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n° 72

ISSN : 1298-020X

Jan. 2003

Site internet : http://www-leibniz.imag.fr/LesCahiers/

A generalization of the Pentomino Exclusion Problem : the Δ-dislocation in graphs.

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January 10, 2003

Abstract

In this paper we investigate the Pentomino Exclusion Problem, due to Golomb. We solve this problem on the $5 \times n$ grid and we give some lower and upper bounds for the $k \times n$ grid for all k, n.
A generalization of this in graphs leads to a new combinatorial problem, the Δ-*dislocation problem* : find the minimum number of vertices to be removed from the graph so as the remaining connected components have cardinality at most Δ. We investigate the algorithmic aspects of the Δ-dislocation problem : we first prove the problem is NP-Complete, then we give a sublinear algorithm wich solves the problem on a restricted class of graphs which includes the $k \times n$ grid graphs when k is not apart of the input.

1 The Pentomino Exclusion Problem

1.1 Introduction

A *polyomino* is a pattern formed by the connection of a specified number of equal-sized squares along common edges (see [5]). A *pentomino* is a polyomino composed of 5 squares.

Golomb [5] proposed the following *Pentomino Exclusion Problem*, denoted $PEP_{k\times n}$ *: Find the minimum number of unit squares to be placed on a $k \times n$ chessboard so as to exclude all pentominos.*
For convenience, we will denote this number $\kappa_{k\times n}$.
Bosch [1, 2] proposed a Linear Integer Programming approach to compute

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$\kappa_{n\times n}$, and determined this number when $n \leq 12$ [1]. In [6], the authors determined $\kappa_{k\times n}$ for all n and $k \leq 4$ (see Theorem 1) and gave some results about the infinite case ($\mathbb{Z}^2$). Using the asymptotical results in [6], we will here determine $\kappa_{5\times n}$ for all n and give some upper and lower bounds of this number for all k and all n.

Theorem 1

$$\kappa_{k\times n} = \begin{cases} \lfloor \frac{n}{5} \rfloor \ \textit{if } k=1 \\ 2\lfloor \frac{n}{3} \rfloor \ \textit{if } k=2 \\ n \ \textit{if } k=3 \textit{ and } n \geq 2 \\ \lceil \frac{3n}{2} \rceil - 1 \ \textit{if } k=4, n \geq 4 \end{cases} . \quad \square$$

1.2 Case of the grid $5 \times n$

In this section, we investigate the problem $\mathrm{PEP}_{5\times n}$.
We denote by $G_{k,n}$ the $k\times n$ grid. For given k and n, $C_1, \ldots, C_n$ (respectively $R_1, \ldots, R_k$) denote the *columns* (resp. the *rows*) of $G_{k,n}$. The squares of $G_{k,n}$ are denoted by $s_{i,j}$ where $\{s_{i,j}\} = R_i \cap C_j$.

Lemma 1 $\kappa_{5\times n} \geq 2n-2$.

Proof:
The proof works by induction on n.
The cases $n \leq 2$ are obvious.
Assume now that $n \geq 3$. Let S be a solution of $PEP_{5\times n}$.

If $|S \cap C_1| \geq 2$, then by the induction hypothesis applied on $\cup_{i=2\ldots n} C_i$ we obtain $|S| \geq 2 + 2(n-1) - 2 = 2n-2$.
Assume now, $|S \cap C_1| = 1$.
Let $j \geq 2$ the smallest integer such that $|S \cap C_j| \neq 2$.
If $\mid S \cap C_j \mid \geq 3$, then $\mid S \cap (C_1 \cup C_2 \cup \ldots \cup C_j) \mid \geq 2j$, and by the induction hypothesis applied on $\cup_{i=j+1,\ldots,n} C_i$ we have $\mid S \cap (C_{j+1} \cup C_{j+2} \cup \ldots \cup C_n) \mid \geq 2(n-j) - 2$, so that $\mid S \mid \geq 2n-2$.
Assume now that $\mid S \cap C_j \mid = 1$

If $j=2$, then it is easy to see that $S \cap C_1 = \{s_{3,1}\}$ and $S \cap C_2 = \{s_{3,2}\}$ (see Figure 1). So we obtain $\mid S \cap C_3 \mid \geq 4$, hence $\mid S \mid \geq 1+1+4+2(n-3)-2 = 2n-2$.

If $j=n$, then $\mid S \mid = 1 + 2(n-2) + 1 = 2n-2$.

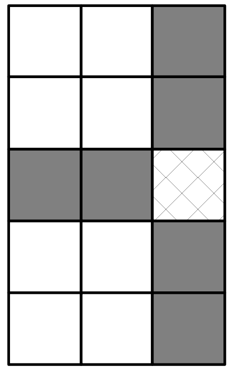

Figure 1: The case $|S \cap C_1| = |S \cap C_2| = 1$

If $2 < j < n$, then we have to study the position of the element in $S \cap C_j$.

