D-modules

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Use these notes at your own risk, I don't claim the proofs are anywhere near optimal nor original, it is just how I understand things.

Please reach out by email or in person if you have any questions or remarks, I'll be happy to discuss. If you happen to have the answer to any of the question I left pending, please reach out !

Conventions and recollections

If X is a topological space, we use capital letters U, V, W to denote open sets. We try to keep Ω and related notations for differential forms. Sheaves and notations related to sheaves are denoted by capital italic letters, $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{O}$... We denote by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})$$

the *sheaf Hom* of $\mathcal{F}$ and $\mathcal{G}$, and we denote by

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

the set of homomorphism of sheaves between $\mathcal{F}$ and $\mathcal{G}$. Since the sheaf Hom $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ can be confusing, we recall it is defined on every open U as

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{G}_{|U})$$

so a section $D \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$ is the data for every $W \subset V \subset U$ of maps $D(V)$ and $D(W)$ compatible with restriction i.e. making the following diagram commute

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "D(V)"] \arrow[d, "\mathrm{res}^V_W"] & \mathcal{G}(V) \arrow[d, "\mathrm{res}^V_W"] \\ \mathcal{F}(W) \arrow[r, "D(W)"] & \mathcal{G}(W) \end{tikzcd} .$$

If we have $V \subset U$ then from $D \in \mathcal{H}om(F, G)(U)$ we get a section $D_{|V} \in \mathcal{H}om(F, G)(V)$ defined for all $W \subset V \subset U$ by

$$D_{|V}(W) = D(W) : \mathcal{F}(W) \longrightarrow \mathcal{G}(W).$$

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If our topological space comes with a sheaf of associative algebras say $\mathcal{R}$ then we may consider sheaves of left modules over $\mathcal{R}$. Sheaves of right modules over $\mathcal{R}$ are by definition sheaves of left modules over the opposite ring $\mathcal{R}^{\mathrm{op}}$. The corresponding linearities will be indicated in notations as subscripts. For instance if X is a complex manifold we will consider the sheaves of rings $\underline{\mathbb{C}}$ (or simply $\mathbb{C}$) of locally constant holomorphic functions which is a subsheaf of the sheaf of holomorphic functions $\mathcal{O}_X$. Both are sheaves of commutative rings, and for instance

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{F}, \mathcal{G}) \text{ (resp. } \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$$

denotes the $\mathbb{C}$-linear (resp. $\mathcal{O}_X$-linear) sheaf Hom between two sheaves of $\mathbb{C}$-vector spaces (resp. $\mathcal{O}_X$-modules) $\mathcal{F}$ and $\mathcal{G}$ on X. Later, after having properly defined the sheaf of differential operators $\mathcal{D}_X$, we will deal with sheaves of left and right $\mathcal{D}_X$-modules.

To compute in algebras of differential operators, we use the multi-index notation. If $n \in \mathbb{N}$, we typically use $\alpha \in \mathbb{Z}^n$ to denote a multi-index. That is $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}^n$. If $z = (z_1, ..., z_n)$ (thought as coordinates on an open U of $\mathbb{C}^n$) then

$$\partial_z^\alpha = \partial_{z_1}^{\alpha_1} ... \partial_{z_n}^{\alpha_n} = \frac{\partial^{\alpha_1 + ... \alpha_n}}{\partial_{z_1}^{\alpha_1} ... \partial_{z_n}^{\alpha_n}}.$$

It is very useful in explicit computation to use the following convention

$$\partial_z^m = 0 \text{ if } m < 0.$$

We recall a few useful formula concerning Lie brackets, Let A be an associative $\mathbb{C}$-algebra, it is naturally a Lie algebra by means of the bracket given by the commutator

$$[x, y] = x \cdot y - y \cdot x$$

for all $x, y \in A$. As such it satisfies the Jacobi identity, for all $x, y, z \in A$, we have

