \; \alpha_{n_k}(x) > d + 2^{-k}\}) \leq (2 + d)\, 2^{-k} \; .$$

By applying Borel Cantelli's lemma, we obtain that for almost every $x \in [0, 1)^D$, there exists $k_0 \geq 1$ such that for every $k \geq k_0$,

$$\alpha_{n_k}(x) \leq d + 2^{-k} \; .$$

Then we have

$$\limsup_{k \to +\infty} \alpha_{n_k}(x) \leq d \quad dm \; a.s. \; .$$

On the other hand,

$$d = \underline{\alpha}(x) \leq \liminf_{k \to +\infty} \alpha_{n_k}(x) \quad dm \; a.s. \; .$$

Thus, we have proved that the subsequence $(\alpha_{n_k})_{k \geq 1}$ converges almost surely to d. □

To prove theorem 2.2, we introduce the function

$$\bar{\alpha}(x) = \limsup_{n \to +\infty} \alpha_n(x) \; .$$

As proved in [Heu98] and [Fal97], we have :

$$\mathrm{Dim}^*(m) = \inf(\{\alpha \geq 0 \; ; \; \bar{\alpha} \leq \alpha \quad dm \; a.s.\}) = \sup \mathrm{ess}(\bar{\alpha}) \; .$$

The inequality $h^*(m) \leq \mathrm{Dim}^*(m)$ is a consequence of Fatou's lemma applied to the sequence $\phi - \alpha_n$. The study of the equality case uses the same ideas as before.

4 An example where $\dim(m) < h_*(m)$ and $\mathrm{Dim}(m) > h^*(m)$.

In this section, we take $D = 1$ and $\ell = 2$. We begin with the construction of a family of auxiliary measures which are of the type described in example 2.6. Let a and b be two real numbers with $0 < a, b < 1$ and fix a sequence of integers $(T_k)_{k\geq 1}$ such that

$$T_1 = 1, \quad T_k < T_{k+1} \quad \text{and} \quad \lim_{k\to+\infty} \frac{T_{k+1}}{T_k} = +\infty \ .$$

Then, we define the family of parameters $p_{a\,b}(i)$:

$$p_{a\,b}(i) = a \ \text{ if } \ T_{2n-1} \leq i < T_{2n} \quad \text{and} \quad p_{a\,b}(i) = b \ \text{ if } \ T_{2n} \leq i < T_{2n+1} \ .$$

If $(Y_i)_{i\geq 1}$ is a sequence of independant random variables such that

$$\mathbb{P}(Y_i = 0) = p_{a\,b}(i) \quad \text{and} \quad \mathbb{P}(Y_i = 1) = 1 - p_{a\,b}(i) \ ,$$

we denote by $m_{a\,b}$ the law of the random variable

$$\sum_{i=1}^{+\infty} 2^{-i} Y_i \ .$$

The choice of the integers T_k and the identities (5) ensure that for $dm_{a\,b}$-almost every $x \in [0,1)$, we have :

$$\begin{cases} \liminf\limits_{n\to+\infty} \dfrac{\log m_{a\,b}(I_n(x))}{-n\,\log 2} = \inf(s(a), s(b)) \\ \limsup\limits_{n\to+\infty} \dfrac{\log m_{a\,b}(I_n(x))}{-n\,\log 2} = \sup(s(a), s(b)) \end{cases} . \tag{8}$$

We can now construct our counterexamples. Let us fix a parameter $\beta \in (0, 1/2]$ and two real numbers p and $\tilde{p}$ such that $0 < p < \tilde{p} \leq 1/2$. If $F(t)$ is defined by

$$F(t) = 2t \ \text{ when } \ t \in [0, 1/2) \quad \text{and} \quad F(t) = 2t - 1 \ \text{ when } \ t \in [1/2, 1),$$

we are interested in the measure m_β defined by

$$m_\beta(A) = \beta\, m_{p\,\tilde{p}}(F([0,1/2) \cap A)) + (1-\beta)\, m_{\tilde{p}\,p}(F([1/2,1) \cap A)) \ .$$

In other words, the measure m_β assigns the mass β (resp. $1-\beta$) to the interval $[0, 1/2)$ (resp. $[1/2, 1)$) and is a copy, in this set, of the measure $m_{p\,\tilde{p}}$ (resp. $m_{\tilde{p}\,p}$)

Measures m_β are examples of unidimensional measures whose dimension can not be calculated with an entropy formula. More precisely, we have the following.

Proposition 4.1 *The measure m_β satisfies the following properties :*

1. $\dim_*(m_\beta) = \dim^*(m_\beta) = s(p)$
2. $\mathrm{Dim}_*(m_\beta) = \mathrm{Dim}^*(m_\beta) = s(\tilde{p})$
3. $h_*(m_\beta) = \beta\, s(\tilde{p}) + (1-\beta)\, s(p)$ *and* $h^*(m_\beta) = \beta\, s(p) + (1-\beta)\, s(\tilde{p})$.