If $S \cap C_j \in \{s_{1,j}; s_{5,j}\}$, then $|S \cap C_{j-1}| \geq 4$, which is a contradiction with the definition of j.
If $S \cap C_j \in \{s_{2,j}; s_{4,j}\}$, then since $|S \cap C_{j-1}| = 2$, it is easy to see that it implies that $|S \cap C_{j+1}| \geq 4$. Then by induction hypothesis applied on $\cup_{i=j+2 \ldots n} C_i$, we obtain $|S| \geq 1 + 2(j-2) + 1 + 4 + 2(n-j-1) - 2 = 2n - 2$.

Thus, we may assume that $S \cap C_j = \{s_{3,j}\}$.

We have $|S \cap (R_1 \cup R_2) \cap (C_{j-2} \cup C_{j-1} \cup C_j)| \geq 2$ and $|S \cap (R_4 \cup R_5) \cap (C_{j-2} \cup C_{j-1} \cup C_j)| \geq 2$. Since $\mid C_{j-1} \cap S \mid = 2$, we get also $\mid C_{j-2} \cap S \mid = 2$, which implies that $j - 2 > 1$ and $(C_{j-2} \cup C_{j-1}) \cap R_3 = \emptyset$.
The squares $s_{2,j-1}$, $s_{4,j-1}$ must belong to S, otherwise the set $\{s_{1,j}, s_{2,j}, s_{2,j-1}, s_{3,j-1}, s_{3,j-2}, s_{4,j-1}, s_{4,j}, s_{5,j}\}$ contains a pentomino. From now, an easy exhaustive exploration of cases (see Figure 2) shows that $|S \cap C_{j-3}| \geq 3$, which is a contradiction with the definition of j. $\square$

Theorem 2

$$\kappa_{5\times n} = \begin{cases} 2n-1 \textit{ if } n \in \{1 \ldots 4\} \\ 2n-2 \textit{ if } n \geq 5 \end{cases}$$

Proof: The case $n \in \{1 \ldots 4\}$ holds by Theorem 1. To conclude we exhibit solutions for $n \geq 5$ (see Figure 3) satisfying $\mid S \mid = 2n - 2$. $\square$

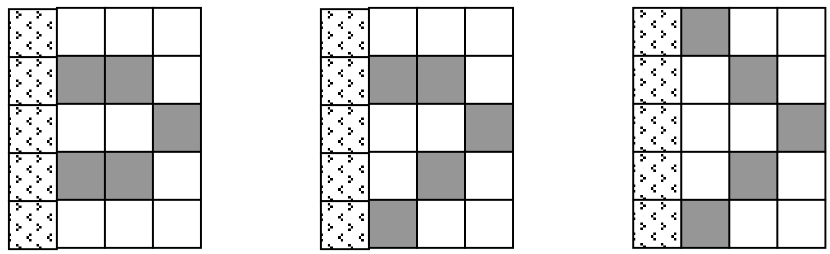

Figure 2: The case $S \cap C_k = \{s_{3,k}\}$.

1.3 General Bounds

We need the following Theorem due to S. Gravier and C. Payan [6] :

Theorem 3 *The density[1] of an optimal solution of PEP$_{\mathbb{Z}^2}$ is $\frac{3}{7}$.* $\square$

Theorem 4 *For all $k, n \geq 5$ we have :*

$$\frac{3}{7}kn + \frac{3}{7}(k+n) - (\left\lceil \frac{4n}{5} \right\rceil + \left\lceil \frac{4k}{5} \right\rceil) - \frac{4}{7} \leq \kappa_{k\times n} \leq \frac{3}{7}(k-4)n + 22\lceil \frac{n}{14} \rceil$$

Proof:

For the upper bound, we know, by Theorem 3, that the optimal density of PEP$_{\mathbb{Z}^2}$ is $\frac{3}{7}$. In Figure 4, we give such a solution $\mathcal{T}$. This solution is periodic and can be described as a translation of a given row. Here we will use a more precise description. For a periodic solution $\mathcal{S}$ of PEP$_{\mathbb{Z}^2}$, each row R_i can be encoded by the *upper-word* $u_1 \dots u_t$ (respectively the *down-word* $d_1 \dots d_t$) where the integer u_s (resp. d_s) means that the corresponding square belongs to a polyomino of size u_s (resp. d_s) in $\cup_{j \geq i} R_j \setminus \mathcal{S}$ (resp. $\cup_{j \leq i} R_j \setminus \mathcal{S}$). Therefore, a square belongs to $\mathcal{S}$ if and only if it is labelled 0.

For instance, the solution $\mathcal{T}$ in Figure 4 admits the upper-word UW = 01040403303301 and the down-word DW = 04010103303304.