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

It is easily checked that

$$[x, yz] = [x, y]z + y[x, z],$$
$$[xy, z] = [x, z]y + x[y, z],$$

in particular if y commutes with $[x, y]$, i.e. if $[y, [x, y]] = 0$, we have

$$[x, y^2] = [x, y]y + y[x, y] = 2y[x, y]$$

and more generally if $[y, [x, y]] = 0$ then for all $n \in \mathbb{N}$

$$[x, y^n] = ny^{n-1}[x, y].$$

1 The sheaf of differential operators

Let X be an n-dimensional complex manifold and $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. For each $x \in X$, we have an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{O}_X \simeq \mathbb{C}\{z_1, ..., z_n\}$$

whenever we choose local coordinates $(z_1, ..., z_n)$ around x where $\mathbb{C}\{z_1, ..., z_n\}$ is the ring of power series in n variables with a nonzero convergence radius.

We want to define what the sheaf of differential operators $\mathcal{D}_X$ on X. Because it is a sheaf of operators, it should be a subsheaf of

$$\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

First, it is clear that $\mathcal{O}_X$ is a subsheaf of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$, we see this by defining the map

$$m : \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X),$$

by defining for each open set U the map

$$m(U) : \mathcal{O}_X(U) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$$

that assigns to a $f \in \mathcal{O}_X(U)$ the morphism of sheaves defined for all $V \subset U$ and all $g \in \mathcal{O}_X(V)$ by

$$m(U)(f)(g) = f\big|_V \cdot g \in \mathcal{O}_X(V).$$

It is clear that m is injective.

Let's turn to operators that are slightly more complicated. Denote by Θ_X the sheaf of vector fields. We can think of vector fields as derivations, it is either a definition or a property, depending on one's background. For any $\mathcal{O}_X$-module $\mathcal{F}$ we can define

$$\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{F}) \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{F})$$

to be the subsheaf given on every open U by

$$\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{F})(U) = \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{F}_{|U}) \,|\, D(V)(f \cdot g) = fD(V)(g) + gD(V)(f) \text{ for all } V \subset U \text{ and all } f, g \in \mathcal{O}_X(V)\}.$$

In particular, again it is either a definition or a property, we have

$$\Theta_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X) \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Note that this can be rewritten

$$\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)(U) = \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) \,|\, [D(V), m(V)(f)] = m(V)(D(V)(f)) \text{ in } \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_X(V), \mathcal{O}_X(V)) \text{ for all } V \subset U \text{ and all } f \in \mathcal{O}_X(V)\}.$$

It can be convenient to think that the Lie derivative construction realizes this isomorphism, i.e. whenever we pick a vector field X on an open U, and a function $f \in \mathcal{O}_X(U)$, we can differentiate f along X at each $p \in U$ by following the flow. That yields a map of sheaves of $\mathcal{O}_X$-modules

$$\Theta_X \longrightarrow \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

that turns out to be an isomorphism. From now one, **we think of vector fields as derivations**, that is we have an equality

$$\Theta_X = \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X).$$

Remark 1. *We are not saying that for a given open set U there exists an isomorphism between $\mathbb{C}$-linear derivations of the $\mathbb{C}$-algebra $\mathcal{O}_X(U)$ and vector fields on U. We can think of $\mathbb{P}^1_{\mathbb{C}}$ where there is a fair amount of global vector fields but no non-trivial derivations of global holomorphic functions (as they are constant functions).*

Question 1. *Can one formulate a sufficient condition so that it holds ?*

This motivates our first definition of $\mathcal{D}_X$, we look at the assignement

$$U \longrightarrow \langle \mathcal{O}_X(U), \Theta_X(U) \rangle \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$$

where $\langle \mathcal{O}_X(U), \Theta(U) \rangle$ denotes the $\mathbb{C}$-algebra generated by $\mathcal{O}_X(U)$ and $\Theta(U)$ in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$. This clearly yields a presheaf, but it needs not be a sheaf. We denote by

$$\langle \mathcal{O}_X, \Theta_X \rangle$$

its sheafification. The following Lemma shows that sheafification is not too "harmful" in our context.

