ON THE CLASSIFICATION OF PARTITION QUANTUM GROUPS

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Abstract. This is a survey on some results obtained recently in the classification of compact quantum groups associated to partitions, with a focus on the non-crossing case. We take a global look at the main results in the subject and highlight some key features of the methods used. We conclude by several suggestions for pushing further the classification.

1. Introduction

It has been known since the foundational works of T. Banica [Ban96], [Ban97] that some families of compact quantum groups are closely linked to very important combinatorial objects, like Temperley-Lieb algebras, as well as to free probability theory. These fundamental connections were unified around ten years ago by T. Banica and R. Speicher in [BS09] into a general setting based on the notion of a *category of partitions*, paving the way for more systematic investigations. One crucial aspect of the problem is, quite naturally, the classification of these categories of partitions and of the corresponding compact quantum groups, called *partition quantum groups*. This classification program has rapidly grown into a subject of its own, giving rise to new ideas and techniques which have reciprocally impact on the general study of quantum groups. It also has proven to have consequences on other subjects, like free probability [BCS12] or, more recently, Deligne categories [FM20] and random walks on trees [Wah20]. This comes from the fact that, despite its quantum group origin, the classification problem can be expressed using only elementary combinatorial objects. This makes it potentially appealing to a broad range of mathematicians.

There has been spectacular progress conerning the classification during the last decade, as a result of many works including [RW14], [RW15] and [RW16], leading to a complete solution of the original problem. However, at the same time, several extended settings were developed, and it became clear that the first methods would not be sufficient to deal with them. Interestingly, the classification of the so-called orthogonal *easy quantum groups* also revealed some classes of quantum groups with an "almost" or "skew" partition structure (see for instance [GW21] or [Maa19]), thus broadening the class of examples of compact quantum groups beyond the current knowledge.

At the present time, the subject keeps developing at a rapid pace, and some of the basic techniques have been considerably refined to deal with the ever more subtle problems posed by the classification program. However, it seems to us that there are some fundamental ideas that underpine much of these works. Isolating and formalizing these principles may be the key to a better understanding of the whole subject, as well as to devising future directions.

The purpose of this work is therefore twofold. First, we will review several results concerning the classification of partition quantum groups. By doing so, we will try to put stress on the various types of *invariants* appearing in the literature, as well as on their use for proving classification results. In order to keep things to a reasonable length, and because this is already an extremely rich world, we have decided to focus mainly on *non-crossing* partitions. We will nevertheless make excursions into the general setting to mention important results and connections.

Second, we will introduce an operation on partitions, that we call *grafting*. The idea is that graftings enable for a short and simple statement of all the classification results for non-crossing partition quantum groups obtained so far. As a consequence, it is very easy and tempting to state general conjectures concerning the structure of all non-crossing partition quantum groups using graftings. Moreover, these conjectures can be naturally split into many smaller problems, the solution of which may be important steps towards a complete understanding of non-crossing partition quantum groups.

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2. What are partition quantum groups?

Partition quantum groups can be thought of as objects whose invariant theory admits a combinatorial description which generalizes that of orthogonal groups given by R. Brauer in [Bra37]. One could even say that they are the most general setting in which one can deal with invariant theory using the combinatorics of partitions of finite sets. The purpose of this section is to make this precise to the non-expert reader, while avoiding as much as possible the technical aspects of the problem to focus on the combinatorial part.

2.1. The setting.

The setting is that of compact quantum groups in the sense of S.L. Woronowicz but one of the beauties of the subject is that it can be treated in a purely combinatorial way. Nevertheless, for the sake of completeness as well as to motivate some of the tools and ideas, we will recall some basic facts concerning compact quantum groups. The reader may refer to the books [Tim08] and [NT13] for detailed treatments of the subject. For an introductory text including the combinatorial approach to compact quantum groups, see [Web17].

Definition 2.1. A *compact matrix quantum group* $\mathbb{G}$ is given by a C*-algebra $C(\mathbb{G})$ together with a matrix $u \in M_N(C(\mathbb{G}))$ such that

- The matrix u is unitary, as well[1] as the matrix ${}^tu$,
- The coefficients of u generate $C(\mathbb{G})$,
- There exists a (necessarily unique) $*$-homomorphism $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ (where the tensor product is the spatial one) such that for all $1 \leqslant i, j \leqslant N$,

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

The matrix u is called the *fundamental representation* of $\mathbb{G}$.

Our prototypical classical example will be the group O_N of all $N \times N$ orthogonal matrices, where u_{ij} is the function sending a matrix to its (i, j)-th coefficient. The algebra of functions from O_N to the complex numbers $\mathbf{C}$ is then, by virtue of Weierstrass' Theorem, dense in the algebra $C(O_N)$ of all continuous functions. Moreover, translating the formula for matrix products via the isomorphism

$$C(O_N \times O_N) \simeq C(O_N) \otimes C(O_N)$$

shows the existence of the $*$-homomorphism Δ.

The notion of a finite-dimensional representation has a natural analogue here: this is simply a matrix $v \in M_n(C(\mathbb{G}))$ for some integer n called its dimension, such that for any $1 \leqslant i, j \leqslant n$,

$$\Delta(v_{ij}) = \sum_{k=1}^{n} v_{ik} \otimes v_{kj}.$$

The foundational work of S.L. Woronowicz shows that much of the representation theory of compact groups carry over to this setting (including for instance Peter-Weyl theory), and particularly the categorical aspects. For instance, using these combinatorial aspects, S.L. Woronowicz introduced in [Wor88] some diagrammatic calculus to study deformations of $SU(N)$.

[1] This condition is stronger than what is needed to make the theory work, but is suited to our purpose. In technical terms, we are here defining *compact quantum groups of Kac type*.

We are however interested in another type of combinatorics, coming from partitions of finite sets. To motivate it, let us dwell on the example of O_N. The fundamental representation is nothing but its defining representation ρ on $V = \mathbf{C}^N$. It is well-known that the whole representation theory of O_N can be recovered from the data of the spaces

$$\mathrm{Fix}_{O_N}(V^{\otimes k}) = \left\{\xi \in V^{\otimes k} \mid \rho^{\otimes k}(g)(\xi) = \xi \text{ for all } g \in O_N\right\}$$

of fixed points of $V^{\otimes k}$: this is Tannaka-Krein duality. Equivalently, one can consider instead the dual spaces, which are the algebraic invariants of the orthogonal group. One fundamental work on these invariants is due to R. Brauer in [Bra37], where he gave a useful generating family the construction of which we will now sketch.

The basic idea is that the inner product of V is, by definition, an invariant of O_N. As a consequence, if k is even, on may just pair copies of V and apply the inner product to each of them. Concretely, if p is a *pair partition* of the set $\{1, \cdots, 2k\}$, that is to say a partition into subsets of cardinality two, then

$$f_p : x_1 \otimes \cdots \otimes x_{2k} \mapsto \prod_{\{a,b\}\in p} \langle x_a, x_b\rangle$$

is an invariant of $V^{\otimes 2k}$. Now, the intuition is that since the invariance of the inner product is the only condition defning O_N, there should not be any additional invariant. That this is indeed the case is the fundamental result of R. Brauer. It is usually stated using fixed vectors instead of invariant linear maps, hence we will use f_p^* which can be seen as a vector in $V^{\otimes 2k}$.

Theorem 2.2 (Brauer). *Denoting by $P(2k)$ the set of all pair partitions, we have*

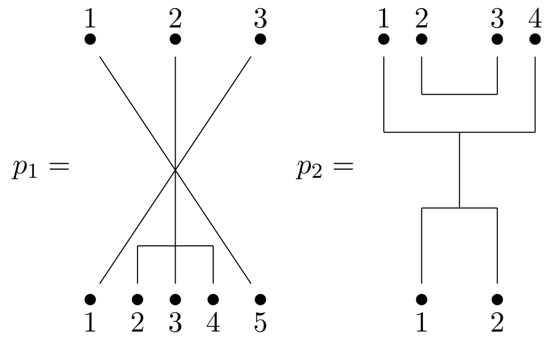
$$\mathrm{Fix}_{O_N}(V^{\otimes 2k}) = \mathrm{Span}\{f_p^* \mid p \in P(2k)\}$$
$$\mathrm{Fix}_{O_N}(V^{\otimes 2k+1}) = \{0\}$$

The motivation behind partition quantum groups is to study all compact groups and quantum groups which admit a similar description of their invariants involving partitions. One fundamental contribution of T. Banica and R. Speicher in [BS09] was to isolate the necessary and sufficient features that a set of partitions must satisfy for a corresponding compact quantum group to exist. This is how the central notion of a category of partitions emerged.

Remark 2.3. We have chosen here an approach through fixed vectors, but one could equivalently write things in terms of self-intertwiners of tensor powers of V. These spaces are commutants of the image of the group under the representation, and the study of such commutants is known as *Schur-Weyl duality.*

2.2. **Categories of partitions.** There has been several generalisations of the aforementioned result of R. Brauer, involving for instance the symmetric group S_N in [Jon93] and [Mar94]. When trying to deal with groups of matrices with complex coefficients, it was soon recognized that one needs to use colours to distinguish between the fundamental representation and its conjugate. This was done for instance for the unitary group U_N in [GvW89] or for reflection groups in [Tan97]. Later on, quantum analogues of these results were obtained by T. Banica in [Ban96], [Ban97], [Ban99] and alors [BV09] together with R. Vergnioux. Combining the idea of using colours and the notion of a category of partitions from T. Banica and R. Speicher leads to the general setting of [Fre17], which covers all the cases mentioned. We will therefore work from the beginning within this setting, which we now introduce from scratch, forgetting quantum groups for a moment.

A *partition* is given by two integers k and ℓ and a partition p of the set $\{1, \dots, k+\ell\}$ and we denote by P the set of all partitions. It is convenient to represent such partitions as diagrams, in particular for computational purposes. A diagram consists in an upper row of k points, a lower row of ℓ points, and some strings connecting these points if and only if they belong to the same subset of the partition. Let us consider for instance the partitions $p_1 = \{\{1,8\},\{2,6\},\{3,4\},\{5,7\}\}$ and $p_2 = \{\{1,4,5,6\},\{2,3\}\}$. If we see p_1 as an element of $P(3,5)$ and p_2 as en element of $P(4,2)$, then their diagram representations are:



When manipulating partitions, the crucial notion is that of a block.

