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Chap 2 – Examples of non-commutative vertex algebras

§1 – Affine vertex algebras

$\leftarrow$ gothic a

Let $\mathfrak{a}$ be a finite-dimensional Lie algebra, equipped with an invariant bilinear form K:

K symm. and bilinear $\quad K : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathbb{C}$

and : $\quad K([x,y],z) = K(x,[y,z]) \qquad \forall x,y,z \in \mathfrak{a}$

Let $\hat{\mathfrak{a}} \overset{= \hat{\mathfrak{a}}_K}{:=} \underbrace{\mathfrak{a}[t,t^{-1}]}_{\mathfrak{a} \otimes \mathbb{C}[t,t^{-1}]} \oplus \mathbb{C}\mathbb{1}$ be the Kac–Moody affinization of $\mathfrak{a}$

with Lie bracket:

$$[\mathbb{1}, \hat{\mathfrak{a}}] = 0$$

$$[\underbrace{xt^m}_{x \otimes t^m}, y t^n] = [x,y] t^{m+n} + m \delta_{m,-n} K(x,y) \mathbb{1}$$

$x,y \in \mathfrak{a}$

$m,n \in \mathbb{Z}$

$\hat{\mathfrak{a}}$ is a central extension of $\overset{\mathfrak{a} \otimes \mathbb{C}[t,t^{-1}]}{\overset{\shortparallel}{\mathfrak{a}[t,t^{-1}]}}$ by $\mathbb{C}.\mathbb{1}$

$$0 \longrightarrow \mathbb{C}\mathbb{1} \longrightarrow \hat{\mathfrak{a}} \longrightarrow \mathfrak{a} \longrightarrow 0$$

In general, when you have such central extension, with $\mathbb{1}$ is central

$$[x,y]_{\hat{\mathfrak{g}}} = [x,y]_{\mathfrak{g}[t,t^{-1}]} + \underline{c(x,y)}\,\mathbb{1}$$

$x,y \in \mathfrak{g}[t,t^{-1}]$

↑ this must satisfy

$$c(y,x) = -c(x,y)$$

$$c(x,[y,z]) + \dots = 0 \quad \text{"Jacobi"}$$

$$c : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{C}.$$

It is the special choice where

$$c(xt^m, yt^n) = \underbrace{m\,\delta_{m,-n}}\, k(x,y)$$

$$-\operatorname{Res}_{t=0}\left(f(t)\,\partial_t\, g(t)\right) \qquad f(t) = xt^m,\ g(t) = yt^n.$$

An $\hat{\mathfrak{g}}$-module M is called <u>smooth</u> if for all $m\in M$, $\forall x \in \mathfrak{g}$

$$xt^n.\,m = 0 \qquad n \gg 0$$

$(\Longleftrightarrow \forall m \in M,\ \exists N$ st $xt^n.\,m = 0$ for $n \geq N)$

Equivalently,

xt^n viewed it as an endomorphism on M.

$$x(z) := \sum_{n\in\mathbb{Z}} \widetilde{(xt^n)}\, z^{-n-1}$$

is a field on M

<u>Lemma</u>: For any smooth $\hat{\mathfrak{g}}$-module M, the fields $x(z)$ and $y(z)$, for $x,y\in\mathfrak{g}$, are mutually local, and we have

$$x(z)\,y(w) \sim \frac{1}{z-w}[x,y](w) + \frac{k(x,y)}{(z-w)^2}$$

proof: We have to show that

$$[x(z), y(w)] = \underline{\underline{[x,y](w)}}\,\delta(z-w) + K(x,y)\,\partial_w \delta(z-w) \qquad (*)$$

$$[x(z), y(w)] = \sum_{n,m} [x_{(m)}, y_{(n)}]\, z^{-m-1} w^{-n-1}$$

$$\sum_m \overset{\shortparallel}{\underbrace{xt^m}_{\overset{\shortparallel}{x_{(m)}}}} z^{-m-1}$$

$$= \sum_{n,m} [x,y]_{(m+n)}\, z^{-m-1} w^{-n-1} + \underbrace{\sum_m m K(x,y)\, z^{-m-1} w^{m-1}}_{K(x,y)\,\partial_w \delta(z-w)}$$

$$= \sum_j [x,y]_j \cdot \left(\sum_m z^{-m-1} w^m\right) w^{-j-1}$$

$$= [x,y](w)\,\delta(z-w) + K(x,y)\,\partial_w \delta(z-w) \qquad ☒$$

Rem: $(*)$ is equivalent to the Lie bracket on $\hat{\mathfrak{g}}$.

$$\longrightarrow V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbb{1})} \mathbb{C}$$

Recalls on enveloping algebras

Let $\mathfrak{g}$ be a Lie alg.

$T\mathfrak{g} = \bigoplus_{i=0}^{\infty} T^i\mathfrak{g}$, $T^0\mathfrak{g} = \mathbb{C}$, $T^1\mathfrak{g} = \mathfrak{g}$, . . , $T^i\mathfrak{g} = \underbrace{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}}_{i \text{ times}}$.

Let J be the two-sided ideal of $T\mathfrak{g}$ generated by elements

$x\otimes y - y\otimes x - [x,y]$, $x,y \in \mathfrak{g}$

$U\mathfrak{g} := T\mathfrak{g}/J$ unital assoc. (non comm in general) $\mathbb{C}$-algebra.

(universal) *enveloping algebra of $\mathfrak{g}$* .

Prop Let A be an associative (unital) $\mathbb{C}$-alg, $[a,b] = ab - b.a$

then: for all Lie algebra homomorphism $\theta : \mathfrak{g} \longrightarrow A$, there exists a unique alg. homomorphism $\Phi : U(\mathfrak{g}) \to A$ s.t the diagram commutes:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\sigma"] \arrow[dr, "\theta"'] & U(\mathfrak{g}) \arrow[d, dashed, "\Phi"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & A
\end{tikzcd}$$

$\Phi\circ\sigma = \theta$

and $U(\mathfrak{g})$ is unique, up to isom, which satisfies this properties.

PBW thm

$U\mathfrak{g}$ is naturally filtered by $U_m(\mathfrak{g}) = \pi(T_m(\mathfrak{g}))$ where $\pi : T\mathfrak{g} \to U(\mathfrak{g})$ and $T_m(\mathfrak{g}) = \bigoplus_{i=0}^{m} T^i\mathfrak{g}$

Clear : $U_m(\mathfrak{g}) . U_p(\mathfrak{g}) \subset U_{m+p}(\mathfrak{g})$.

$\mathrm{gr}\, U\mathfrak{g} = \bigoplus_{i=0}^{\infty} \mathrm{gr}^n U\mathfrak{g}$ $\qquad$ $U_0\mathfrak{g} \simeq \mathbb{C}$

Key point: $\mathrm{gr}\, U(\mathfrak{g})$ is commutative, and there is a surjective homom.

$$S(\mathfrak{g}) \twoheadrightarrow \mathrm{gr}\, U(\mathfrak{g}),$$

where $S(\mathfrak{g}) := T\mathfrak{g}/I$, I = ideal of $T\mathfrak{g}$ gen by $x\otimes y - y\otimes x$.