Lemma 1. *Let X be a topological space, let $\mathcal{G}$ be a sheaf on X. Let $\mathcal{F}$ be a pre-subsheaf of $\mathcal{G}$. Then $\mathcal{F}^{sh}$ can be described on every open U as*

$$\begin{aligned}
\mathcal{F}^{sh}(U) &= \{g \in \mathcal{G}(U) \,|\, g_x \in \mathcal{F}_x \textit{ for all } x \in U\} \\
&= \{g \in \mathcal{G}(U) \,|\, \textit{For all } x \in U \textit{ there exists } x \in V \subset U \textit{ such that } g_{|V} \in \mathcal{F}(V)\} \\
&= \{g \in \mathcal{G}(U) \,|\, \textit{There exists a covering } U = \bigcup_i U_i \textit{ such that for all } i \in I,\ g_{|U_i} \in \mathcal{F}(U_i)\}.
\end{aligned}$$

In particular, it is clear in that case that $\mathcal{F}$ is a sub-presheaf of $\mathcal{F}^{sh}$ which is itself a subsheaf of $\mathcal{G}$.

Proof. When you glue the sections in some bigger sheaf, which you can always do, it doesn't have to remain in the presheaf, precisely because it is not a sheaf. But clearly it is in the presheaf locally, so you add all the people obtained by this procedure and can show this is enough. □

We get the following explicit description, for every U an element $D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$ belongs to $\langle \mathcal{O}_X, \Theta_X \rangle(U)$ if and only if around every point $x \in U$ a neighboorhood V such that $D_{|V} \in \langle \mathcal{O}_X(V), \Theta_X(V) \rangle$.

Remark 2. *For some open sets U, typically with infinitely many connected components, it may happen that*

$$\langle \mathcal{O}_X(U), \Theta_X(U) \rangle \neq \langle \mathcal{O}_X, \Theta_X \rangle(U).$$

It may also happen for "non-trivial" reasons.

Question 2. *Find a "non-trivial" counterexample.*

Proposition 1. *Let U be an open of X and $D \in \langle \mathcal{O}_X, \Theta_X \rangle(U)$. Then for all $x \in U$ there exists a neighborhood $V \subset U$ of x that can be chosen to be a coordinate chart*

$$z = (z_1, ..., z_n) : V \longrightarrow \mathbb{C}^n$$

such that

$$D_{|V} = \sum_\alpha m(V)(a_\alpha(z))\partial_z^\alpha = \sum_\alpha a_\alpha(z)\partial_z^\alpha$$

for some ***uniquely determined*** *functions $a_\alpha(z) \in \mathcal{O}_{\mathbb{C}^n}(V)$.*

Proof. The existence of such a V is clear and we leave the reader to check the unicity of the writing. Because V is a coordinate chart, it is clear that elements of $\Theta_X(V)$ are given by elements of the form

$$\sum_{i=1}^n m(V)(f_i(z))\partial_{z_i},$$

where the $(f_i(z))_{1 \le i \le n}$ are holomorphic functions on V. So it is clearly enough by linearity to show for instance that an element of the form

$$m(V)(f(z))\partial_{z_1}^2 m(V)(g(z))\partial_{z_2} \ f, g \in \mathcal{O}_X(V)$$

can be written in the correct way. It is a straightforward computation, we just move the partial derivatives to the right :

$$\begin{aligned} f(z)\partial_{z_1}^2 g(z)\partial_{z_2} &= f(z)g(z)\partial_{z_1}^2\partial_{z_2} + f(z)[\partial_{z_1}^2, g(z)]\partial_{z_2} \\ &= f(z)g(z)\partial_{z_1}^2\partial_{z_2} + f(z)\frac{\partial g}{\partial z_1}(z)\partial_{z_1}\partial_{z_2} + f(z)\partial_{z_1}\frac{\partial g}{\partial z_1}(z)\partial_{z_2} \\ &= f(z)g(z)\partial_{z_1}^2\partial_{z_2} + f(z)\frac{\partial g}{\partial z_1}(z)\partial_{z_1}\partial_{z_2} + f(z)\frac{\partial g}{\partial z_1}(z)\partial_{z_1}\partial_{z_2} + f(z)\frac{\partial^2 g}{\partial z_1^2}(z)\partial_{z_2}. \end{aligned}$$

