03/02/2021

Introduction

What is a vertex algebra (roughly)?

[Borcherds, 88] A vertex algebra is a complex vector space V together with some data:

* $|0\rangle \in V$ (vacuum)
* a linear map

$$V \longrightarrow (\operatorname{End} V)[[z, z^{-1}]]$$

$$a \longmapsto a(z) = \sum_{n \in \mathbb{Z}} \underbrace{a_{(n)}}_{\in \operatorname{End} V} z^{-n-1}$$

s.t. $\forall\, b \in V$, $\underline{a_{(n)} b = 0}$ $\quad n >> 0$

+ axioms

In particular (locality axiom):

$\forall a, b \in V$,

$$(z-w)^N a(z) b(w) = (z-w)^N b(w) a(z) \qquad \text{in } (\operatorname{End} V)[[z^{\pm}, w^{\pm 1}]]$$

for some $N > 0$

< Wightman axioms in QFT (quantum field theory)

Refs :

[Frenkel - Ben Zvi, Vertex alg and Algebraic curves]

[Kac : Vertex algebras for beginners] ↔ physical point of view

[Arakawa - Moreau : Arc spaces and vertex algebras]

my webpage (soon)

Chap 1 - Definition of vertex algebras

§1 - Notation

R is a $\mathbb{C}$-algebra with unit, $n \in \mathbb{N}^*$

$R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$: the vector space of R-valued formal power series in variables $z_1, \dots, z_n$:

$$a(z_1, \dots, z_n) = \sum_{i_1 \in \mathbb{Z}} \dots \sum_{i_n \in \mathbb{Z}} \underbrace{a_{i_1, \dots, i_n}}_{\in R} z_1^{i_1} \dots z_n^{i_n} \qquad (*)$$

If $a \in R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$, $b \in R[[w_1^{\pm 1}, \dots, w_m^{\pm 1}]]$, $m \in \mathbb{N}^*$, then

ab is well-defined in $R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}, w_1^{\pm 1}, \dots, w_m^{\pm 1}]]$

⚠ if both $a, b \in R[[z_1^{\pm 1}, \dots, z_n^{\pm 1}]]$, then ab does NOT always makes sense!

because individual coeff of the product may be infinite sums of the coeff...

However: it is ok if one multiplies by a Laurent polynomial, i.e

an element of the form $(*)$ with $a_{i_1 \dots i_n} = 0$

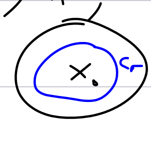
but finitely n-uplets $(i_1 \dots i_n)$.

. if $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in R[[z^{\pm 1}]]$

$$\boxed{\mathrm{Res}_{z=0}\, a(z) := a_{-1}}$$

. if $R = \mathbb{C}$ and $a(z)$ is the Laurent series of a meromorphic function defined on

$D^*(0,R)$, then $\mathrm{Res}_{z=0}\, a(z) = \frac{1}{2i\pi} \int_{C_r} a(z)\, dz$



$0 < r < R$

§2 - Formal delta function

Define the formal delta function by

$$\delta(z-w) := \frac{1}{z}\sum_{n\in\mathbb{Z}}\left(\frac{w}{z}\right)^n = \sum_{n\in\mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[\![z^{\pm 1}, w^{\pm}]\!]$$

Let us introduce two embeddings of algebras:

- $\tau_{z,w} : \mathbb{C}[z^{\pm 1}, w^{\pm 1}, \frac{1}{z-w}] \longrightarrow \mathbb{C}((z))((w))$

$$\frac{1}{z-w} \longmapsto \underbrace{\frac{1}{z}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n}_{\text{expansion of } \frac{1}{z-w} \text{ on the domain } |z|>|w|} = \sum_{n\geq 0}(z^{-n-1})\, w^n$$

(indeed : $\frac{1}{z-w} = \frac{1}{z(1-\frac{w}{z})} = \frac{1}{z}\sum_{n\geq 0}\left(\frac{w}{z}\right)^n$)

Rem : $R[\![z]\!]$, $R((z)) \subset \mathbb{C}[\![z, z^{-1}]\!]$.

- $\tau_{w,z} : \mathbb{C}[z^{\pm 1}, w^{\pm 1}, \frac{1}{z-w}] \longrightarrow \mathbb{C}((w))((z))$

$$\frac{1}{z-w} \longmapsto \underbrace{-\frac{1}{z}\sum_{n>0}\left(\frac{z}{w}\right)^n}_{\text{expansion of } \frac{1}{z-w} \text{ in the domain } |w|>|z|}$$

$$\frac{1}{z-w} = -\frac{1}{w-z} = -\frac{1}{w(1-\frac{z}{w})} = -\frac{1}{w}\sum_{n\geq 0}\left(\frac{z}{w}\right)^n = -\sum_{n\geq 0} z^{n-1} w^{-n}$$

then : $\delta(z-w) = \tau_{z,w}\left(\frac{1}{z-w}\right) - \tau_{w,z}\left(\frac{1}{z-w}\right)$

Lemma 1 : $\forall$ any $\mathbb{C}$-algebra R, $\forall f \in R[\![z, z^{-1}]\!]$,

$$f(z)\,\delta(z-w) = f(w)\,\delta(z-w)$$

($\rightarrow$ motivates the terminology "delta function") proof: $f(z)-f(w)$ is divisible by $z-w$, and

$$(z-w)\,\delta(z-w) = (z-w)\left(\tau_{z,w}\left(\frac{1}{z-w}\right) - \tau_{w,z}\left(\frac{1}{z-w}\right)\right) = \tau_{z,w}(1) - \tau_{w,z}(1)$$
$$= 0 \qquad \square$$

Hence

$$\tau_{z,w}\Big(\underbrace{\partial_w^n\left(\frac{1}{z-w}\right)}_{=\,\frac{n!}{(z-w)^{n+1}}}\Big) - \tau_{w,z}\left(\partial_w^n\left(\frac{1}{z-w}\right)\right)$$

$$\Big\| \quad (z-w)^{n+1}\,\frac{1}{n!}\,\partial_w^n\,\delta(z-w) = 0 \qquad \forall n \geq 0$$

Rem: ∂_w and ∂_z commute with $\tau_{z,w}$ and $\tau_{w,z}$ (exercise)

Lemma 2 $\quad \forall m, n \geq 0$

$$\Big|\quad \mathrm{Res}_{z=0}\left((z-w)^m\,\frac{1}{n!}\,\partial_w^n\,\delta(z-w)\right) = \delta_{m,n}$$

dém $\quad (z-w)^m\,\frac{1}{n!}\,\partial_w^n\,\delta(z-w) = \overbrace{\tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right)}^{1} - \overbrace{\tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right)}^{2}$