[1]The density of a solution S in a finite set T is given by $d(S,T) = \frac{|S \cap T|}{|T|}$. Roughly speaking, the density of a solution S in $\mathbb{Z}^2$ is the limit (when there exists one) of $d(S,T)$ when T 'grows'. For a more precise definition we refer the reader to [6].

Now, from $\mathcal{T}$, we will construct a solution of $\text{PEP}_{k\times n}$ for any $k, n \geq 5$, as follows : first replace the rows $\cup_{j\leq 2} R_j$ by two rows corresponding to a solution of $\text{PEP}_{2\times\mathbb{Z}}$ where the row R_2 has a down-word $DW2 = 10401040101040$ (see Figure 5). The density of $R_1 \cup R_2$ is $\frac{11}{14}$ which is better than the density of $\mathcal{T}$ $(\frac{12}{14})$.

Observe that this gives a solution $\mathcal{T}'$ (see Figure 5) of $\text{PEP}_{\mathbb{Z}^+\times\mathbb{Z}}$. Indeed, each coordinate of the word $UW + DW2$ is smaller than 5.

Moreover, since for the solution $\mathcal{T}$ each polymino in $\mathbb{Z}^2 \setminus \mathcal{T}$ has cardinality 4, then every row R_j of $\mathcal{T}'$ with $j \geq 3$ has an down-word for which each coordinate is smaller or equal to the 'corresponding' coordinate of DW. In fact, in our solution of $\text{PEP}_{\mathbb{Z}^+\times\mathbb{Z}}$, the down-word of R_j is again DW whenever $j \geq 4$ and the down-word of R_3 is 01010103303301.

Let $k \geq 5$ be an integer and let DW' be the down-word of $\mathcal{T}'$ associated to R_{k-2}.

Replace the rows $\cup_{j\geq k-2} R_j$ by two rows corresponding to the solution of $\text{PEP}_{2\times\mathbb{Z}}$ where the row R_{k-1} has the upper-word $UW2 = 10401040101040$ (see Figure 5).

This construction gives a solution S' of $\text{PEP}_{k\times\mathbb{Z}}$, since each coordinate of the word $DW' + UW2$ is smaller than 5.

Now, to obtain a solution S^* for $\text{PEP}_{k\times n}$ it is enough to take n consecutive columns $C_1, \ldots, C_n$ of S' for which the density of $(R_3 \cup \ldots \cup R_{k-2}) \cap (C_1 \cup \ldots \cup C_n)$ is $\leq \frac{3}{7}$.

Finally, S^* has cardinality :

- In the first (last) 2 rows, less or equal to $11\lceil\frac{n}{14}\rceil$.
- In $(R_3 \cup \ldots \cup R_{k-2}) \cap (C_1 \cup \ldots \cup C_n)$, less or equal to $\frac{3}{7}(k-4)n$.

This gives the desired upper bound :

$$\kappa(k \times n) \leq \frac{3}{7}(k-4)n + 22\lceil\frac{n}{14}\rceil.$$

For the lower bound, let S be an optimal solution of $\text{PEP}_{k\times n}$. First we claim that :

$$\text{We may assume that there is at most 2 consecutive elements of } S \text{ in } (C_1, C_n, R_1, R_k). \tag{1}$$

Indeed, each solution S of the $\mathrm{PEP}_{k\times n}$ can be transformed into another solution S' satisfying the claim by the following transformation : given 3 consecutive elements $s_{1,i-1}, s_{1,i}, s_{1,i+1}$ of $S \cap R_1$; set $S' = S \cup \{s_{2,i}\} \setminus \{s_{1,i}\}$. Clearly, repeating this process in C_1, C_n, R_1, R_k we will obtain a solution S' of $\mathrm{PEP}_{k\times n}$ satisfying Claim 1 and such that $\mid S' \mid \leq \mid S \mid$.

Let S be an optimal solution of $\mathrm{PEP}_{k\times n}$ satisfying Claim 1. We will construct a solution S^* of $\mathrm{PEP}_{(2k+2)\times(2n+2)}$ such that $\{s+(a.(2n+2), b.(2k+2)) \mid s \in S^*, a, b \in \mathbb{Z}\}$ is a solution of $\mathrm{PEP}_{\mathbb{Z}^2}$.
We set $S^* = S \cup S_1 \cup S_2 \cup S_3 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6 \cup B_7 \cup B_8 \cup T$ (see Figure 7), where :

- $B_1 = \{s_{i,n+1} | i \in \{1, \ldots, k\}, s_{i,n} \notin S\}$, and $B_2 = \{s_{k+1,j} | j \in \{1, \ldots, n\}, s_{k,j} \notin S\}$. Due to Claim 1, B_1 and B_2 satisfy the pentomino exclusion property (and even the trimino exclusion property).

- S_1 is the symmetric of S according to axis B_1, i.e. $S_1 = \{s_{i,2n+2-j} | s_{i,j} \in S\}$. Analogously, define S_2 as the symmetric of S according to axis B_2, i.e. $S_2 = \{s_{2k+2-i,j} | s_{i,j} \in S\}$.