□

Using this definition, we could define the order of $D \in \langle \mathcal{O}_X, \Theta_X \rangle(U)$ at $x \in U$ to be the maximal order of derivative that appears when restricted to a local chart around x and check this doesn't depend on the chart.

We'll give another description of the sheaf of differential operators that will make the order clearly coordinate-independant.

To do so, we define recursively for sub-presheaves $\mathcal{D}_X^{\leq m}$ of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ on each U via the conditions

$$\begin{aligned}
\mathcal{D}_X^{\leq -1}(U) &= 0, \\
\mathcal{D}_X^{\leq 0}(U) &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) \,|\, [D(V), m(V)(f)] = 0 \\
&\quad \text{for all } V \subset U \text{ and all } f \in \mathcal{O}_X(V)\}, \\
\mathcal{D}_X^{\leq m}(U) &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) \,|\, [D(V), m(V)(f)] \in \mathcal{D}_X^{\leq m-1}(V) \\
&\quad \text{for all } V \subset U \text{ and all } f \in \mathcal{O}_X(V)\}, \text{ if } m \geq 0.
\end{aligned}$$

They turn out to be subsheaves. It is clear by definition that

$$0 = \mathcal{D}_X^{\leq -1} \subset \mathcal{D}_X^{\leq 0} \subset \mathcal{D}_X^{\leq 1} \subset \dots.$$

The first two steps can be very explicitely described as follows.

Proposition 2. *The image of the map of $\mathcal{O}_X$-modules*

$$m : \mathcal{O}_X \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

lies in $\mathcal{D}_X^{\leq 0}$. Conversely evaluating at the constant function equal to 1 yields a map of $\mathcal{O}_X$-modules

$$ev_1 : \mathcal{D}_X^{\leq 0} \longrightarrow \mathcal{O}_X.$$

These two maps are inverses of each other, in particular the map m identifies $\mathcal{O}_X$ with $\mathcal{D}_X^{\leq 0}$.

Proof. Let's be explicit about how ev_1 is defined. Let U be an open and $D \in \mathcal{D}_X^{\leq 0}(U)$, we can evaluate $D(U)$ on the constant function $1_U \in \mathcal{O}_X(U)$ which yields $D(U)(1_U) \in \mathcal{O}_X(U)$, it is easy to see it defines a map of sheaves of $\mathcal{O}_X$-modules

$$\mathrm{ev}_1 : \mathcal{D}_X^{\leq 0} \longrightarrow \mathcal{O}_X.$$

I claim that it is straightforward to check $m \circ \mathrm{ev}_1 = \mathrm{id}_{\mathcal{D}_X^{\leq 0}}$ and $\mathrm{ev}_1 \circ m = \mathrm{id}_{\mathcal{O}_X}$.[1] ☐

Let's now turn to the next part of our filtration. Now that we know $\mathcal{D}_X^{\leq 0} \simeq \mathcal{O}_X$, it is straightforward from the definition of Θ_X to check that

$$\Theta_X \subset \mathcal{D}_X^{\leq 1}.$$

Let $U \subset X$ and $D \in \mathcal{D}_X^{\leq 1}(U)$, by our definition, for all $V \subset U$ and $f \in \mathcal{O}_X(V)$ we have

$$[D(V), m(V)(f)] \in \mathcal{D}^{\leq 0}(V).$$

[1] This is not any deeper than checking the following fact : let R be a commutative $\mathbb{C}$-algebra and $M \in \mathrm{End}_{\mathbb{C}}(R)$, if M commutes with all operators of multiplication $m_f : r \in R \mapsto f \cdot r \in R$ for $f \in R$ then $M = m_{M(1)}$ and conversely $m_f = m_{m_f(1)}$ for all $f \in R$.