Thm (Poincaré – Birkhoff – Witt : PBW)

the above morphism is an isomorphism:

$$\mathrm{gr}\, U(\mathfrak{g}) \simeq S(\mathfrak{g}).$$

As a consequence: $\mathfrak{g} \longrightarrow U(\mathfrak{g})$ is injective. Moreover, if $\{x_i, i\in I\}$ is a basis of $\mathfrak{g}$, then the unit 1 together with:

$$x_{i_1}^{r_1} \dots x_{i_n}^{r_n}, \quad n>0, \quad i_j \in I, \quad i_1 < i_2 < \dots < i_n, \quad r_j > 0$$

form a basis of $U(\mathfrak{g})$. (PBW basis).

If $\mathfrak{h}$ the subalg of $\mathfrak{g}$, with basis $\{h_i, i\in J\}$, completed in a basis $\{h_i, i\in J\} \cup \{x_i, i\in I\}$ of $\mathfrak{g}$, then $U(\mathfrak{h}) \longrightarrow U(\mathfrak{g})$ induced from $\mathfrak{h} \hookrightarrow \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ is injective, and $U(\mathfrak{g})$ is a free $U(\mathfrak{h})$-module with basis consisting of 1 and elements

$$x_{i_1}^{r_1} \dots x_{i_n}^{r_n}, \quad n>0 \quad \dots$$

Back to $\hat{\mathfrak{g}}$

Set $V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\otimes\mathbb{C}\mathbb{1})} \mathbb{C}$,

where $\mathbb{C}$ is a one-dimensional representation of $\underline{\underline{\mathfrak{g}[t]\oplus\mathbb{C}\mathbb{1}}}$ on which $\mathfrak{g}[t]$ acts trivially $(\to 0)$ and $\mathbb{1}$ acts as the identity.

$$\hat{\mathfrak{g}} = \underline{(\mathfrak{g}\otimes} t^{-1}\mathbb{C}[t^{-1}]) \oplus \mathfrak{g}[t]\oplus\mathbb{C}\mathbb{1}$$

We get

$$U(\hat{\mathfrak{g}}) \simeq U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{g}[t]\oplus\mathbb{C}\mathbb{1})$$

then $V^k(\mathfrak{g}) \simeq U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])$ as vector spaces over $\mathbb{C}$.

and $V^k(\mathfrak{g})$ admits the following PBW basis:

$$\underbrace{x^{i_1}_{(-n_1)} \cdots\cdots x^{i_r}_{(-n_r)}\cdot\mathbb{1}}_{v} \;,\quad n_1 \le \cdots \le n_r \text{ and if } n_i = n_{i+1}\;\; i_j \le i_{j+1}.$$

$\{x^i : i\in I\}$ basis of $\mathfrak{g}$.

. The space $V^k(\mathfrak{g})$ is naturally graded: $V^k(\mathfrak{g}) = \bigoplus_{\Delta\in\mathbb{Z}_{\geq 0}} V^k(\mathfrak{g})_\Delta$

$$\deg(v) = \sum_{i=1}^{r} n_r \;,\quad \deg(1) = 0.$$

. $V^k(\mathfrak{g})_0 = \mathbb{C}$, $V^k(\mathfrak{g})_1 \simeq \mathfrak{g}$ via the correspondence

$$\mathfrak{g} \ni x \longmapsto \underset{x t^{-1}\cdot}{x_{(-1)}\cdot 1}$$

Prop: there is a unique VA structure on $V^k(\mathfrak{a})$ st $|0\rangle = $ "$1 \otimes 1$",

$$Y(x, z) = Y(xt^{-1}|0\rangle, z) = x(z) = \sum_{n\in\mathbb{Z}} \underset{\overset{\shortparallel}{xt^n}}{x_{(n)}} z^{-n-1}$$

and T is given by: $T|0\rangle = 0$,

$$[T, xt^n] = -n\,xt^{n-1}, \quad x \in \mathfrak{a}, \quad n \in \mathbb{Z}.$$

Proof: directly follows from reconstruction theorem. ◻

Def: $V^k(\mathfrak{a})$ is called the universal affine vertex algebra associated with $\mathfrak{a}$ and k

Rem: $V^k(\mathfrak{a})$ is smooth $\hat{\mathfrak{a}}$-module.

Let M be any smooth $\hat{\mathfrak{a}}$-module on which the central element $\mathbb{1}$ acts as the identity (ex: $V^k(\mathfrak{a})$)

By lemma, $\underbrace{\langle x(z) : x \in \mathfrak{a} \rangle_M}_{\overset{\shortparallel}{\langle x(z)\rangle_M}}$ has a structure of a vertex algebra.

Moreover, the correspondence

$$\hat{\mathfrak{a}} \ni x_{(n)} = xt^n \longrightarrow x(z)_{(n)} \in \mathrm{End}(\langle x(z)\rangle_M)$$

$$\mathbb{1} \longmapsto \mathrm{Id}_M$$

gives an $\hat{\mathfrak{a}}$-module structure on $\langle x(z)\rangle_M$.

[Indeed: $\underline{x(z)_{(n)}\, y(z)_{(m)}}\, c(z) - y(z)_{(m)}\, x(z)_{(n)}\, c(z) = \sum_j (x(z)_{(j)} y(z))_{(m+n-j)} c(z)$

$$[x_{(m)}, y_{(n)}] = [x,y]_{(m+n)} + m\,\delta_{m,-n}\, k(x,y)\,\mathbb{1} \ldots \quad]$$

this $\hat{n}$-module ($\langle n(z)\rangle_M$) is generated by Id_M, and satisfies the condition that $n[t]. \mathrm{Id}_M = 0$

<u>By Frobenius reciprocity</u>

$$\mathrm{Hom}_{U(g)}(U(g)\otimes_{U(b)} M, N) \simeq \mathrm{Hom}_{U(b)}(M, N)$$

where g Lie alg, b Lie subalg of g

M is a b-module, N is a g-module

Here: $\mathrm{Hom}_{U(\hat{n})}(V^k(n) = U(\hat{n})\otimes_{U(n[t]\oplus \mathbb{C}\mathbb{1})} \mathbb{C}, \langle n(z)\rangle_M)$

$$\simeq \underline{\mathrm{Hom}_{U(n[t]\oplus\mathbb{C}\mathbb{1})}(\mathbb{C}, \langle n(z)\rangle_M)}$$

<u>Conclusion</u>: this shows there is a homomorphism of $\hat{n}$-modules

$$\| \quad V^k(n) \longrightarrow \langle n(z)\rangle_M$$

<u>Claim</u> (proof later using Li's filtration)

| this map is surjective. so that $\langle n(z)\rangle_M$ is a quotient of $V^k(n)$)

In addition it is a homomorphism of VA.