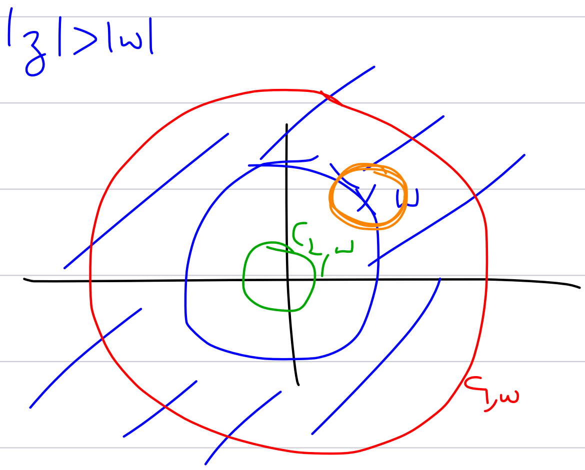
Consider the meromorphic function $f_w(z) = \frac{1}{(z-w)^{n-m+1}}$ with poles in $\{0, w, \infty\}$

f_w admits the following Laurent series expansions:

$$f_w(z) = \begin{cases} \tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right) = \sum_{n\in\mathbb{Z}} a_n(w)\, z^n, & |z|>|w| \\[2ex] \tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right) = \sum_{n} b_n(w)\, z^n, & |w|>|z| \end{cases}$$

Note that $\mathrm{Res}_{z=0}\left(\iota_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right)\right) = a_{-1}(w) = \frac{1}{2\pi i}\int_{C_{1,w}} \frac{dz}{(z-w)^{n-m+1}}$

and $\mathrm{Res}_{z=0}\left(\iota_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right)\right) = b_{-1}(w) = \frac{1}{2\pi i}\int_{C_{2,w}} \frac{dz}{(z-w)^{n-m+1}}$



Now $a_{-1}(w) - b_{-1}(w) = \frac{1}{2\pi i}\int_{C_{1,w}} \frac{dz}{(z-w)^{n-m+1}} - \frac{1}{2\pi i}\int_{C_{2,w}} \frac{dz}{(z-w)^{n-m+1}}$

$$= \frac{1}{2\pi i}\int_{C_w} \frac{dz}{(z-w)^{n-m+1}} \underset{\text{clear!}}{=} \delta_{n,m}$$

by the residue theorem applied to f_w ☒

Rem: one can also prove the lemma purely combinatorially

$$(z-w)^m \frac{1}{n!}\partial_w^n \delta(z-w) = \begin{cases} \frac{\partial_w^{n-m}\delta(z-w)}{(n-m)!} = \sum_{k\in\mathbb{Z}} \binom{k}{n-m} w^{k-n+m} z^{-k-1} & \text{if } n \geq m \geq 0 \\ 0 & \text{if } m > n \end{cases}$$

then $\mathrm{Res}_{z=0}(\quad) = \delta_{n,m}$

§3 – Locality and operator product expansion

Let V be a vector space $/\mathbb{C}$

Elements of $(\mathrm{End}\,V)[[z,z^{-1}]]$ are called <u>series on V</u>

For $a(z) \in (\mathrm{End}\,V)[[z,z^{-1}]]$, we set

$$a_{(n)} := \mathrm{Res}_{z=0}\, z^n a(z)$$

so that $a(z) = \sum_{n\in\mathbb{Z}} \underbrace{a_{(n)}}\, z^{-n-1}$ $\qquad a_{(0)} = \mathrm{Res}_{z=0}\, a(z)$

a Fourier mode of $a(z)$

$$a(z)\, b := \sum_{n\in\mathbb{Z}} (a_{(n)} b)\, z^{-n-1} \in V[[z,z^{-1}]]$$

$a(z)$ series on V, $b \in V$

<u>Def</u>: A series $a(z) \in (\mathrm{End}\,V)[[z,z^{-1}]]$ is called a <u>field</u> on V if

for any $b \in V$, $a(z)\,b \in V((z))$, i.e.: $\forall b \in V$, $a_{(n)} b = 0 \quad n \gg 0$

$\mathcal{F}(V)$ = space of all fields on V.

⚠ $a(z), b(z) \in \mathcal{F}(V)$, $a(z)b(z)$ does not make sense in general

<u>Def</u>: the <u>normally ordered product</u> of $a(z)$ and $b(z)$

$$:a(z)\,b(z): \;= a(z)_+\, b(z) + b(z)\, a(z)_- , \qquad \in \mathcal{F}(V) \text{ (exercise)}$$

where:

$$a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}, \quad a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-n-1}$$

this makes sense!!!

⚠ $: :$ is neither commutative, nor associative !!

By definition,

$$:a(z)\,b(z)\,c(z): \;=\; :a(z) :b(z)\,c(z)::$$

$a(z)\,b(w)$ is well-defined in $(\mathrm{End}\,V)[[z^{\pm 1}, w^{\pm 1}]]$

Still we introduce

$$:a(z)\,b(w): \;=\; a(z)_+\,b(w) + b(w)\,a(z)_-$$

Note that $:a(z)\,b(w):v \in V[[z,w]][z^{-1},w^{-1}]$

while $a(z)\,\underbrace{b(w)\,v} \in V((z))((w))$

$b(w)\,a(z)\,v \in V((w))((z))$

the intersection of $V((z))((w)) \cap V((w))((z)) = V[[z,w]][z^{-1},w^{-1}]$

Def : We say that $a(z)$ and $b(z)$ are mutually local ($a(z), b(z) \in \mathcal{F}(V)$)

if $(z-w)^N\,a(z)\,b(w) = (z-w)^N\,b(w)\,a(z)$ in $(\mathrm{End}\,V)[[z^{\pm 1}, w^{\pm 1}]]$

ie $(z-w)^N\,[a(z), b(w)] = 0$

for some $N = N_{a,b}$

⚠ : A field $a(z)$ no need to be local with itself !

<u>Proposition</u> $a(z), b(w) \in \mathcal{F}(V)$. The following assertions are equivalent:

(i) $a(z)$ and $b(z)$ are mutually local: i.e.: $\exists N \in \mathbb{Z}_{\geq 0}$

$$(z-w)^N [a(z), b(w)] = 0 \quad \text{in } (\mathrm{End}\, V)[[z^{\pm 1}, w^{\pm 1}]]$$

(ii) $\exists\, c_0(w), c_1(w), \dots, c_{N-1}(w) \in \mathcal{F}(V)$ s.t.

$$[a(z), b(w)] = \sum_{n=0}^{N-1} c_n(w) \frac{1}{n!} \partial_w^n \delta(z-w) \quad \text{in } (\mathrm{End}\, V)[[z^{\pm 1}, w^{\pm 1}]]$$

(iii) $\exists\, c_0(w), c_1(w), \dots, c_{N-1}(w) \in \mathcal{F}(V)$ s.t.

$$a(z) b(w) = \sum_{n=0}^{N-1} c_n(w)\, \tau_{z,w}\left(\frac{1}{(z-w)^{n+1}}\right) + \underline{:a(z) b(w):}$$

and in $(\mathrm{End}\, V)[[z^{\pm 1}, w^{\pm 1}]]$

$$b(w) a(z) = \sum_{n=0}^{N-1} c_n(w)\, \tau_{w,z}\left(\frac{1}{(z-w)^{n+1}}\right) + \underline{:a(z) b(w):}$$

Proof: (iii) $\Rightarrow$ (ii) clear: $\delta(z-w) = \tau_{z,w}(\ ) - \tau_{w,z}(\ )$

(ii) $\Rightarrow$ (i) $(z-w)^{n+1} \frac{1}{n!} \partial_w^n \delta(z-w) = 0$

<u>(i) $\Rightarrow$ (iii)</u>

- $\underset{\substack{\| \\ a(z)_+ + a(z)_-}}{a(z)}\, b(w) - \overbrace{:a(z) b(w):}^{a(z)_+ b(w) + b(w) a(z)_-} = [a(z)_-, b(w)]$

- $b(w) a(z) - :a(z) b(w): = [b(w), a(z)_-]$.