Definition 2.4. Let p be a partition.

- A maximal set of points which are all connected (i.e. one of the subsets defining the partition) is called a *block* of p,
- If moreover the block consists only of neighbouring points, then it is called an *interval*,
- If b contains both upper and lower points (i.e. the subset contains an element of $\{1, \dots, k\}$ and an element of $\{k+1, \dots, k+\ell\}$), then it is called a *through-block*,
- Otherwise, it is called a *non-through-block*.

The total number of through-blocks of the partition p is denoted by $t(p)$.

Even though we will mention some aspects of the general case, this text mainly focuses on the specific family of non-crossing partitions in the following sense:

Definition 2.5. Let p be a partition. A *crossing* in p is a tuple $k_1 < k_2 < k_3 < k_4$ of integers such that:

- k_1 and k_3 are in the same block,
- k_2 and k_4 are in the same block,
- The four points are *not* in the same block.

If there is no crossing in p, then it is said to be a *non-crossing* partition. The set of non-crossing partitions will be denoted by NC.

As an example, the partition p_1 above has a crossing, while the partition p_2 is non-crossing. To generalize this setting, the idea is to further colour the points of the partitions with elements of a fixed set. These colours will allow for a combinatorial decomposition of the fundamental representation into irreducible components, as well as help distinguish representations from their conjugates.

Definition 2.6. A *colour set* is a set $\mathcal{A}$ together with an involution denoted by $x \mapsto x^{-1}$. An $\mathcal{A}$-coloured partition is a partition together with an element of $\mathcal{A}$ attached to each point. A coloured partition is said to be non-crossing if the underlying uncoloured partition is non-crossing. The set of all $\mathcal{A}$-coloured partitions will be denoted by $P^{\mathcal{A}}$ while the set of non-crossing ones will be denoted by $NC^{\mathcal{A}}$.

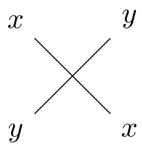
Let p be an $\mathcal{A}$-coloured partition. Reading from left to right, we can associate to the upper row of p a word w over $\mathcal{A}$ and to its lower row (again reading from left to right) a word w' over $\mathcal{A}$. For a set of partitions $\mathcal{C}$, we will denote by $\mathcal{C}(w, w')$ the subset of all partitions in $\mathcal{C}$ such that the upper row is coloured by w and the lower row is coloured by w' and we will denote by $|w|$ the length of a word w. There are several fundamental operations available on partitions called the *category operations*:

- If $p \in \mathcal{C}(w, w')$ and $q \in \mathcal{C}(z, z')$, then $p \otimes q \in \mathcal{C}(w.z, w'.z')$ (where $w.z$ denotes the concatenation of words) is their *horizontal concatenation*, i.e. the first $|w|$ of the $|w| + |z|$ upper points are connected by p to the first $|w'|$ of the $|w'| + |z'|$ lower points, whereas q connects the remaining $|z|$ upper points with the remaining $|z'|$ lower points.
- If $p \in \mathcal{C}(w, w')$ and $q \in \mathcal{C}(w', w'')$, then $qp \in \mathcal{C}(w, w'')$ is their *vertical concatenation*, i.e. $|w|$ upper points are connected by p to $|w'|$ middle points and the lines are then continued by q to $|w''|$ lower points. This process may produce loops in the partition. More precisely, consider the set L of elements in $\{1, \dots, |w'|\}$ which are not connected to an upper point of p nor to a lower point of q. The lower row of p and the upper row of q both induce a partition of the set L. For $x, y \in L$, let us set $x \sim y$ if x and y belong either to the same block of the partition induced by p or to the one

induced by q. The transitive closure of $\sim$ is an equivalence relation on L and the corresponding partition is called the *loop partition* of L, its blocks are called *loops* and their number is denoted by $\mathrm{rl}(q, p)$. To complete the operation, we remove all the loops. Note that we can only perform this vertical concatenation if the words associated to the lower row of p and the upper row of q match.
- If $p \in \mathcal{C}(w, w')$, then $p^* \in \mathcal{C}(w', w)$ is the partition obtained by reflecting p with respect to an horizontal axis between the two rows (without changing the colours).
- If $w = w_1 \dots w_n$, $w' = w'_1 \dots w'_k$ and $p \in \mathcal{C}(w, w')$, then rotating the extreme left point of the lower row of p to the extreme left of the upper row and changing its colour to its inverse yields a partition $q \in \mathcal{C}((w'_1)^{-1} w_1 \dots w_n, w'_2 \dots w'_k)$. The partition q is called a *rotated version* of p. One can also perform rotations on the right and from the upper to the lower row.

Let us say that for an element $x \in \mathcal{A}$, the *x-identity partition* is the partition $| \in \mathcal{C}(x, x)$ coloured with x on both ends. We are now ready for the definition of a category of coloured partitions, the fundamental object of this text.

Definition 2.7. A *category of $\mathcal{A}$-coloured partitions* $\mathcal{C}$ is the data of a set of $\mathcal{A}$-coloured partitions $\mathcal{C}(w, w')$ for all words w and w' over $\mathcal{A}$, which is stable under all the category operations and contains the x-identity partition for all $x \in \mathcal{A}$. If $\mathcal{C}$ moreover contains the partition



for all $x, y \in \mathcal{A}$, then it is said to be *symmetric*.

Such data gives rise to a compact quantum group, but this requires some notations. Let N be a given integer and consider for each colour $x \in \mathcal{A}$ a copy V^x of $\mathbf{C}^N$. For a word $w = w_1 \cdots w_n$ over $\mathcal{A}$, we set

$$V^w = V^{w_1} \otimes \cdots \otimes V^{w_n}.$$

Given representation $(u^x)_{x \in \mathcal{A}}$ and a word w over $\mathcal{A}$, one defines in the same way the tensor product representation u^w on V^w. With these notations, for $p \in \mathcal{C}(w) = \mathcal{C}(w, \emptyset)$ we can define a linear map

$$f_p : V^w \to \mathbf{C}$$

generalizing R. Brauer's construction as follows: denoting by $(e_i^x)_{1 \leqslant i \leqslant N}$ the canonical basis of V^x,

$$f_p(e_{i_1}^{w_1} \otimes \cdots \otimes e_{i_n}^{w_n}) = 1$$

if whenever two points are connected by p, the corresponding indices are equal, and f_p vanishes on all the other basis vectors. Here is now the precise existence statement, which is a slight generalization of T. Banica and R. Speicher's result [BS09, Thm 3.9] (see [Fre17, Thm 3.2.3] for a proof in our setting) :

Theorem 2.8. *Let $\mathcal{A}$ be a colour set, let $\mathcal{C}$ be a category of $\mathcal{A}$-coloured partitions and let N be an integer. Then, there exists a compact quantum group $\mathbb{G}$ together with unitary representations $(u^x)_{x \in \mathcal{A}}$ of dimension N such that*
- *Any finite-dimensional representation of $\mathbb{G}$ is equivalent to a subrepresentation of the tensor product u^w for some word w over $\mathcal{A}$,*
- *For any word w over $\mathcal{A}$,*

$$\mathrm{Fix}_{\mathbb{G}}(u^w) = \mathrm{Span}\{f_p^* \mid p \in \mathcal{C}(w)\}.$$

The compact quantum group $\mathbb{G}$ is called the partition quantum group *associated to $\mathcal{C}$ and N and is denoted by $\mathbb{G}_N(\mathcal{C})$. Moreover, $\mathbb{G}_N(\mathcal{C})$ is a classical compact group if and only if $\mathcal{C}$ is symmetric.*

Remark 2.9. The definition above only depends on the sets $\mathcal{C}(w, \emptyset)$ of partitions lying on one line. However, using rotations it is easily seen that these sets contain all the information concerning the category of partitions $\mathcal{C}$. This the combinatorial counterpart to Frobenius reciprocity at the compact quantum group level.

Remark 2.10. One may wonder about the assumption that all the representations u^x should have the same dimension. Removing it may lead to trouble depending on the partitions in $\mathcal{C}$, but in some cases it is possible. The first appearance of this phenomenon is in the work of D. Gromada and M. Weber [GW21] which we will not discuss here, despite its great interest.

Remark 2.11. One important source of examples in quantum group theory is twisting. It is indeed possible to twist the maps T_p to produce new examples. Some particular twistings have been studied in detail by T. Banica, and the question of finding all possible ways of twisting this construction is still open, see for instance [Ban19].

Now that the stage is set, we can start discussing the problem of classifying categories of partitions.

3. Take it easy

We will now discuss classification results in the original setting of [BS09], consisting in one colour, or two colours which are mutual inverses. The corresponding compact quantum groups are named *easy quantum groups*.

3.1. **Orthogonal easy quantum groups.** The first case to consider is obviously that of a colour set reduced to one point, which must be its own inverse. This means that we are working with quantum subgroups of the *free orthogonal quantum group* O_N^+, which is the reason why they are called *orthogonal*. That object dates from founding work of S. Wang in [Wan95], but for our purpose it suffices to say that it corresponds to the category NC_2 of all non-crossing pair partitions.

3.1.1. *The non-crossing case.* Let us consider the non-crossing case, which should be the simplest one. Because our aim is to discuss the general classification problem, we will start by stating a classification theorem, and then discuss the key features of the proof. The result we will consider is due to the joint efforts of T. Banica and R. Speicher in [BS09] and M. Weber in [Web13]. We will state it as a list of compact quantum groups, the definition of which comes from various prior works. Even though this may seem mysterious to the non-expert reader, it shows some structures involving the group $\mathbf{Z}_2$ which are important to understand the general classification strategy.

Theorem 3.1 (Banica–Speicher, Weber)**.** *There exist exactly seven non-crossing quantum groups on one colour:*

$$O_N^+,\ B_N^+,\ H_N^+,\ S_N^+,\ B_N^+ * \mathbf{Z}_2,\ B_N^+ \times \mathbf{Z}_2,\ S_N^+ \times \mathbf{Z}_2.$$

Proof. The first thing one may try when classifying structures, is to build invariants, and hope that they will completely capture the structure in question. In our case, there are two natural invariants which are easy to define for categories of partitions. The first one is the *Block Size*

$$BS(\mathcal{C}) = \{n \in \mathbf{N} \mid \exists\, p \in \mathcal{C} \text{ with a block of size } n\}.$$

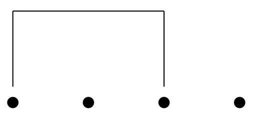
This invariant can take four different values: $\{2\}$, $\{1, 2\}$, $2\mathbf{N}$ and $\mathbf{N}$. These values each define a family of compact quantum groups which will be respectively denoted by $\mathscr{O}$, $\mathscr{B}$, $\mathscr{H}$ and $\mathscr{S}$. Since it only depends on the blocks of $\mathcal{C}$, we may call BS a *local invariant*.

