# Integrability of integrable connections

Jean-Benoît Bost

15/11/2024

## Basic integrability results

- $(E, \nabla)$ MIC

$$\mathbb{E} := \mathcal{E}^{\nabla = 0} = \mathrm{Ker}(\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X)$$

$$\underline{R = 0} \Rightarrow [\nabla_{\theta_1}, \nabla_{\theta_2}] = \nabla_{[\theta_1, \theta_2]}$$

We want to prove that $\mathbb{E} \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{E}$

$$s \otimes \lambda \longmapsto s\lambda$$

$\Updownarrow$ Equivalent to proving that

- $(E, \nabla)$ MIC, locally there exists a frame $\underline{s}$ of $\mathcal{E}$ such that $\nabla \underline{s} = 0$

$$\begin{tikzcd}
\underline{s} : \mathcal{O}_{|U}^{\oplus e} \arrow[r, "\sim"] \arrow[d] & \mathcal{E}_{|U} \arrow[d, "\nabla"] \\
\Omega^{1\,\oplus e}_{X|U} \arrow[r, "d", "\sim"'] & \mathcal{E} \otimes \Omega^1_{X|U}.
\end{tikzcd}$$

$$\mathcal{E}_{|U}^{\nabla = 0} \xhookleftarrow{\sim} \underline{\mathbb{C}}_U^{\oplus e} : \underline{s}$$

- $0 \in U \hookrightarrow \mathbb{C}^n$ ~~→ looking at the~~

$\Omega \in M_e(\Omega^1(U))$ : $\quad d\Omega + \Omega^2 = 0 \Rightarrow \exists\, 0 \in V \hookrightarrow U$, (connected.)

$g \in GL_e(\mathcal{O}_X(V))$

s.t $\Omega = g^{-1} dg$.

$g$ is unique if you impose $g(0) = I_e$.

$\mathcal{E} = \mathcal{O}_U^{\oplus e}$

$\nabla = d + \Omega$

after a gauge transformation it becomes $d$.

# ① Formulae for the parallel transport along a $\mathcal{C}^1$-path.

$X \circlearrowleft \mathbb{E} \qquad \gamma : [0,1] \longrightarrow X \quad \mathcal{C}^0$

$\mathbb{E}_{\gamma(0)} \xrightarrow{\sim} \mathbb{E}_{\gamma(1)}$ $\qquad \gamma^{-1}\mathbb{E}$ local system on $[0,1]$,

depend only on the $[\gamma]$

may be trivialized on $[0,1]$. as a local system

There are nice formulae in the case of the theorem

$\mathbb{E} = \mathcal{E}^{\nabla = 0} \qquad \gamma$ $\mathcal{C}^1$ piecewise.

(Lapo-Danilewski
Dyson expansional... )

$(E, \nabla) \qquad \gamma^*(E,\nabla)$ vector bundle with flat connection on $[0,1]$.

$G^{\oplus e}_{[0,1]} \xrightarrow[\Delta]{\sim} \gamma^* E$

$\nabla = d + \Omega.$

Matrix of one forms $M_e(\Omega^1([0,1]))$

$\Omega = A(t)dt$, $A \in M_e(\mathcal{C}^\infty([0,1]))$

$\underline{\Delta}$ Res frame for $\gamma^*(E,\nabla)$

$g \in GL_e(\mathcal{C}^\infty([0,1]))$.

$$\begin{cases} \dfrac{dg(t)}{dt} + A(t)\, g(t) = 0 \\ g(0) = I_e. \end{cases}$$

general solution.

$$g(t) = I_e + \int_{0 \le t_1 \le t} (-A(t_1)\,dt_1) + \dots$$

$$+ (-1)^n \int_{0 \le t_1 \le \dots \le t_n \le t} A(t_n) \dots A(t_1)\, dt_1 \dots dt_n.$$

$$A(t) = A \quad : \quad 1 - tA \dots + (-1)^n \frac{t^n}{n!} A^n \qquad = e^{-tA}.$$

+ even get the $\mathcal{C}^\infty$ dependance on parameters.

...othing over a square:

$[0,1]^2 \quad d + A_1 dx_1 + A dx_2$

$$\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} + [A_1, A_2] = 0.$$

$\Rightarrow$ Show ~~___~~ independance of choice of path.

② ~~Proof~~ $\rightsquigarrow$ ~~___~~ non-linear p.d.e. & implicit function theorem

$0 \in U \hookrightarrow \mathbb{C}^n$

all products are wedge products.

$$GL_e(\mathcal{O}_X(U)) \to \{\Omega \in M_e(\Omega^1(U)) \mid d\Omega + \Omega^2 = 0\}$$

$$g \longmapsto g^{-1}dg.$$

$\cancel{d(g^{-1}dg) = d(g^{-1})dg + g^{-1}ddg.}$

$\cancel{= -(g^{-1}dg)^2.}$

$$d(g^{-1}dg) = -g^{-1}dg\, g^{-1}dg = -(g^{-1}dg)^2$$

__observation__: If $\Omega$ is given on $U$, $\varepsilon > 0$

$R_\varepsilon : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $\varepsilon$ small, $R_\varepsilon^*\Omega$ is define over $\overline{B}(0,1)$ in $\mathbb{C}^n$.

$\quad z \longmapsto \varepsilon z$ + arbitrary small.

$$\{\Omega \in M_e(\Omega^1_{\mathbb{C}^n}(\overline{B})) \mid d\Omega + \Omega^2 = 0\} \ni g^{-1}dg$$

$$\uparrow$$

$$\{g \in GL_e(\mathcal{O}_{\mathbb{C}^n}(\overline{B})) \mid g(0) = I_e\} \ni g$$

closed balls are there to get Banach spaces.

$E$ Banach space

$\Pi \ni 0 \quad \Pi \cap \overset{\circ}{U} \quad U \hookrightarrow E$

$\downarrow s \qquad \downarrow s$

$V \cap (E_1 \oplus 0) \quad V \hookrightarrow E_1 \oplus E_2$

$$L^1(\mathbb{R}/\mathbb{Z})$$
$$\cup$$
$$H^1 = \{f \in L^1 \mid \forall n < 0, \hat{f}(n) = 0\}$$