Hence: $[a(z), b(w)] = [a(z)_-, b(w)] - [b(w), a(z)_-]$

(i) $\Rightarrow$

$$\underbrace{(z-w)^N [\overset{\cdots z^{-1} + z^{-2} \cdots}{\overset{\|}{a(z)_-}}, b(w)]}_{\text{no term greater than } N-1 \text{ in } z} = \underbrace{(z-w)^N [b(w), a(z)_+]}_{\text{no term of negative degree in } z}$$

so there exist $c_0(w), c_1(w), \dots, c_{N-1}(w) \in (\mathrm{End}\, V)[[w, w^{-1}]]$ s.t.

$$(z-w)^N [a(z)_-, b(w)] = \sum_{j=0}^{N-1} c_j(w)(z-w)^{N-j-1}$$

for each $v \in V$ $[a(z)_-, b(w)]v = (a(z)b(w) - :a(z)b(w):)v \in \underbrace{V((z))((w))}$

vector sp. on $\underline{\underline{\mathbb{C}((z))((w))}}$

$$\cdot\ [a(z)_-, b(w)]v = \tau_{z,w}\left(\frac{1}{(z-w)^N}\right) \times (z-w)^N [a(z)_-, b(w)]v$$

$$= \tau_{z,w}\left(\frac{1}{(z-w)^N}\right) \sum_{j=0}^{N-1} c_j(w) v (z-w)^{N-j-1} = \sum_{j=0}^{N-1} \tau_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right) c_j(w) v$$

this is true for every v $\longrightarrow$ the first equality of (iii)

similarly we get the second one of (iii).

$\underline{\text{It remains to show that } c_j(w) \text{ is a field on } V \text{ for } i=0,\dots,N-1}$

We have obtained that

$$\mathrm{Res}_{z=0}\left([a(z), b(w)] = \sum_{j=0}^{N-1} c_j(w) \frac{1}{j!} \partial_w^j \delta(z-w)\right)$$

<u>Recall</u> (lemma 2) : $\mathrm{Res}_{z=0}\left((z-w)^m \frac{1}{n!} \partial_w^n \delta(z-w)\right) = \delta_{m,n}$

$$\Rightarrow c_j(w) = \mathrm{Res}_{z=0}\left((z-w)^j [a(z), b(w)]\right)$$

As both $a(z)$ and $b(w)$ are fields on V, $c_j(w)$ is a field on V. (exercise)

$\boxtimes$

<u>Notation</u> (physicists notation)

$$a(z)b(w) \sim \sum_{n=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}} \qquad (\ast)$$

<u>Def</u>: this relation is called operator product expansion (OPE) of $a(z)$ and $b(w)$

Prop the OPE $(\star)$ is equivalent to:

$$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} (c_j)_{(n+m-j)} \qquad m, n \in \mathbb{Z}$$

Here $\binom{m}{j} = \frac{m(m-1)\times\cdots\times(m-j+1)}{j(j-1)\times\cdots\times 1}$ if $j \geq 0$ $m \in \mathbb{Z}$

Proof "$\Leftarrow$" easier (exercise)

$(\star) \Rightarrow \underbrace{[a_{(m)}, b_{(n)}]}_{} = \ldots$

$a_{(n)} = \mathrm{Res}_{z=0}\, z^n a(z)$ $\qquad = \mathrm{Res}_{w=0}\left(w^n \,\mathrm{Res}_{z=0}\left(z^m [a(z), b(w)]\right)\right)$

As in lemma 2, we get

$$\mathrm{Res}_{z=0}\left(z^m [a(z), b(w)]\right) = \sum_{j=0}^{N-1} \mathrm{Res}_{z=w}\left(\frac{z^m}{(z-w)^{j+1}} c_j(w)\right)$$

$$= \sum_{j=0}^{N-1} \binom{m}{j} c_j(w)\, w^{m-j}$$

$\mathrm{Res}_{w=0}\left(\quad \right) \longrightarrow$ result $\square$

Hence, OPE encodes all brackets between mutually local fields $a(z), b(w)$ on V.

Example

$\mathcal{B}$ = unital associative algebra generated by elements b_n, $n \in \mathbb{Z}$, with relations:

$$[b_n, b_m] = m\,\delta_{m+n,0} \qquad m, n \in \mathbb{Z}$$

A $\mathcal{B}$-module M is called smooth if for each $m \in M$, $\exists N$ s.t.

$b_n \cdot m = 0 \qquad n > N$

If M is smooth, then

$$b(z) := \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$$

is a field on M.

$$[b(z), b(w)] = \sum_{m,n \in \mathbb{Z}} [b_m, b_n] z^{-m-1} w^{-n-1} = \sum_{m \in \mathbb{Z}} m z^{-m-1} w^{m-1}$$
$$= \partial_w \delta(z-w)$$

Hence: $b(z)$ is local to itself and

$$b(z) b(w) \sim \frac{1}{(z-w)^2} \qquad (N=2)$$

(Conversely: from $b(z) b(w) \sim \frac{1}{(z-w)^2} \longrightarrow [b_n, b_m] = m\,\delta_{m,-n}$

$N=2$, $c_0(w) = 0$, $c_1(w) = 1 \implies (c_1)(k) = \delta_{k,-1} \ldots$)

§4. Definition of a vertex algebra

Def: A vertex algebra is a vector space V over $\mathbb{C}$ equipped with the following data:

* (vacuum vector) $|0\rangle \in V$
* (vertex operator) a linear map

$$Y : V \longrightarrow \mathcal{F}(V) \quad , \quad a \longmapsto \underline{Y(a,z)} = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} ,$$

* (translation operator): a linear map $T : V \longrightarrow V$

these data are subject to the following axioms:

(i) (vacuum axiom) $Y(|0\rangle, z) = \mathrm{Id}_V$ $\quad (|0\rangle_{(-1)} = \mathrm{Id}_V \ , \ |0\rangle_{(n)} = 0 \quad n \neq -1)$

and: $\forall a \in V$, $Y(a,z)|0\rangle \in V[[z]]$ and $\lim_{z\to 0} Y(a,z)|0\rangle = a$

($\Rightarrow$ Y is injective !!)