One must now find a way of distinguishing, say, $S_N^+ \times \mathbf{Z}_2$ from S_N^+. This can be done using the *Odd Block Number*

$$BN(\mathcal{C}) = \{n \mid \exists\, p \in \mathcal{C} \text{ with } n \text{ blocks of odd size }\}$$

which can be either $\{0\}$, $\mathbf{N}$ or $2\mathbf{N}$. By contrast with the block size, we will call this a *global invariant*.

These two invariants together are not sufficient to obtain a full classification, since they cannot distinguish $B_N^+ * \mathbf{Z}_2$ from $B_N^+ \times \mathbf{Z}_2$. The final step is however not done by introducing a new invariant, but by considering the presence or absence of a peculiar partition, called the *positioner partition*



The presence of this partition does is usually[2] not translated into a numerical invariant, but rather into a *commutation property*: using it and the category operations, we can move singletons around without leaving the category of partitions. □

[2]There is a way of encoding the presence of the positioner partition into a global property of the partitions, see for instance item (5) in [Web13, Prop 2.7]. This is however not as elementary as the previous numerical invariants. A characterization through a local invariant, which was suggested to us by the anonymous referee, will be given below.

We can now give a list in terms of categories of partitions instead of compact quantum groups. We will do this in table, specifying the invariants and the family ($\mathscr{O}$, $\mathscr{B}$, $\mathscr{H}$, $\mathscr{S}$) to which each category belongs. In the same order as in the statement of Theorem 3.1, we have

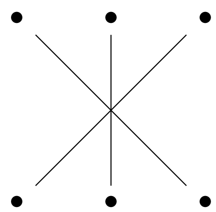
$\mathcal{C}$	Description	$BS(\mathcal{C})$	$BN(\mathcal{C})$	Family
NC_2	Non-crossing pair partitions	$\{2\}$	$\{0\}$	$\mathscr{O}$
$NC_{2,1}$	Non-crossing partitions with blocks of size at most 2	$\{1,2\}$	$\mathbf{N}$	$\mathscr{B}$
NC_{even}	Non-crossing partitions with blocks of even size	$2\mathbf{N}$	$\{0\}$	$\mathscr{H}$
NC	Non-crossing partitions	$\mathbf{N}$	$\mathbf{N}$	$\mathscr{S}$
$NC'_{2,1}$	Partitions in $NC_{2,1}$ with an even number of blocks of odd size	$\{1,2\}$	$2\mathbf{N}$	$\mathscr{B}$
$NC^*_{2,1}$	Smallest category containing the positioner	$\{1,2\}$	$2\mathbf{N}$	$\mathscr{B}$
NC'	Ppartitions in NC with an even number of blocks of odd size	$\mathbf{N}$	$2\mathbf{N}$	$\mathscr{S}$

Remark 3.2. From Theorem 3.1 one may also obtain a classification of the classical easy orthogonal compact groups. Indeed, given a symmetric category of partitions $\mathcal{C}$, it follows from the definitions that $\mathcal{C} \cap NC$ is a category of non-crossing partitions. Conversely, given a category of non-crossing partitions, one can add the crossing partition $\{\{1,3\},\{2,4\}\} \in P(4,0)$ and this generates a symmetric category of partitions. These operations are not inverse to one another, since $B_N^+ * \mathbf{Z}_2$ and $B_N^+ \times \mathbf{Z}_2$ both collapse to $B_N \times \mathbf{Z}_2$, but at least it yields all possible classical groups (see for instance [TW18, Lem 8.2]).

3.1.2. *The full classification.* It was already known at the time of the aforementioned works that there exist orthogonal easy quantum groups which are neither classical nor non-crossing, the first examples being the so-called *half-liberations*. These are obtained by adding the relation

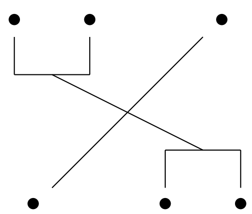
$$abc = cba$$

for all triples (a, b, c) of generators, which is easily seen to be given by the following partition:



M. Weber was able to show that any *non-hyperoctahedral* (see [Web13, Thm 3.12] for the definition) easy quantum group is either classical, non-crossing, or obtained from those through half-liberation, making a total of thirteen easy quantum groups. It therefore remains to classify hyperoctahedral quantum groups, i.e. those whose category of partitions contain the four-block (the partition consisting in four points which are all connected) but not the double singleton (the partition consisting in two points which are not connected).

The hyperoctahedral case was then completely classified by S. Raum and M. Weber in a series of papers (we refer the reader to [RW16] and references therein for more details). Again, the strategy relies on the definition of a specific partition which helps distinguishing fundamental properties of the associated quantum groups. This is the *pair positioner partition*:



It leads to the following dichotomy in the hyperoctahedral case:

- If $\mathcal{C}$ contains the pair positioner partition then it is called *group-theoretical.* It can be decomposed as a kind of semi-direct product of a discrete group acting on the algebra $C(S_N)$ of complex-valued functions on S_N. Once again, this can be seen as a commutation relation: any four-block partition can be moved along the points of any partition in $\mathcal{C}$.
- If $\mathcal{C}$ does not contain the pair positioner partition, then $\mathcal{C}$ is called an *interpolating category.* One then defines a numerical invariant, the maximum of the *wdepth* of $\mathcal{C}$, and shows that it completely

classifies the interpolating categories. In other words, there is just one extra series

$$H_N^{[s]},\ 1 \leqslant s \leqslant \infty$$

besides non-hyperoctahedral and group-theoretical quantum groups. Note that the maximum of the wdepth is a local invariant which can be encoded by a partition π_k which we will not draw here.

Remark 3.3. There is a precise characterization in [RW15] of those compact quantum groups of semi-direct product type which are easy. The non-easy ones exhibit an interesting "skew-easy" structure investigated by L. Maaßen in [Maa19]. To understand their structure, L. Maaßen introduces a variant of categories of partitions called *skew categories of partitions*. Using a suitable adaptation of the definition of the operators T_p, she is then able to completely classify all compact quantum groups of semi-direct product type.

3.2. **Unitary easy quantum groups.** It is tempting, in view of the full classification of orthogonal easy quantum groups achieved by S. Raum and M. Weber, to hope for a similar classification in the unitary case, i.e. with the colour set $\mathcal{A} = \{\circ, \bullet\}$ with $\circ^{-1} = \bullet$. Things turn out, however, to be more complicated.

3.2.1. *The non-crossing case.* A starting idea is to try to refine the invariants $BS(\mathcal{C})$ and $BN(\mathcal{C})$ from the orthogonal case. P. Tarrago and M. Weber did this in [TW18] by defining three new numerical invariants called the *colouring parameters*[3]. Let us start with a convenient definition.

Definition 3.4. Let p be a partition of $\{1, \cdots, k\}$. A sub-partition q (i.e. a union of blocks of p) is said to be *full* if, up to a rotation of p, it is a partition of $\{a, \cdots, a+b\}$ for some $1 \leqslant a \leqslant a+b \leqslant k$.

Definition 3.5. Let $\mathcal{C}$ be a category of non-crossing partitions coloured with $\{\circ, \bullet\}$. For a partition $p \in \mathcal{C}$ lying on one line, let us denote by $c(p)$ the difference between the number of white points and the number of black points, called the *colour sum* of p. Then,

- The *global colouring parameter* of $\mathcal{C}$ is the minimum $k(\mathcal{C})$ of the numbers $|c(p)|$ for all partitions in $p \in \mathcal{C}$ lying on one line,
- The *first local colouring parameter* of $\mathcal{C}$ is the minimum $d_{\circ\bullet}(\mathcal{C})$ of the numbers $c(p)$ for all full sub-partitions p appearing between two connected points with different colours of a partition in $\mathcal{C}$ lying on one line,
- The *second local colouring parameter* of $\mathcal{C}$ is the minimum $d_{\bullet\bullet}(\mathcal{C})$ of the numbers $c(p)$ for all full sub-partitions p appearing between two connected points with the same colour of a partition in $\mathcal{C}$ lying on one line.

Using these, they were able to classify categories of non-crossing unitary easy quantum groups

Theorem 3.6 (Tarrago-Weber)**.** *Let us say that a category of coloured partitions is* globally colourised *if it is stable under inversion of colours in the partitions, and* locally colourised *otherwise. Then, the quadruple*

$$(BS(\mathcal{C}), k(\mathcal{C}), d_{\circ\bullet}(\mathcal{C}), d_{\bullet\bullet}(\mathcal{C}))$$

is a complete invariant for globally colourised categories of non-crossing partitions, as well as for locally colourised ones.

Proof. The proof is of course extremely involved and out of our scope. Let us simply mention the rough strategy. Given the invariants, one builds partitions "encoding" them and proves that they must belong to $\mathcal{C}$. Then, one shows that $\mathcal{C}$ is in fact generated by these partitions. □

Interestingly, P. Tarrago and M. Weber showed in [TW17] that the previous combinatorial invariants translate into operations generalizing the free complexification operation introduced by T. Banica in [Ban08].

Definition 3.7. Let $\mathbb{G}$ be a compact quantum group with a fundamental representation u and let d be an integer. The *d-free complexification* of $\mathbb{G}$ is the compact quantum group given by the C*-subalgebra of $C(\mathbb{G}) * C(\mathbf{Z}_d)$ generated by the coefficients of uz (where z denotes the fundamental representation of $\mathbf{Z}_d$) together with the restriction of the coproduct. The *d-tensor complexification* is obtained similarly using $C(\mathbb{G}) \otimes C(\mathbf{Z}_d)$. Eventually, the image of the d-free complexification in the quotient of $C(\mathbb{G}) * C(\mathbf{Z}_d)$ by the relations

$$(u_{ij}z^r)^* = u_{ij}z^r$$

for $r \mid d$, is called the *r-self-adjoint d-free complexification.*

[3]The names and notations here are ours and differ from those given in [TW18].

