In search of a quantum unitary Brownian motion

A. Freslon †

These are notes based on supports for various talks given by the author on the works [FFS22] and [FFS24]. They are intended for a general audience with background in operator algebras.

1 Why Brownian motion ?

Our purpose in the sequel is to study a specific family of random processes on free unitary quantum groups called *Gaussian processes*. But why should we bother about random processes in the first place ? One motivation is the search for a geometrical theory of compact quantum groups. What we mean by this is that as we will see, free unitary quantum groups are defined as analogues of the compact Lie groups U_N, so that they should have an underlying differential geometric structure, hopefully expressible in the setting of non-commutative geometry (by which we mean spectral triples or other objects in the spirit of [Con94]). Alas, no such structure has appeared so far in a natural way, mainly due to the lack of a quantum analogue of the corresponding Lie theory (see for instance [Wan97]). Therefore, we will rather start by looking for "shadows" of that geometric structure expressed in a language which we already know how to translate to the quantum setting. Probability theory has a non-commutative counterpart which has been well-developped (see for instance [Par92]) – under the name of *quantum probability* – due to its fundamental connection with the mathematical foundations of quantum mechanics. This makes it a good starting points.

1.1 From geometry to probability

We want to find some trace of the geometry of a compact Lie group in probability theory. There is a way of doing this, which is the theory of *Lévy processes*. We will now explain how these are related, but first we need to clarify what we are talking about.

Definition 1.1. Let G be a compact group. A *Lévy process* on G is a family of G-valued random variables $(X_t)_{t\in\mathbf{R}_+}$ such that

i) $X_{t_2}X_{t_1}^{-1},\cdots,X_{t_n}X_{t_{n-1}}^{-1}$ are independent for all $n\in\mathbf{N}$ and $0\leqslant t_1<\cdots<t_n$;

ii) $\mathrm{Law}\left(X_tX_s^{-1}\right)=\mathrm{Law}(X_{t-s})$ for all $t\geqslant s\geqslant 0$;

iii) $X_t\to X_0$ when $t\to 0$ in probability[1].

We see from the properties above that setting $\mu_t=\mathrm{Law}(X_t)$, all the probabilistic information about the process is contained in the family of measures $(\mu_t)_{t\in\mathbf{R}_+}$. Moreover, the compatibility between the group structure and the Lévy process translates into the following equality at the level of measures:

Lemma 1.2. *Let $t,s\in\mathbf{R}_+$. Then, for any Borel subset A of G,*

$$(\mu_t\otimes\mu_s)\left(\left\{(g,h)\in G\times G\mid gh\in A\right\}\right)=\mu_{t+s}(A).$$

†. A. Freslon, Université Paris-Saclay, CNRS UMR 8628, Laboratoire de Mathématique d'Orsay, 91405 Orsay CEDEX, France.
e-mail : amaury.freslon@universite-paris-saclay.fr
url : https://www.imo.universite-paris-saclay.fr/~freslon/

Proof. The left-hand side is the probability that $X_tX_s \in A$, while the right-hand side is the probability that $X_{t+s} \in A$. Multiplying by X_s^{-1} on the right, we can use the properties of a Lévy process to compute

$$\begin{aligned}\mathbb{P}(X_tX_s \in A) &= \mathbb{P}(X_t \in AX_s^{-1}) \\ &= \mathbb{P}(X_{t+s}X_s^{-1} \in AX_s^{-1}) \\ &= \mathbb{P}(X_{t+s} \in A).\end{aligned}$$

■

The left-hand side defines the *convolution* $\mu_t * \mu_s$ of the two measures, so that we conclude that we have a *convolution semigroup* of probability measure :

$$\mu_t * \mu_s = \mu_{t+s}.$$

The natural question is of course whether one can construct such a process out of the structure of G. This is not clear in general, but can be done assuming that G has some differential structure. To see how, we need to connect the associated convolution semigroup with harmonic analysis on the group. To make things simpler, we will work from now on with $G = SU_N$, which is a closed subgroup of $GL_N(\mathbf{C})$, hence a *Lie group*[2]. The core idea is the following fact : for any $f \in \mathscr{C}^2(SU_N)$, the quantity

$$\frac{1}{t}\left(\int f(g)\mathrm{d}\mu_t - f(\mathrm{I}_N)\right)$$

converges as $t \to 0$ to a limit denoted by $L(f)$ (we refer the reader to [Lia04] for a complete treatement of the subject). This defines a linear map $L : \mathscr{C}^2(SU_N) \to \mathbf{C}$ – called the *infinitesimal generator* of $(\mu_t)_{t\in\mathbf{R}_+}$ – which satisfies the following properties :

- $L(1) = 0$ (*normalization*) ;
- $L(\overline{f}) = \overline{L(f)}$ (*hermitianity*) ;
- $L(f) \geqslant 0$ if $f \geqslant 0$ and $f(\mathrm{I}_N) = 0$ (*conditional positivity*).

We will now give a name to such functionals, but it will be better to define them on a suitable subalgebra[3] of $\mathscr{C}^2(SU_N)$. More precisely, since SU_N is a group of matrices, we can consider the $*$-algebra $\mathscr{O}(SU_N)$ of functions on SU_N which are polynomial in the coefficients. Note that $\mathscr{O}(SU_N) \subset \mathscr{C}^2(SU_N)$.

DEFINITION 1.3. A linear map on $\mathscr{O}(SU_N)$ satisfying the properties above is called a *generating functional* on SU_N.

As it turns out, this is enough to recover the convolution semigroup of probability measures. Indeed, given two linear functionals L_1, L_2 on $\mathscr{O}(SU_N)$, $L_1 \otimes L_2$ is defined[4] on

$$\mathscr{O}(SU_N \times SU_N) = \mathscr{O}(SU_N) \otimes \mathscr{O}(SU_N)$$

so that one may define their convolution product as

$$(L_1 * L_2)(f) = (L_1 \otimes L_2)((g,h) \mapsto f(gh)).$$

It is easily checked that this defines an associative product, so that the formula (with the convention $L^{*0} = \mathrm{ev}_{\mathrm{I}_N}$)

$$\sum_{k=0}^{+\infty} \frac{t^k}{k!} L^{*k}(f)$$

unambiguously defines a linear map $\varphi_t : \mathscr{O}(SU_N) \to \mathbf{C}$. Moreover, we have the following straightforward properties[5] :

- $\varphi_t(\mathrm{I}_N) = 1$;
- $\varphi_t(f) \geqslant 0$ if $f \geqslant 0$;

- $\varphi_t * \varphi_s = \varphi_{t+s}$.

