Jean-Benoît Bost : GAGA for curves

$$\begin{tikzcd}[column sep=small]
\text{pre-gaga: } \left\{\begin{array}{c}\text{smooth proj.}\\ \text{connected curves}/\mathbb{C}\\ \text{non est.}\end{array}\right\} \arrow[rr, "\text{an.}"] \arrow[dr] & & \left\{\begin{array}{c}\text{compact connected}\\ \text{R. surface}\\ \text{non est.}\end{array}\right\} \arrow[dl] \\
& \left\{\begin{array}{c}\mathbb{C} \subset K \text{ finite}\\ \deg_{\mathbb{C}}(K) = 1\end{array}\right\} &
\end{tikzcd}$$

$C \longmapsto \mathbb{C}(C) \xrightarrow{\sim} \mathcal{M}(S) \longleftarrow S$

GAGA

$C \subset C^{an}$ as above

comparison: for any vector bundle E over $\mathbb{C}$

$$H^i(C, E) \xrightarrow{\sim} H^i(C^{an}, E^{an})$$

coherence doesn't add a lot

Existence: Analytic vector bundles on C^{an} is isom to E^{an} for some alg. v. b of E over $\mathbb{C}$.

Known fact: S compact Riemann surface, E vector bundle on S.

$$+\begin{cases} \dim_{\mathbb{C}} H^i(S, E) < +\infty. & \text{(theorem of CARTAN-SERRE).} \\ \text{Serre duality}: \quad H^0(S, E) \times H^1(S, E^\vee \otimes \Omega^1_S) & \\ \qquad\qquad \downarrow & \text{perfect pairing.} \\ \qquad H^1(S, \Omega^1_S) \xrightarrow[\mathrm{Tr}]{\sim} \mathbb{C}.. & \end{cases}$$

How to get pre-GAGA: S compact connected.

$H^1(S, \mathcal{O}_S)$ finite dimensional $\Rightarrow$ for any $P \in S$, $\exists f \in \mathcal{O}(S) \setminus \mathbb{C}$, non constant whose only pole is at P.

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(nP) \to \mathcal{O}_S(nP)|_P \to 0.$$

$\Rightarrow$

$$H^0(\qquad \to \mathcal{O}_S(nP)|_P \underset{\mathbb{C}}{=} \quad \text{--- --- ---}$$

$n \to +\infty$ the order of this grows.

$$\longrightarrow H^1(S, \mathcal{O}_S)$$

$f = \pi : S \xrightarrow{\text{degree } d} \mathbb{P}^1(\mathbb{C})$ finite morphism / ramified covering of $\mathbb{P}^1_{\mathbb{C}}$.

Then $\mathbb{C}(X) \simeq \mathcal{M}(\mathbb{P}^1(\mathbb{C}))$ easy. • degree $\leq d \Rightarrow$ degree d.

$$\pi^* : \mathbb{C}(X) \xrightarrow{\sim} \mathcal{M}(\mathbb{P}^1(\mathbb{C})) \longrightarrow \mathcal{M}(S)$$

$g \in \mathcal{M}(S)$, $\mathrm{Tr}_\pi g : \underset{\mathbb{P}^1(\mathbb{C})}{\overset{\cap}{Q}} \longrightarrow \underbrace{\int g \, \delta_{\pi^* Q}}_{\text{finite sum...}} \in \mathcal{M}(\mathbb{P}^1(\mathbb{C})) = \mathbb{C}(X)$ "local computation".

$$\prod_{P \in \pi^* Q} (X - g(P)) = X^d + \sum \underbrace{a_i(Q)}_{\substack{\text{rational functions} \\ \in \pi^* \mathbb{C}(X)}} X^{d-i} \qquad a_1 = -\mathrm{Tr}_\pi g.$$

$Q \in \mathbb{P}^1(\mathbb{C})$

$\mathcal{M}(S) = \mathbb{C}(f)[g]$; $g \in \mathcal{M}(S)$ separating the d-points $\pi^{-1}(Q)$ for some Q.