The classification of [TW18] can then be restated in the following way:

Theorem 3.8 (Tarrago-Weber)**.** *The non-crossing partition unitary quantum groups are everything which can be obtained from orthogonal easy quantum groups and free wreath products by applying (r-self-adjoint) d-free and d-tensor complexifications.*

3.2.2. *Crossing pair partitions.* If crossings are allowed, then the situation becomes much more involved, even in the simplest instance $BS(\mathcal{C}) = \{2\}$. Indeed, in the orthogonal case there are only three corresponding quantum groups, namely O_N, O_N^+ and the half-liberation O_N^* (see [BV10] for details) while in the unitary case the classification of all easy quantum groups sitting in between U_N and U_N^+ was only recently obtained by A. Mang and M. Weber in [MW19a] and [MW19b].

To see the difficulty, simply wonder at the following question: what is the analogue of half-liberation for non-self-adjoint generators? Should one consider the relation

$$abc = cba$$

just for the generators u_{ij} or also for their adjoints? And what about the relations

$$ab^*c = cb^*a$$

or other variants? The solution is to think about the problem from another angle. Let us get back to numerical invariants and, writing $\mathcal{P}_2^{\circ,\bullet}$ for the set of coloured pair partitions, define the following:

Definition 3.9. Let us say that a *sector* in a partition $p \in \mathcal{P}_2^{\circ\bullet}$ lying on one line is a full sub-partition whose endpoints are connected. We define the *sector colour number* $\sigma(\mathcal{C})$ of $\mathcal{C}$ to be the minimum of the numbers $|c(p)|$ for all sectors p of partitions in $\mathcal{C}$ such that $c(p) \neq 0$. If there is no such partition, we set $\sigma(\mathcal{C}) = 0$.

A. Mang and M. Weber showed in [MW19a, Prop 8.1] that $\sigma(\mathcal{C})$ in a sense classifies "half" of the easy quantum groups between U_N and U_N^+. To classify the other half, one needs an additional invariant:

Definition 3.10. The *colour semi-group* of $\mathcal{C} \subset \mathcal{P}_2^{\circ\bullet}$ is the set $D(\mathcal{C})$ of all values of $c(p)$, where p is a full sub-partition of a partition in $\mathcal{C}$ whose endpoints belong to two blocks which cross.

As the name indicates, $D(\mathcal{C})$ is a sub-semi-group of $(\mathbf{N}, +)$ as proven in [MW19b, Prop 7.14]. Moreover, by [MW19a, Prop 8.1] and [MW19b, Thm 8.3], this is enough to complete the classification:

Theorem 3.11 (Mang-Weber)**.** *The pair $(\sigma(\mathcal{C}), D(\mathcal{C}))$ is a complete invariant for categories of partitions $NC_2^{\circ\bullet} \subset \mathcal{C} \subset \mathcal{P}_2^{\circ\bullet}$.*

Proof. Let us simply mention that, once again, the spirit of the proof is to translate the invariants into partitions. Here, this gives rise to a large family of so-called *bracket partitions* with a rich combinatorial structure. One very nice feature of these partitions is that they are rotations of partitions giving half-liberation-type relations. In a sense, this quarter turn rotation was the key to the classification. □

There are also classification results beyond pair partitions, for instance by D. Gromada for the so-called *globally colourised* categories of partitions in [Gro18]. It may even be hoped that within a reasonable time a complete classification will be obtained.

3.3. **A brief summary.** We have encountered a number of tools so far for proving classification theorems, and we would like to pause a moment to look at their main features. We have somehow three types of tools: local invariants, global invariants and specific partitions implementing "commutation relations". Let us work one-by-one using the translation of these invariants into relations in the compact quantum group as explained in [TW18].

- In the unitary case, global invariants consist in the global colouring property, the uncoloured invariant BN and the global colouring number $k(\mathcal{C})$. The latter is intimately linked to the group of one-dimensional representations of the compact quantum group. The latter is usually isomorphic to $\mathbf{Z}_{k(\mathcal{C})}$ or to a quotient of it.
- As for local invariants, they first give through BS a splitting into four subclasses in which we can work separately. Then, $d_{\circ\bullet}(\mathcal{C})$ gives commutation relations: the subgroup of the group of one-dimensional representations commuting with the fundamental representation is exactly $\mathbf{Z}_{d_{\circ\bullet}(\mathcal{C})}$. And the invariant $d_{\bullet\bullet}(\mathcal{C})$ gives r-self-adjointness, which is a kind of twisted commutation relation.

- We are left with the positioner partition which is somehow by definition a commutation relation. Note that it can nevertheless be exchanged for a local invariant in the non-crossing case in the following way[4] : define the *Interval Number* $IN(\mathcal{C})$ of a category of partition $\mathcal{C}$ to be the set of integers n such that there exists a partition p in $\mathcal{C}$ with two connected points separated by n points. Then, $IN(\mathcal{C})$ equals either $2\mathbf{N}$ or $\mathbf{N}$ and the second case occurs if and only if $\mathcal{C}$ contains the positioner partition.

As the term *easy* suggests, one reason for the definition of easy quantum groups is that they should be more amenable to classification because we can resort to the rich combinatorics of partitions. One may nevertheless wonder for other classes of quantum groups to classify. It turns out that even building non-easy quantum groups is not a simple task in general and only recently were new families of quantum groups defined and classified.

More generally, any compact quantum group $S_N \subset \mathbb{G} \subset O_N^+$ is determined by a family of linear combinations of partitions. It is extremely difficult to find explicit linear combinations which do not yield a genuine category of partitions in the end, as the following open problem shows:

Question. *Is there a compact quantum group $S_N \subset \mathbb{G} \subset S_N^+$ for*[5] *$N \geqslant 6$? In other words, given all non-crossing partitions and a linear combination of crossing ones, can one always build all partitions?*

The first progress in this direction is a very recent work of D. Gromada and M. Weber [GW19], which is doubly interesting. First, it describes quantum groups which are not easy but with explicit linear combinations of partitions generating their intertwiners, paving the way for a deeper study of these objects. Second, the examples were obtained through computer assisted computations, and this suggests to further investigate the potential of computers in the study of such combinatorial quantum groups.

4. Two colours, but not the same

In this section, we will deal with a problem which, although it looks quite close to the previous one, yields an apparently different type of classification. It is concerned with partition quantum groups associated to categories of non-crossing partitions coloured by a set $\mathcal{A} = \{x, y\}$ with $x^{-1} = x$ and $y^{-1} = y$ (and some general results for an arbitrary colour set). This means that these quantum groups are naturally quotients of $O_N^+ * O_N^+$, and we will see that they form a large family with many interesting new examples. More importantly, the method is different from the ones explained in the previous section, and we believe that it may complement them in the general classification program (see Section 5 for more details).

4.1. **From partitions to representation theory.** The strategy heavily relies upon the work [FW16] linking non-crossing partitions to the representation theory of the corresponding compact quantum group. We therefore start by briefly reviewing these results, the fundamental object being the following:

Definition 4.1. A partition p is said to be *projective* if $pp = p = p^*$. The corresponding map T_p is then a scalar multiple of a projection.

An important fact, proved in [FW16, Prop 2.18], is that for any partition r, r^*r (hence also rr^*) is always projective. Based on this fact and the analogy with projections in Hilbert spaces, we will say that a projective partition p is *equivalent* to another projective partition q if there exists a third partition r such that

$$p = r^*r \text{ and } q = rr^*.$$

In [FW16, Sec 4], given a category of coloured non-crossing partitions, we associate to any projective partition $p \in \mathcal{C}$ and integer N a unitary representation u_p of $\mathbb{G}_N(\mathcal{C})$ in such a way that the following properties are satisfied (see [FW16, Prop 4.15, Thm 4.18 and Prop 4.22] and [Fre14, Lem 5.1] for proofs):

1. u_p is irreducible for all p,
2. Any irreducible representation of $\mathbb{G}_N(\mathcal{C})$ is equivalent to u_p for some p,
3. u_p is one-dimensional if and only if $t(p) = 0$, where $t(p)$ denotes the number of *through-blocks* of p,
4. $u_p \sim u_q$ if and only if $p \sim q$.

Remark 4.2. There is also an explicit formula for the fusion rules, given in [FW16, Thm 4.27], but we will not use it in the sequel.

[4] We are grateful to the anonymous referee for suggesting that invariant to us

[5] The answer is negative for $N \leqslant 3$ because $S_N^+ = S_N$ in that case, for $N = 4$ by [BB09] and for $N = 5$ by [Ban21].

4.2. **The classification.**

4.2.1. *Three constructions.* To explain our strategy, let us go back to the easy orthogonal case. The local invariant BS gives us a set of four base cases, namely

$$\mathcal{S} = \{O_N^+, B_N^+ * \mathbf{Z}_2, H_N^+, S_N^+ \times \mathbf{Z}_2\}.$$

By this, we mean that these are the largest partition quantum groups whose category of partitions has a prescribed BS invariant. Then, we refine with the global invariant BN which distinguishes whether the copy of $\mathbf{Z}_2$ disappears, which is equivalent to deciding whether a certain one-dimensional representation is trivial. This suggests to consider more generally relations at the level of one-dimensional representations. Let us give it a name for convenience:

Definition 4.3. A compact quantum group $\mathbb{H}$ is said to be a quotient of a compact quantum group $\mathbb{G}$ *by group-like relations* if $C(\mathbb{H})$ is a quotient of $C(\mathbb{G})$ by a closed ideal generated by elements of the form

$$x - 1,$$

where $x \in C(\mathbb{G})$ is a group-like element.