But we have just shown that $\mathcal{D}_X^{\leq 0} \simeq \mathcal{O}_X$. Explicitely,

$$[D(V), m(V)(f)] = m(V)([D(V), m(V)(f)](1_V)) = m(V)(D(V)(f) - fD(V)(1_V)),$$

hence we have built a morphism of $\mathcal{O}_X$-modules

$$p : \mathcal{D}_X^{\leq 1} \longrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{D}_X^{\leq 0}) = \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

that (all identifications done and being loose on the open sets) associates to a $D \in \mathcal{D}_X^{\leq 1}$ the map $f \in \mathcal{O}_X \mapsto [D, f](1) \in \mathcal{O}_X$. We can say a bit more, let $f, g \in \mathcal{O}_X(V)$ then

$$\begin{aligned} [D(V), m(V)(fg)] &= [D(V), m(V)(f)m(V)(g)] \\ &= [D(V), m(V)(f)]m(V)(g) + m(V)(g)[D(V), m(v)(f)] \end{aligned}$$

that is exactly the same as saying, loosely :

$$p(D)(fg) = p(D)(f)g + fp(D)(g)$$

i.e. the image of p lies in $\Theta_X \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

Remark 3. *This whole discussion can feel very formal, but we are just saying that from the order one differential operator $z^4 \frac{\partial}{\partial z} + 12z$ to recover the operator $z^4 \frac{\partial}{\partial z}$ it is enough to do the bracket with the operator of multiplication by f, which yields the operator of multiplication by $z^4 \frac{\partial f}{\partial z}$ and evaluate on the unit.*

Proposition 3. *The following diagram of sheaves of $\mathcal{O}_X$-modules is commutative with exact rows and columns*

$$\begin{tikzcd} & \Theta_X \arrow[d] \arrow[dr, "id"] & \\ \mathcal{D}_X^{\leq 0} \arrow[r, "f"] & \mathcal{D}_X^{\leq 1} \arrow[r, "p"] \arrow[d, "ev_1"] & \Theta_X \\ & \mathcal{O}_X \arrow[ul, "m"] & \end{tikzcd} .$$

In particular the short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{D}_X^{\leq 0} \longrightarrow \mathcal{D}_X^{\leq 1} \xrightarrow{p} \Theta_X \longrightarrow 0,$$

and

$$0 \longrightarrow \Theta_X \longrightarrow \mathcal{D}_X^{\leq 1} \xrightarrow{ev_1} \mathcal{O}_X \longrightarrow 0$$

are split and after identifying $\mathcal{O}_X$ with $\mathcal{D}_X^{\leq 0}$ via m we have, as sheaves of $\mathcal{O}_X$-modules, the equality

$$\mathcal{D}_X^{\leq 1} = \mathcal{D}_X^{\leq 0} \oplus \Theta_X = \mathcal{O}_X \oplus \Theta_X.$$

Proof. Exercice. $\square$

Now that we have a bit more intuition as to why this inductive definition makes sense, we define the following presheaf

$$\mathcal{D}_X^{pre} : U \longmapsto \bigcup_{n\geq 0} \mathcal{D}_X^{\leq n}(U).$$

Proposition 4. *For all $m, m' \geq 0$ and all open $U \subset X$ we have*

$$\mathcal{D}_X^{\leq m}(U) \circ \mathcal{D}_X^{\leq m'}(U) \subset \mathcal{D}_X^{\leq m+m'}(U),$$
$$[\mathcal{D}_X^{\leq m}(U), \mathcal{D}_X^{\leq m'}(U)] \subset \mathcal{D}_X^{\leq m+m'-1}(U).$$

In particular, $\mathcal{D}_X^{\leq 1}$ is a sheaf of Lie algebras and $D_X^{pre}(U) = \bigcup_{m>0} \mathcal{D}_X^{\leq m}(U)$ is an $\mathcal{O}_X(U)$-filtered algebra that is quasi-commutative, that is to say such that the graded $\mathcal{O}_X(U)$-algebra

$$\bigoplus_{m\geq 0} D_X^{\leq m}(U)/D_X^{\leq m-1}(U)$$

is commutative.