In other words, we have a convolution semigroup of *states* on $\mathcal{O}(G)$. Now, positivity implies that φ_t extends to the algebra $C(G)$ of all continuous functions on G, and the RIESZ REPRESENTATION THEOREM therefore implies that there exist a probability measure μ_t such that for all $f \in C(G)$,

$$\varphi_t(f) = \int f(g)\mathrm{d}\mu_t(g).$$

In other words, we have built a convolution semigroup of probability measures !

The conclusion of that story is that all the information is encapsulated in the generating functional, so that we can forget everything else. We are now ready to produce a Lévy process out of the geometry of G. Indeed, the theory of continuous semi-groups of operators also provides us with generating functionals, and on a Lie group we can build such semi-groups as heat kernels of differential operators. Here is a rough outline of the procedure for SU_N :

1. Since SU_N has a manifold structure, take its tangent space at the identity $T_{\mathrm{I}_N}SU_N$;
2. We know that $T_{\mathrm{I}_N}(SU_N) = \mathfrak{su}_N$ has a Lie algebra structure, hence in particular a distinguished bilinear form called the *Killing form* ;
3. Because SU_N is semisimple, the Killing form is negative definite, hence its oppposite defines an inner product on $\mathfrak{su}_N$;
4. Because a Lie group is parallelisable, this extends to a Riemannian structure on SU_N ;
5. With a Riemannian structure comes a Laplace-Beltrami operator Δ defined on $\mathcal{O}(SU_N)$ (and even on $\mathscr{C}^2(SU_N)$) ;
6. For $f \in \mathcal{O}(SU_N)$, set $L_B(f) = \Delta(f)_{|\mathrm{I}_N}$.

The subscript B in the definition is meant to connect this with the probabilistic interpretation of the associated stochastic process on SU_N : this is the *Brownian motion* on SU_N ! To make this more convincing, let us make some computation on the real line.

Proposition 1.4. *Let $(\mu_t)_{t\in\mathbf{R}_+}$ be the convolution semi-group of probability measure on $\mathbf{R}$ corresponding to the Brownian motion. Then, for any function $f \in \mathscr{C}^2(\mathbf{R})$ which goes to 0 at infinity*[6],

$$\frac{1}{t}\left(\int_{\mathbf{R}} f(x)\mathrm{d}\mu_t(x) - f(0)\right) \underset{t\to 0}{\longrightarrow} \frac{f''(0)}{.}$$

Proof. By definition, μ_t is Gaussian with mean 0 and variance t. Then, if $f \in \mathscr{C}^2(\mathbf{R})$, we can write it as

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + g(x).$$

Remembering that the first moment of μ_t vanishes and that the second one is t, we have

$$\begin{aligned}
\frac{1}{t}\left(\int_{\mathbf{R}} f(x)\mathrm{d}\mu_t(x) - f(0)\right) &= \frac{1}{t}\left(\int_{\mathbf{R}} f(x)\mathrm{d}\mu_t(x) - \int_{\mathbf{R}} f(0)\mathrm{d}\mu_t(x)\right) \\
&= \frac{1}{t}\int_{\mathbf{R}} (f(x) - f(0))\mathrm{d}\mu_t(x) \\
&= \frac{f'(0)}{t}\int_{\mathbf{R}} x\mathrm{d}\mu_t(x) + \frac{f''(0)}{2t}\int_{\mathbf{R}} x^2\mathrm{d}\mu_t(x) + \frac{1}{t}\int_{\mathbf{R}} g(x)\mathrm{d}\mu_t(x) \\
&= \frac{f''(0)}{2} + \int_{\mathbf{R}} g(x)\frac{\mathrm{d}\mu_t}{t}(x).
\end{aligned}$$

Because the derivative of the density of μ_t is just the same function divided by $-t$, we can integrate by parts :

$$\begin{aligned}\frac{1}{t}\left(\int_{\mathbf{R}} f(x)\mathrm{d}\mu_t(x) - f(0)\right) &= \frac{f''(0)}{2} + \left[-g(x)\frac{e^{-x^2/2t}}{\sqrt{2\pi t}}\right]_{-\infty}^{+\infty} + \int_{\mathbf{R}} g'(x)\mathrm{d}\mu_t(x)\\ &= \int_{\mathbf{R}} g'(x)\mathrm{d}\mu_t(x),\end{aligned}$$

where we used the fact that since f goes to 0 at infinity, $g(x) = O(x^2)$ at infinity. The proof is now concluded using the weak convergence of the Gaussian measures to δ_0. ■

1.2 To the quantum world

Now that we know how to build random processes out of the geometric structure of a compact Lie group, we want to define analogous probabilistic objects on compact quantum groups. But instead of introducing the general theory, we will simply work directly with free unitary quantum groups since these are the objects that we will focus on.

As explained above, the important objects can all be defined on the algebra of polynomial functions, so that we will only define this. We refer the reader to [Fre23] for a detailed treatment of the theory of compact quantum groups in terms of such $*$-algebras.

DEFINITION 1.5. For $N \in \mathbf{N}$, let $\mathcal{O}(U_N^+)$ be the universal $*$-algebra generated by N^2 elements $(u_{ij})_{1\leqslant i,j\leqslant N}$ such that for all $1 \leqslant i,j \leqslant N$,

$$\sum_{k=1}^{N} u_{ik}^* u_{jk} = \delta_{ij} = \sum_{k=1}^{N} u_{ki}^* u_{kj} \quad \& \quad \sum_{k=1}^{N} u_{ik} u_{jk}^* = \delta_{ij} = \sum_{k=1}^{N} u_{ki} u_{kj}^*.$$

As we see, this is the largest algebra of polynomial functions in the coefficients of unitary matrices, except that they are not even require to commute with one another. One concrete way of seeing this is that the abelianization map yields a surjective $*$-homomorphism

$$\pi_{U_N} : \mathcal{O}(U_N^+) \to \mathcal{O}(U_N)$$

sending u_{ij} to the function c_{ij} which associates to any matrix in U_N its (i,j)-th coefficient.

This is of course not enough to generalize the previous setting, but we saw that the crucial feature of polynomial functions on Lie groups was that there is a convolution product. To define a convolution product on $\mathcal{O}(U_N^+)$, we need an analogue of the group law, which for matrices is easily expressed.