The key observation is that this operation leaves the class of non-crossing partition quantum groups invariant, as shown in [Fre19b, Prop 3.8]:

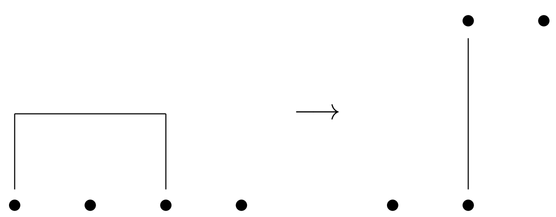
Proposition 4.4. *If $\mathbb{G}$ is a non-crossing partition quantum group, then any quotient of $\mathbb{G}$ by group-like relations is again a non-crossing partition quantum group.*

Proof. If x is a group-like element that we want to make trivial, let us take a partition p such that $u_p = x$. Because $t(p) = 0$, we can find a partition b lying on one line such that $p = b^*b$. Then simply consider the category of partitions generated by that of $\mathbb{G}$ and b. □

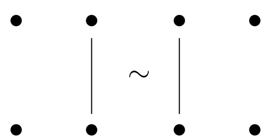
This yields the first step of our strategy. Let $\mathcal{C}$ be a category of non-crossing coloured partitions, and let $\mathcal{C}' \subset \mathcal{C}$ be a subcategory containing all the projective partitions of $\mathcal{C}$. Then, if $p \in \mathcal{C} \setminus \mathcal{C}'$, we can rotate it on one line to obtain a new partition p', and consider $q = p'^*p'$. By assumption, $q \in \mathcal{C}'$ and adding p to $\mathcal{C}'$ is the same as adding the relation $u_q = 1$ to $\mathbb{G}_N(\mathcal{C}')$. Thus, we can recover $\mathbb{G}_N(\mathcal{C})$ from $\mathbb{G}_N(\mathcal{C}')$ by adding group-like relations.

This reduces the classification to categories of partitions which are generated by their projective partitions. Instead of trying to list them, we will push the previous idea further and try to find other constructions which preserve the class of partition quantum groups so that we could reduce the classification to a *generating set* like $\mathcal{S}$.

Let us consider again the orthogonal easy case to get some inspiration. The last ingredient in the classification was the positioner partition and we already mentioned that it corresponds to a kind of commutation property of the category of partitions. To make this rigorous, let us rotate it



This rotated version implements an equivalence



exactly meaning that the fundamental representation $u_{|}$ of B_N^+ commutes with the non-trivial one-dimensional representation $u_{:}$ corresponding to $\mathbf{Z}_2$. Abstracting the idea yields to the following definition:

Definition 4.5. A compact quantum group $\mathbb{H}$ is said to be a quotient of a compact quantum group $\mathbb{G}$ *by commutation relations* if $C(\mathbb{H})$ is a quotient of $C(\mathbb{G})$ by a closed ideal generated by elements of the form

$$xv_{ij} - v_{ij}x$$

for all $1 \leqslant i, j \leqslant \dim(v)$, where $x \in C(\mathbb{G})$ is a group-like element and v is a representation of $\mathbb{G}$.

Let us highlight the difference with group-like relations. If $\mathcal{C}$ denotes the category of representations of $B_N^+ * \mathbf{Z}_2$ and p is the positioner partition, then

$$\langle \mathcal{C}, p\rangle = \langle \mathcal{C}, pp^*\rangle$$

so that this new category of partitions is still generated by its projective partitions, even though the partition we added to it is not projective. The reason for this is that any category of partitions containing pp^* must contain p, hence u_{pp^*} must be the trivial representation there. As a consequence, adding p does not produce a quotient by group-like relations. The important fact is of course that, like for group-like relations, commutation relations preserve the partition structure.

Proposition 4.6. *If $\mathbb{G}$ is a non-crossing partition compact quantum group, then any quotient of $\mathbb{G}$ by commutation relations is again a non-crossing partition compact quantum group.*

Proof. Given v and x, we take partitions p and q such that $u_p \sim v$ and $u_q = x$. Writing $q = b^*b$ with b lying on the upper row, the commutation relation is equivalent to the fact that $T_{b\otimes p\otimes b^*}$ is an intertwiner, hence the quotient is given by the category of partitions

$$\mathcal{C}' = \langle \mathcal{C}, b\otimes p\otimes b^*\rangle .$$

□

We have formalized and generalized now both the global invariants (group-like relations) and the commutation partitions (commutation relations), but this is not enough. Indeed, these constructions do not increase the number of colours in the colour set, so that starting with $\mathcal{S}$ we will remain in the class of orthogonal easy quantum groups. To be able to build new objects, we must combine elements of $\mathcal{S}$. One of the most natural ways of doing this is through free product constructions, and it turns out that we can even throw in amalgamation in a broad sense.

Assume that we are given two compact quantum groups $\mathbb{G}_1$ and $\mathbb{G}_2$ together with a third compact quantum group $\mathbb{H}$ and embeddings

$$i_k : C(\mathbb{H}) \hookrightarrow C(\mathbb{G}_k)$$

intertwining the coproducts. One can then construct the amalgamated free product by quotienting $C(\mathbb{G}_1) * C(\mathbb{G}_2)$ by the closed ideal generated by $i_1(C(\mathbb{H})) - i_2(C(\mathbb{H}))$ (see [Wan95] for details). But if the (non-amalgamated) free product $\mathbb{G}_1 * \mathbb{G}_2$ has one-dimensional representations, then one may twist, say, $i_1(C(\mathbb{H}))$ before identifying it with $i_2(C(\mathbb{H}))$, leading to the following notion:

Definition 4.7. With the previous notations, a compact quantum group $\mathbb{G}$ is said to be a *twisted amalgamated free product* if it is the quotient of $C(\mathbb{G}_1) * C(\mathbb{G}_2)$ by a closed ideal generated by

$$xi_1(C(\mathbb{H}))x^{-1} - i_2(C(\mathbb{H}))$$

for some group-like element $x \in C(\mathbb{G}_1) * C(\mathbb{G}_2)$.

An example (without twisting) will be treated in detail in Section 4.3.1. Once again, this operation can be encoded with partitions as soon as the embedding satisfies some kind of compatibility.

Proposition 4.8. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be non-crossing partition compact quantum groups and let $\mathbb{H}$ be a common quantum subgroup. Assume that there is a generating set $X = \{v_1, \cdots, v_n\}$ of irreducible representations of $\mathbb{H}$, and partitions $p_1, \cdots p_n, p'_1, \cdots p'_n$ such that for all $1 \leqslant i \leqslant n$, $t(p_i) = t(p'_i)$ and*

$$i_1(v_i) = u_{p_i} \text{ and } i_2(v_i) = u_{p'_i}.$$

Then, any twisted amalgamated free product over $\mathbb{H}$ is a non-crossing partition compact quantum group.

Proof. Let us first mention that to build the free product, one takes disjoint copies $\mathcal{A}_1$ and $\mathcal{A}_2$ of the colour sets of $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively, and then considers the category of partitions generated by $\mathcal{C}_1^{\mathcal{A}_1}$ and $\mathcal{C}_2^{\mathcal{A}_2}$ in $P^{\mathcal{A}_1\sqcup\mathcal{A}_2}$.

Now for a fixed $1 \leqslant i \leqslant n$, because $t(p_i) = t(p_i')$, we can "graft"[6] p_i and p_i' by gluing the upper row of the former to the lower row of the latter. If r_i denotes the resulting partition, and if $q = b^*b$ represents x, then the twisted amalgamated free product corresponds to the category of partitions

$$\mathcal{C}' = \langle \mathcal{C}, b \otimes r_1 \otimes b^*, \cdots, b \otimes r_n \otimes b^* \rangle .$$

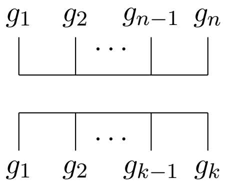
□

Remark 4.9. The condition in the statement is necessary for the construction of the proof to work. It turns out moreover that if the factors are in $\mathcal{S}$, then it is always satisfied. It remains open however whether it is always satisfied under sole assumption that $\mathbb{G}_1$ and $\mathbb{G}_2$ are non-crossing partition quantum groups.

The surprising fact is that the three operations above are in a sense enough to describe all non-crossing partition quantum groups on the colour set $\mathcal{A}$. More precisely, we claim that all non-crossing partition quantum groups on two self-inverse colours can be built from the three operations. Of course, such a statement is vacuous as long as the generating set is not given. The best one may hope is certainly to start with $\mathcal{S}$. This is in fact almost true, except that we miss one family which is a generalization of the free wreath product construction. Let us introduce it first.

4.2.2. *Free wreath products of pairs and the classification.* Given a discrete group Γ and a symmetric generating set $S \subset \Gamma$ containing the neutral element, one may consider the category $\mathcal{C}_{\Gamma,S}$ of all non-crossing partitions coloured by S (with the involution given by inversion in the group) such that in each block, the product of the elements in the upper row equals the product of the elements in the lower row. It was proven by F. Lemeux in [Lem15] that the corresponding quantum group is the *free wreath product* $\widehat{\Gamma} \wr_* S_N^+$ introduced by J. Bichon in [Bic04]. The representation theory of these objects is well-known, and in particular they have no non-trivial one-dimensional representation. There is however a rather natural way of adding one-dimensional representations.

Let $\lambda \in \Gamma$, which can be written as $\lambda = g_1 \cdots g_n$, on the generators in S and consider the partition

$\beta_\lambda =$ 

If we apply Theorem 2.8 to the category of partitions generated by β_λ and $\mathcal{C}$, then we get a quotient of the free wreath product with a non-trivial one-dimensional representation, namely u_{β_λ}. Doing this for all the elements of a fixed subgroup $\Lambda \subset \Gamma$ produces the *free wreath product of the pair* (Γ, Λ), introduced in [Fre19b] and denoted by $H_N^{++}(\Gamma, \Lambda)$. By construction, these are partition quantum groups and cannot be obtained by the previous operations. For instance, consider the free wreath product

$$(\mathbf{Z}_2 * \mathbf{Z}_2) \wr_* S_N^+ \simeq H_N^+ \underset{S_N^+}{*} H_N^+.$$

Any subgroup Λ of the infinite dihedral group $\mathbf{Z}_2 * \mathbf{Z}_2$ gives rise to a free wreath product of a pair which is a non-crossing partition quantum group on two self-inverse colours and whose C*-algebra is a quotient of that of a free product. However, since there is no one-dimensional representation in the free product to quotient by, and amalgamation would just yield another free wreath product, this object cannot be obtained by the constructions of Section 4.2.1.

In fact, free wreath products of pairs form a closed family of compact quantum groups in a strong sense, as the following result proven in [Fre19b, Prop 3.18] shows:

[6]We will make this process more formal later on in Section 5 under the term "grafting" and use it to explore possible extensions of these results.

