Diffusion of the quantum orthogonal Brownian motion

Amaury Freslon, Lucas Teyssier & Simeng Wang

université PARIS-SACLAY

Département de Mathématiques d'Orsay

The classical Brownian recipe

Fig. 1 – *A classical Brownie*

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Ingredients :

- A compact Lie group G ;
- An *ad-invariant* Lévy process $(X_t)_{t\in\mathbb{R}_+}$ on G.

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Steps :

- Peal off X_t to keep only $\mu_t = \mathrm{Law}(X_t)$;
- Boil down to the *infinitesimal generator*

$$L : f \in \mathcal{O}(G) \longmapsto \lim_{t\to 0} \frac{f * \mu_t - f * \mu_0}{t}$$

- By a theorem of M. Liao, $L = b\Delta + \text{Lévy}$ (with $b > 0$). Chop off the Lévy part to get the Brownian motion.

The quantum orthogonal Brownian recipe

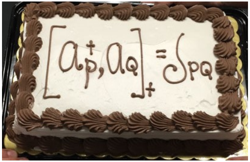
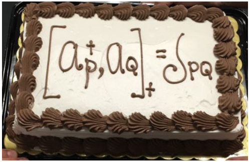


Fig. 2 – *A quantum Brownie*

The quantum orthogonal Brownian recipe

Ingredients :

- The compact quantum group O_N^+ ;
- A convolution semigroup $(\varphi_t)_{t\in\mathbb{R}_+}$ of *central* states on $\mathscr{O}(O_N^+)$.



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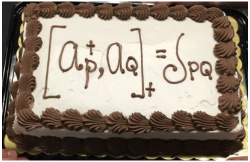

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- Remove the surplus to keep only the restriction $\widetilde{L}$ to $\mathcal{O}(O_N^+)_{\text{central}}$.

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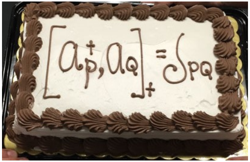

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Theorem (Cipriani-Franz-Kula)

There is a decomposition $\widetilde{L} = b\widetilde{\psi} + \text{Lévy}$*, with* $b > 0$ *and* $\widetilde{\psi}(P_n) = \dfrac{P_n'(N)}{P_n(N)}$.

$P_0(X) = 1$, $P_1(X) = X$ and $XP_n(X) = P_{n+1}(X) + P_{n-1}(X)$.

Mixing time

Fig. 3 – *A Brownian mixing*

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THEOREM : Setting $t_N = N\ln(N)$, then for any $\epsilon > 0$,

$$\lim_{N\to+\infty} \|\psi_{(1-\epsilon)t_N} - h\|_{C_{\max}(O_N^+)^*} = 1$$
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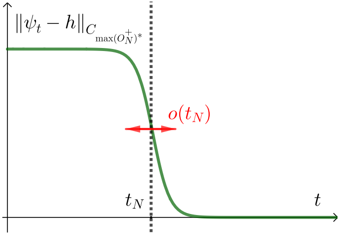

This is the **cutoff phenomenon** !

Cutoff profile

Fig. 4 – *A Brownie cutter*

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Theorem (F.-TEYSSIER-WANG)

For any $c \in \mathbb{R}$,

$$\lim_{N \to +\infty} \|\psi_{N\ln(N)+cN} - h\| = \left\|\mathrm{Poiss}^+(e^{2c}, -e^{-c}) \boxplus \delta_{e^c + e^{-c}} - \mathrm{SC}\right\|_{TV}$$
$$= \left\|\mathrm{Meix}^+(-e^{-c}, 0) \boxplus \delta_{e^{-c}} - \mathrm{Meix}^+(0,0)\right\|_{TV}$$

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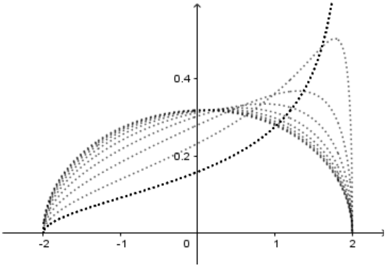

Fig. 5 – *Free Meixner laws*

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- We can **cook up** an $\widetilde{N}(t) \in [-N, N]$ such that

$$m_t^{(N)} = \alpha(t)\delta_{\widetilde{N}(t)} + \sum_{n=0}^{+\infty} \left[e^{-t\widetilde{\psi}(P_n)} P_n(N) - \alpha(t) P_n\left(\widetilde{N}(t)\right)\right] P_n \mathrm{dSC}$$

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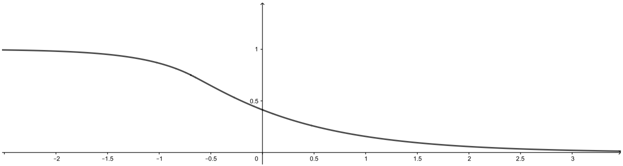

Fig. 6 – *The complete profile*

Thanks for your attention