(ii) (translation axiom) $T|0\rangle = 0$ and for all $a \in V$,

$$[T, Y(a,z)] = \partial_z Y(a,z)$$

(iii) (locality axiom): $\forall a, b \in V$, $Y(a,z)$ and $Y(b,z)$ are mutually local:

$$(z-w)^N \, [Y(a,z), Y(b,w)] = 0 \qquad N \gg 0.$$

05/02/2021

lecture 2

Remarks on the proof of lemma 2

I have slightly modified the proof of the lemma and add a remark!

$$\mathrm{Res}_{z=0}\left((z-w)^m \frac{\partial_w^n \delta(z-w)}{n!} \right) = \delta_{m,n}$$

(1) You can prove it combinatorially (Kac's book)

$$\delta(z-w) = \sum_n w^n z^{-n-1}$$

$$\frac{\partial_w^n \delta(z-w)}{n!} = \iota_{z,w}\left(\frac{1}{(z-w)^{n+1}}\right) - \iota_{w,z}\left(\frac{1}{(z-w)^{n+1}}\right)$$

$$\| $$

$$\sum_{\ell \in \mathbb{Z}} \binom{\ell}{n} w^{\ell-n} z^{-\ell-1}$$

$$\Rightarrow (z-w)^m \frac{\partial_w^n \delta(z-w)}{n!} = \begin{cases} \frac{\partial_w^{n-m} \delta(z-w)}{(n-m)!} & \text{if } n \geq m \geq 0 \\ 0 & \text{otherwise } (m>n) \end{cases}$$

$$\Rightarrow \mathrm{Res}_{z=0}(\ ) = \delta_{m,n} \dots \qquad \boxtimes$$

(2) $f(z) = \dfrac{z^j w^k}{(z-w)^m}$ poles at $\{z=w, z=0, \infty\}$

$$f(z) = \begin{cases} \sum_n a_n z^n, & |z|>|w| \\ \sum_m b_m z^m & |w|>|z| \end{cases}$$

$$\frac{1}{2i\pi}\int_{C_w} f(z)\,dz = a_{-1}, \qquad \frac{1}{2i\pi}\int_{C_w} f(z)\,dz = b_{-1}$$

$$\frac{1}{2i\pi}\int_{C_{1,w}}(\quad) - \frac{1}{2i\pi}\int_{C_{2,w}}(\quad) = \frac{1}{2i\pi}\int_{C_w} f(z)\,dz$$

$$\operatorname{Res}_{z=w} f(z) \;\left(= \delta_{n,w}\right)$$

$\square$

Back to §4 – Definition of a vertex algebra

Rem: vacuum:

$$Y(a,z)\,|0\rangle \in V[[z]] \quad \text{and} \quad \underbrace{\lim_{z\to 0} Y(a,z)\,|0\rangle}_{Y(a,z)|0\rangle|_{z=0}} = a$$

<u>ie</u>: $a_{(n)}\,|0\rangle = 0 \quad \forall\, n \geq 0$ and $\boxed{a_{(-1)}\,|0\rangle = a}$

$\Rightarrow$ the vertex operator Y is injective.

Def A <u>vertex algebra homomorphism</u> between two vertex algebras V, W:

a linear map $\Phi : V \longrightarrow W$ s.t.: $\Phi(|0\rangle) = |0\rangle$,

$\Phi(Ta) = T(\Phi a)$, and $\Phi(a_{(n)} b) = \Phi(a)_{(n)}\, \Phi(b)$. $\forall a, b \in V$

$\forall n \in \mathbb{Z}$

Exercise. V, W vertex algebras. Show that $V \otimes W$ is a vertex algebra with

vacuum $|0\rangle \otimes |0\rangle$, translation operator $T \otimes 1 + 1 \otimes T$ and

$$Y(a \otimes b, z) = Y(a,z)\, Y(b,z)$$

First examples: commutative vertex algebras

A vertex alg. is called commutative if $\underbrace{[Y(a,z), Y(b,w)]}_{\text{in } (\mathrm{End}\,V)[[z^{\pm 1}, w^{\pm 1}]]} = 0$ $\forall\, a, b \in V$

$\Leftrightarrow\ [a_{(m)}, b_{(n)}] = 0 \qquad \forall\, a, b \in V, \quad m, n \in \mathbb{Z}$

Assume that V is commutative

then $Y(a,z) \in (\mathrm{End}\,V)[[z]]$ ie $\forall\, b \in V$, $\underline{Y(a,z)\,b \in V[[z]]}$

Indeed: $b \in V$

$$Y(a,z)\,b \underset{\substack{\uparrow \\ \text{vacuum axiom}}}{=} Y(a,z)\,Y(b,w)\,|0\rangle\big|_{w=0} \underset{\substack{\uparrow \\ V \text{ is comm}}}{=} \underbrace{Y(b,w)\,\underline{Y(a,z)\,|0\rangle}}_{\text{no negative power in } z}\big|_{w=0}$$

$\Rightarrow\ Y(a,z)\,b \in V[[z]]$ ✓

(ie: $a_{(n)} = 0 \quad \forall\, n \geq 0$)

Conversely, if $\underline{Y(a,z) \in (\mathrm{End}\,V)[[z]]}$ for all $a \in V$

fix $b \in V$

$$\underbrace{(z-w)^N\, Y(a,z)\, Y(b,w)\,c}_{\in V[[z,w]]} = \underbrace{Y(b,w)\,Y(a,z)\,c\,(z-w)^N}_{\in V[[z,w]]} \qquad \text{(locality axiom)} \quad \forall\, c \in V$$

$\Rightarrow \quad Y(a,z)Y(b,w)c = Y(b,w)Y(a,z)c \qquad \forall\ c \in V$

hence: V is commutative.

<u>Cor</u>: V is comm $\Leftrightarrow$ $a_{(n)} = 0 \quad \forall\ n \in \mathbb{Z}_{\geq 0}$.

<u>Exercise</u>: Show that a commutative algebra R (with unit) equipped with a derivation ∂ (= a <u>differential algebra</u>) carries a commutative vertex algebra with vacuum the unit, and

$$Y(a,z)b = (e^{z\partial}a)\,b := \sum_{n\geq 0} \frac{z^n}{n!}(\partial^n a)\,b \qquad \forall\ a,b \in R$$

and $T = \partial$

<u>Rem</u>: we will see later that the converse is true

§5 – GODDARD uniqueness theorem and aplications

<u>Thm</u>: Let V be a VA and $A(z)$ a field on V

Suppose that exists $a \in V$ s.t.

$$A(z)|0\rangle = Y(a,z)|0\rangle$$

and $A(z)$ is local with any $Y(b,z)$, $b \in V$,

then $\underline{A(z) = Y(a,z)}$

<u>proof</u>: (Let $c \in V$)

By hyp and locality axioms, for $N \gg 0$, the following equalities hold in $V[[z^{\pm 1}, w^{\pm 1}]]$.

$$(z-w)^N \underline{A(z)} Y(b,w) |0\rangle = (z-w)^N Y(b,w) A(z) |0\rangle = (z-w)^N Y(b,w) Y(a,z)|0\rangle$$

$$= (z-w)^N \underline{Y(a,z)} Y(b,w) |0\rangle$$

Vacuum axioms $\overset{w=0}{\Longrightarrow}$ $z^N \underline{A(z)\, b} = z^N \underline{Y(a,z)\, b}$ $\quad \forall b \in V$ ☒

<u>Corollary</u>: V is a VA

$\forall a \in V, \quad Y(Ta,z) = \partial_z Y(a,z) \quad (= [T, Y(a,z)])$

<u>proof</u>: Since $Y(a,z)|0\rangle \in V[[z]]$, $\partial_z^n Y(a,z)|0\rangle \in V[[z]]$ $\quad \forall n \geq 0$

Moreover, $\partial_z Y(a,z)$ is local with $Y(b,z)$ $\quad \forall b \in V$ $\quad$ (exercise)