Proposition 4.10. *Let $\mathbb{G}$ be a non-crossing partition quantum group which is a quotient of $H_N^{++}(\Gamma, \Lambda)$, then there exists a group $\Lambda \subset \widetilde{\Lambda} \subset \Gamma$ and a normal subgroup $\Lambda_0 \subset \widetilde{\Lambda}$ such that*

$$\mathbb{G} \simeq H_N^{++}(\Gamma/\Lambda_0, \widetilde{\Lambda}/\Lambda_0).$$

The previous result leads to the second best statement one can make, and this one happily holds and is the content of the article [Fre19b].

Theorem 4.11. *Any non-crossing partition quantum group on two self-inverse colours is either*

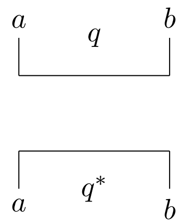
- *Obtained from the set $\mathcal{S} = \{O_N^+, B_N^+ * \mathbf{Z}_2, H_N^+, S_N^+ \times \mathbf{Z}_2\}$ using twisted amalgamation, commutation relations and group-like relations,*
- *Or a free wreath product of a pair.*

Sketch of proof for a special case. The proof is extremely involved and covers the entire article [Fre19b], we will therefore not explain it here. Let us nevertheless illustrate how it works in a simple case. Consider a category of non-crossing partitions $\mathcal{C}$ with $BS(\mathcal{C}) = \{1, 2\}$ and containing double singletons coloured both by x and y. The corresponding compact quantum group is a quotient of $(B_N^+ * \mathbf{Z}_2) * (B_N^+ * \mathbf{Z}_2)$ and we will denote by $\mathcal{C}_0 \subset \mathcal{C}$ the category of partitions of this free product.

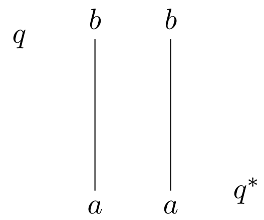
Let $p \in \mathcal{C} \setminus \mathcal{C}_0$ be a projective partition. A straightforward induction shows that p is a horizontal concatenation of the form

$$p = (b_0^* b_0) \otimes | \otimes (b_1^* b_1) \otimes | \otimes \cdots \otimes (b_{n-1}^* b_{n-1}) \otimes | \otimes (b_n^* b_n)$$

An easy lemma (see [Fre19b, Prop 3.5]) shows that $b_i^* b_i \in \mathcal{C}$ for all $0 \leqslant i \leqslant n$. Moreover, up to decomposing again using horizontal concatenations, we may assume that b_i is a sector, so that $b_i^* b_i$ has the form



for some partition q lying on the upper row. Rotating then yields



As a consequence, adding $b_i^* b_i$ to $\mathcal{C}_0$ is the same as either quotienting $C(\mathbb{G}_N(\mathcal{C}_0))$ by commutation relations (if $a = b$) or performing a twisted amalgamation (if $a \neq b$). One then only has to prove that p can be reconstructed from the partitions $b_i^* b_i$ (see for instance the proof of [Fre19b, Thm 5.10]) to conclude that there exists a category of partitions $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}$ containing p such that it is obtained from the initial free product by our operations. Iterating this construction, we end up with a category of partitions

$$\mathcal{C}_0 \subset \widetilde{\mathcal{C}} \subset \mathcal{C}$$

satisfying the following properties:

- Any projective partition of $\mathcal{C}$ lies in $\widetilde{\mathcal{C}}$,
- $\mathbb{G}_N(\widetilde{\mathcal{C}})$ is obtained from $\mathbb{G}_N(\mathcal{C}_0)$ by twisted amalgamation and quotienting by commutation relations.

Now, any additional partition in $\mathcal{C} \setminus \widetilde{\mathcal{C}}$ gives a group-like relation, and the proof is complete. □

The previous result can be turned, with some extra work (see Section 4.3 for a glimpse), into a list of all possible categories of partitions. This is based on the fact that we know all quantum subgroups of elements of $\mathcal{S}$ and that the groups of one-dimensional representations are always dihedral groups, of which all the subgroups are known. That list, given in [Fre19b, Thm 8.1], is roughly one page long and not very enlightening. On the contrary, Theorem 4.11 suggests the following question:

Question. *Can any non-crossing partition quantum group on an arbitrary number of colours which are all their own inverses be obtained from the set $\mathcal{S}$ using twisted amalgamation, commutation relations and group-like relations as soon as it is not a free wreath product of a pair?*

We will come back to this question in Section 5.

4.2.3. *To unitarity and beyond.* What if we now tried this approach for the general case? If we allow different colours to be inverse to one another, we need at least an additional ingredient: the free complexification operations of P. Tarrago and M. Weber explained in Section 3.2.1. This leads to the following question:

Question. *Can any non-crossing partition quantum group be obtained from the set $\mathcal{S}$ and the free wreath products of pairs using twisted amalgamation, commutation relations, group-like relations and complexifications?*

Let us see what we can precisely say in the two-colour case. Considering the categories of partitions given by P. Tarrago and M. Weber in [TW18] we first see that in the globally colourised case, everything can indeed be obtained by taking the tensor complexification and adding group-like relations. Moreover, one cannot add commutation relations since the group of one-dimensional representations is already central. As for the locally colourised case, here is a series of observations:

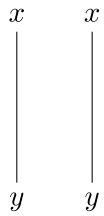
(1) In the case $BS(\mathcal{C}) = \{2\}$, the only possibility is U_N^+, which is the free complexification of O_N^+.
(2) The case $BS(\mathcal{C}) = \{1, 2\}$ involves the compact quantum group C_N^+ whose category of partitions is generated by the singletons. Let us first consider the category generated by the double white singleton. The corresponding compact quantum group is easily seen to be isomorphic to $U_{N-1}^+ * \mathbf{Z}$ and we obtain C_N^+ by adding the group-like relation making the $\mathbf{Z}$ factor trivial. Everything is then obtained by adding commutation relations, group-like relations and a twisted commutation relation corresponding to the r-self-adjointness.
(3) In the case $BS(\mathcal{C}) = 2\mathbf{N}$, we already know that we have the quantum reflection groups $H_N^{s+} = \mathbf{Z}_s \wr_* S_N^+$, which are free wreath products of pairs with Λ being trivial. Moreover, the comparison of the categories of partitions of these compact quantum groups, given in [TW18, Sec 7], and the categories of partitions of a free wreath product of a pair recalled in Section 4.2.2 shows that we have the following isomorphisms for $d \mid k$:
$$\left(\mathbf{Z}_d \wr_* S_N^+\right) \tilde{\times} \mathbf{Z}_k \simeq H_N^{++}(\mathbf{Z}_k, d\mathbf{Z}_k).$$
Hence, all complexifications of free wreath products are free wreath products of pairs. Note that free complexification is also possible but yields the same compact quantum group except for H_N^+.
(4) Eventually, in the case $BS(\mathcal{C}) = \mathbf{N}$, one simply starts with $S_N^+ \times \mathbf{Z}_2$ and after freely complexifying it, adds commutation relations and group-like relations.

4.3. **What about invariants?** In this exposition, we have purposely emphasised the difference between the methods of [Fre19b] and the ones of previous similar works, based on invariants. This does not mean however that invariants disappeared. For instance, the classification in [Fre19b] starts by splitting into four cases according to the local invariant BS, and this is even refined according to the sizes of blocks with only one colour.

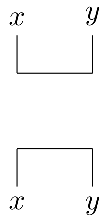
Furthermore, if one wants a finer understanding of specific examples, then it is necessary to go back to the language of invariants. This is the strategy used in [Fre19a] to compute the representation theory of some non-crossing partition quantum groups and we will now explain it to illustrate our point.

4.3.1. *A global invariant.* Let us start with the free product $O_N^+ * O_N^+$ and consider the common quantum subgroup PO_N^+ generated by the tensor square of the fundamental representation. Performing amalgamation yields the compact quantum group
$$O_N^{++} = O_N^+ \underset{PO_N^+}{*} O_N^+.$$
Note that the amalgamation cannot be twisted since O_N^+ has no non-trivial one-dimensional representations. However, O_N^{++} does have such representations. To see this, let us first denote by x the colour corresponding to

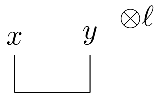
the first copy of O_N^+ and by y the colour corresponding to the second one. Following the proof of Proposition 4.8, we see that the category of partitions of O_N^{++} is generated by



which after rotation yields



The latter partition implements a non-trivial one-dimensional representation s which is easily proven to have infinite order (see [Fre19a, Lem 3.2]). By Theorem 4.11, all we can build from this are the quotients by the group-like relations $s^\ell = 1$ for $\ell \in \mathbf{N}$, yielding compact quantum groups denoted by $O_N^{++}(\ell)$. This is of course equivalent to adding the partition



If one wants to compute the representation theory of $O_N^{++}(\ell)$, the description of the category of partitions with generators is not practical, since we need to classify projective partitions up to equivalence. It would be better to describe this category through a global invariant.

This is doable once one realizes that global invariants should not be numbers but rather groups or semi-groups. Indeed, the block numbers form a sub-semi-group of $\mathbf{N}$. Even better, the global colouring of partitions in the unitary case, that is to say the number of white points minus the number of black points, form a subgroup of $\mathbf{Z}$ and the global colouring parameter is just a generator of this subgroup. Indeed, given a word w on $\{\circ\bullet\}$, we can define an element $\varphi(w)$ of $\mathbf{Z}$ by sending $\circ$ to 1 and $\bullet$ to -1. If a partition p has upper colouring w and lower colouring w', we then set

$$\varphi(p) = \varphi(w)\varphi(w')^{-1}.$$

The same idea works here, except that because $x^{-1} = x$ and $y^{-1} = y$, the invariant will be a subgroup of $\mathbf{Z}_2 * \mathbf{Z}_2$.

Definition 4.12. To a word w on $\{x, y\}$ we associate an element $\varphi(w) \in \mathbf{Z}_2 * \mathbf{Z}_2$ by sending x to the first generator and y to the second one. If a partition p has upper colouring w and lower colouring w', we then set

$$\varphi(p) = \varphi(w)\varphi(w')^{-1}.$$

Eventually, we define $\mathcal{D}_\ell$ to be the category of all non-crossing pair partitions coloured with $\{x, y\}$ such that

$$\varphi(p) \in \left\langle (xy)^\ell \right\rangle \subset \mathbf{Z}_2 * \mathbf{Z}_2.$$

One easily checks that $\mathcal{D}_\ell$ is indeed a category of partitions and, by construction, $\mathcal{C}_\ell \subset \mathcal{D}_\ell$. In view of the classification given in Theorem 4.11, they should be equal, and this is indeed the case, see for instance [Fre19a, Cor 3.7]. The point, however, is that we *do not need* this global invariant at any point in the classification.