<u>Prop</u>: the category of $V^k(n)$-modules is the same as the category of that of smooth $\hat{n}$-modules on which $\mathbb{1}$ acts as the identity.

proof: Let M be a $V^k(\mathfrak{a})$-module. then M is smooth $\hat{\mathfrak{a}}$-module using the correspondence $x_{(n)} \longmapsto \mathrm{Res}_{z=0}\left(z^n \overset{M}{x}(z)\right)$

Conversely, if M is a smooth $\hat{\mathfrak{a}}$-module, $\mathbb{1} \to \mathrm{Id}_M$,
then: $V^k(\mathfrak{a}) \longrightarrow \langle x(z)\rangle_M$ VA morphism
So M is a $V^k(\mathfrak{a})$-module
this correspondence is compatible with morphisms. ☒

Example: Assume that $\mathfrak{a} = \mathfrak{h}$ is a commutative Lie alg.
(ie: a $\mathbb{C}$-vector space viewed as a comm. Lie algebra), equipped with any symm. bilinear form on $\mathfrak{h}$.
then $V^k(\mathfrak{h})$ is the *Heisenberg vertex algebra* associated with $\mathfrak{h}$ and k

In the special case where $\mathfrak{h} \simeq \mathbb{C}$ and $k \neq 0$, $V^k(\mathfrak{h}) \simeq \pi$
(exercise).

§2 – Affine vertex algebras associated with simple Lie algebra

Assume that $\mathfrak{a} = \mathfrak{g}$ is a simple Lie algebra $/ \mathbb{C}$, and

$$\kappa = \frac{k}{2h^\vee} \times \text{Killing form of } \mathfrak{g}.$$

$h^\vee$: dual Coxeter number of $\mathfrak{g}$, $k \in \mathbb{C}$.

Digression on nondegenerate bilinear forms on $\mathfrak{g}$

They are all proportional to the Killing form

$$\text{Killing } (x,y)_{\mathrm{Kil}_\mathfrak{g}} = \mathrm{tr}(\mathrm{ad}\,x\ \mathrm{ad}\,y)$$

More generally, if $\rho_M : \mathfrak{g} \longrightarrow \mathrm{End}(M)$ faithful repr (M fin-dimensional)

$$(x,y)_M = \mathrm{tr}(\rho_V(x)\,\rho_V(y))$$

EX: $\mathfrak{g} = \mathfrak{sl}_n$. $\quad (x,y)_{\mathrm{Kil}} = 2n\, \underline{\underline{\mathrm{tr}(x,y)}}$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$

(ex: $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{h} = \left\{\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right\} \subset \mathfrak{g}$

Fix $(.|.)$ a nondegenerate bilinear form of $\mathfrak{g}$

It is known that $(\ |\ )_{|\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate.

$\rightsquigarrow$ you can define a non degenerate bilinear form on $\mathfrak{h}^* \times \mathfrak{h}^*$

One wants to choose $(.|.)$ so that $(\theta|\theta) = 2$, where θ is the highest positive root. $\quad \Delta = \Delta(\mathfrak{g},\mathfrak{h}) \rightsquigarrow \Delta_+ = \{\text{positive roots}\}$

wrt to a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$$

$$(\ |\ ) = \frac{1}{2h^\vee}(\ |\ )_{kil}$$

$$h^\vee = ht(\theta^\vee)+1$$

<u>Ex</u> : sl_n $\quad \underbrace{\circ - \circ - \circ \cdots - \circ}_{n-1}$ $\quad \theta = \alpha_1 + \cdots + \alpha_{n-1}$ $\quad \{\alpha_1 \cdots \alpha_{n-1}\}$ simple roots

$\underline{h^\vee = n}$

$\kappa = \frac{k}{2h^\vee}$ killing form of $\mathfrak{g}$.

We write $V^k(\mathfrak{g}) = V^\kappa(\mathfrak{g})$

Note that it is the same as

$$\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K \quad ;$$

$$[K, \hat{\mathfrak{g}}] = 0$$

$$[xt^m, yt^n] = [x,y]t^{m+n} + m\delta_{m,-n}(x|y)K$$

where $(x|y) = \frac{1}{2h^\vee}(x|y)_{kil}$.

$$V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k$$

where $\mathbb{C}_k$ is a one-dimensional repr. of $\mathfrak{g}[t]\oplus\mathbb{C}K$, where $\mathfrak{g}[t]$ acts trivially and K acts as $k\,\mathrm{Id}_{\mathbb{C}_k}$.

. $V^k(\mathfrak{g})$ is highest weight representation of $\hat{\mathfrak{g}}$.

triangular decomposition on $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$$

where $\hat{\mathfrak{n}}_+ = (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] + \mathfrak{n}_+ \otimes \mathbb{C}[t] = \mathfrak{n}_+ + t\mathfrak{g}[t]$

$$\hat{\mathfrak{n}}_- = \mathfrak{n}_- \oplus t^{-1}\mathfrak{g}[t^{-1}]$$

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \quad \leftarrow \text{Cartan subalgebra of } \hat{\mathfrak{g}}$$

The highest weight of $V^k(\mathfrak{g})$ is $k\Lambda_0$, where Λ_0 is the highest weight of the *basic representation* $(k=1)$, the highest weight vector is $v_k = |0\rangle$

Def: A $\hat{\mathfrak{g}}$-representation M is called of *level k*, $k \in \mathbb{C}$, if K acts as $k\,\mathrm{Id}$ on it.

Ex: $V^k(\mathfrak{g})$ is of *level k*

$V^k(\mathfrak{g})$ is the (universal) *affine vertex algebra associated with $\mathfrak{g}$ of level k*

Recall: $V^k(\mathfrak{g})$ is graded

Any (proper) graded ideal of $V^k(\mathfrak{g})$ does not contain the vacuum.

So there exists a unique simple graded quotient of $V^k(\mathfrak{g})$.

We denote it by $L_k(\mathfrak{g})$

Rem: $L_k(\mathfrak{g})$ is a simple $\hat{\mathfrak{g}}$-module, it is a hw repr of level k.

Exercise Let V be a vertex algebra, and suppose that there exists a VA homomorphism

$$\Phi : V^k(\mathfrak{g}) \longrightarrow V \simeq \langle Y(a,z) ; a \in V \rangle_V \subset \mathcal{F}(V)$$

(thus V has a $\hat{\mathfrak{g}}$-module structure)

Show that

$$\mathrm{Com}\left(\Phi(V^k(\mathfrak{g})), V\right) \simeq V^{\mathfrak{g}[t]},$$

where: $V^{\mathfrak{g}[t]} = \{ v \in V : \mathfrak{g}[t] . v = 0 \}$

§3 – Virasoro vertex algebras

Let $\mathrm{Vir} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$ be the Virasoro Lie algebra, with commutation relations:

$$[C, \mathrm{Vir}] = 0$$

$$[L_n, L_m] = (n-m)\,L_{n+m} + \frac{n^3-n}{12}\,\delta_{m,-n}\,C$$

Rem: $\mathrm{Vir} = \underset{\mathbb{C}((t))\partial_t}{\underset{\parallel}{\mathrm{Der}\,\mathbb{C}((t))}} \oplus \mathbb{C}C$ $\qquad L_n = -t^{n+1}\partial_t$

$$[f(t)\partial_t, g(t)\partial_t] = (fg' - f'g)\partial_t - \frac{1}{\boxed{12}}\,\mathrm{Res}_{t=0}(f g''' dt)\,C$$

. A Vir-module M is called smooth if

$$L(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{\underline{-n-2}} \qquad\qquad L_{(n)} = L_{n-1}$$

is a field on M.