By Goddard's thm, it suffices to show that

$$\partial_z Y(a,z)|0\rangle = Y(Ta,z)|0\rangle$$

<u>ie</u>: $\partial_z^{n+1} Y(a,z)|0\rangle|_{z=0} = \partial_z^n Y(Ta,z)|0\rangle|_{z=0} \quad \forall n\geq 0$

$\partial_z Y(a,z)|0\rangle|_{z=0} \underset{\text{translation axiom}}{=} [T, Y(a,z)]|0\rangle|_{z=0} \underset{T|0\rangle=0}{=} TY(a,z)|0\rangle|_{z=0} = Ta$

By induction,

$$\partial_z^{n+1} Y(a,z)|0\rangle|_{z=0} = T^n a \qquad \forall n\geq 0 \qquad \text{(true for arbitrary } a)$$

therefore

$$\partial_z^n Y(Ta,z)|0\rangle|_{z=0} = T^n(Ta) = T^{n+1}a = \partial_z^{n+1} Y(a,z)|0\rangle|_{z=0}$$

$\square$

<u>By corollary</u>, $(Ta)_{(n)} = -n\, a_{(n-1)}$

<u>Hence</u>: $\boxed{T^n a = n!\, a_{(-n-1)}|0\rangle}$

(<u>In particular</u>, $\boxed{Ta = a_{(-2)}|0\rangle}$)

<u>Induction</u>: $T^n a = T^{n-1}(Ta) = (n-1)!\,(Ta)_{(-n)}|0\rangle = n!\, a_{(-n-1)}|0\rangle$ ✓

$$\boxed{Y(a,z)|0\rangle = \sum_{n\geq 0} \frac{1}{n!}(T^n a) z^n =: e^{zT} a}$$

§6 - n-th product of fields

Aim: to understand what is $Y(\underbrace{a_{(n)}b}_{\in V}, z)$?? $Y(a,z)_{(n)}Y(b,z)$!! ...

In particular $Y(a_{(-1)}b, z)$?? we will see that is

$:Y(a,z)\,Y(b,z):$

Df: V a vector space, $a(z), b(z)$ two fields on V, mutually local.

We define: $a(z)_{(n)}b(z) := \mathrm{Res}_{w=0}\left((w-z)^n\,[a(w), b(z)]\right)$ $\quad n \geq 0$

It is a field on V since $a(z)$ and $b(z)$ are.

Rem: The OPE of $a(z)$ and $b(z)$ is expressed as follows:

$$a(z)\,b(w) \sim \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{(z-w)^{j+1}}$$

(and $a(w)_{(j)}b(w) = 0 \quad \forall\, j \geq N$)

In fact $a(z)_{(n)}b(z)$ makes sense for $n \in \mathbb{Z}$

where $\mathrm{Res}_{w=0}\left((w-z)^n\,[a(w), b(z)]\right)$ means

$\mathrm{Res}_{w=0}\left(\tau_{w,z}((w-z)^n)\,\underbrace{a(w)\,b(z)}_{\in V((w))((z))}c\right) - \mathrm{Res}_{w=0}\left(\tau_{z,w}((w-z)^n)\,\underbrace{b(z)\,a(w)}_{V((z))((w))}c\right)$

Explicitly, we have (exercise)

$\neq 0$ only if $l+i<0$, $l<-i$ $\qquad i\geq 0$

$$(\Phi_{(n)}) \qquad \Big[\, a(z)_{(n)} b(z) = \sum_{l\in\mathbb{Z}} \sum_{i\geq 0} (-1)^i \binom{n}{i} \Big(a_{(n-i)}\, b_{(l+i)} - (-1)^n\, b_{(n+l-i)}\, a_{(i)} \Big) z^{-l-1}$$

only positive powers in z

Def The field $a(z)_{(n)} b(z)$ is called <u>the n-th product of $a(z)$ and $b(z)$</u>

<u>Rem</u>: (1) $a(z)_{(-1)} b(z) = :a(z)\, b(z):$

(2) $$a(z)_{(-n)} \mathrm{Id}_V = \begin{cases} \frac{1}{(n-1)!} \partial_z^{(n-1)} a(z) & \text{if } n>0 \\ 0 & \text{if } n\leq 0 \end{cases}$$

[rmk $a(z)_{(-n)} \mathrm{Id}_V = \mathrm{Res}_{w=z} \left(\frac{a(w)}{(w-z)^n} \right) := \sum_{k\in\mathbb{Z}} a_{(k)} \mathrm{Res}_{w=z} \frac{w^{-k-1}}{(w-z)^n}$]

(3) $(\mathrm{Id}_V)_{(-n)} a(z) = \delta_{n,-1}\, a(z)$

⚠ The nth product is NOT associative

So $a(z)_{(m)} b(z)_{(n)} c(z) \neq a(z)_{(m)} (b(z)_{(n)} c(z))$

<u>Lemma</u> (<u>Dong's lemma</u> / Haisheng Li)

If $a(z), b(z), c(z)$ are mutually local fields on V, then $a(z)_{(n)} b(z)$ and $c(z)$ are mutually local for any $n\in\mathbb{Z}$

proof: By assumption, $\exists\, N>0$ s.t.

(1) $(z-w)^N a(z)\, b(w) = (z-w)^N b(w)\, a(z)$

(2) $(z-u)^N a(z)\, c(u) = (z-u)^N c(u)\, a(z)$

(3) $(w-u)^N b(w)\, c(u) = (w-u)^N c(u)\, b(w)$

Aim: $\exists ?\ M>0$

$$(w-u)^M a(w)_{(n)} b(w)\, c(u) = (w-u)^M c(u)\, a(w)_{(n)} b(w)$$

$a(w)_{(n)}b(w)$: $\mathrm{Res}_{z=0}(\tau_{z,w}(\dots))$; $w-u = w-z+z-u$

Enough to show: $\exists M>0$

$$(*)\quad \begin{cases} (w-u)^M \left(\tau_{z,w}\left((z-w)^n a(z) b(w)\right) - \tau_{w,z}\left((z-w)^n b(w) a(z)\right)\right) c(u) \\ \quad = (w-u)^M c(u) \left(\dots\dots\right) \end{cases}$$

I claim it is true for $M=4N$ if $N+n\geq 0$ (one can assume it is)

Indeed

$$(w-u)^{4N} = (w-u)^N \sum_{s=0}^{3N} \binom{3N}{s} (z-u)^s \underline{(w-z)^{3N-s}}$$

* If $0\leq s\leq N$, then

$$(w-z)^{3N-s}\tau_{z,w}\left((z-w)^n\right) = (-1)^{3N-s}\tau_{z,w}(z-w)^{3N-s+n}$$

thus: $(3N-s+n\geq N)$

$$(w-z)^{3N-s}\left(\tau_{z,w}\left((z-w)^n a(z) b(w)\right) - \tau_{w,z}(\dots)\right) = 0 \quad \text{by (1)}$$

So the left-hand-side of $(*)$ is equal to:

$$\sum_{s=N+1}^{3N} (w-u)^N (z-u)^s (w-z)^{3N-s} (\tau_{z,w}\dots)\, c(u)$$

Similarly, the right-hand-side of $(*)$ is equal to:

$$\sum_{s=N+1}^{3N} (w-u)^N (z-u)^s (w-z)^{3N-s} c(u)\, \tau_{z,w}(\dots)$$

But these are equal by (2) and (3) $\square$

Exercise: let $a(z)$, $b(z)$ two mutually local fields on V

then $\partial_z (a(z)_{(n)} b(z)) = \partial_z a(z)_{(n)} b(z) + a(z)_{(n)} \partial_z b(z)$