- In the same way, $B_3 = \{s_{k+1,j} | j \in \{n+2, \ldots, 2n+1\}, s_{k,j} \notin S_1\})$ and $B_4 = \{s_{i,n+1} | i \in \{k+2, \ldots, 2k+1\}, s_{k,j} \notin S_2\})$.

- S_3 is the symmetric of S_1 according to axis B_3, i.e. $S_3 = \{s_{2k+2-i,j} | s_{i,j} \in S_1\}$. Notice that, by construction, S_3 is also the symmetric of S_2 according to axis B_4.

- $B_5 = \{s_{i,2n+2} | i \in \{1, \ldots, k\}, s_{i,2n+1} \notin S_1\}$, $B_6 = \{s_{i,2n+2} | i \in \{k+2, \ldots, 2k+1\}, s_{i,2n+1} \notin S_3\}$, $B_7 = \{s_{2k+2,j} | j \in \{1, \ldots, n\}, s_{2k+1,j} \notin S_2\}$ and $B_8 = \{s_{2k+2,j} | j \in \{n+2, \ldots, 2n+1\}, s_{2k+1,j} \notin S_3\}$.

- $T = \{s_{k+1,n+1}, s_{k+1,2n+2}, s_{2k+2,n+1}, s_{2k+2,2n+2}\}$.

The set S^* is a solution of $\mathrm{PEP}_{(2k+2)\times(2n+2)}$. The corresponding $(2k+2) \times (2n+2)$ rectangle tiles $\mathbb{Z}^2$ by translation of vectors $(0, 2k+2)$ and $(2n+2, 0)$ which gives a solution of $\mathrm{PEP}_{\mathbb{Z}^2}$. As the optimal density for $\mathrm{PEP}_{\mathbb{Z}^2}$ is $\frac{3}{7}$, then the density of S^* is greater or equal to $\frac{3}{7}$.

Since the first and last rows and columns of S have density at least $\frac{1}{5}$, then the B_i's have density at most $\frac{4}{5}$. As the S_i's are symmetric of S then $|S_i| = |S| = \kappa_{k,n}$ for all i. Moreover $|T| = 4$, so finally we have :

$$\frac{3}{7}(2k+2)(2n+2) \leq |S^*| \leq 4\kappa_{k,n} + 4(\left\lceil \frac{4n}{5} \right\rceil + \left\lceil \frac{4k}{5} \right\rceil) + 4$$

Which leads to the desired bound :

$$\kappa_{k,n} \geq \frac{3}{7}kn + \frac{3}{7}(k+n) - (\left\lceil \frac{4n}{5} \right\rceil + \left\lceil \frac{4k}{5} \right\rceil) - \frac{4}{7}.$$

Observe that in the proof of the lower bound we need only the assumption that $k, n \geq 2$. □

Let us mention as a direct consequence of the previous theorem, a maybe more explicit corollary which shows how close are our bounds :

Corollary 1 *For all $k, n \geq 5$. There are two constant C_1 and C_2, such that :*

$$\frac{3}{7}kn - \frac{26}{70}(k+n) - C_1 \leq \kappa_{k \times n} \leq \frac{3}{7}kn - \frac{5}{70}(k+n) + C_2. \quad \square$$

2 The Δ-dislocation problem

2.1 Introduction

In this section we present a generalization of the PEP.

For a graph $G = (V, E)$ and a positive integer Δ, we will say that a subset S of vertices of G is a Δ-*dislocation set* of G if and only if all the connected components of $G - S$ have at most Δ vertices.

One can notice that if G is the grid $k \times n$ and $\Delta = 4$, then a Δ-dislocation set of G is a solution of $\mathrm{PEP}_{k\times n}$.

Since $S = V(G)$ is always a set of Δ-dislocation of G for any Δ, then the corresponding optimisation problem is to find an S of minimum cardinality:

Δ-DISLOCATION

Instance : a graph G and an integer k

Question : is there a Δ-dislocation set S in G of cardinality $\leq k$?

Remark that for a graph G and $\Delta = 1$, a Δ-dislocation set S of G is a transversal of G, i.e. the complementary of S is an independent set of G. Since the problem TRANSVERSAL is NP-Complete[4], then for $\Delta = 1$ the problem Δ-DISLOCATION is NP-Complete. In the next subsection we will

prove that for any fixed Δ this problem is still NP-Complete.
Motivated by the PEP$_{k\times n}$ problem, we present in subsection 3 a polynomial time algorithm for solving Δ-dislocation problem in the grid graph $G_{k,n}$ whenever k is not apart of the input. More generally, our algorithm works for a largest class of graphs namely fasciagraphs. This algorithm was inspired by the work of Žerovnik and Klavžar on fasciagraphs [7].