. For any smooth Vir-module M, the field $L(z)$ is local to itself,

$$L(z)L(w) \sim \frac{\partial_w L(w)}{(z-w)} + \frac{2L(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4}$$

Rem: analogy of the OPE of the field on π

$$L(z) = \frac{1}{2} :b(z)^2: + \alpha\,\partial_z b(z).$$

- A Vir-module M is said to be of central charge $c \in \mathbb{C}$ if the central element C acts as $c\,\mathrm{Id}_M$.

Let M be a smooth Vir-module of central charge c.

Then $\langle L(z) \rangle_M$ is a smooth Vir-module of central charge c by the action

$$L_{(n+1)} = L_n \longmapsto L(z)_{(n+1)}$$

$$\left(L(z)_{(n+1)} L(z)_{(m+1)} c(z) - L(z)_{(m+1)} L(z)_{(n+1)} c(z) \quad \dots \right)$$

$\langle L(z) \rangle_M$ is generated by Id_M, and we have $L(z)_{(n)} \cdot \mathrm{Id}_M = 0 \quad n \geq 0$

Set

$$\mathrm{Vir}^c := U(\mathrm{Vir}) \otimes_{U(\bigoplus_{n\geq -1} L_n \oplus \mathbb{C}C)} \mathbb{C}_c ,$$

where $\mathbb{C}_c$ is a one-dimensional representation of $\bigoplus_{n\geq -1} L_n \oplus \mathbb{C}C$, on which L_n acts trivially, $n \geq -1$, and C acts as $c\,\mathrm{Id}$.

- We have a Vir-module homomorphism

$$\mathrm{Vir}^c \longrightarrow \langle L(z) \rangle_M .$$

It is a surjective homomorphism (proof: later using Li's filtration)

- By PBW's theorem, Vir^c has a PBW basis of the form:

$$L_{j_1} \dots L_{j_m} |0\rangle , \quad \text{where } j_1 \leq \dots \leq j_m \leq -2,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c.

<u>Prop</u>: there is a unique VA structure on Vir^c s.t $|0\rangle = \overline{1 \otimes 1}$ is the vacuum and $Y(\omega, z) = L(z)$ where $\omega = L_{-2}|0\rangle$ $(= L_{(-1)}|0\rangle)$

Moreover, there is a surjective homomorphism $\mathrm{Vir}^c \longrightarrow \langle L(z) \rangle_M$ of VA for any smooth Vir-module of central charge c.

Vir^c is called the <u>(universal) Virasoro vertex algebra with central charge c</u>

<u>dem</u>: follows from reconstruction theorem.

<u>Rmk</u>: Note that $T = L_{-1}$ on Vir^c : the translation operator is "inner"

since $L(z)_{(0)} L(z) = \partial_z L(z)$

$$\overset{\shortparallel}{(L_{(0)}L)(z)} = (L_{-1} L_{-2}|0\rangle)(z) = (L_{-3}|0\rangle)(z) = (T\omega)(z) = \partial_z L(z)$$

<u>Rmk</u>: Vir^c is $\mathbb{Z}_{\geq 0}$-graded $\quad \mathrm{Vir}^c = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} \mathrm{Vir}^c_\Delta$

$$\deg(L_{j_1} \cdots L_{j_m}|0\rangle) = -\sum_{\ell=1}^{m} j_\ell \qquad j_\ell \leq -2$$

$\mathrm{Vir}^c_0 = \mathbb{C}|0\rangle$, $\mathrm{Vir}^c_1 = 0$, $\mathrm{Vir}^c_2 = \mathbb{C}\,\underset{L_{-2}|0\rangle}{\overset{}{\omega}}$

this grading is given by L_0 (exercise)

$$L_0(L_{-2}|0\rangle) = 2 L_{-2}|0\rangle \quad , \quad L_0(L_{j_1}|0\rangle) = (-j_1) L_{j_1}|0\rangle \qquad j_1 \leq -2$$

$$\mathrm{Vir}^c_\Delta = \{ v \in \mathrm{Vir}^c : L_0 v = \Delta v \}$$

the unique simple graded quotient of Vir^c is called the simple Virasoro vertex algebra with central charge c, and is denoted $\mathrm{Vir}_c = \mathrm{Vir}^c/\ldots$

Prop: The category of Vir^c-modules is the same as that of smooth representations of Vir of central charge c

§4 - Conformal vertex algebras

A Hamiltonian of a VA V is a semisimple operator H on V satisfying

$$[H, a_{(n)}] = -(n+1)a_{(n)} + (Ha)_{(n)} \qquad \text{for all } a \in V,\ n \in \mathbb{Z}.$$

Ex: (1) $V = Vir^c$, $H = L_0$

- $[L_0, L_{(n)}] = [L_0, L_{n-1}] = (-n+1)L_{n-1}$
- $(HL)_{(n)} = (L_0\omega)_{(n)} = (L_0 L_{-2}|0\rangle)_{(n)} = 2L_{(n)} = 2L_{n-1}$

$\Rightarrow \quad -(n+1)L_{(n)} + (HL)_{(n)} = (-n+1)L_{n-1}$ ✓

(2) $V = V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k$.

Define an operator D by $[D, x_{(n)}] = n\,x_{(n)}$

$H = -D$.

Better way to say:

$$\tilde{\mathfrak{g}} = \underset{= \mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}K}{\hat{\mathfrak{g}}} \oplus \mathbb{C}D \qquad \text{extended affine Kac-Moody Lie algebra.}$$

$[D, K] = 0$, $\quad [D, xt^n] = n\,xt^n \qquad x \in \mathfrak{g},\ n \in \mathbb{Z}$

$V^k(\mathfrak{g}) \simeq U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K\oplus\mathbb{C}D)} \mathbb{C}_k$ $\quad \hookrightarrow$ $\mathfrak{g}[t]\oplus\mathbb{C}D$ act trivially, $K \mapsto k\,\mathrm{Id}$

$\underline{\underline{H = -D}}$

Def: A VA equipped with a Hamiltonian H is called graded. In that case, set

$V_\Delta = \{a \in V : Ha = \Delta a\}$, $\Delta \in \mathbb{C}$, $\quad V = \bigoplus_{\Delta\in\mathbb{C}} V_\Delta$.

A graded VA is called conical if $\exists\ m \in \mathbb{N}$ st

$$V = \bigoplus_{\Delta \in \frac{1}{m}\mathbb{Z}_{\geq 0}} V_\Delta . \quad \text{and} \quad V_0 \simeq \mathbb{C} .$$

($\longrightarrow$ later : corresponding associated variety is conical).

Ex : Vir^c, $V^k(\mathfrak{g})$, Vir_c, $L_k(\mathfrak{g})$ are conical.

When V is graded, set $a_n = a_{(n+\Delta_a-1)}$ if $a \in V_{\Delta_a}$ $\leftarrow$ conformal weight of a.