Lemma 1.6. *There is a unique $*$-homomorphism $\Delta : \mathcal{O}(U_N^+) \to \mathcal{O}(U_N^+) \otimes \mathcal{O}(U_N^+)$ such that for all $1 \leqslant i,j \leqslant N$,*

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

Proof. Simply set

$$v_{ij} = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$$

and apply the universal property. ■

Observe that through the isomorphism $\mathcal{O}(U_N) \otimes \mathcal{O}(U_N) = \mathcal{O}(U_N \times U_N)$, we have for any $1 \leqslant i,j \leqslant N$ that

$$c_{ij}(gh) = \sum_{k=1}^{N} c_{ik}(g)c_{kj}(h) = \sum_{k=1}^{N} (c_{ik} \otimes c_{kj})(g,h)$$

so that the formula above generalizes that of the matrix product in SU_N.

With this in hand, we can define a convolution product for linear functionals L_1, L_2 on $\mathcal{O}(U_N^+)$ through the formula

$$L_1 * L_2 = (L_1 \otimes L_2) \circ \Delta,$$

and therefore set for $t \in \mathbf{R}_+$

$$\varphi_t = \sum_{k=0}^{+\infty} \frac{t^k}{k!} L^{*k}.$$

To prove that this yields states, we first need a notion of conditional positivity. But this was defined by refering to functions vanishing at the identity. To generalize this to U_N^+, we therefore need an analogue of the evaluation $\mathrm{ev}_{\mathrm{I}_N}$ at the neutral element.

DEFINITION 1.7. The *counit* of U_N^+ is the unique $*$-homomorphism $\varepsilon : \mathcal{O}(U_N^+) \to \mathbf{C}$ such that for all $1 \leqslant i, j \leqslant N$,

$$\varepsilon(u_{ij}) = \delta_{ij}.$$

Remark 1.8. The uniqueness of ε is clear since we know it on all generators. As for existence, it simply follows from the fact that the matrix $(\delta_{ij})_{1\leqslant i,j\leqslant N}$ satisfies the generating relations of $\mathcal{O}(U_N^+)$, together with the universal property.

With this in hand, we can give a natural definition of generating functional.

DEFINITION 1.9. A *generating functional* on U_N^+ is a linear map $L : \mathcal{O}(U_N^+) \to \mathbf{C}$ such that

- $L(1) = 0$ (normalization) ;
- $L(x^*) = \overline{L(x)}$ (hermitianity) ;
- $L(x^*x) \geqslant 0$ for all $x \in \ker(\varepsilon)$ (conditional positivity).

It is then clear that φ_t is a state for all $t \in \mathbf{R}_+$, and that we have a convolution semi-group :

$$\varphi_t * \varphi_s = \varphi_{t+s}.$$

In the non-commutative philosophy, states are analogues of integration with respect to a probability measure. We are therefore back to convolution semi-groups of measures !

As a conclusion, all we have to do is to find suitable generating functionals on $\mathcal{O}(U_N^+)$. But where to start ? A fundamental idea, due tu Schürmann, is that the functional $L_B = \Delta_{|\mathrm{I}_N}$ has the property that it vanishes as soon as f has a zero of order at least 3 at I_N. But such functions – or at least the polynomial ones – are easy to characterize algebraically : they are the products of three polynomials vanishing at I_N ! To express this more asbtractly, let us set $K_1 = \ker(\varepsilon)$ and

$$\begin{aligned} K_{n+1} &= K_1.K_n \\ &= \mathrm{Span}\{xy \mid x \in K_1, y \in K_n\} \\ &= \mathrm{Span}\{x_1 \cdots x_n \mid x_1, \cdots, x_n \in K_1\}. \end{aligned}$$

Then, we want to consider generating functionals vanishing on K_3. Because of their connection to the Laplacian, and therefore to the Gaussian distribution on $\mathbf{R}$, these are called *Gaussian* generating functionals[7].

DEFINITION 1.10. A generating function L on U_N^+ is called *Gaussian* if it vanishes on K_3.

2 GAUSSIAN FUNCTIONALS ON FREE UNITARY QUANTUM GROUPS

In the remainder of this text, we will study Gaussian generating functionals on U_N^+, from two different points of view. First, we will investigate the global problem of how much information on U_N^+ the set of Gaussian functionals contains. Second, we will give a somewhat explicit description of all these functionals.

2.1 GLOBAL STUDY : THE GAUSSIAN PART

Before going further in the study of Gaussian processes on U_N^+, we should stop and ask ourselves a question : are we looking at the correct object ? The question is not about the functionals, but about the quantum group. Indeed, U_N^+ is an analogue of U_N, which is *not* semisimple. In particular, the Killing form on the Lie algebra $\mathfrak{u}_N$ is degenerate and one must restrict it to $\mathfrak{su}_N$ to get an inner product. But then, the corresponding functional L_B only depends on the restriction of functions to the subgroup $SU_N \subset U_N$. Therefore, it is natural to wonder whether we should rather be working on some quantum subgroup of U_N^+ analogous to SU_N.

The question is tricky, because there is no clear way of defining some kind of determinant at the level of $\mathcal{O}(U_N^+)$. But we can take the problem the other way round : is there a "quantum subgroup" such that any Gaussian functional only depends on the restriction to that subgroup ? This requires some definition.

DEFINITION 2.1. A *quantum subgroup* $\mathbb{G}$ of U_N^+ is a $*$-algebra $\mathcal{O}(\mathbb{G})$ together with a surjective $*$-homomorphism $\pi_{\mathbb{G}} : \mathcal{O}(U_N^+) \to \mathcal{O}(\mathbb{G})$ and a $*$-homomorphism $\Delta_{\mathbb{G}} : \mathcal{O}(\mathbb{G}) \to \mathcal{O}(\mathbb{G}) \otimes \mathcal{O}(\mathbb{G})$ satisfying

$$(\pi \otimes \pi) \circ \Delta = \Delta_{\mathbb{G}} \circ \pi.$$

The quantum subgroup $\mathbb{G}$ is said to be *strict* if π is not injective.