Proof. Exercice, this follows either from the explicit formulas for the principal symbol or the Jacobi identity. $\square$

We define $\mathcal{D}_X$ as the sheafification of $\mathcal{D}_X^{pre}$. To put it "explicitely" :

$$\begin{aligned}\mathcal{D}_X(U) &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) | \text{For all } x \in U, \text{ there exists } m_x \in \mathbb{N} \text{ such that } D_x \in (\mathcal{D}_X^{\leq m_x})_x\} \\ &= \{D \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U) | \text{For all } x \in U, \text{ there exists } m_x \in \mathbb{N} \text{ and } x \in V \subset U \text{ s.t. } D_{|V} \in \mathcal{D}_X^{\leq m_x}(V)\}.\end{aligned}$$

We will write loosely

$$\mathcal{D}_X = \bigcup_{m\geq 0} \mathcal{D}_X^{\leq m}.$$

It follows from the previous proposition that for all $m, m' \in \mathbb{N}$ we have well defined multiplication maps of sheaves of $\mathcal{O}_X$-modules

$$\mathcal{D}_X^{\leq m} \times \mathcal{D}_X^{\leq m'} \longrightarrow \mathcal{D}_X^{\leq m+m'}$$

that make

$$\mathcal{D}_X = \bigcup_{m\geq 0} \mathcal{D}_X^{\leq m}$$

into a sheaf of $\mathbb{Z}$-filtrered associative $\mathcal{O}_X$-algebras.

Remark 4. *The order of a differential operator on an open U i.e. of an element of $\mathcal{D}_X(U)$ is only defined locally on U. In fact let $D \in \mathcal{D}_X(U)$, then for all $x \in U$ by assumption there exists a unique $m_x \in \mathbb{N}$ such that $D_x \in (\mathcal{D}_X^{\leq m_x})_{n_x}$ and $D_x \notin (\mathcal{D}_X^{\leq m_x - 1})_x$. We say that m_x is the order of P at x. We'll see later that the order defines a locally constant function in particular, sheafification doesn't change anything on a connected open set U i.e. we have*

$$\mathcal{D}_X(U) = \bigcup_{m\geq 0} \mathcal{D}_X(U) = \mathcal{D}_X^{pre}(U).$$

We now explain how to derive a commutative object from $\mathcal{D}_X$. Define for each $m \in \mathbb{N}$ the sheaf of $\mathcal{O}_X$-modules $\mathrm{Gr}^m(\mathcal{D}_X)$ to be the quotient sheaf of $\mathcal{D}_X^{\leq m}$ by its subsheaf $\mathcal{D}_X^{\leq m-1}$. By definition it fits in an exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{D}_X^{\leq m-1} \longrightarrow \mathcal{D}_X^{\leq m} \longrightarrow \mathrm{Gr}^m(\mathcal{D}_X) \longrightarrow 0.$$

We recall that by definition $\mathrm{Gr}^m(\mathcal{D}_X) = \mathcal{D}_X^{\leq m}/\mathcal{D}_X^{\leq m-1}$ is the sheafification of the presheaf

$$U \longmapsto \mathcal{D}_X^{\leq m}(U)/\mathcal{D}_X^{\leq m-1}(U).$$

Remark 5. *For an open set U, it might happen that*

$$Gr^m(\mathcal{D}_X)(U) \neq \mathcal{D}^{\leq m}(U)/\mathcal{D}^{\leq m}(U).$$

This fact has to do with sheaf cohomology as we'll see later.