$a_n V_\Delta \subset V_{\Delta-n}$

then we write : $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-\Delta_a}$ standard notation in physics

ex : $L(z) = (\overset{\overset{\omega}{\shortparallel}}{\underset{\underset{\Delta_\omega = 2}{\uparrow}}{L_{-2}|0\rangle}})(z) = \sum L_n z^{-n-2}$

§5 – Conformal vertex algebras

Def: A graded VA V is called conformal if there is a vector $\overset{\in V}{\omega}$, called the stress tensor, or the conformal vector, and that the corresponding field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n^V z^{-n-2} =: T(z)$$

satisfying the following conditions:

(1) $[L_n^V, L_m^V] = (n-m) L_{n+m}^V + \frac{n^3-n}{12}\delta_{n,-m}\overset{=c^V}{c}$, where c is a certain complex number called the central charge of V.

(2) $\omega_{(0)} = L_{-1}^V = T$.

(3) $\omega_{(1)} = L_0^V = H$ — Hamiltonian, that is, $L_0^V|_{V_\Delta} = \Delta\, \mathrm{Id}_V$ for all $\Delta \in \mathbb{C}$.

If V is conformal of central charge c, there is a VA homomorphism

$$Vir^c \longrightarrow V, \quad \omega \longmapsto \omega = \omega^V$$

Ex: (1) Clearly, Vir^c is conformal of central charge c, with conformal vector $\omega = L_{-2}|0\rangle$!

(2) π is a conformal vertex algebra with $\omega_\alpha = L_{-2}|0\rangle$

$$L(z) = \frac{1}{2} :b(z)^2: + \underline{\underline{\alpha}}\, \partial_z b(z)$$

with central charge $c_\alpha = 1 - 12\alpha^2$

$$L(z)L(w) = \frac{\partial_w L(w)}{z-w} + \frac{2L(w)}{(z-w)^2} + \frac{\overset{=c/2}{(1-12\alpha^2)/2}}{(z-w)^4}$$

(3) More complicated: $V^k(\mathfrak{g})$ is conformal provided that $k \neq -h^\vee$

[next time]

Exercises session

Ex: $(R, \partial) \longrightarrow$ comm VA structure ✓

Ex: V $\mathbb{C}$-vector space

(1) if V VA, $Ta = a_{(-2)}|0\rangle$

(2) V is endowed with $|0\rangle \in V$ and assume that vacuum axiom + locality axioms are satisfied.

We want to show that $V \xrightarrow{\Phi} V$ satisfies the translation axiom.

$$a \longmapsto a_{(-2)}|0\rangle$$

(i) $\Phi|0\rangle = 0$ ✓.

$[\Phi, a(z)] = \partial_z a(z)$?

Key pt: we haven't use translation axiom to prove Goddard uniqueness theorem.

Use G' thm to show $[\Phi, a(z)] = (\underbrace{\Phi a}_{\in V})(z)$

$$(\underbrace{\Phi a}_{\in V})(z) = \partial_z a(z) = \partial_z Y(a, z)$$

- $\overbrace{(\Phi a)(z)|0\rangle}^{\in V[[z]]}|_{z=0}$ $\qquad$ $\overbrace{\partial_z a(z)}^{\in V[[z]].}|0\rangle|_{z=0}$? $= a_{(-2)}|0\rangle$

$$\underbrace{A(z)|0\rangle}_{\in V[[z]]} = \sum_n \frac{1}{n!} \partial_z^n A(z)|0\rangle$$

Hence: $(\Phi^k a)(z)|0\rangle|_{z=0} = \Phi(\Phi^{k-1}a)(z)|0\rangle = \partial_z(\Phi^{k-1}a)(z)|0\rangle$

$$\partial_z^k(\Phi a)(z)|0\rangle|_{z=0} = \Phi^{k+1}(a)(z)|0\rangle|_{z=0}.$$

- $[\Phi, a(z)]|0\rangle = \Phi a(z)|0\rangle = \sum \Phi a_{(n)} z^{-n-1}$

Exercise: About $\pi = \mathbb{C}[b_{-1}, b_{-2}, \dots]$

let $\omega = \frac{1}{2} b_{-1}^2 + \alpha b_{-2} \in \pi$

$Y(\omega, z) \underset{?}{=} \frac{1}{2} :b(z)^2: + \alpha\, \partial_z b(z)$ (after reconstruction thm)

$=: L(z)$

$Y(b_{-1}|0\rangle, z) = \sum b_n z^{-n-1}$

$Y(b_{-n}|0\rangle, z) = \frac{1}{n!} \partial_z^n b(z)$

$Y(b_{-1}^2|0\rangle, z) = :b(z)^2:$

1) $L(z)L(w) \sim \frac{(1-12\alpha^2)/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}$

2) $b(z)\, L(w) \sim \overset{(b_1\omega)(w)}{\overset{\|}{\frac{b(w)}{(z-w)^2}}} + \overset{(b_2\omega)(w)}{\overset{\|}{\frac{2\alpha}{(z-w)^3}}} + \overset{(b_0\omega)(w) = 0.}{\frac{0}{(z-w)^2}}$ $\leftarrow$ easier!

$\Leftrightarrow$ $b_0\omega = 0$, $b_1\omega = b_{-1}$, $b_{-2}\omega = 2\alpha$

3) Show: $L_{-1} = T$ on π

In other words π is a conformal vertex algebra with c.c. $1-12\alpha^2$

Wick formulas

$$:\partial_z^{n_1} a^{i_1}(z)\cdots\partial_z^{n_r} a^{i_r}(z):\; :\partial_z b^{j_1}(z)\cdots\partial_z b^{j_m}(z):$$

Ex: $b(z)b(w) \sim \frac{1}{(z-w)^2} \Rightarrow \partial_z b(z)\, b(w) \sim \frac{-2}{(z-w)^3}$

$b(z)\,\partial_w b(w) \sim \frac{2}{(z-w)^3}$

⚠ $b(z)L(w) \sim \frac{b(w)}{(z-w)^2} + \frac{2\alpha}{(z-w)^3}$

$L(z)b(w) \sim \frac{-2\alpha}{(z-w)^3} + \frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)}$

We have to understand is

(*) $\underbrace{:a^1(z)\dots a^r(z):}_{:a^1(z)\dots\, :a^{l-1}(z)a^l(z)::}\; :b^1(w)\dots b^n(w):$ $\quad n, m \in \mathbb{Z}$

$$\Big(\, :a(z)b(z)c(z): = :a(z)_+\big(b(z)_+(c(z)_+ + c(z)_-) + (c(z)_+ + c(z)_-)b(z)_-\big)$$
$$+ \big(\quad\big) a(z)_- \quad \Big)$$

(*) = sum of 2^l terms of the form:

$a^{i_1}(z)_+ a^{i_2}(z)_+ \cdots\; a^{j_1}(z)_- a^{j_2}(z)_-$

$i_1 < i_2 < \dots$, $j_1 > j_2 \dots$ partition of $\{1..l\}$.