2.2 Complexity of the Δ-dislocation problem

In this section we investigate the complexity of the Δ-dislocation problem.

Theorem 5 *For any Δ, we have that Δ-DISLOCATION problem is NP-Complete.*

Proof: The problem is clearly in NP. We will reduce it to TRANSVERSAL.
TRANSVERSAL
Instance : A graph G and an integer k
Question : Is there an transversal T in G of size $\leq k$?

Given a graph G, let us construct a graph G' such that G has a transversal of size $\leq k$ if and only if G' has a set of Δ-dislocation of size $\leq k$.
G' is constructed in the following way :

- To each vertex $v_i \in V(G)$ corresponds a clique K_i on Δ vertices in G'.
- In each clique K_i in G' let us choose one vertice v_i^*
- To each edge $v_i v_j \in E(G)$ corresponds one edge $v_i^* v_j^* \in E(G)$

If S is a subset $\{v_i\}_i$ of $V(G)$, then S^* will denote the corresponding subset $\{v_i^*\}_i$ of $V(G')$. Conversely for a subset $S^* = \{v_i^*\}_i$ of $V(G')$, S will denote the subset $\{v_i\}_i$ of $V(G)$.

$\star$ Assume that G has a transversal T of size $\leq k$. Then T^* is a set of Δ-dislocation of G'. Indeed, as T is a transversal of G, then the connected components of G' are :

- the $K_i - v_i^*$ for all $v_i^* \in T^*$, of cardinality $\Delta - 1$
- the K_j for all $v_j^* \notin T^*$, of cardinality Δ

This shows that T^* is a set of Δ-dislocation of G.

$\star$ On the other hand, assume that G' has a set S of Δ-dislocation of size $\leq k$. Let us show that G has an transversal of size $\leq k$.
Let us consider the set $T^* \subseteq V(G')$ defined by :

$$v_i^* \in T^* \iff K_i \cap S \neq \emptyset$$

As S is of size $\leq k$, then T^* is of size $\leq k$.
Consider the set T in $V(G)$ corresponding to T^*. Then we claim that T is a transversal of G. Indeed, if there exist an edge $v_i^* v_j^* \in E(G')$ such that $v_i^*, v_j^* \notin T^*$, it means that $K_i \cap T = K_j \cap T = \emptyset$, so that $K_i \cup K_j$ is contained in a connected component of $G' - T^*$ of size $\geq 2\Delta$, a contradiction. $\square$

2.3 Case of the Fasciagraphs

2.3.1 Fasciagraphs : definition

Let $G_1, \ldots, G_n$ be disjoints graphs and $X_1, \ldots, X_n$ a sequence of sets of edges such that an edge of X_i joins a vertex of $V(G_i)$ with a vertex of $V(G_{i+1})$, where G_{n+1} denotes the graph G_1.
A *polygraph* $\Omega_n(G_1, \ldots, G_n; X_1, \ldots, X_n)$ over the *monographs* $G_1, \ldots, G_n$ is defined in the following way :

- $V(\Omega_n) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_n)$
- $E(\Omega_n) = E(G_1) \cup X_1 \cup E(G_2) \cup \ldots \cup X_{n-1} \cup E(G_n) \cup X_n$

The monographs G_j are also called the *fibers* of the polygraph.

Assume that for all $i = 1 \ldots n$, G_i is isomorphic to a fixed graph G. In addition, let the sets X_i, $i = 1 \ldots n-1$ be equal to a fixed edge set X and $X_n = \emptyset$. Then we call the polygraph a *fasciagraph* and we denote it $\Psi_n(G, X)$.

2.3.2 Algorithm for fasciagraphs

Let G be a graph on k vertices $v_1, \ldots, v_k$ and let $\Psi_n(G, X)$ be a fasciagraph of fiber G. We will denote v_i^j the vertex v_i in the j-th fiber G_j of $\Psi_n(G, X)$. In this section we denote by Ψ_j, $j = 1 \ldots n$ the subgraph of $\Psi_n(G, X)$ induced by the j-th first fibers $G_1, \ldots, G_j$ of $\Psi_n(G, X)$.

For a fixed $j \leq n$, a word $(a_1, a_2, \ldots, a_k)$ of $\{0, \ldots, \Delta\}^k$ is said to be *valid* (for G_j) if there exists a set S of Δ-dislocation of Ψ_j such that :

$a_i = t$ iff v_i^j belongs to a connected component of cardinality k of $\Psi_j - S$

We will construct an auxiliary digraph $\mathcal{G}(G, X)$.

- The vertices of $\mathcal{G}(G, X)$ are the valid words of $\{0, \ldots, \Delta\}^k$, plus two additional vertices B and E, that we call respectively *begin* and *end* vertices of $\mathcal{G}(G, X)$.
- There is an arc between B and any valid word $V \neq E$ for G_1.
- There is an arc between any vertex $V \neq B$ and E.
- There is an arc between two valid words V and $W \neq B, E$ if and only if there exists a set S of Δ-dislocation and an integer j for which V is valid for G_j and W is valid for G_{j+1}.