4.3.2. *Local invariants.* As an example of a local invariant, we now consider free wreath products of pairs. Once again, the numerical invariants can be seen as groups. This is also true for the local colouring parameters of P. Tarrago and M. Weber, as explained in [TW17, Lem 2.14]. Here is the local invariant we need.

Definition 4.13. Let $\mathcal{C}$ be a category of coloured non-crossing partitions. Its *local subgroup invariant* is the subgroup generated by $\varphi(p)$ for all full sub-partitions p of a partition in $\mathcal{C}$.

Here we mean that this invariant is a subgroup of the free product of one copy of $\mathbf{Z}_2$ for each self-inverse colour, and one copy of $\mathbf{Z}$ for each other pair of mutually inverse colours, with the obvious extension of the map φ. The key idea is that if p_λ denotes the upper block of β_λ, then $\varphi(p_\lambda) = \lambda$, so that one may hope to recover Λ as the local subgroup invariant. This is the case and the proof uses the following definition:

Definition 4.14. Let Γ be a discrete group with a symmetric generating set S and let $\Lambda \subset \Gamma$ be a subgroup. We define $\mathcal{D}_{\Gamma,\Lambda,S}$ to be the set of all partitions $p \in NC^S(w, w')$ such that

- $\varphi(w) = \varphi(w')$ as elements of Γ,
- For any full sub-partition of p, $\varphi(p) \in \Lambda$.

Note that it is not obvious that $\mathcal{D}_{\Gamma,\Lambda,S}$ is a category of partitions, because one has to prove that the local condition is preserved under vertical concatenation. This was proven in [Fre19a, Lem 4.2] and one deduces from this the expected result (see [Fre19a, Cor 4.5]):

Proposition 4.15. *The compact quantum group associated to $\mathcal{D}_{\Gamma,\Lambda,S}$ is the free wreath product of the pair (Γ, Λ).*

5. Prospects

The results of the previous sections can suggest conjectures for the case of more colours. However, nothing guarantees for the moment that we will not need more operations than just twisted amalgamated free products, commutation relations and group-like relations. Moreover, the dichotomy between free wreath products of pairs and the rest is somewhat annoying.

We will introduce in this section an operation on partitions called the *grafting operation*. It is very simple, but as we will see, it enables to restate all the classification results for non-crossing partition quantum groups in a synthetic way. As will soon become apparent, it also offers a great deal of flexibility, which is practical for further investigations in the classification program.

5.1. Grafting partitions.

5.1.1. *The definition.* From now on, $\mathcal{C}$ denotes a category of coloured partitions and $N \geqslant 4$ is a fixed integer. We will use the following elementary operation on projective partitions, inspired from the proof of Proposition 4.8:

Definition 5.1. Let $p, q \in \mathcal{C}$ be projective partitions with $t(p) = t(q)$.

- The *graft* of p and q is the non-crossing partition $r(p, q)$ obtained by gluing the upper row of p with the lower row of q,
- A category of non-crossing coloured partitions $\mathcal{D}$ is said to be obtained by *grafting* from $\mathcal{C}$ if it is generated by $\mathcal{C}$ and grafts of partitions in $\mathcal{C}$.

We can make the definition more formal using the notion of *through-block decomposition* in the sense of [FW16]. Indeed, it was proven in [FW16, Prop 2.9] that any non-crossing partition s decomposes uniquely as $s_l^* s_u$ with s_l and s_u being *building partitions*. The latter means that they are non-crossing partitions such that each upper point belongs to a different through-block. With this in hand, we have

$$r(p, q) = q_l^* p_u.$$

Let us now make a crucial observation: any non-crossing coloured partition is a graft.

Proposition 5.2. *For any partition s, we have*

$$s = r(s^* s, s s^*).$$

Proof. It follows from [FW16, Lem 2.11] that if $s = s_l^* s_u$ is the through-block decomposition of s, then $s^* s = s_u^* s_u$ and $s s^* = s_l^* s_l$. □

This suggests that any category of non-crossing partitions can be obtained from "basic" ones using graftings and the classification problem can therefore be split into two parts:

(1) Find a set $\mathcal{S}$ of categories of partitions from which all non-crossing partition quantum groups can be obtained by grafting free products of elements of $\mathcal{S}$,
(2) Classify all possible such graftings.

This is a purely combinatorial classification program for categories of non-crossing coloured partitions, which ressembles the quantum group program outlined in Section 4. The reason for that is that many graftings can easily be translated into operations at the quantum group level, for instance commutation relations or free products. Thus, they provide a bridge between the two lines of thought surveyed above. Let us illustrate this with concrete examples.

5.1.2. *Examples.* We will now show how the grafting operation generalizes all the operations used so far in the classification of non-crossing partition quantum groups.

Lemma 5.3 (Group-like relations)**.** *Quotients by group-like relations are graftings.*

Proof. Consider two group-like elements that we want to make equal. They can be written as u_p and u_q for projective partitions p and q satisfying $t(p) = 0 = t(q)$. Adding the group-like relation $u_p = u_q$ is then equivalent to adding $r(p, q)$. □

Remark 5.4. Even though group-like relations are graftings, it may be necessary to keep them apart from the other operations, due to their peculiar status in the proofs of the classification statements. See Definition 5.13 and the comments afterwards.

Lemma 5.5 (Commutation relations)**.** *Quotients by commutation relations are graftings.*

Proof. With the notations of Proposition 4.4, the partition that we need to add to the category of partitions is $b\otimes p\otimes b^*$. If $\overline{b}$ denotes the partition obtained by rotating b upside down, then the partition we are interested in is the grafting of $(b^*b)\otimes p$ and $p\otimes(\overline{b}^*\overline{b})$. □

Lemma 5.6 (Amalgamation)**.** *Twisted amalgamated free products are graftings.*

Proof. With the notations of Proposition 4.8, we need to add the partitions $b\otimes r(p_i, p_i')\otimes b^*$. Just as above this is the grafting of $(b^*b)\otimes p_i$ and $p_i'\otimes(\overline{b}^*\overline{b})$. □

Graftings are also efficient to deal with free wreath products of pairs. To explain this, first recall that among quantum reflection groups, we denote by $H_N^{\infty+}$ the free wreath product of $\widehat{\mathbf{Z}}$ by S_N^+.

Proposition 5.7. *The free wreath product of pairs are all graftings of free products of copies of $H_N^{\infty+}$.*

Proof. First observe that, denoting by $\mathbf{F}_n$ the free group on n generators, we can write the corresponding free wreath product as an n-fold free product with amalgamation over S_N^+:

$$\widehat{\mathbf{F}}_n \wr_* S_N^+ = \left(H_N^{\infty+}\right)^{\underset{S_N^+}{*}n},$$

which is itself a grafting of $\left(H_N^{\infty+}\right)^{*n}$.

Let us introduce a notation : for two words w and w' over a colour set $\mathcal{A}$, we denote by $\pi(w, w')$ the one-block partition with upper colouring w and lower colouring w'. If now Γ is a group generated by n elements $g_1, \cdots, g_n$, and if S is the corresponding symmetric generating set, then there are reduced words $w_1, \cdots, w_\ell$ on the generators such that Γ is the quotient of $\mathbf{F}_n$ by the relations $w_i = 1$ for $1 \leqslant i \leqslant \ell$. The category of partitions of $\widehat{\Gamma}\wr_* S_N^+$ is then generated by that of $\widehat{\mathbf{F}}_n \wr_* S_N^+$ and the partitions $\pi(w_i, \emptyset)$. This can be seen as a graft for instance by rotating one point to the lower row. More precisely, if $w_i = a_1\cdots a_{n_i}$ then $\pi(w_i, \emptyset)$ can be rotated to become

$$\pi(a_2\cdots a_{n_i}, a_1^{-1}) = r\left(\pi(a_2\cdots a_{n_i}, a_2\cdots a_{n_i}), \pi(a_1^{-1}, a_1^{-1})\right).$$

If now $\Lambda\subset\Gamma$ is a subgroup, consider an element $\lambda\in\Lambda$. It can be written as a word $g_1\cdots g_n$ on S and the corresponding partition in $\mathcal{C}_{\Gamma,\Lambda,S}$ is given by b^*b where b is the one-block partition on the upper row coloured by $(g_1, \cdots, g_n)$. Rotating this yields the graft between projective partitions $\pi(g_ng_n^{-1}, g_ng_n^{-1})$ and $\pi(g_{n-1}^{-1}\cdots g_1^{-1}g_1\cdots g_{n-1}, g_{n-1}^{-1}\cdots g_1^{-1}g_1\cdots g_{n-1})$. Therefore, by successively grafting for all elements of Λ, we obtain $H_N^{++}(\Gamma, \Lambda)$. □

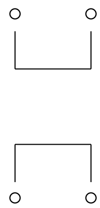
In that case we can even say more. Indeed, consider a category of partitions obtained from $\widehat{\mathbf{F}}_n \wr_* S_N^+$ by graftings. Because it is still non-crossing and contains the category of partitions of a free wreath product, it is by Proposition 4.10 the category of partitions of a free wreath product of pairs. In other words, the class of free wreath products of pairs is closed under graftings.

5.1.3. *Complexifications.* Our aim now is to see how one may use the grafting operation to restate classification results in a way more amenable to generalisation. To do this, we first need to briefly investigate the connection between graftings and the complexification operations. For this, it is simpler to consider each of the four families $\mathscr{O}$, $\mathscr{B}$, $\mathscr{H}$ and $\mathscr{S}$ given by the invariant BS separately. All our claims below follow from the full classification given in [TW18, Thm 7.1 and 7.2].

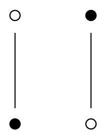
Let us start with categories of non-crossing pair partitions. Then, the only possibilities are the free unitary quantum group U_N^+ (which is the free 0-complexification of O_N^+) and the tensor d-complexifications.

Proposition 5.8. *The d-tensor complexifications of O_N^+ are graftings of U_N^+.*

Proof. The 0-tensor complexification of O_N^+ is obtained by adding the partition



Rotating it yields the partition



and this is the graft of the partitions $\mathrm{id}_\circ \otimes \mathrm{id}_\bullet$ and $\mathrm{id}_\bullet \otimes \mathrm{id}_\circ$, where id_x denotes the x-identity partition.