(hint: $\mathrm{Res}_{w=0} (\partial_w (\ \ )) = 0$

Prop: $a(z), b(z), c(z) \in \mathcal{F}(V)$ mutually local, then

$$a(z)_{(m)} b(z)_{(n)} c(z) - b(z)_{(n)} a(z)_{(m)} c(z) = \sum_{j \geq 0} \binom{m}{j} \left(a(z)_{(j)} b(z) \right)_{(m+n-j)} c(z)$$

(compare with $[a_{(m)}, b_{(n)}] = \dots$)

Proof: (admitted for the moment: similar technics)

We are in a position to prove

Lemma: Let V be a vertex algebra, $a, b \in V$, $n \in \mathbb{Z}$. then

$$Y(a_{(n)} b, z) = Y(a, z)_{(n)} Y(b, z)$$

In particular $Y(a_{(-1)} b, z) = \, :Y(a,z) Y(b,z):$

Proof: By Dong's lemma, $Y(a,z)_{(n)} Y(b,z)$ is local to all $Y(v,z)$ $\forall v \in V$

Hence by Goddard's uniqueness theorem, it is sufficient to prove:

$$\underbrace{Y(a_{(n)} b, z)}_{\in V[[z]]} |0\rangle = \underbrace{Y(a,z)_{(n)} Y(b,z)}_{\in V[[z]] \text{ by } (\clubsuit_n)} |0\rangle$$

(Rem: if $A(z) \in \mathcal{F}(V)$ st $A(z)|0\rangle \in V[[z]]$, then

$$A(z)|0\rangle = \sum_{k \geq 0} z^k \left(\frac{1}{k!} \partial_z^k A(z) |0\rangle \right)\Big|_{z=0} \ )$$

We have

$$Y(a,z)_{(n)} Y(b,z)|0\rangle \,\Big|_{z=0} \overset{(\star_n)}{=} a_{(n)} b_{(-1)} |0\rangle = a_{(n)} b$$

Also:

$$T a_{(n)} b = [T, a_{(n)}] b + a_{(n)} T b = -n\, a_{(n-1)} b + a_{(n)} T b$$

and

$$\partial_z \left(Y(a,z)_{(n)} Y(b,z) \right) = + \Big(\underbrace{\partial_z Y(a,z)}_{-n Y(a,z)_{(n-1)}}\Big)_{(n)} Y(b,z) + Y(a,z)_{(n)} \partial_z Y(b,z)$$

By induction:

$$Y(\underbrace{T^k a_{(n)} b}_{\partial_z^k a_{(n)} b}, z)|0\rangle \,\Big|_{z=0} = \partial_z^k\, Y(a,z)_{(n)}\, Y(b,z)|0\rangle \,\Big|_{z=0}$$

$\square$

Thm: (Borcherds identities)

- $*$ $[a_{(m)}, b_{(n)}] = \sum_{j\geq 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}$ (OK OPE + $Y(a,z)_{(j)} Y(b,z) = Y(a_{(j)} b, z)$)
- $*$ $(a_{(m)} b)_{(n)} = \sum_{j\geq 0} (-1)^j \left(\binom{m}{j} a_{(m-j)} b_{(n+j)} - (-1)^m b_{(m+n-j)} a_{(j)} \right)$ $m, n \in \mathbb{Z}$

Exercise: V $\mathbb{C}$-vector space

(1) V VA, $a \in V$. Show that $Ta = a_{(-2)}|0\rangle$

(2) Conversely, verify that if a vector space V is equipped with $|0\rangle \in V$ and a linear map $V \to F(V)$ satisfying vacuum and locality axioms

then: $V \to V$, $a \mapsto a_{(-2)}|0\rangle$ verifies the translation axiom.

Back to commutative vertex algebras

Let V be a comm. vertex algebra

Borcherds identities with $m=n=-1$

$$(a_{(-1)}b)_{(-1)} = a_{(-1)}b_{(-1)}$$

ie: A comm. VA has the structure of a unital comm alg with product:

$$a \cdot b = a_{(-1)}b \qquad (:a(z)b(z): = (a_{(-1)}b)(z))$$

unit $|0\rangle$

Moreover, T acts as a derivation on it:

$$T(a.b) = (Ta).b + a.(Tb).$$

10/02/2021

Commutative vertex algebras

Let V be a comm. vertex algebra.

$$(a_{(-1)}b)_{(-1)} = a_{(-1)}b_{(-1)} \quad \text{(second Borcherds identity)}$$

$$a \cdot b \overset{\text{def}}{=} a_{(-1)} b \qquad :a(z)\,b(z): \;=\; (a_{(-1)}b)(z)$$

unit: $|0\rangle$

the translation operator T acts as a derivation on this product:

$$T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb) = \big(\underset{\overset{\shortparallel}{Ta}}{a_{(-2)}|0\rangle}\big)_{(-1)} b + a_{(-1)}\big(b_{(-2)}|0\rangle\big)$$

Indeed:

$$T(a_{(-1)}b) = (\underset{m}{a_{(-1)}} b)_{\underset{n}{(-2)}}|0\rangle \underset{\text{Borcherds (2)}}{=} \sum_{j \geq 0} (-1)^j \binom{-1}{j} a_{(-1-j)}\, b_{\underset{n}{(-2+j)}}|0\rangle + 0 \qquad -2+j<0$$

$$= \underbrace{a_{(-1)} b_{(-2)}}_{a \cdot (Tb)}|0\rangle + \underbrace{a_{(-2)} \overbrace{b_{(-1)}|0\rangle}^{b}}_{=(a_{(-2)}|0\rangle)_{(-1)} b}$$

Therefore: a comm VA has the structure of a <u>diff alg</u> with unit.

Conversely, we have seen (exercise) that a diff alg (R, ∂) has the structure of a comm VA.

Thm (Borcherds)

The category of comm. VA is the same as the category of diff algebras

Example :

Let $X = \operatorname{Spec} R$ be an affine scheme.

We will define the arc space $J_\infty X$ of X, this an affine scheme of infinite type

Roughly : what it is ??

For example, assume that $X = \{x^2 + yz = 0\} = \operatorname{Spec}\left(\mathbb{C}[x,y,z]/(x^2+yz)\right) \subset \mathbb{A}^3$

$J_\infty X$ is defined by equations :

$$x(t)^2 + y(t)\, z(t) = 0$$

$$x(t) = x_0 + x_1 t + \dots = \sum_{i \geq 0} x_i t^i$$

$$y(t) = \sum_{i \geq 0} y_i t^i \qquad z(t) = \sum_{i \geq 0} z_i t^i$$

You get :
$$\begin{cases} x_0^2 + y_0 z_0 = 0 & \text{"}t=0\text{"} \\ 2x_0 x_1 + y_0 z_1 + y_1 z_0 = 0 &] \\ \dots \end{cases}$$

$\mathbb{C}[J_\infty X]$: Here is a derivation on this alg. " ∂ : $x_i \longmapsto i x_{i+1}$

$\quad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y_i \longmapsto i\, y_{i+1}$

$\left(\mathbb{C}[x_i, y_i, z_i, i \geq 0]\right) / (x_0^2 + y_0 z_0 = 0, \dots) \qquad z_i \longmapsto i\, z_{i+1}$ "

$(\mathbb{C}[J_\infty X], \partial)$ a comm. VA.