Notice that there is a one to one mapping between the sets of Δ-dislocation of $\Psi_n(G, X)$ and the pathes on $n+2$ vertices from B to E of $\mathrm{G}(G, X)$.

In addition, the graph is labeled in the following way :

- For any V and any $W \neq E$, the arc (V, W) is labeled by the number of zeros of the vector W.
- For any V, the arc (V, E) is labeled by 0.

Remark that for a given path on $n+2$ vertices $P = (B, V_1, V_2, \ldots, V_n, E)$, the length of P is the sum of the number of zeros of $V_1, V_2, \ldots, V_n$. So the length of the path is the cardinality of the corresponding set of Δ-dislocation. Then, determine the minimum cardinality of a Δ-dislocation set of $\Psi_n(G, X)$ is equivalent to compute the minimum length of a (B, E)-path on $n+2$ vertices in $\mathcal{G}(G, X)$. This can be done in $O(K \log n)$ time, where K depends only on the size of $\mathcal{G}(G, X)$ (see for example [3] or [8]).

Theorem 6 *For a fixed k, one can compute the Δ-dislocation number of a fasciagraph $\Psi_n(G, X)$ in $O(\log n)$ time whenever $|V(G)| \leq k$.*

Proof: First observe that the size of $\mathcal{G}(G, X)$ depends only on k and Δ, since $\mathcal{G}(G, X)$ has at most $(\Delta + 1)^k$ vertices.

It remains, now, to prove that one can construct $\mathcal{G}(G,X)$ in $O(\log n)$ time. This comes from the fact that determining if a word of $\{0,\ldots,\Delta\}^k$ is valid, and checking if two valid words are adjacent, can be done by computing all the Δ-dislocation sets of $\Psi_\Delta(G,X)$. Indeed, for $j > \Delta$, a word w is valid for G_j in $\Psi_n(G,X)$ if and only if w is valid for G_Δ in $\Psi_\Delta(G,X)$. $\square$

3 Open Problems

Roughly speaking, Theorem 4 shows that $\kappa_{k\times n}$ is approximatively $\frac{3}{7}$ times the area of the $k \times n$ grid minus an improvment on the boundary $(k+n)$. For $k = 5$ and when 14 divides n, the upper bound of Theorem 4 gives $\kappa_{5\times n} \leq 2n$. Moreover, Theorem 2 tolds us that $\kappa_{5\times n} = 2n - 2$. Therefore, one may ask :

Conjecture 1 *For any k and n 'sufficiently large' (with $n \geq k$), we have*

$$\kappa_{k\times n} = \lfloor \frac{3}{7}(k-4)n \rfloor + 22\lceil \frac{n}{14} \rceil - O(1).$$

From the algorithmic point of view, one may consider a weaker question related to the grid graph. Theorem 6 proves that determining $\kappa_{k\times n}$ for a grid graph is a polynomial task whenever k is fixed. So a weaker version of Conjecture 1 is :

Conjecture 2 *It is polynomially solvable to decide whether $\kappa_{k\times n} \leq P$ for given integers P, k and n.*