For instance, U_N (hence also SU_N by restriction of the functions) is a strict quantum subgroup of U_N^+ : we already defined π_{U_N} as the abelianization map, and the coproduct is given on a polynomial function $f \in \mathcal{O}(SU_N)$ by the formula

$$\Delta_{SU_N}(f) : (g,h) \mapsto f(gh),$$

through the isomorphism $\mathcal{O}(SU_N \times SU_N) \cong \mathcal{O}(SU_N) \otimes \mathcal{O}(SU_N)$. A generating function $L : \mathcal{O}(U_N^+) \to \mathbf{C}$ is said to *factor through* $\mathbb{G}$ if there exists a linear functional $L' : \mathcal{O}(\mathbb{G}) \to \mathbf{C}$ such that $L = L' \circ \pi$. With these notions, it is easy to express the question whether U_N^+ is the "correct object" to study Gaussian generating functionals :

Question. *Does there exist a strict quantum subgroup of U_N^+ through which* all *Gaussian generating functionals factor ? If not, then we will say that U_N^+ is* Gaussian.

To get a better grasp at that question, let us make a few algebraic observations. If there is a quantum subgroup $\mathbb{G}$ such that $\ker(\pi_{\mathbb{G}}) \subset K_3$, then all Gaussian functionals will factor through $\mathbb{G}$. But $I = \ker(\pi_{\mathbb{G}})$ is not any ideal of $\mathcal{O}(U_N^+)$: the definition of a quantum subgroup implies that it satisfies the following property :

$$\Delta(I) \subset I \otimes \mathcal{O}(U_N^+) + \mathcal{O}(U_N^+) \otimes I.$$

It turns out that there is an ideal in K_3 which satisfies this, namely

$$K_\infty = \bigcap_{n \geqslant 1} K_n.$$

We should therefore start by checking that this one is trivial. This first requires a little aparte. If G is a compact group of matrices, setting $\varepsilon_G = \mathrm{ev}_{\mathrm{I}_N}$ we can define ideals K_n and K_∞ inside $\mathcal{O}(G)$. Similarly, if Γ is a discrete group, setting $\varepsilon_\Gamma(g) = 1$ for all $g \in \Gamma$ we can define ideals K_n and K_∞ inside the group algebra $\mathbf{C}[\Gamma]$. We then have the following two facts :

- For a compact group of matrices G, $K_\infty = \{0\}$ if and only if G is connected ;
- For a discrete group Γ, $K_\infty = \{0\}$ if and only if Γ is residually torsion-free nilpotent.

It is tempting to term the property $K_\infty = \{0\}$ "connectedness" because of the first property. Unfortunately, there is already a notion of connectedness for compact quantum groups [Wan09], which is weaker than this one. We therefore call it *strong connectedness*. It turns out that the question of strong connectedness can be solved for U_N^+.

Proposition 2.2 (FRANZ-F.-SKALSKI). *The quantum group U_N^+ is strongly connected, i.e. we have $K_\infty = \{0\}$.*

Proof. Consider the following two quantum subgroups of U_N^+ :

- $\pi_{U_N} : \mathcal{O}(U_N^+) \to \mathcal{O}(U_N)$ is the abelianization map and

$$\Delta_{U_N}(f) : (g,h) \mapsto f(gh);$$

- $\pi_{\mathbb{F}_N} : \mathcal{O}(U_N^+) \to \mathbf{C}[\mathbb{F}_N]$ is the quotient by the relations $u_{ij} = 0$ for all $1 \leqslant i \neq j \leqslant N$ and

$$\Delta_{\mathbb{F}_N}(g) = g \otimes g$$

  for all $g \in \mathbb{F}_N$.

The proof relies on two main facts. First, because we have the inclusions (coming from the surjectivity of the $*$-homomorphisms in the definition of a quantum subgroup)

$$\pi_{U_N}(K_\infty) \subset K_\infty \quad \& \quad \pi_{\mathbb{F}_N}(K_\infty) \subset K_\infty$$

and both right-hand sides vanish by the results mentionned above, we have

$$K_\infty \subset \ker(\pi_{U_N}) \cap \ker(\pi_{\mathbb{F}_N}).$$

Second, it is known from [Chi20] that the previous intersection does not contain any non-trivial ideal I satisfying $\Delta(I) \subset I \otimes \mathcal{O}(U_N^+) + \mathcal{O}(U_N^+) \otimes I$. ■

The technique used in the previous proof is called *topological generation* : we use the fact that U_N^+ is in fact "generated" as a compact quantum group by its quantum subgroups SU_N and $\widehat{\mathbb{F}}_N$.

Remark 2.3. Let $\mathcal{O}(O_N^+)$ be the quotient of $\mathcal{O}(U_N^+)$ by the relations $u_{ij} = u_{ij}^*$ for all $1 \leqslant i,j \leqslant N$. It is easy to see that this is a quantum subgroup – called the *free orthogonal quantum group* – but to this day it is not known whether it is strongly connected.

Back now to the question of Gaussianity, we can also characterize it for classical groups and duals of discrete groups.

THEOREM 2.4 (FRANZ-F.-SKALSKI) A classical group G is Gaussian if and only if it is connected. The dual of a discrete group Γ is Gaussian if and only if $\mathbb{G}$ is torsion-free and 2-step nilpotent.

Remark 2.5. Let us briefly remark here that this shows that our original motivation for the study of the Gaussian part, involving the lack of semi-simplicity of U_N, was not a good one. Indeed, the previous theorem says that Gaussian processes (or rather their trajectories) do reach all of U_N (or at least a dense subset).

So how about the original question ? We do not have an answer to this date, but we can give a sufficient criterion relying on similar ideas. This requires some extra vocabulary though. We will denote by Γ_2 the quotient of the free group $\mathbb{F}_2$ by the normal subgroup generated by all the elements $[g,[h,k]]$. This is the free 2-step nilpotent group on two generators. We can then see it as a quantum subgroup of U_N^+ by defining π_{Γ_2} to be the composition of $\pi_{\mathbb{F}_2}$ and the quotient map.

Proposition 2.6 (FRANZ-F.-SKALSKI). *If* $\ker(\pi_{U_N}) \cap \ker(\pi_{\Gamma_2})$ *does not contain any non-trivial ideal I satisfying*

$$\Delta(I) \subset I \otimes \mathcal{O}(U_N^+) + \mathcal{O}(U_N^+) \otimes I,$$

then U_N^+ is Gaussian for all $N \geqslant 2$.