$\leadsto$ see chap 3

Motivations

We will see later that any vertex algebra V is naturally filtered.

Moreover, $\underline{\mathrm{gr}\, V}$ is a comm. VA

$$\begin{tikzcd} V \arrow[r, mapsto] & \mathrm{gr}\, V \arrow[r, leftrightarrow, "\text{link}"] & \mathbb{C}[J_\infty X_V] =: \underline{\underline{J_\infty R_V}} \\ VA \arrow[r] \arrow[dr] & \text{comm. VA} & \\ & \text{Com alg.} & R_V \arrow[u, mapsto] \end{tikzcd}$$

On the other hand: to any VA one can attach a comm. alg:

$$R_V := V / \mathrm{Span}\langle a_{(-2)} b : a, b \in V \rangle$$ is a comm alg ("Poisson")

$\underline{X_V = \mathrm{Spec}\, R_V}$

Commutant and center

Let V be a VA (not necessary commutative)

$W \subset V$ a subspace is called a vertex subalgebra if $|0\rangle \in W$, $TW \subset W$ and $a_{(n)} b \in W \quad \forall a, b \in W, n \in \mathbb{Z}$.

If $W \subset V$ is a vertex subalg, we set

$$\mathrm{Com}(W, V) = \{ v \in V : [w_{(n)}, v_{(m)}] = 0 \text{ for all } w \in W, m, n \in \mathbb{Z} \}$$
$$= \{ v \in V : w_{(n)} v = 0 \quad \forall w \in W, n \in \mathbb{Z}_{\geq 0} \}$$

see the argument for comm. vertex algebras.

another way to see is to use Borcherds identities.

It is easy to see that $\mathrm{Com}(W, V)$ is a vertex subalgebra of V, called the commutant of W in V.

Rem: $\mathrm{Com}(W, V) = \{ v \in V : v_{(n)} w = 0 \quad \forall w \in W, n \geq 0 \}$

Def: The center of V is $\mathrm{Com}(V, V) =: Z(V)$.

§8 – Vertex algebras of local fields and reconstruction theorem

Thm (4.i) Let M be a vector space, and let $\mathcal{V}$ be a subspace of $\mathcal{F}(M)$ s.t. :

(i) $a(z)$ and $b(z)$ are mutually local for all $a(z)$ and $b(z) \in \mathcal{V}$,

(ii) $\mathrm{Id}_M \in \mathcal{V}$

(iii) $a(z)_{(n)} b(z) \in \mathcal{V}$ for all $a(z), b(z) \in \mathcal{V}$, $n \in \mathbb{Z}$.

then $\mathcal{V}$ has a VA structure with vacuum Id_M, $T = \partial_z$,

and $$Y(a(z), \zeta) = \sum_{n\in\mathbb{Z}} \underbrace{a(z)_{(n)}} \zeta^{-n-1},$$

where $a(z)_{(n)}$ acts on $\mathcal{V}$ by $a(z)_{(n)} : \mathcal{V} \longrightarrow \mathcal{V}$

$$b(z) \longmapsto a(z)_{(n)} b(z)$$

proof: Recall :

$$a(z)_{(-n)} \mathrm{Id}_M = \begin{cases} \frac{1}{(n-1)!} \partial_z^{n-1} a(z) & \text{if } n>0 \\ 0 \text{ if } n<0 \end{cases} \qquad (1)$$

$$(\mathrm{Id}_M)_{(n)} a(z) = \delta_{n,-1} a(z) \qquad (2)$$

. By (1) and (2), vacuum axioms are satisfied.

. By (1) with $n=2$, $a(z)_{(-2)} \mathrm{Id}_M = \partial_z a(z)$

$\mathcal{V}$ is stable by ∂_z

$$[\partial_z, Y(a(z), \zeta)] = \sum_n \partial_z a(z)_{(n)} \zeta^{-n-1} - \sum_n a(z)_{(n)} \partial_z \zeta^{-n-1}$$

$$\underset{\partial_\zeta Y(a(z), \zeta)}{\overset{\shortparallel\, ?}{}} \qquad = \sum_n (\partial_z a(z))_{(n)} \zeta^{-n-1} \qquad = Y(\partial_z a(z), \zeta)$$

$$\partial_z (a(z)_{(n)} b(z)) = (\partial_z a(z))_{(n)} b(z) + a(z)_{(n)} \partial_z b(z)$$

Since $\mathrm{Res}_{w=\infty}\left(\partial_w\left((w-z)^n[a(w), b(z)]\right)\right) = 0$, we get:

$$\underbrace{\mathrm{Res}_{w=\infty}\left(n(w-z)^{n-1}[a(w), b(z)]\right)}_{n\ a(w)_{(n-1)}} + \underbrace{\mathrm{Res}_{w=\infty}\left((w-z)^n[\partial_w a(w), b(z)]\right)}_{(\partial_z a(z))_{(n)}} = 0$$

Hence: $Y(\partial_z a(z), z) = \partial_z Y(a(z), z)$

. locality axiom holds by

$$\underbrace{a(z)_{(m)} b(z)_{(n)} a(z) - b(z)_{(n)} a(z)_{(m)} c(z)}_{[a(z)_{(m)},\ b(z)_{(n)}]\ c(z)} = \sum_{j\geq 0}\binom{m}{j}\left(a(z)_{(j)} b(z)\right)_{(m+n-j)}$$

+ OPE's proposition. $\boxed{\cdot}$

Let $\mathcal{S}$ be set of pairwise mutually local fields on a vector space M.

$\langle \mathcal{S}\rangle_M =$ subspace of $\mathcal{F}(M)$ spanned by the fields constructed by successive application of the n-th product, as well as Id_M.

By the theorem + Dong's lemma $\Rightarrow$ $\langle \mathcal{S}\rangle_M$ has a VA structure, called the <u>vertex algebra of the local fields generated by $\mathcal{S}$</u>

<u>Lemma</u> (State field correspondence).

Let V be a vertex algebra, $\mathcal{S} = \{Y(a,z) : a \in V\} \subset \mathcal{F}(V)$.

Then the linear map : $V \longrightarrow \langle \mathcal{S}\rangle_V$, $a \longmapsto Y(a,z)$

is an isomorphism of vertex algebras.

proof: It is a vertex algebra homomorphism by

$$Y(a,z)_{(n)}Y(b,z) = Y(a_{(n)}b, z)$$

$$+\ Y(|0\rangle, z) = \mathrm{Id}_V \quad + \quad \underline{\partial_z Y(a,z) = Y(Ta, z)}$$ (sque of translation operator axioms).

Isomorphism, whose inverse map: $Y(a,z) \longmapsto Y(a,z)|0\rangle|_{z=0}$ ☒

thm (Reconstruction theorem)

Let V be a vector space, $|0\rangle$ a nonzero vector of V, and $T \in \mathrm{End}(V)$.