Write $\underbrace{\langle a^i, a^j \rangle} = a^i(z)a^j(w) - :a^i(z)a^j(w):$

contraction

"singular part"

OPE

$= \underline{[a^i(z)_-, a^j(w)]}$

thm (Wick's formula)

$a^1(z) — a^m(z)$, $b^1(z) — b^n(z)$ fields s.t.:

"singular part" OPE

$$[a^i(z)_-, a^j(w)] = a^i(z)a^j(w) - :a^i(z)a^j(w):$$

(i) $[\underbrace{\langle a^i, a^j \rangle}, a^l(z)_\pm] = 0$ $\quad\forall\, i, j, l$

$\qquad\qquad / \; b^l(z)_\pm$

(ii) $[a^i(z)_\pm, b^j(z)_\pm] = 0$

then

$$:a^1(z)\dots a^m(z): :b^1(w)\dots b^n(w): \qquad (*)$$

$$= \sum_{s=0}^{\min(n,m)} \sum_{\substack{i_1<\dots<i_s \\ j_1\neq\dots\neq j_s}} \langle a^{i_1}, b^{j_1}\rangle \dots \langle a^{i_s}, b^{j_s}\rangle :a^1(z) — a^m(z)\, b^1(w)\dots b^n(w):_{(i_1 - i_s; j_1 \dots j_s)}$$

you remove
$a^{i_1}(z)\dots a^{i_s}(z)$
$b^{j_1}(z)\dots b^{j_s}(z)$.

Ex: (i) and (ii) are satisfied for $b(z)$, $\partial_z b(z)$ … in π.

$$[b_m, b_n] = m\,\delta_{m,-n}$$

proof: the typical term of the LHS of (*) is

$$(a^{j_1}(z)_+ a^{j_2}(z)_+ \dots \underbrace{a^{i_1}(z)_- a^{i_2}(z)_- \dots}) (b^{k_1}(w)_+ b^{k_2}(w)_+ \dots b^{l_1}(w)_- b^{l_2}(w)_- \dots)$$

we want to move the $a^i(z)_-$ across the $b^j(w)_+$

(ii) $\Rightarrow a^i(z)_- b^j(w)_+ = b^j(w)_+ a^i(z)_- + \langle a^i, b^j \rangle$

(i) $\Rightarrow \langle a^i, b^j \rangle$ commutes with all fields $b^j(w)_+$, $b^l(w)_-$

Hence you can move to the left. ☒

Application: OPE of $L(z)L(w)$ in π

$$L(z) = \frac{1}{2} :b(z)^2: + \alpha\, \partial_z b(z) \qquad \alpha \in \mathbb{C}$$

$* \; :b(z)^2: \, :b(w)^2:$

$$= \underset{n=0}{:b(z)^2 b(w)^2:} + 4 \overset{\frac{1}{(z-w)^2}}{\overbrace{\underset{n=1}{\langle b, b\rangle}}} \underline{:b(z)\, b(w):} + \underbrace{2\langle b, b\rangle^2}_{\frac{2}{(z-w)^4}}$$

$$\langle b, b\rangle \sim \frac{1}{(z-w)^2}$$

$$:b(z)\, b(w): \; = \; :b(w)^2: + (z-w) :\partial_w b(w)\, b(w): + (z-w)^2(\,\cdot\,)$$

$$:b(z)^2::b(w)^2: \sim \frac{4 :b(w)^2:}{(z-w)^2} + \frac{4 :\partial_w b(w)\, b(w):}{(z-w)^3} + \frac{2}{(z-w)^4}.$$

$$* \; \partial_z b(z) :b(w)^2: \underset{\text{Wick}}{=} :\partial_z b(z)\, b(w)^2: + 2 \underbrace{\langle \partial_z b(z), b(w)\rangle}_{\partial_z\left(\frac{1}{(z-w)^2}\right) = \frac{-2}{(z-w)^3}} \underline{\underline{b(w)}}$$

$$\sim \frac{-4 b(w)}{(z-w)^3}$$

$$* \; :b(z)^2: \partial_w b(w) \underset{\text{Wick}}{=} :b(z)^2 b(w): + 2 \underbrace{\langle b(z), \partial_w b(w)\rangle}_{= \frac{2}{(z-w)^3}} \underbrace{b(z)}$$

$$b(w) + (z-w)\partial_w b(z) + \frac{(z-w)^2}{2} \partial_w^2 b(w) + (z-w)^3(\ldots)$$

$$\sim \frac{4 b(w)}{(z-w)^3} + \frac{4\, \partial_w b(w)}{(z-w)^2} + \frac{2\, \partial_w^2 b(w)}{(z-w)}$$

To summarize, we get

$$L(z)L(w) = \frac{1}{(z-w)^4}\overbrace{\left(\frac{1}{2}-6\alpha^2\right)}^{(1-12\alpha^2)/2} + \frac{1}{(z-w)^2}\overbrace{\left(:b(w)^2: + 2\alpha\,\partial_w b(w)\right)}^{2L(w)}$$

$$+ \frac{1}{z-w}\left(\underbrace{:\partial_w b(w)\, b(w): + \alpha\,\partial_w^2 b(w)}_{\partial_w L(w) \quad (\text{check})}\right).$$

Therefore L_n satisfies the Virasoro relations, with central charge $c_\alpha = 1-12\alpha^2$.

It remains to show that $L_{-1}=T$, L_0 gives the grading.

It suffices to check:

$b_0 w = 0, \quad b_1 w = b_{-1}, \quad b_2 w = 2\alpha$

$L_{-1} = T$

Let us compute $L(z)\,b(w)$ $\;b(z)\,L(w)$ (exercise: finish to check that π is a conformal vertex algebra)

Rem: about the surjectivity of $\pi \longrightarrow \langle b(z)\rangle_M$ for any smooth $\mathcal{B}$-module M...

In fact: it follows from the facts that $\mathcal{B} \longrightarrow \mathrm{End}(\langle b(z)\rangle_M)$ homom of $\mathcal{B}$-modules.

($b(z)_{(m)}\, b(z)_{(n)}\, c = 0 \Leftarrow b_{(m)}\, b_{(n)}\, c = 0$ ...)

Similarly, we can check that

$V^k(\mathfrak{n}) \longrightarrow \langle x(z) : x \in \mathfrak{n} \rangle_M$ for any smooth $\mathfrak{n}$...

$Vir^c \longrightarrow \langle L(z) \rangle_M$ "

are surjective.

Recall that if V is a conformal vertex algebra then there exists a nontrivial VA homom. $Vir^c \longrightarrow V$

One can show (later!) that

Vir^c is simple $\Leftrightarrow c \neq c(p,q)$, where $c(p,q) = 1 - 6(p-q)^2/pq$

$p, q \in \mathbb{Z}_{\geq 1}$ $(p,q) = 1$.

x If $c = c(p,q)$, then Vir^c has a unique simple quotient Vir_c

Rem: More generally, any conical vertex algebra, that is, a $\mathbb{Z}_{\geq 0}$-graded VA such that $V_0 \simeq \mathbb{C}|0\rangle$ has a unique simple graded quotient.

Indeed, any proper graded ideal does not contain the vacuum $|0\rangle$

And the sum of two graded ideals is still a graded ideal which does not contain the vacuum (due to the conical assumption)

Hence V admits a maximal proper graded ideal.

. If $c \neq c(p,q)$ then $Vir^c \hookrightarrow VA$

Hence V has Vir^c as vertex subalgebra.

. If $c \neq c(p,q)$, then Vir^c or Vir_c is subalgebra of V.