2.2 LOCAL STUDY : A CLASSIFICATION

To get a better understanding of Gaussian generating functionals on U_N^+, one may try to find explicit formulæ describing them. Quite surprisingly, this is possible. The starting point is a simple observation concerning derivations. Let us recall what we mean by this.

DEFINITION 2.7. A *derivation* on $\mathcal{O}(U_N^+)$ is a linear map $D : \mathcal{O}(U_N^+) \to \mathbf{C}$ such that for any $x, y \in \mathcal{O}(U_N^+)$,

$$D(xy) = D(x)\varepsilon(y) + \varepsilon(x)D(y).$$

Clearly, a derivation is completely determined by the images of the generators, which form a matrix A with coefficients $A_{ij} = D(u_{ij})$ for all $1 \leqslant i,j \leqslant N$. The observation is that any matrix $A \in M_N(\mathbf{C})$ can appear.

Lemma 2.8. *For any matrix $A \in M_N(\mathbf{C})$, there is a unique derivation $D_A : \mathcal{O}(U_N^+) \to \mathbf{C}$ such that $D_A(u_{ij}) = A_{ij}$.*

Proof. Uniqueness is clear, so that we have to prove existence. Because the generators of $\mathcal{O}(U_N^+)$ are linearly independent[8], we can define a linear map D_A^1 on their linear span sending u_{ij} to A_{ij}. Next we extend it the linear span of these elements and their adjoints by setting

$$D_A(u_{ij}^*) = -D_A(u_{ji}).$$

This then extends to a derivation on the free algebra $\mathscr{A}$ generated by these elements, and all we have to do is to prove that it vanishes on the ideal J generated by the defining relations.

To do this, observe that by definition $D(1) = 0$ for any derivation, and that

$$\begin{aligned} D_A\left(\sum_{k=1}^N u_{ik}u_{jk}^*\right) &= \sum_{k=1}^N D(u_{ik})\varepsilon(u_{jk}^*) + \varepsilon(u_{ik})D(u_{jk}^*) \\ &= D(u_{ij}) + D(u_{ji}^*) \\ &= 0. \end{aligned}$$

A similar computation works for the other three types of relations, so that D_A vanishes on a set generating the ideal J. Let now $x \in J$ and observe that this implies $\varepsilon(x) = 0$. It then follows that if $D_A(x) = 0$, we also have

$$D_A(xy) = 0 = D_A(yx)$$

for any $y \in \mathcal{O}(U_N^+)$. All in all, we have proven that D_A vanishes on J. But then, it defines a linear map on the quotient vector space of $\mathscr{A}$ by J, that is to say on $\mathcal{O}(U_N^+)$, and the proof is complete. ∎

This has no reason to be a generating functional in general. For instance, we have seen in the proof above that

$$D_A(u_{ij}^*) = -A_{ji} = \overline{D_{-A^*}(u_{ij})},$$

so that A would have to satisfy $A = -A^*$. The good news however is that this is the only obstruction.

Lemma 2.9. *Let H be an anti-hermitian matrix. Then, D_H is a Gaussian generating functional.*

Proof. By definition of a derivation, we must have $D_H(1) = 0$. Hermitianity is guaranteed by the assumptions, so that we have to check conditional positivity. But if $x \in \ker(\varepsilon)$, we have by the derivation property

$$D_H(x^*x) = D_H(x^*)\varepsilon(x) + D_H(x)\varepsilon(x^*) = 0.$$

As for Gaussianity, observe that by the derivation property, $D_H(xy) = 0$ as soon as $x, y \in \ker(\varepsilon)$ so that D_H vanishes on $K_2 \supset K_3$. ∎

We will give a name to these particular generating functional since they are very specific.

DEFINITION 2.10. A Gaussian generating functional L is a *drift* if it vanishes on K_2, which exactly means[9] that $L = D_H$ for some anti-hermitian matrix H.

Remark 2.11. A direct computation shows that $D_H * D_K - D_K * D_H = D_{[H,K]}$, so that there is a Lie algebra isomorphism between drifts on U_N^+ and $\mathfrak{u}_N$.

To go further, we need to understand the defect of a general Gaussian generating functional from being a derivation. To do this, let us introduce the *coboundary* of L to be the map

$$\partial L : \mathcal{O}(U_N^+) \otimes \mathcal{O}(U_N^+) \to \mathbf{C}$$

defined by

$$\partial L(x \otimes y) = L(xy) - L(x)\varepsilon(y) - L(y)\varepsilon(x).$$

This makes sense for any linear functional, but in the Gaussian case we have a nice description due to Schürmann (see the book [Sch93] for details).

THEOREM 2.12 (SCHÜRMANN) If L is a Gaussian generating functional on U_N^+, then there is a finite-dimensional Hilbert space V and a derivation $\eta : \mathcal{O}(U_N^+) \to V$ such that for all $x, y \in \mathcal{O}(U_N^+)$,

$$\partial L(x \otimes y) = \langle \eta(x^*), \eta(y) \rangle.$$

Remark 2.13. Equivalently, η is a 1-cocyle with value in V equipped with the trivial representation of $\mathcal{O}(U_N^+)$. Therefore, the understanding of Gaussian functionals on U_N^+ is tightly connected to the computation of its Hoschild cohomology groups.

Now if d is the dimension of V and $(e_1, \cdots, e_d)$ is an orthonormal basis, then η decomposes as

$$\eta = \sum_{r=1}^{d} \eta_r e_r$$

with $\eta_r : \mathcal{O}(U_N^+) \to \mathbf{C}$ a derivation. In other words, we have matrices, $A_1, \cdots, A_d \in M_N(\mathbf{C})$ such that

$$\eta = \sum_{r=1}^{d} D_{A_r} e_r.$$

What we need is to find conditions on these matrices ensuring that η is a coboundary. There is at least a necessary one that can be obtained quite easily.

