

Integrability of integrable connections

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Basic integrability results

(E, ∇) PIC

$$E := E^{\nabla=0} = \text{Ker}(\nabla: E \rightarrow E \otimes_{G_X} \Omega^1_X)$$

$$\underline{R}=0 \Rightarrow [\nabla_{\theta_1}, \nabla_{\theta_2}] = \nabla[\theta_1, \theta_2]$$

We want to prove that $E \otimes_{\mathbb{C}} G_X \longrightarrow E$

$$\rho \otimes \lambda \longmapsto \rho \lambda$$

Equivalent to proving that

(E, ∇) PIC, locally there exists a frame $\underline{\Delta}$ of E such that $\nabla_{\underline{\Delta}}=0$

$$\underline{\Delta}: G_U^{\otimes e} \xrightarrow{\sim} E|_U$$

$$\begin{array}{ccc} \downarrow \otimes e & & \downarrow \nabla \\ \Omega^1_X|_U & \xrightarrow{\underline{d}} & E \otimes \Omega^1_X|_U \end{array}$$

$$E|_U^{\nabla=0} \xleftarrow{\sim} \underline{C}_U^{\otimes e} : \underline{\Delta}$$

$0 \in U \hookrightarrow \mathbb{C}^n$ ~~connected at the~~

$$\Omega \in \Pi_e(\Omega^1(U)) : d\Omega + \Omega^2 = 0 \Rightarrow \exists 0 \in V \hookrightarrow U, g \in GL_e(G_X(V))$$

s.t. $\Omega = g^{-1}dg$.

g is unique if you impose $g(0) = I_e$.

$$E = G_U^{\otimes e}$$

$$\nabla = d + \Omega$$

after a gauge transformation it becomes d .

① Formulae for the parallel transport along a \mathcal{C}^1 -path.

$x \in \mathbb{E} \quad \gamma: [0,1] \longrightarrow X \subset \mathcal{E}^0$

$\mathbb{E}_{\gamma(0)} \xrightarrow{\sim} \mathbb{E}_{\gamma(1)}$
 \uparrow
 depend only on the $[\gamma]$

$\gamma^{-1}\mathbb{E}$ local system on $[0,1]$, may be initialized on $[0,1]$ as a local system

These are nice formulae in the case of the theorem
 $\mathbb{E} = \mathcal{E}^{\nabla=0}$ $\gamma \in \mathcal{C}^1$ piecewise.

(depo-Darboux/Dyson expansion...)

(E, ∇) $\gamma^*(E, \nabla)$ vector bundle with flat connection on $[0,1]$.

$G_{[0,1]}^{\oplus e} \xrightarrow{\sim} \gamma^*E$
 \cong

$\nabla = d + \Omega$

Matrix of one form
 $\Omega \in \mathcal{A}^1(\mathcal{E}^{\infty}([0,1]))$

$\Omega = A(t)dt, A \in \mathcal{A}^0(\mathcal{E}^{\infty}([0,1]))$

\cong Path frame for $\gamma^*(E, \nabla)$
 $g \in \mathcal{G}_e(\mathcal{E}^{\infty}([0,1]))$

$\begin{cases} \frac{dg(t)}{dt} + A(t)g(t) = 0 \\ g(0) = I_e \end{cases}$

\cong general solution.

$g(t) = I_e + \int_{0 \leq t_1 \leq t} (-A(t_1)dt_1) + \dots$
 $+ (-1)^n \int_{0 \leq t_1 \leq \dots \leq t_n \leq t} A(t_n) \dots A(t_1) dt_1 \dots dt_n$

$A(t) = A : 1 - tA + (-1)^n \frac{t^n}{n!} A^n = e^{-tA}$

+ even get the \mathcal{E}^{∞} dependence on parameters.

working over a square:

$$\Sigma(0,1]^2 \quad d + A_1 dx_1 + A_2 dx_2$$

=> Show linear independence of choice of path.

$$\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1, A_2] = 0.$$

② Proof -> de Rham non-linear p.d.e. & implicit function theorem

$$0 \in U \hookrightarrow \mathbb{C}^n$$

$$GL_e(G_x(0)) \rightarrow \{ \Omega \in \Pi_e(\Omega'(U)) \mid d\Omega + \Omega^2 = 0 \}$$

$$g \mapsto g^{-1}dg.$$

all products are wedge products.

~~$$d(g^{-1}dg) = d(g^{-1})dg + g^{-1}ddg$$

$$= -(g^{-1}dg)^2.$$~~

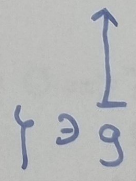
$$d(g^{-1}dg) = -g^{-1}dg g^{-1}dg = -(g^{-1}dg)^2$$

Observation: If Ω is given on U , $\epsilon > 0$

$R_\epsilon: \mathbb{C}^n \xrightarrow{\quad} \mathbb{C}^n$, ϵ small, R_ϵ^* is defined over $\bar{B}(0,1)$ in \mathbb{C}^n .
 $\mathbb{Z} \xrightarrow{\quad} \epsilon\mathbb{Z}$, \rightarrow arbitrary small.

$$\{ \Omega \in \Pi_e(\Omega'_{\mathbb{C}^n}(\bar{B})) \mid d\Omega + \Omega^2 = 0 \} \ni g^{-1}dg$$

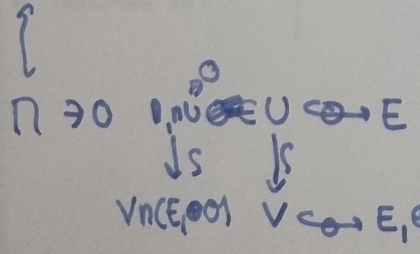
$$\{ g \in GL_e(G_{\mathbb{C}^n}(\bar{B})) \mid g(0) = I_e \}$$



closed balls are used to get Banach spaces.

E Banach space

$$L^1(\mathbb{R}/\mathbb{Z})$$



$$H^1 = \{ f \in L^1 \mid \forall n < 0, \hat{f}(n) = 0 \}$$