Conformal structure on $V^k(\mathfrak{g})$ with $\mathfrak{g}$ a simple Lie algebra, for $k = -h^\vee$

Set $\quad S = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x^i_{(-1)} x_{i(-1)} |0\rangle \quad$, called Sugawara vector

where $\{x^i\}_i$ and $\{x_i\}$ are dual basis of $\mathfrak{g}$ with respect to $(-|-)$

$(-|-) = \frac{1}{2h^\vee}$ Killing form of $\mathfrak{g}$.

$$Y(S,z) = \frac{1}{2} \sum :x^i(z) x_i(z):$$

Claim: for $k \neq -h^\vee$, $L = \frac{S}{k+h^\vee}$ is a conformal vector for $V^k(\mathfrak{g})$ with central charge $c(k) = \frac{k \dim \mathfrak{g}}{k+h^\vee}$

We will devote the rest of the lecture to the proof.

Exercise: Show that we have the following isom. of comm. $\mathbb{C}$-alg

$$\underset{\substack{\nearrow \\ \text{comm structure is the normally ordered product.}}}{Z(V^k(\mathfrak{g}))} = \mathrm{End}_{\mathfrak{g}}(V^k(\mathfrak{g}))$$

⟶ gives a description of the center which has nothing to do with VA structure

Indic: Frobenius reciprocity $\mathrm{Hom}_{U(\mathfrak{g})}(\overbrace{U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k}^{V^k(\mathfrak{g})}, V^k(\mathfrak{g}))$

$k = -h^\vee$ is called critical

Rem: One can easily show that $Z(V^k(\mathfrak{g})) = \mathbb{C}|0\rangle$ if $\underset{\nearrow}{k \neq -h^\vee}$ using the conformal vector.

What about $k=-h^\vee$?!

The center is huge !! $Z(V^{-h^\vee}(\mathfrak{g})) =: \mathfrak{z}(\hat{\mathfrak{g}})$ Feigin–Frenkel center

We will see : $\mathrm{gr}\,\mathfrak{z}(\hat{\mathfrak{g}}) \simeq \mathcal{O}(J_\infty(\mathfrak{g}/\!/G))$, $\mathfrak{g}/\!/G := \mathrm{Spec}\,\mathbb{C}[\mathfrak{g}]^G$

Lie $G = \mathfrak{g}$ (G: adjoint group of $\mathfrak{g}$).

Analogy with Casimir operator

$$\Omega_{\mathfrak{g}} = \sum_{i=1}^{\dim\mathfrak{g}} x^i x_i \quad \in \quad U(\mathfrak{g})$$

Ex : $\mathfrak{g} = \mathfrak{sl}_2$ with basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$

$(x|y) = \mathrm{tr}(xy)$

take (e,h,f) as basis of $\mathfrak{g}$: dual basis $f, \frac{h}{2}, e$

$\mathrm{tr}(ef) = \mathrm{tr}(fe) = 1$

$\mathrm{tr}(h^2) = 2$

$$\Omega_{\mathfrak{sl}_2} = ef + \frac{h^2}{2} + fe = 2ef + h^2 + h .$$

Claim : $\Omega_{\mathfrak{g}} \in Z(U(\mathfrak{g}))$ i.e : $\Omega_{\mathfrak{g}} u = u\,\Omega_{\mathfrak{g}}$ for all $u \in U(\mathfrak{g})$

Note that if $\{\tilde{x}^i\}$ and $\{\tilde{x}_i\}$ are dual basis wrt to the Killing form then

$$\sum x^i x_i = 2h^\vee \overbrace{\sum \tilde{x}^i \tilde{x}_i} .$$

By the definition of the Killing form $\sum_i \tilde{x}^i \tilde{x}_i$ acts as identity on $\mathfrak{g}$.

Hence $\Omega_{\mathfrak{g}}$ acts as $2h^\vee \, \mathrm{Id}_{\mathfrak{g}}$ on $\mathfrak{g}$.

Lemma: (characterization of conformal vertex algebras)

A $\mathbb{Z}_{\geq 0}$-graded VA is conformal of central charge c

iff it contains a nonzero $\omega \in V_2$ st the corresponding field

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfies: $L_{-1} = T$, $\underline{L_0 \text{ gives the } \mathbb{Z}_{\geq 0}\text{-grading}}$ (in particular $L_0\omega = 2\omega$)

and: $L_2\omega = \frac{c}{2}|0\rangle$

In that case: $\exists!$ VA homomorphism $Vir^c \longrightarrow V$, $|0\rangle \longmapsto |0\rangle$

$L_{-2}|0\rangle = \omega \longmapsto \omega$

Pf: "$\Rightarrow$" clear.

"$\Leftarrow$" Then $\omega \in V_2$

Hence: $\omega_n\omega = L_n\omega \in V_{2-n}$ Hence if $2-n<0$

then $\underbrace{\omega_n}_{\omega_{(n+1)}}\omega = 0$

$$Y(\omega,z)\,Y(\omega,w) \sim ?$$

no term in $\frac{1}{(z-w)^n}$, $n>4$

By the hypothesis:

$$Y(\omega,z)\,Y(\omega,w) \sim \frac{\overset{\checkmark}{c/2\; Y(|0\rangle,w)}}{(z-w)^4} + \frac{\boxed{Y(L_1\omega,w)}}{(z-w)^3} + \frac{\overset{\checkmark}{2Y(\omega,w)}}{(z-w)^2} + \frac{\overset{\partial_w Y(\omega,w)}{\overset{\shortparallel}{Y(T\omega,w)}}}{(z-w)}$$

$\underline{\text{We have to show that } Y(L_1\omega,w) = 0}$

Exchange z and w:

- On one hand:

$$Y(\omega,w)\,Y(\omega,z) \sim \frac{\overset{c/2}{\overset{\shortparallel}{c/2\; Y(|0\rangle,z)}}}{(z-w)^4} - \frac{\overset{Y(L_1\omega,w)+(z-w)(\,\cdot\cdot\,)}{\overset{\shortparallel}{Y(L_1\omega,z)}}}{(z-w)^3} + \frac{2Y(\omega,z)}{(z-w)^2} - \frac{\partial_z Y(\omega,z)}{z-w}$$

- On the other hand

$Y(\omega,w)Y(\omega,z)$ has the same "singular part" of $Y(\omega,z)Y(\omega,w)$

therefore $Y(L_1\omega,w)=0$, ie $L_1\omega=0$

then: $Y(\omega,z)Y(\omega,w)$ has Virasoro OPE.