Lemma 2.14. *With the notations above, we must have the equality*

$$\sum_{r=1}^{d} A_r A_r^* = \sum_{r=1}^{d} A_r^* A_r.$$

Proof. We start with the equality

$$\begin{aligned}
0 &= L(1) \\
&= L\left(\sum_{k=1}^{N} u_{ik}^* u_{jk}\right) \\
&= L(u_{ij}^*) + L(u_{ji}) + \sum_{k=1}^{N} \langle \eta(u_{ik}), \eta(u_{jk}) \rangle \\
&= L(u_{ij}^*) + L(u_{ji}) + \sum_{k=1}^{N} \sum_{r=1}^{d} \overline{A}_{ik} A_{jk} \\
&= L(u_{ij}^*) + L(u_{ji}) + \sum_{r=1}^{d} \sum_{k=1}^{N} \overline{A}_{ik} A_{jk} \\
&= L(u_{ij}^*) + L(u_{ji}) + \sum_{r=1}^{d} (A_r^* A_r)_{ji}
\end{aligned}$$

and then perform a similar one with another of the relations, namely

$$\begin{aligned}
0 &= L(1) \\
&= L\left(\sum_{k=1}^{N} u_{ki} u_{kj}\right) \\
&= L(u_{ji}) + L(u_{ij}^*) + \sum_{k=1}^{N} \langle \eta(u_{ki}^*), \eta(u_{kj}^*) \rangle \\
&= L(u_{ji}) + L(u_{ij}^*) + \sum_{k=1}^{N} \sum_{r=1}^{d} A_{ki} \overline{A}_{jk} \\
&= L(u_{ji}) + L(u_{ij}^*) + \sum_{r=1}^{d} \sum_{k=1}^{N} A_{ki} \overline{A}_{jk} \\
&= L(u_{ji}) + L(u_{ij}^*) + \sum_{r=1}^{d} (A_r A_r^*)_{ji}
\end{aligned}$$

Therefore, the equality in the statement holds. ■

The main result of this section is that the condition above is in fact sufficient.

THEOREM 2.15 (FRANZ-F.-SKALSKI) Let $A_1, \cdots, A_d \in M_N(\mathbf{C})$, with $d \leqslant N^2$, be such that

$$\sum_{r=1}^{d} A_r A_r^* = \sum_{r=1}^{d} A_r^* A_r.$$

Then, there is a unique Gaussian generating functional L on U_N^+ such that for all $1 \leqslant i, j \leqslant N$,

$$L(u_{ij}) = \frac{1}{2}\left(\sum_{r=1}^{d} A_r A_r^*\right)_{ij}.$$

Moreover, we can always assume $d \leqslant N$, and the corresponding cocycle is

$$\eta = \sum_{r=1}^{d} D_{A_r} e_r.$$

Remark 2.16. Note that even though L is only defined on the generators in the statement above, the Gaussian property and the link with the cocyle show that it is in fact uniquely defined on the whole of $\mathcal{O}(U_N^+)$.

To conclude, let us mention that according to this result and to the last theorem of the previous subsection, it would in principle be enough to "understand" all quadruples (A_1, A_2, A_3, A_4) of matrices in $M_2(\mathbf{C})$ such that

$$\sum_{r=1}^{4} A_r A_r^* = \sum_{r=1}^{4} A_r^* A_r.$$

to determine whether U_N^+ equals its Gaussian part for all $N \geqslant 2$. The problem of course is in the word "understand", because it means determining through which quantum subgroup each process can factor.

3 A CENTRAL HOPE

The picture is for the moment quite disappointing. We do not really understand globally Gaussian processes on U_N^+, and there is no specific one that singles out as a candidate for a replacement of the Laplace-Beltrami operator. Yet, there is hope, if we throw in another important property.

Indeed, the generating functional L_B on $\mathcal{O}(SU_N)$ has a remarkable property : it is invariant under the adjoint action of the group on itself. More precisely, if $g \in SU_N$ and $f \in \mathcal{O}(SU_N)$, then $f \circ \mathrm{Ad}_g$ is again a polynomial function, and we have

$$L_B(f \circ \mathrm{Ad}_g) = L_B(f).$$

This is because invariance under the adjoint action is built in the whole construction from the start since it is satisfied already by the Killing form.

This suggests to look for Gaussian generating functionals on U_N^+ which are invariant under the adjoint action ... once we make sense of the latter. But the good news is that this is not very complicated if we take a more functional point of view. By this, we mean that if $L_1, L_2 : \mathcal{O}(SU_N) \to \mathbf{C}$ are linear maps and L_1 is invariant under the adjoint action, then we can conjugate by any element the first variable so that

$$\begin{aligned}(L_1 * L_2)(f) &= (L_1 \otimes L_2)((g,h) \mapsto f(gh)) \\ &= (L_1 \otimes L_2)((g,h) \mapsto f((hgh^{-1}h)) \\ &= (L_1 \otimes L_2((g,h) \mapsto f(hg)) \\ &= (L_2 \otimes L_1)((g,h) \mapsto f(gh)).\end{aligned}$$

In other words, L_1 is central for the convolution product. This leads to the following definition :

DEFINITION 3.1. A linear map $L : \mathcal{O}(U_N^+) \to \mathbf{C}$ is said to be *central* if it is central for the convolution product.

Remark 3.2. One can define a notion of adjoint action for compact quantum groups and check that a linear map is central if and only if it is invariant under that adjoint action in a suitable sense, see [CFK14].

So the question now is : what are the central Gaussian generating functionals ? The answer is unfortunately not satisfying, all such functionals factor through a very small classical subgroup. More precisely, let z denote the identity functional on the group U_1 of complex numbers of modulus one. Then the coefficients of the matrix $z\mathrm{I}_N$ satisfy the defining relations of $\mathcal{O}(U_N^+)$, so that there is a surjective $*$-homomorphism

$$\pi_{U_1} : \mathcal{O}(U_N^+) \to \mathcal{O}(U_1)$$

sending u_{ij} to $z\delta_{ij}$. Therefore, any Gaussian generating functional L on U_1 lifts to a Gaussian generating functional $\widetilde{L} = L \circ \pi_{U_1}$ on U_N^+. Of course, since U_1 is commutative, all its functionals are central. That this remains true when they are pulled back to U_N^+ comes from the fact that π_{U_1} is *co-central* in the sense that the following equality holds :

$$(\pi_{U_1} \otimes \mathrm{id}) \circ \Delta = \Sigma \circ (\mathrm{id} \otimes \pi_{U_1}) \circ \Delta,$$

where

$$\Sigma : \mathcal{O}(U_N^+) \otimes \mathcal{O}(U_N^+) \to \mathcal{O}(U_N^+) \otimes \mathcal{O}(U_N^+)$$

denotes the flip map sending $x \otimes y$ to $y \otimes x$. This gives us a source of central Gaussian functionals on U_N^+, and we in fact get all of them in that way.