In particular, $\dim(m_\beta) < h_*(m_\beta) \le h^*(m_\beta) < \mathrm{Dim}(m_\beta)$.

Remark. For such a measure m_β, unidimensionality ensures that for almost every point x, there exists a subsequence n_k such that $\alpha_{n_k}(x)$ converges to $\dim_*(m)$. Nevertheless, according to theorem 2.1, we can not find a subsequence n_k such that α_{n_k} converges almost surely to $\dim_*(m)$. A similar remark can be made concerning $\mathrm{Dim}^*(m)$.

If we consider two real numbers $\beta_1 \neq \beta_2$ and if $\mu = m_{\beta_1}$ and $\nu = m_{\beta_2}$ we also have the following corollary :

Corollary 4.2 *We can construct two unidimensional measures μ and ν such that :*

1. $\exists c > 0$; $\frac{1}{c}\,\nu \le \mu \le c\,\nu$
2. $h_*(\mu) \neq h_*(\nu)$ *and* $h^*(\mu) \neq h^*(\nu)$.

This result indicates that entropy is a bad concept of dimension (even for unidimensional measures) ; for a good concept of dimension, it is indeed reasonable to demand that two equivalent measures should have the same dimension.

Proof of proposition 4.1. Properties 1 and 2 are immediate consequences of (8). We only have to prove property 3. Let us put $p_i = p_{p\,\tilde{p}}(i)$ and $\tilde{p}_i = p_{\tilde{p}\,p}(i)$. We have

$$\begin{aligned}
\tau_n(t) &= \frac{1}{n}\log_2\left(\sum_{I\in\mathcal{F}_n} m_\beta(I)^t\right) \\
&= \frac{1}{n}\log_2\left(\beta^t\prod_{i=1}^{n-1}(p_i^t + (1-p_i)^t) + (1-\beta)^t\prod_{i=1}^{n-1}(\tilde{p}_i^t + (1-\tilde{p}_i)^t)\right) .
\end{aligned}$$

We can easily deduce that

$$\begin{aligned}
& h_n(m_\beta) = -\tau_n'(1) = \\
&= \frac{1}{n}\left(\beta\left(-\log_2\beta + \sum_{i=1}^{n-1}s(p_i)\right) + (1-\beta)\left(-\log_2(1-\beta) + \sum_{i=1}^{n-1}s(\tilde{p}_i)\right)\right) . \quad (9)
\end{aligned}$$

If $n_0(n)$ is the number of integers $i \le n-1$ such that $p_i = p$ and if $n_1(n) = n-1-n_0(n)$, we can write

$$h_n(m_\beta) = \frac{n_0(n)}{n}\,(\beta\, s(p) + (1-\beta)\, s(\tilde{p})) + \frac{n_1(n)}{n}\,(\beta\, s(\tilde{p}) + (1-\beta)\, s(p)) + O(\frac{1}{n}) .$$

In particular, we have

$$\beta\, s(\tilde p) + (1-\beta)\, s(p) + O(\frac{1}{n}) \le h_n(m_\beta) \le \beta\, s(p) + (1-\beta)\, s(\tilde p) + O(\frac{1}{n}) \;.$$

To conclude, it suffices to remark that

$$\limsup_{n\to+\infty} \frac{n_0(n)}{n} = 1 \quad \text{and} \quad \limsup_{n\to+\infty} \frac{n_1(n)}{n} = 1 \;.$$

5 An example where $\dim(m) = h_*(m)$ and $\mathrm{Dim}(m) > h^*(m)$.

As observed in example 2.6, the equalities $\dim_*(m) = h_*(m)$ and $\mathrm{Dim}^*(m) = h^*(m)$ hold as soon as the sequence of random variables

$$X_n(x) = \log\left(\frac{m(I_n(x))}{m(I_{n-1}(x))}\right)$$

satisfies the strong law of large numbers

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - \mathbb{E}(X_i)) = 0 \quad dm \ a.s. \;. \tag{10}$$

We are going to construct a unidimensional measure m for which $\dim_*(m) = h_*(m)$ but $\mathrm{Dim}^*(m) > h^*(m)$. Of course, such a measure does not satisfy (10).