- Finally to define the map:

$$Vir^c \ni L_{j_1}\cdots L_{j_m}|0\rangle \longmapsto L^\vee_{j_1}\cdots L^\vee_{j_m}|0\rangle \in V \qquad \boxtimes\,.$$

We intend to apply this lemma to ω: conformal vector of degree 2

Write $Y(\omega,z)=\frac{1}{k+h^\vee}Y(S,z) = \sum_n L_n z^{-n-2}$

It suffices to show:

$$S_{-1}|0\rangle = 0,\quad S_0|0\rangle=0 \quad \text{and} \quad \Big[\frac{1}{k+h^\vee}S_n, \overset{xt^m}{\overset{\shortparallel}{x_{(m)}}}\Big] = -m\, x_{(m+n)} \qquad \begin{matrix} m,n\in\mathbb{Z} \\ x\in\mathfrak{g}\end{matrix}$$

(in particular we get what we want for $n=0,1$)

$Y(S,z)=\sum_n S_n z^{-n-2}$ $\quad S_n|0\rangle=0 \quad n\geq -1 \implies S_{-1}|0\rangle=0$

$S_0|0\rangle=0.$

Let us show that $\Big[\frac{1}{k+h^\vee}S_n, x^j_{(m)}\Big] = -m\, x^j_{(m+n)}$

It is equivalent to compute OPE $S(z)x^j(w)$, ie $S_{(n)}x^j$, $n\geq 0$

$S_n x^j \quad n\geq -1$

Rem: $S_n x^j_{(-1)} |0\rangle = 0$ if $n > 1$

Other argument: $S_n x^j \in V_{1-n}$ so $S_n x^j = S_n x^j_{(-1)}|0\rangle = 0$

if $1-n < 0$.

We need to compute only $S_{-1} x^j_{(-1)}|0\rangle$, $S_0 x^j_{(-1)}|0\rangle$ $S_1 x^j_{(-1)}|0\rangle$

$$* \quad S_1 x^j_{(-1)}|0\rangle = \frac{1}{2}\sum_i \Big(\underbrace{x^i_{(0)} x_{i,(1)} x^j_{(-1)}|0\rangle}_{\textcircled{1}} + \underbrace{x^i_{(1)} x_{i,(0)} x^j_{(-1)}|0\rangle}_{\textcircled{2}} \Big)$$

$$\Big[:x^i(z) x_i(z): = x^i(z)_+ x_i(z) + x_i(z) x^i(z)_-$$

$$= \sum_{\substack{n<0 \\ m\in\mathbb{Z}}} \underbrace{x^i_{(n)} x_{i,(m)} z^{-(n+m)-2}}_{\substack{n+m=i \\ \to \text{ coeff in } S_i}} + \sum_{\substack{n\geq 0 \\ m\in\mathbb{Z}}} x_{i,(m)} x^i_{(n)} z^{-(n+m)-2}$$

S_i : $x^i_{(n)} x_{i,(m)} x^j_{(-1)}|0\rangle$

$n+m=1$ $n<0$ $\Rightarrow$ $m>1$ $\Rightarrow$ $x_{i,(m)} x^j_{(-1)}|0\rangle = 0$.

$n+m=1$ $n\geq 0$ $x_{i,(m)} x^i_{(n)} x^j_{(-1)}|0\rangle \neq 0$ $\Rightarrow$ $\boxed{\begin{matrix} n=0 & \text{or} & n=1 \\ m=1 & & m=0 \end{matrix}}$ $\Big]$

① gives $k \sum_i x^{(i)}_0 (x_i | x^j) |0\rangle = 0$

$$\textcircled{2} = \frac{1}{2}\sum_i x^i_{(1)} [x_i, x^j]_{(-1)}|0\rangle = \frac{k}{2} \underbrace{\sum_i (x^i | [x_i, x^j])}_{=0 \text{ because of the}} |0\rangle$$

invariance of $(\_|\_)$. (classical extension on simple lie alg).

$= 0$

$$\boxed{S_1 x^j_{(-1)}|0\rangle = 0}$$

$$* \quad S_0\, x^j_{(-1)}|0\rangle = \sum_i \Big(\tfrac{1}{2} \underbrace{x^i_{(0)}\, x_{i(0)}}_{①} + \underbrace{x^i_{(-1)}\, x_{i(1)}}_{②} \Big)\, x^j_{(-1)}|0\rangle$$

① : is equal to $\frac{1}{2}(2h^\vee)\, x^j_{(-1)}|0\rangle = h^\vee\, x^j_{(-1)}|0\rangle$

② $= k \sum_i \underbrace{(x_i|x^j)}_{\delta_{ij}}\, x^i_{(-1)}|0\rangle = k\, x^j_{(-1)}|0\rangle$

$$S_0\, x^j_{(-1)}|0\rangle = (k+h^\vee)\, x^j_{(-1)}|0\rangle$$

* A similar computation gives:

$$S_{-1}\, x^j_{(-1)}|0\rangle = (k+h^\vee)\, \underbrace{x^j_{(-2)}|0\rangle}_{''\; T x^j_{(-1)}|0\rangle.}$$

To summarize:

$$S(z)\, x^j(w) \sim (k+h^\vee)\left(\frac{x^j(w)}{(z-w)^2} + \frac{\partial_w x^j(w)}{z-w} \right)$$

Rem: if $k = -h^\vee$ $S(z)x^j(w)$ is regular $\Longrightarrow$ $[S_n, x_{(m)}] = 0$ $\forall x \in \mathfrak{g}$ $\forall n, m \in \mathbb{Z}$

S_n are central in "some" completed universal enveloping alg.

. For $k \neq -h^\vee$

$$\frac{1}{k+h^\vee}\, S(z)\, x^j(w) \sim \frac{x^j(w)}{(z-w)^2} + \frac{\partial_w x^j(w)}{z-w}$$

$\Longleftrightarrow$ commutation relations $\left[\frac{1}{k+h^\vee} S_n,\ x_{(m)}\right] = \dots$

$\Longrightarrow$ $L_{-1} = T$, L_0 gives the grading.

to complete the proof, we need to compute $L_2 L$ (or $S_2 S$)

Using commutation relations, we get

$$S_2 S = S_2 \cdot \frac{1}{2}\Big(\sum_i x^i_{(-1)} x_{i(-1)} |0\rangle \Big) = \frac{k+h^\vee}{2} \sum_{i=1}^{\dim g} \overbrace{x_{i(-1)} x^i_{(-1)}}^{k} |0\rangle$$

$$= \frac{k+h^\vee}{2} \times (k \dim g) |0\rangle$$

$$c(k) = \frac{k \dim g}{k+h^\vee}$$

therefore: $L_2 \omega = \frac{c(k)}{2} |0\rangle$ $\square$

Modules over conformal VA

M is a V-modules

$$\begin{tikzcd} Vir^c \arrow[r, bend left] & M \end{tikzcd} \qquad \begin{tikzcd} Vir^c \arrow[r] \arrow[rr, bend left] & V \arrow[r] & \langle Y(a,z) : a \in V \rangle_M \end{tikzcd}$$

$\Longrightarrow$ $M = \bigoplus_d M_d$ $\quad M_d = \{ m \in M : L_0 m = d\, m \}$.

If $\dim M_d < \infty$ + grading is bounded from below

typically $M = \bigoplus_{d \in \mathbb{Z}_{\geq}} M_d$, $\dim M_d < \infty$

then one can normalized character of M

$$\chi_M(q) = tr_M\big(q^{L_0 - c/24}\big) = q^{-c/24} \sum_d (\dim M_d)\, q^d$$ is well-defined

$q = e^{2i\pi\tau}$, $\tau \in \{ z \in \mathbb{C} \mid \operatorname{Im} \tau > 0 \}$