Passing to the d-tensor complexification amounts to making the one-dimensional representation u_{b^*b} of order $d/2$ (d-tensor complexification is only possible for even d in that case). This is a group-like relation, hence a grafting. □

The second case is that of categories of non-crossing partitions where all blocks have size at most two. The central object this time will be the free complexification of $B_N^+ * \mathbf{Z}_2$, which we will denoted by $\widetilde{B}_N^{+\sharp}$. Following the notations of [TW17], let us say that the quantum groups in this class are of type $\mathscr{B}$.

Proposition 5.9. *All non-crossing unitary easy quantum groups of type $\mathscr{B}$ are graftings of $\widetilde{B}_N^{+\sharp}$.*

Proof. The category of partitions of $\widetilde{B}_N^{+\sharp}$ is generated by the double singleton partition ∘•. Rotating this yields a projective partition p generating a group of one-dimensional representations which is in fact isomorphic to the copy of $\mathbf{Z}$ coming from the free complexification. As a consequence, the d-free complexification is obtained by adding the group-like relation $u_p^d = 1$, hence is a grafting. Moreover, the r-self-adjoint d-free complexification corresponds to the relation

$$u \otimes u_p^r = u_p^{*r} \otimes u,$$

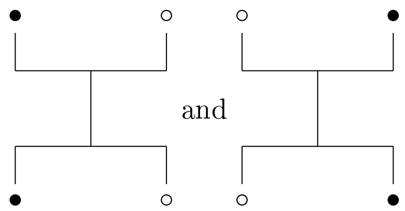
which is given by a graft.

As for the d-tensor complexification, as in the case of O_N^+ it is obtained by adding the graft of the partitions $\mathrm{id}_\circ \otimes \mathrm{id}_\bullet$ and $\mathrm{id}_\bullet \otimes \mathrm{id}_\circ$, hence is once again a grafting. One may add also the commutation relation between u_p and u, which is given by a graft. □

Next is the case where all blocks of the partitions have even size. The starting point here is the free complexification $\widetilde{H}_N^+$ of the quantum hyperoctahedral group H_N^+.

Proposition 5.10. *All quantum reflection groups and their complexifications are graftings of $\widetilde{H}_N^+$.*

Proof. Recall that the category of partitions of $\widetilde{H}_N^+$ consists in all non-crossing partitions such that each block has even size and, once rotated on one line, alternating colouring. To obtain the quantum reflection group $\widehat{\mathbf{Z}} \wr_* S_N^+$, one has to add the one-block partition on four points lying on one line with colouring $\circ \circ \bullet\bullet$. Up to rotation, this is the graft of the partitions



By Proposition 5.7, one can then recover the free wreath product of any cyclic group, i.e. all quantum reflection groups.

There is no free complexification of quantum reflection groups except for $\widetilde{H}_N^+$, and this one was our starting point. As for the k-tensor complexification, we already mentioned that it was indeed a free wreath product of pair. One can also see directly that it can be obtained by adding the partition $b_d^*b_d$ (where b_d is a block of size d lying on the upper row with all points coloured in white) to the category of partitions of $\widehat{\mathbf{Z}}_k \wr_* S_N^+$ (for $d \mid k$) and this operation is again a grafting. □

The final step is simply all the remaining cases, and the basic object is the free complexification of $S_N^+ \times \mathbf{Z}_2$, which will be denoted by $\widetilde{S}_N^{+\prime}$. These are called of type $\mathscr{S}$ in [TW17].

Proposition 5.11. *All free unitary quantum groups of type $\mathscr{S}$ are graftings of $\widetilde{S}_N^{+\prime}$.*

Proof. The category of partitions of $\widetilde{S}_N^{+\prime}$ is generated by that of $\widetilde{H}_N^+$ and the double white singleton. Moreover, the d-free complexification is obtained by adding a group-like relation, and the d-tensor complexification corresponds to further adding a commutation relation and the same partition as for O_N^+. □

5.2. **The classification program.** This final section is rather speculative and aims at discussing some possible directions to push further classification results for non-crossing partition quantum groups. Let us start by gathering all the observations above into a statement:

Theorem 5.12. *Let $\mathcal{S} = \{U_N^+, \widetilde{B}_N^{+\sharp}, \widetilde{H}_N^+, \widetilde{S}_N^{+\prime}\}$. Then, non-crossing partition quantum groups on two colours are graftings of free products of elements of $\mathcal{S}$.*

Proof. Let us first note that given a non-crossing unitary easy quantum group, one may add the partition $|$ coloured with $\circ$ and $\bullet$ to produce a non-crossing orthogonal easy quantum groups, and this partition is the graft of the partitions $\mathrm{id}_\circ$ and $\mathrm{id}_\bullet$ respectively. Using this, and all the propositions above, the result follows. □

This is a rather crude form since it does not give any information on the concrete graftings needed to obtain all the objects. It nevertheless immediately suggests a first general conjecture:

Conjecture. *All non-crossing partition quantum groups are graftings of free products of elements of $\mathcal{S}$.*

One virtue of Theorem 5.12 is to put the emphasis on the freely complexified cases. Indeed, passing from them to the orthogonal ones is conceptually simpler than going the other way round. We will take this observation further below, but we first come back to the distinction between group-like relations and the other ones.

5.2.1. *Projective categories of partitions.* As we already mentioned in the sketch of proof of Theorem 4.11, the proofs of classification statements in [Fre19b] usually split into two parts. Given a category of non-crossing partitions $\mathcal{C}$, one first builds another category of partitions $\mathcal{C}' \subset \mathcal{C}$ which contains all the projective partitions in $\mathcal{C}$. Then, $\mathbb{G}_N(\mathcal{C})$ is a quotient of $\mathbb{G}_N(\mathcal{C}')$ by group-like relations. This strategy naturally leads to the following definition:

Definition 5.13. A category of partitions is said to be *projective* if it is generated by projective partitions.

Given a category of partitions $\mathcal{C}$, there is always a largest projective category of partitions contained in it, namely

$$\mathcal{C}_{\mathrm{proj}} = \langle p \mid p \in \mathcal{C} \text{ projective } \rangle .$$

Then, $\mathbb{G}_N(\mathcal{C})$ is by definition a quotient of $\mathbb{G}_N(\mathcal{C}_{\mathrm{proj}})$ by group-like relations. Moreover, such quotients turn out to be usually easy to compute, in particular as far as representation theory is concerned, and there may be a general phenomenon here:

Question. *If $\mathcal{C}$ is projective, is there a general procedure to compute the representation theory of the quotient of $\mathbb{G}_N(\mathcal{C})$ by group-like relations in terms of the representation theory of $\mathbb{G}_N(\mathcal{C})$?*

5.2.2. *Symmetric graftings.* A consequence of the previous remarks is that the main problem for classification is certainly to classify projective categories of partitions. A close look at the above results then shows that a specific form of graftings is especially important.

Let us recall a notation introduced in the proof of Lemma 5.5 : for a partition p, we denote by $\overline{p}$ the partition obtained by rotating p upside-down. It follows from [FW16] that the representation $u_{\overline{p}}$ is conjugate to u_p. The results of Section 5.1.3 show that graftings involving p and $\overline{p}$ are particularly useful. Let us give them a name for convenience.

Definition 5.14. A grafting is said to be *symmetric* if it is of the form $r(p, \overline{p})$.

A look at the proofs of Proposition 5.8, 5.9, 5.10, 5.11 and Lemma 5.3, 5.5, 5.6 gives examples of symmetric graftings that we gather in a statement for convenience:

Proposition 5.15. *Any complexification of an orthogonal easy quantum group can be obtained from $\mathcal{S}$ through symmetric graftings. Moreover, any quotient by commutation relations can be obtained from a free product of elements of $\mathcal{S}$ through symmetric graftings.*

This suggests, as a first step, to investigate more closely symmetric graftings. Noticing that r-self-adjoint complexifications are given by symmetric graftings which translate into conjugate commutation relations, the following question may be a good starting point for those investigations:

Question. *Does any symmetric grafting correspond to a commutation relation or to a conjugate commutation relation?*

This seems likely, and would provide a unification of parts of the settings of [TW17] and [Fre19b].

5.2.3. *Low order grafts.* The proof of a positive answer to the previous question would probably require extending one fundamental idea from [Fre19b] on the decomposition of a category of partitions into a series of graftings. Let us assume that we have two projective partitions $p, q \in \mathcal{C}(k, k)$ which decompose as

$$p = p_1 \otimes \cdots \otimes p_n \text{ and } q = q_1 \otimes \cdots \otimes q_n,$$

where for each $1 \leqslant i \leqslant n$, $t(p_i) = t(q_i)$. Obviously, we have the corresponding decomposition of grafts

$$r(p, q) = r(p_1, q_1) \otimes \cdots \otimes r(p_n, q_n)$$

so that

$$\langle \mathcal{C}, r(p, q) \rangle \subset \langle \mathcal{C}, r(p_i, q_i) \mid 1 \leqslant i \leqslant n \rangle.$$

It is natural to wonder whether that inclusion is an equality, i.e. whether the grafts of p_i and q_i can be recovered from the graft of p and q. This is often the case for two colours, but will be more conveniently expressed through the following notion:

Definition 5.16. The *order* of the graft $r(p, q)$ is the number $t(p) = t(q)$.

A technique used repeatedly in [Fre19b] is to reduce the problem to grafts with order at most two and then to classify the latter. In particular, we have the following consequence of Theorem 5.12 :

Corollary 5.17. *Let $\mathcal{S} = \{U_N^+, \widetilde{B}_N^{+\sharp}, \widetilde{H}_N^+, \widetilde{S}_N^{+\prime}\}$. Then, non-crossing partition quantum groups on two colours are obtained from free products of elements of $\mathcal{S}$ by graftings of order at most two.*

This raises several questions whose answers could be important steps in a tentative classification using graftings, namely:

- Can we obtain any non-crossing partition quantum group from free products of elements of $\mathcal{S}$ using only graftings of a bounded order?
- If yes, is there a bound independent from the number of colours (and is it equal to two)?
- Otherwise, can we build examples of categories of non-crossing partitions which cannot be obtained with graftings of order less than n for any integer n?

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