We can define $\mathrm{Gr}(\mathcal{D}_X)$ to be the sheaf associated with the presheaf of $\mathcal{O}_X$-modules

$$U \mapsto \bigoplus_{m \geq 0} \mathcal{D}^{\leq m}(U)/\mathcal{D}^{\leq m-1}(U).$$

So that we have, and there is a little something to check I think, as sheaves of $\mathcal{O}_X$-modules, the equality

$$\mathrm{Gr}(\mathcal{D}_X) = \bigoplus_{m \geq 0} \mathrm{Gr}^m(\mathcal{D}_X).$$

Proposition 5. *For all pair of integers $m, m' \in \mathbb{Z}$, the multiplication map*

$$\mathcal{D}_X^{\leq m} \times \mathcal{D}_X^{\leq m'} \longrightarrow \mathcal{D}_X^{\leq m+m'}$$

defines a map of sheaves of $\mathcal{O}_X$-modules

$$Gr^m(\mathcal{D}_X) \times Gr^{m'}(\mathcal{D}_X) \longrightarrow Gr^{m+m'}(\mathcal{D}_X).$$

These maps in turn define a structure of a sheaf of a $\mathbb{Z}$-graded ***commutative*** *$\mathcal{O}_X$-algebra on*

$$Gr(\mathcal{D}_X) = \bigoplus_{m \geq 0} Gr^m(\mathcal{D}_X).$$

Remark 6. *Again, for an open set U, it might happen that*

$$Gr(\mathcal{D}_X)(U) \neq \bigoplus_{m \geq 0} \mathcal{D}_X^{\leq m}(U)/\mathcal{D}_X^{\leq m-1}(U).$$

It can happen for stupid reasons, for instance if U has an infinite number of connected components then it can clearly almost never work because of the infinite direct sum. And it can happen for deeper reasons, that have to do with the previous remark and are related to sheaf cohomology.

Question 3. *When U is connected, do we need to sheafify the infinite direct sum ? That is do we have the equality*

$$Gr(\mathcal{D}_X)(U) = \bigoplus_{m \geq 0} Gr^m(\mathcal{D}_X)(U) \text{ ?}$$

Answer : No need to sheafify

The following theorem shows that our two definitions of differential operators coincide. In particular the order of a differential operator near a point is a well-defined notion that is independant of coordinates. It will also allows us to describe explicitely the associated graded sheaf.

Theorem 1. *The equality of sheaves of $\mathcal{O}_X$-modules*

$$\mathcal{D}_X = \langle \mathcal{O}_X, \Theta_X \rangle$$

holds inside of $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$.

Proof. We start by the easy inclusion, let $D \in \langle \mathcal{O}_X, \Theta_X \rangle(U)$, then for each $x \in U$ there exists $x \in V \subset U$ such that

$$D_{|V} \in \langle \mathcal{O}_X(V), \Theta_X(V) \rangle$$

so in particular, since $\mathcal{O}_X = \mathcal{D}_X^{\leq 0}(V)$ and $\Theta_X(V) \subset \mathcal{D}_X^{\leq 1}(V)$ then it is clear from Proposition 4 that there exists $n_x \in \mathbb{N}$ such that

$$D_{|V} \in \mathcal{D}_X^{\leq n_x}(V)$$

and so $D \in \mathcal{D}_X(U)$. The reverse inclusion is more subtle, let $D \in \mathcal{D}_X(U)$ then for each $x \in U$ there exists $x \in V \subset U$ and $n_x \in \mathbb{N}$ such that

$$D_{|V} \in \mathcal{D}_X^{\leq n_x}(V),$$

if we show that there exists an open neighbourhood $W \subset V$ of x such that $\mathcal{D}_X^{\leq n_x}(W) \subset \langle \mathcal{O}_X(W), \Theta_X(W) \rangle$ we are done. To do so, we may shrink V as we please and assume for instance that we are on a coordinate chart mapping x to 0

$$(z_1, ..., z_n) : V \longrightarrow \mathbb{C}^n,$$

so the proof reduces to that of the following Lemma.