Proposition 3.3. *The Gaussian functional $\widetilde{L} = L \circ \pi_{U_1}$ is always central. Moreover, all central Gaussian functionals on U_N^+ arise in that way.*

Proof. The first part is a simple computation : for any linear functional L', we have

$$\begin{aligned} \widetilde{L} * L' &= (\widetilde{L} \otimes L') \circ \Delta \\ &= (L \otimes L') \circ (\pi_{U_1} \otimes \mathrm{id}) \circ \Delta \\ &= (L \otimes L') \circ \Sigma \circ (\mathrm{id} \otimes \pi_{U_1}) \circ \Delta \\ &= (L' \otimes L) \circ (\mathrm{id} \otimes \pi_{U_1}) \circ \Delta \\ &= (L' \otimes \widetilde{L}) \circ \Delta \\ &= L' * \widetilde{L}. \end{aligned}$$

As for the second part, let L be a Gaussian generating functional on U_N^+, which by Theorem 2.12 is given by an anti-hermitian matrix H and d matrices $A_1, \cdots, A_d \in M_N(\mathbf{C})$. We will first consider the matrix $L(U) \in M_N(\mathbf{C})$ with coefficients $(L(u_{ij}))_{1 \leqslant i,j \leqslant N}$ and prove that it is central in $M_N(\mathbf{C})$. Indeed, let $M \in M_N(\mathbf{C})$ be any *unitary* matrix. Then, there is by the universal property a $*$-homomorphism $\mathrm{ev}_M : \mathcal{O}(U_N^+) \to \mathbf{C}$ sending u_{ij} to M_{ij}. Then, the equality

$$(L * \mathrm{ev}_M)(u_{ij}) = (\mathrm{ev}_M * L)(u_{ij})$$

translates into the equality

$$\sum_{k=1}^N L(U)_{ik} M_{kj} = \sum_{k=1}^N M_{ik} L(U)_{kj}$$

so that $L(U)$ commutes with M. Now, a matrix commuting with all unitary matrices is a multiple of the identity[10], hence there exists $\lambda \in \mathbf{C}$ such that $L(U) = \lambda \mathrm{I}_N$. Setting for convenience

$$A = \sum_{i=1}^d A_r^* A_r,$$

we conclude that $A + H = \lambda \mathrm{I}_N$, and the fact that A is hermitian and H is anti-hermitian shows that $A = \Re(\lambda)\mathrm{I}_N$ and $H = \Im(\lambda)\mathrm{I}_N$.

Forgetting now about H (we can always remove drifts and still have a Gaussian generating functional), one then proceeds similarly for $L(U \otimes U)$ and concludes that it is in the center of $M_N(\mathbf{C}) \otimes M_N(\mathbf{C})$, yielding the equality

$$\sum_{r=1}^{d} A_r \otimes A_r^* \in \mathbf{C}.\mathrm{I}_N \otimes \mathrm{I}_N.$$

Elementary linear algebra manipulations then show that each A_r must be a multiple of the identity. But that means that the kernel of L contains the kernel of π_{U_1}, so that L comes from a Gaussian generating functional on U_1. ■

It seems like we are back to our starting point, but there is still a slight hope : what if the Brownian motion was not given by a Gaussian generating functional, but by taking a Gaussian generating functional and then making it central ? To explain whay we mean by this, let us see how one can make a generating functional on SU_N central.

The simplest way would be to have a natural way of making functions invariant under the adjoint action. But this can be done, thanks to the existence of the Haar measure[2] : for any $f \in C(SU_N)$, just set

$$\mathbb{E}(f) = \int_{SU_N} f(hgh^{-1}) \mathrm{dHaar}(h).$$

The crucial point now is that the map $\mathbb{E}$ has nice properties, and in particular

- It preserves the unit : $\mathbb{E}(1) = 1$;
- It preserves positivity : $\mathbb{E}(f) \geqslant 0$ for $f \geqslant 0$;
- It preserves invariant functions : $\mathbb{E}(f) = f$ if and only if f is invariant under the adjoint action.

It is then straightforward that if $L : \mathcal{O}(SU_N) \to \mathbf{C}$ is a generating functional, then $L \circ \mathbb{E}$ still is a generating functional which is furthermore central. That the same strategy works for U_N^+ is a standard fact from compact quantum group theory, the proof of which would take us too far away from our topic. We will therefore just state it as a blackbox. In order to do this, let us observe that the algebra of polynomial functions on SU_N which are invariant under the adjoint action is the same as the algebra of characters of finite-dimensional representation, and this is generated as a $*$-algebra by the character of the defining representation as matrices[11]. In terms of the functions $c_{ij} \in \mathcal{O}(SU_N)$ sending a matrix to its (i,j)-th coefficient, this character is simply

$$\chi = \sum_{i=1}^{N} c_{ii}.$$

Proposition 3.4. *Let us denote by $\mathcal{O}(U_N^+)_c$ the $*$-subalgebra of $\mathcal{O}(U_N^+)$ generated by the element*

$$\chi = \sum_{i=1}^{N} u_{ii}.$$

Then, there exists a linear map

$$\mathbb{E} : \mathcal{O}(U_N^+) \to \mathcal{O}(U_N^+)_c$$

such that for any generating functional L on U_N^+, $L \circ \mathbb{E}$ is a central generating functional.

We can now investigate functionals of the form $\widetilde{L} = L \circ \mathbb{E}$ with L Gaussian. To do this, let us make a few elementary remarks. First, by construction $\widetilde{L}$ is completely determined by the values of L on $\mathcal{O}(U_N^+)_c$. Second, the Gaussianity property implies that the values of L are completely determined by the values on products of at most two generators and their adjoints. Since, $\mathcal{O}(U_N^+)_c$ is generated by the single element χ, this leaves us with six values to compute. These can be reduced to four values using hermitianity of L, and we will now see that the number of parameters is even smaller.

2. Like all compact groups, SU_N has a unique Borel probability measure μ which is bi-invariant in the sense that for any $g \in SU_N$ and any Borel subset A, $\mu(g.A) = \mu(A) = \mu(A.g)$.