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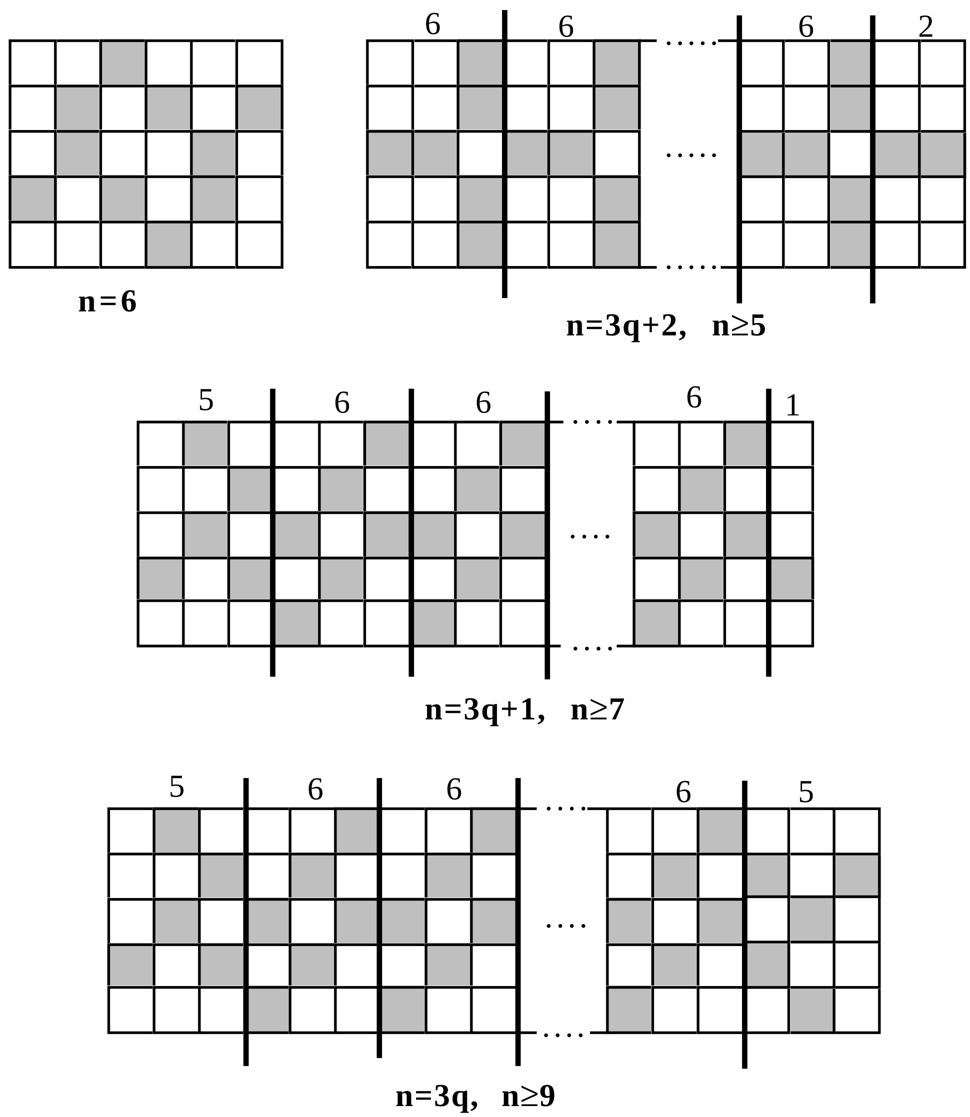

Figure 3: Some solutions of $PEP_{5\times n}$ for $n \geq 5$.

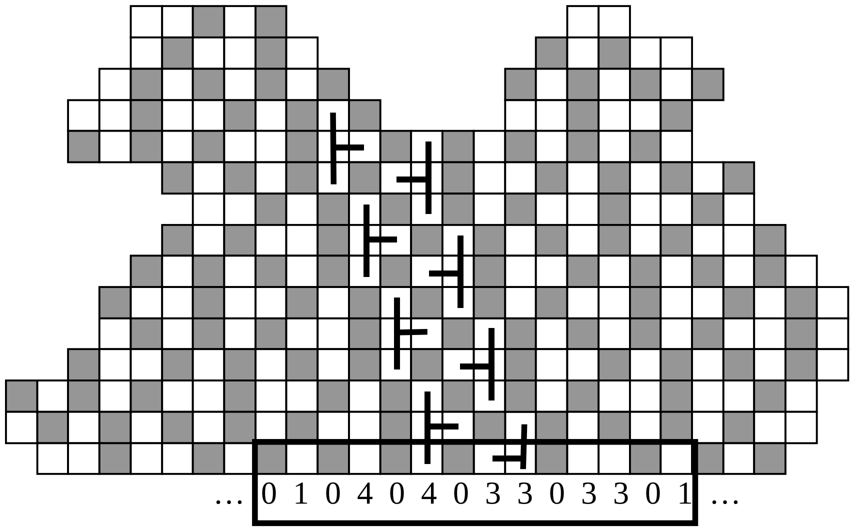

Figure 4: Optimal solution of $\mathbb{Z}^2$.