$\mathbb{C}(X)[Y]$ + algebraic relation on Y. $P(X,Y) = 0$. → smooth proj. model.

$\wr|$ "classical algebraic geometry"

$\mathbb{C}(C)$ $\qquad$ C smooth projective connected curve / $\mathbb{C}$. $\{P(\hat{f}, \hat{g}) = 0$

$$S \xrightarrow[\sim]{\hat{f}, \hat{g}} C.$$

GAGA - Riemann:

① Prove the comparison for H^0 line bundles.

L over C, $L \simeq \mathcal{O}_C(D)$ divisor

$\qquad L^{an} \simeq \mathcal{O}_{C^{an}}(D)$

$$\begin{array}{ccc}
\mathbb{C}(C) & \xrightarrow{\sim} & \mathcal{M}(C^{an}) \\
\uparrow & & \uparrow \\
\Gamma(C, \mathcal{O}_C(D)) & \simeq & \Gamma(C^{an}, \mathcal{O}_{C^{an}}(D)) \\
\wr| & & \| \\
H^0(C, L) & \simeq & H^0(C^{an}, L^{an})
\end{array}$$

2) Existence theorem (Kodaira-Spencer)

E analytic vector bundle on C^{an}, $P \in C$, $E(nP) = E \otimes_{\mathcal{O}_C} \mathcal{O}_{C^{an}}(nP)$

$P \in C$,

$$0 \longrightarrow E^a((n-1)P) \longrightarrow E^a(nP) \longrightarrow \text{"}E^a(nP)|_P\text{"} \longrightarrow 0.$$

$$0 \longrightarrow H^0(C, E((n-1)P)) \longrightarrow H^0(C, E^a(nP)) \longrightarrow E^a(nP)|_P \longrightarrow$$
$$\longrightarrow H^1(C, E(n-1)P) \longrightarrow H^1(C, E^a(nP)) \longrightarrow 0$$

finite dim $=$ for $n \gg 0$.

So $\dim_{\mathbb{C}} H^1(C, E(n-1)P) \geq \dim_{\mathbb{C}} H^1(C, E(nP))$

$\Rightarrow$ for n large $0 \to H^0(C, E((n-1)P)) \to H^0(C, E(nP)) \to E^a(nP)|_P \to 0$

So after twisting $E \otimes \mathcal{O}(n(P_1 + \dots + P_a))$

$$\mathcal{O}_C^{\oplus N} \longrightarrow E \otimes \mathcal{O}_C(n(P_1 + \dots + P_a)) \to 0.$$

$$\ker(\varphi) \to \mathcal{O}_C^{\oplus N}(-n(P_1 + \dots + P_a)) \xrightarrow{\varphi} E \longrightarrow 0.$$

$$\uparrow$$

$$\mathcal{O}_{C^{an}}^{\oplus N}(-n(P_1 + \dots + P_a)) \qquad L_1^{\oplus N_1} \xrightarrow{R} L_2^{\oplus N_2} \longrightarrow E^{an} \longrightarrow 0.$$

$$\mathrm{Hom}_{\mathcal{O}_C^{an}}(L_1^{an}, L_2^{an}) \simeq \Gamma(C^{an}, L_1^{\vee} \otimes L_2^{an})$$

$$R \in \Pi_{N_1 N_2}(\qquad)$$

$$\Rightarrow \quad R \in \Pi_{N_1 N_2}(\Gamma(C^{an}, L_1^{an\vee} \otimes L_2^{an}))$$

$\Vert$ Stage 1.

$$\Pi_{N_1, N_2}(\Gamma(C, L_1^{\vee} \otimes L_2))$$

$\overset{\text{defines}}{\Rightarrow}$ $L_1^{\oplus N_1} \xrightarrow{R} L_2^{\oplus N_2} \longrightarrow \underset{\text{"}}{E} \longrightarrow 0.$ (hidden flatness that is obvious in dimension 1).