Proposition 3.5. *If L is a Gaussian functional on U_N^+, then $\widetilde{L}\circ\mathbb{E}$ is determined by two parameters $\alpha\in\mathbf{C}$ and $\beta\in\mathbf{R}$ with*

$$\Re(\alpha)\geqslant\frac{\beta}{N}\geqslant 0.$$

Proof. Let us set $a=\mathrm{Tr}(A)\in\mathbf{R}_+$, $b=\mathrm{Tr}(H)\in i\mathbf{R}$ and

$$\beta=(\mathrm{Tr}\otimes\mathrm{Tr})\left(\sum_{r=1}^{d}A_r\otimes A_r^*\right)=\sum_{r=1}^{d}|\mathrm{Tr}(A_r)|^2\geqslant 0.$$

Then, using the explicit formulæ from Theorem 2.12, we find that

$$\begin{aligned}L(\chi)&=\frac{a}{2}+ib\\ L(\chi\chi)&=Na+2Nib-\beta\\ L(\chi\chi^*)&=Na+\beta\\ L(\chi^*\chi)&=Na+\beta\end{aligned}$$

Setting $\alpha=a/2+b$, this can be writtent as

$$L(\chi)=\alpha\quad;\quad L(\chi\chi)=2N\alpha-\beta\quad;\quad L(\chi\chi^*)=2N\Re(\alpha)+\beta.$$

The inequality in the satement then follows from the Cauchy-Schwarz inequality : for all $1\leqslant i\leqslant d$,

$$|\mathrm{Tr}(A_r)|^2\leqslant N\mathrm{Tr}(A_r^*A_r).$$

∎

We now hope that the reader is convinced that the two-parameter family of central functionals above is worth studying. Let us conclude by mentioning some recent results concerning them, due to Delhaye in [Del24] :

- Consider the generating functional $L_B^{\mathrm{class}}:\mathcal{O}(U_N)\to\mathbf{C}$ of the classical Brownian motion on the group U_N. Then, the generating functional

  $$L_B^{\mathrm{class}}\circ\pi_{U_N}\circ\mathbb{E}$$

  is of the form given in Proposition 3.5, and the coefficients α and β can be computed explicitely in terms of normalizations of the Laplace-Beltrami operators on SU_N and on the kernel of the Killing form on $\mathfrak{u}_N$. In particular, if we start instead from the generating functional of the Brownian motion on SU_N, then we have $\beta=0$.

- Given some relative growth conditions on α and β (which are in particular satisfied whenever $\beta=0$), the corresponding stochastic process exhibits a *cut-off phenomenon* and the corresponding limit profile is explicitly computed.

REFERENCES

[CFK14] F. CIPRIANI, U. FRANZ & A. KULA – "Symmetries of Lévy processes, their Markov semigroups and potential theory on compact quantum groups", *J. Funct. Anal.* **266** (2014), no. 5, p. 2789–2844.

[Chi20] A. CHIRVASITU – "Topological generation results for free unitary and orthogonal groups", *Internat. J. Math.* **31** (2020), no. 1, p. 2050003.

[Con94] A. CONNES – *Noncommutative geometry*, Academic Press, 1994.

[Del24] J. DELHAYE – "Brownian motion on the unitary quantum group: construction and cut-off", *arXiv preprint* (2024).

[FFS22] U. FRANZ, A. FRESLON & A. SKALSKI – "Connectedness and Gaussian parts for compact quantum groups", *J. Geom. Phys.* (2022), p. 104710.

[FFS24] — , "Gaussian processes on free easy quantum groups", *Arxiv preprint* (2024).

[Fre23] A. FRESLON – *Compact matrix quantum groups and their combinatorics*, LMS Student Texts in Mathematics, Cambridge University Press, 2023.

[Lia04] M. LIAO – "Lévy processes and Fourier analysis on compact Lie groups", *Ann. Probab.* (2004), p. 1553–1573.

[Par92] K. PARTHASARATHY – *An introduction to quantum stochastic calculus*, Monographs in Mathematics, vol. 85, Birkhaüser, 1992.

[Sch93] M. SCHÜRMANN – *White noise on bialgebras*, Lect. Notes Math., vol. 1544, Springer-Verlag, 1993.

[Wan97] S. WANG – "Problems in the theory of quantum groups", *Banach Center Publ.* **40** (1997), no. 1, p. 67–78.

[Wan09] — , "Simple compact quantum groups I", *J. Funct. Anal.* **256** (2009), no. 10, p. 3313–3341.

NOTES

1. For the convenience of the reader, let us recall that this means that for any $\epsilon > 0$, then $\mathbb{P}(|X_t - X_0| > \epsilon) \to 0$ as $t \to 0$.
2. It would be more natural in principle to work with U_N since we want to consider a quantum analogue of the latter. However, its lack of semi-simplicity would be an issue, as we will discuss below.
3. One of the reasons for that is that there is no good notion of $\mathscr{C}^2$-functions on a compact quantum group, precisely because we do not have a (non-commutative) differential structure. Another reason will be given below.
4. Observe that $\mathscr{C}^2(SU_N) \otimes \mathscr{C}^2(SU_N)$ is not equal to $\mathscr{C}^2(SU_N \times SU_N)$, a problem which vanishes when restricting to polynomial functions.
5. Note that we also have the hermitianity property $\varphi_t(\overline{f}) = \overline{\varphi_t(f)}$, but this follows from positivity.
6. Since we are dealing here with a non-compact group, we cannot consider arbitrary functions. The correct notion for a locally compact space is rather that of functions going to 0 at infinity.
7. The term *quadratic* is also used.
8. We are cheating here of course, this is does not follow directly from the definition.
9. Indeed, setting $x' = x - \varepsilon(x)1$ and $y' = y - \varepsilon(y)1$, we have for any $x, y \in \mathscr{O}(U_N^+)$ that
$$\begin{aligned} L(xy) &= L(x'y') + L(x')\varepsilon(y) + \varepsilon(x)L(y') + L(\varepsilon(x)\varepsilon(y)1) \\ &= L(x'y') + L(x)\varepsilon(y) + \varepsilon(x)L(y) \\ &= L(x)\varepsilon(y) + \varepsilon(x)L(y) \end{aligned}$$
10. Observe that if λ is an eigenvalue of $L(U)$ associated to some eigenvector $x \neq 0$, then for any vector x' with the same norm as x, there exists a unitary matrix M such that $Mx = x'$, so that
$$\begin{aligned} L(U)x' &= ML(U)M^*x \\ &= ML(U)x \\ &= \lambda Mx \\ &= \lambda x'. \end{aligned}$$
This means that all vectors are eigenvectors of $L(U)$, which is only possible if the latter is a multiple of the identity matrix.
11. This is a translation of the fact that this representation is faithful.