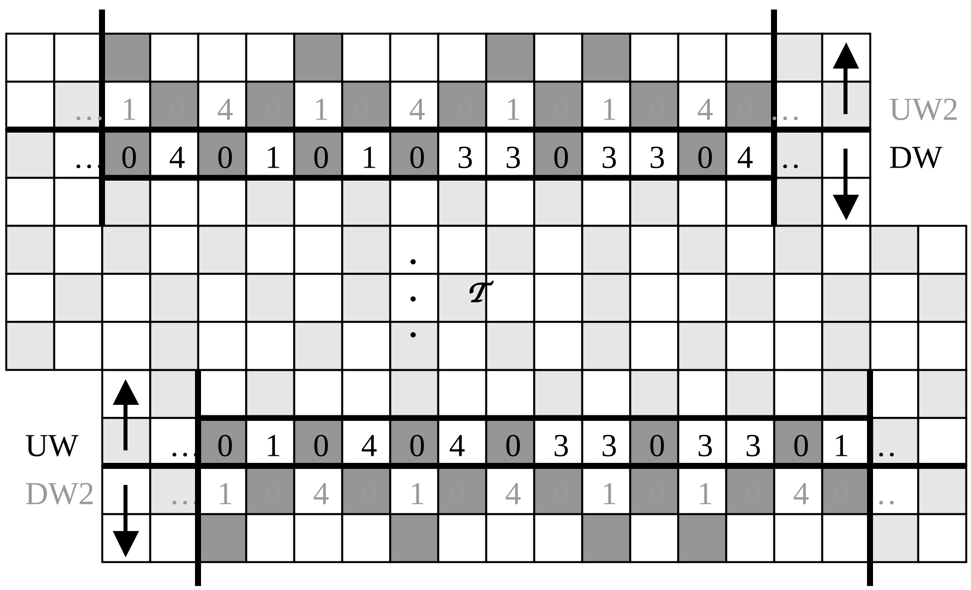

Figure 5: Construction of a solution of $\mathrm{PEP}_{k\times\mathbb{Z}}$.

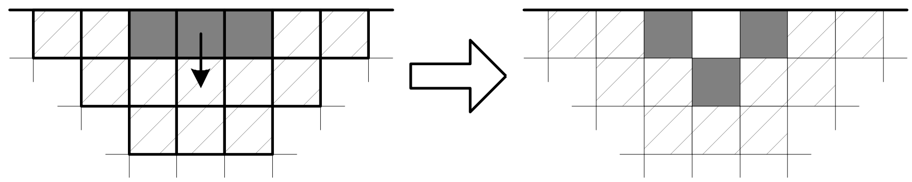

Figure 6: Proof of Claim 1.

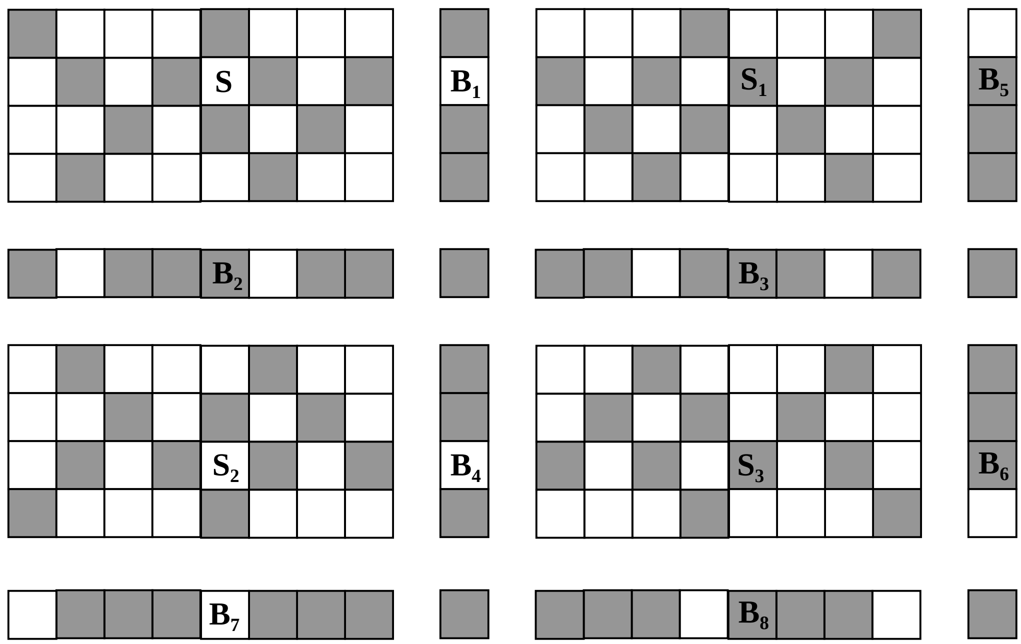

Figure 7: Construction of S^* from S.

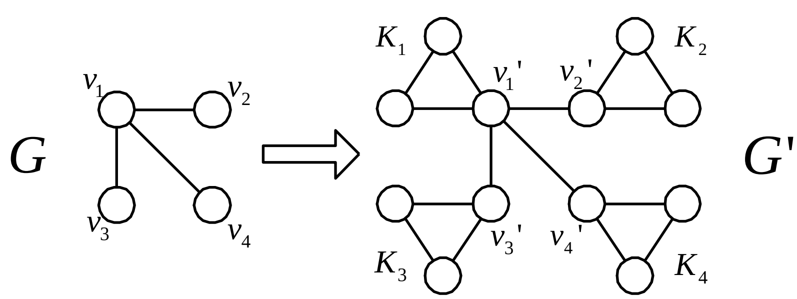

Figure 8: Construction of G' from G.

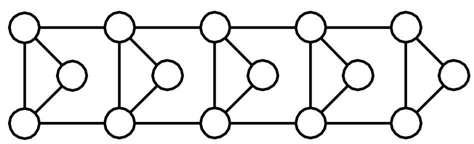

Figure 9: An example of fasciagraph over monographs K_3.

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Directeur de la publication : Nicolas Balacheff
ISSN : 1298-020X