Proposition 5.1 *There exists a probability measure m such that*

$$\dim_*(m) = \dim^*(m) = h_*(m) \quad \textit{and} \quad h^*(m) < \mathrm{Dim}^*(m) = \mathrm{Dim}_*(m) \;.$$

The notations are the same as in the previous section. Let $\beta = 1/2$. The measure $\mu_1 = m_{p\tilde p}$ is the same as before but we modify slightly the measure $m_{\tilde p p}$. In section 4, the measure $m_{\tilde p p}$ was constructed in such a way that

$$\forall n \ge 1, \quad \forall i \in \{T_{2n-1}, \ldots, T_{2n} - 1\} \;, \quad \tilde p_i = p_{\tilde p\, p}(i) = \tilde p \;.$$

We modify the value of $\tilde p_i$ according to the parity of n and define

$$\begin{cases} \tilde p_i = \tilde p \ \text{ if } \ i \in \{T_{2n-1}, \ldots, T_{2n} - 1\} \ \text{ and } \ n = 2k+1 \\ \tilde p_i = p \ \text{ if } \ i \in \{T_{2n-1}, \ldots, T_{2n} - 1\} \ \text{ and } \ n = 2k \end{cases} .$$

We do not modifie the value of $\tilde p_i$ when $i \in \{T_{2n}, \ldots, T_{2n+1} - 1\}$. Let μ_2 be the so constructed measure and put :

$$m(A) = \frac{1}{2}\mu_1(F([0,1/2) \cap A)) + \frac{1}{2}\mu_2(F([1/2,1) \cap A)) \;.$$

Similar computations as those made before ensure that :

$$\dim_*(m) = \dim^*(m) = s(p) \quad \text{and} \quad \mathrm{Dim}_*(m) = \mathrm{Dim}^*(m) = s(\tilde{p}) \ .$$

Let us see the evolutions of entropy. Since $p < \tilde{p}$, the new measure has smaller entropy than the measure $m_{1/2}$ (corresponding to the parameter $\beta = 1/2$) of proposition 4.1. Hence,

$$h^*(m) \leq \frac{1}{2}\left(s(p) + s(\tilde{p})\right) < \mathrm{Dim}^*(m) \ .$$

In fact, it is easy to prove that $h^*(m) = (s(p) + s(\tilde{p}))/2$.

Moreover, using formula (9) for the new measure m, we have :

$$\begin{aligned} h_{T_{4k}}(m) &= \frac{1}{2T_{4k}} \sum_{i=1}^{T_{4k}-1} \left(s(p_i) + s(\tilde{p}_i)\right) \ + O(\frac{1}{T_{4k}}) \\ &= \frac{T_{4k} - T_{4k-1}}{T_{4k}} \,.\, s(p) \ + O(\frac{T_{4k-1}}{T_{4k}}) + O(\frac{1}{T_{4k}}) \ . \end{aligned}$$

It follows that

$$h_*(m) \leq \lim_{k\to+\infty} h_{T_{4k}}(m) = s(p) = \dim_*(m),$$

which gives the non trivial inequality between these two numbers. Let us finaly remark that it is easy to prove that for almost every $x \in [0,1)$,

$$\lim_{k\to+\infty} \frac{\log m\left(I_{T_{4k}}(x)\right)}{-T_{4k}\log 2} = s(p) \ .$$

6 An extension of theorems 2.1 and 2.2.

The ℓ-adic partition $\mathcal{F}_n$ of the cube $[0,1)^D$ is not the only situation where theorems 2.1 and 2.2 make sense. Fix a sequence $(\ell_n)_{n\geq 1}$ of strictly positive real numbers and construct a family $(\mathcal{G}_n)_{n\geq 0}$ in such a way.

1. $\mathcal{G}_0 = \{B_0\}$ where B_0 is a Borel set in $\mathbb{R}^D$ with $B(0,1/c) \subset B_0 \subset B(0,c)$ (we denote by $B(0,r)$ the ball with center 0 and radius r).

2. For every $n \geq 0$, $\mathcal{G}_n$ is a finite family of disjoint Borel sets which are similar to B_0 in the ratio ℓ_n.

3. For every $B \in \mathcal{G}_{n+1}$, there exists a unique $\tilde{B} \in \mathcal{G}_n$ such that $B \subset \tilde{B}$.

Let m be a measure supported by a borel set E and suppose that for every $n \geq 0$, E is a subset of $\bigcup_{B\in\mathcal{G}_n} B$. If we suppose that the sequence $\log(\ell_n)/n$ is bounded, then, coverings using elements of $\bigcup_n \mathcal{G}_n$ are sufficient to calculate the Hausdorff dimension of subsets of E. Conclusions of theorem 2.1 and theorem 2.2 are also true in this situation if we define $h_*(m)$ (resp. $h^*(m)$) as the lim inf (resp. lim sup) of the sequence

$$h_n(m) = \frac{1}{\log \ell_n} \sum_{B \in \mathcal{G}_n} m(B) \log m(B) .$$

In particular, our results can be applied for measures supported by Cantor sets constructed in the same way as those described in [Bat96] or [Bat98].

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