Let $\{a^i\}_{i\in I}$, I a set, a collection of vectors in V.

Suppose also that we have given fields:

$$a^i(z) = \sum_{i\in\mathbb{Z}} a^i_{(n)} z^{-n-1} \in \mathcal{F}(V) \qquad i \in I$$

s.t:

(1) $\forall i \in I$, $a^i(z)|0\rangle = a^i + zV[[z]]$

(2) $T|0\rangle = 0$, and $[T, a^i(z)] = \partial_z a^i(z)$

(3) all fields $a^i(z)$ are pairwise mutually local.

(4) V is spanned by the vectors

$$a^{i_1}_{(n_1)} \cdots a^{i_r}_{(n_r)}|0\rangle \qquad \text{for } n_j < 0$$

then there exists a unique vertex algebra structure on V s.t. $Y(a^i, z) = a^i(z)$ $\forall i \in I$

and the vacuum is $|0\rangle$, and $T = T$.

proof (later: see the draft book)

<u>Rem</u>: the vertex operator is defined by

$$Y(a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)} |0\rangle, z) \qquad n_j \geq 0$$
$$= \frac{1}{n_1! n_2! \cdots n_r!} :(\partial_z^{n_1} a^{i_1}(z))(\partial_z^{n_2} a^{i_2}(z)) \ldots (\partial_z^{n_r} a^{i_r}(z)):$$

<u>Indeed</u>:

- $Y(a^i_{(-1)}|0\rangle, z) = Y(a^i, z) = a^i(z)$

$Y(a^i_{(-n-1)} |0\rangle, z) = \frac{1}{n!} \partial_z^n a^i(z)$

$Y(\underset{\underset{a_{(-2)}|0\rangle}{\shortparallel}}{Ta^i}, z) = \partial_z a^i(z)$ + induction $Y(\underset{\underset{n!\, a_{(-n-1)}|0\rangle}{\shortparallel}}{T^n a^i}, z) = \partial_z^n a(z)$

- $Y(a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} |0\rangle, z) = :Y(a^{i_1}_{(-n_1-1)}|0\rangle, z)\, Y(a^{i_2}_{(-n_2-1)}|0\rangle, z):$

① $Y(\underset{\shortparallel}{a_{(-n-1)} b_{(-1)} |0\rangle}, z)$

$Y(a_{(-n-1)} b, z) = Y((a_{(-n-1)}|0\rangle)_{(-1)} b, z) \underset{\uparrow}{\underset{\text{Borcherds}}{=}} Y(a_{(-n-1)} b_{(-1)} |0\rangle, z)$ ✓

$Y(a_{(-n-1)} b_{(-2)} |0\rangle, z) = Y(a_{(-n-1)}(Tb), z) \underset{\underset{①}{\uparrow}}{=} :Y(a_{(-n-1)}|0\rangle, z)\, Y(\underset{\underset{b_{(-2)}|0\rangle}{\shortparallel}}{Tb}, z):$

+ induction . . .

One example

Take $\pi = \mathbb{C}[b_{-1}, b_{-2}, \dots]$

π is a smooth $\mathcal{B}$-module on which b_n, $n \geq 0$, acts as $n\frac{\partial}{\partial b_{-n}}$

$\mathcal{B}$ = unital $\mathbb{C}$-alg gen by $b_n, n \in \mathbb{Z}$ with relation $[b_m, b_n] = m\,\delta_{m,-n}$

and b_{-n}, $n > 0$, acts as multiplication by b_{-n}.

"smooth": $b_n \cdot f = 0$, $n \gg 0$.

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1} \in \mathcal{F}(\pi)$$

Define: $T = \sum_{n>0} n\, b_{-n-1} \frac{\partial}{\partial b_{-n}} \in End(\pi)$

Then (exercise) $[T, b(z)] = \partial_z b(z)$ on π

Recall: $b(z)\,b(w) \sim \frac{1}{(z-w)^2} \Rightarrow b(z)$ is local with itself.

By reconstruction thm, there exists a unique vertex algebra structure on π s.t. $|0\rangle = 1$

$T = T$, $Y(b_{-1}, z) = b(z)$.
$b_{-1} = b_{-1}\cdot 1$

Exercise: Let M be a $\mathcal{B}$-module

(i) Show that the following correspondence gives the VA $\langle b(z)\rangle_M$ a $\mathcal{B}$-module structure:

$$\mathcal{B} \longrightarrow End(\langle b(z)\rangle_M), \quad b_n \longmapsto b(z)_{(n)}$$

(ii) Show that there is surjective homomorphism of VA $\pi \longrightarrow \langle b(z)\rangle_M$.

Exercise

(i) Let $\omega = \frac{1}{2} b_{-1}^2 + \alpha b_{-2} \in \pi$, $\alpha \in \mathbb{C}$

(then $Y(\omega, z) = \frac{1}{2} :b(z)^2: + \alpha\, \partial_z b(z) =: L(z)$)

Verify that we have the following OPE:

$$L(z)\,L(w) \sim \frac{(1-12\alpha^2)/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}$$

and verify that the above OPE is equivalent to:

$$b_0\,\omega = 0, \quad b_1\,\omega = b_{-1}, \quad b_{-2}\,\omega = 2\alpha$$

(ii) Show that $L_{-1} = T$ on π.

§12. Vertex algebras modules and quotient of vertex algebras

V a VA.

Def A representation M of the vertex algebra V, or a <u>V-module</u>, is a vertex algebra homomorphism from V to a vertex algebra of local fields on M.

We denote by $Y_M(a,z) = a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$ the image of $a\in V$ in $\mathcal{F}(M)$

Lemma (clear)

M is a V-module iff: there exists a linear map $V \longrightarrow \mathcal{F}(M)$, $a \longmapsto Y_M(a,z)$ s.t.

- $Y_M(|0\rangle, z) = \mathrm{Id}_M$
- $[Y_M(a,z), Y_M(b,w)] = \sum_{j\geq 0} Y_M(a_{(j)}b, w) \frac{1}{j!}\partial_w^j \delta(z-w)$
- $Y_M(a_{(n)}b, z) = Y_M(a,z)_{(n)} Y(b,z)$ $\qquad a,b\in V,\ n\in\mathbb{Z}$

Rem: a VA is a module over itself, called the <u>adjoint representation</u>

- By definition, a subspace N of a V-module M is a <u>submodule</u> of M if $a^M_{(n)} N \subset N$ for all $a\in V$, $n\in\mathbb{Z}$
- A <u>T-stable proper submodule</u> of the adjoint representation V is an <u>(vertex) ideal</u> of V.

If $f: V \longrightarrow V'$ a vertex algebra homomorphism, then $\operatorname{Ker} f$ is an ideal of V.

Claim: if I is an ideal of V, then V/I inherit a VA structure from that of V.

[<u>proof</u>: use Reconstruction thm since V/I is generated by the image of $a_{(-1)}|0\rangle$, $a\in V$]

Exercise: Show that the vertex algebra π is simple, that is, there is no non-trivial ideal of π.

(this implies that the VA $\langle b(z) \rangle_{M}$ of local fields of any non trivial smooth $\mathcal{B}$-smooth M is isomorphic to π.)