Lemma 2. *Let V an open neighbourhood of 0 in $\mathbb{C}^n$. Then for all $m \in \mathbb{N}$ and all $D \in \mathcal{D}_{\mathbb{C}^n}^{\leq m}(V)$ there exists an neighbourhood $W \subset V$ of 0 such that*

$$D_{|W} \in \langle \mathcal{O}_{\mathbb{C}^n}(W), \Theta_{\mathbb{C}^n}(W) \rangle.$$

Proof. Let $D \in \mathcal{D}_{\mathbb{C}^n}^{\leq m}(V) \subset \mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_V, \mathcal{O}_V)$, looking at the stalk at 0, it defines a $\mathbb{C}$-linear morphism

$$D_0 : \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow \mathcal{O}_{\mathbb{C}^n,0}.$$

The following equality holds in $\mathrm{End}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0})$

$$[...[[D_0, f_1], f_2, ..., f_{m+1}]] \dots] = 0,$$

for all $f_1, ..., f_{m+1} \in \mathcal{O}_{\mathbb{C}^n,0}$. But since

$$\mathcal{O}_{\mathbb{C}^n,0} \simeq \mathbb{C}\{z_1, ..., z_n\},$$

it is enough to show the following Lemma

Lemma 3. *Let $m \in \mathbb{N}$ and $D \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$ such that for all $P_1, ..., P_{m+1} \in \mathbb{C}\{z_1, ..., z_n\}$ we have the following equality in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$*

$$[...[[D, P_1], P_2], ..., P_{m+1}] = 0.$$

Denote by $\mathcal{D}^{\leq m}(\mathbb{C}\{z_1, ..., z_n\})$ the set of all such D. Then there exists a family $(a_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{C}\{z_1, ..., z_n\}$ such that

$$D = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \partial_z^\alpha$$

and $P_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| > m$.

Proof. We proceed by induction on m. If $m = 0$ this is easy (we've seen it in a more general context, recall the proof of the equality of sheaves $\mathcal{D}_X^{\leq 0} = \mathcal{O}_X$). Let $m \in \mathbb{N}$ such that the statement holds and let $D \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$ such that for all $P_1, ..., P_{m+2} \in \mathbb{C}\{z_1, ..., z_n\}$ we have

$$[...[[D, P_1], P_2], ..., P_{m+2}] = 0.$$

Then clearly if we set for all $i \in \{1, ..., n\}$,

$$D_i = [D, z_i] \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$$

then $D_i \in \mathcal{D}^{\leq m}(\mathbb{C}\{z_1, ..., z_n\})$ and moreover

$$[D_i, z_j] = [[D, z_i], z_j] = [[D, z_j], z_i] = [D_j, z_i]$$

for all $i, j \in \{1, ..., n\}$. Assume momentaneously that there exists

$$\widetilde{D} \in \mathcal{D}^{\leq m+1}(\mathbb{C}\{z_1, ..., z_n\})$$

such that for all $i \in \{1, ..., n\}$ we have

$$[\widetilde{D}, z_i] = D_i.$$

Then

$$[D - \widetilde{D}, z_i] = [D, z_i] - [\widetilde{D}, z_i] = D_i - D_i = 0$$

for all $i \in \{1, ..., n\}$. It follows from Lemma 4 below that $D - D' \in \mathcal{D}^{\leq 0}(\mathbb{C}\{z_1, ..., z_n\}) = \mathbb{C}\{z_1, ..., z_n\}$ then we are done.

We come back to the proof of the existence of $\widetilde{D}$. By induction hypothesis we know that for all $i \in \{1, ..., n\}$ there exists $a^i_\alpha \in \mathbb{C}\{z_1, ..., z_n\}$ such that

$$D_i = [D, z_i] = \sum_{|\alpha| \leq m} a^i_\alpha \partial^\alpha_z.$$

Now forget about D, it is a general fact of differential operators that if a family $(D_i)_{1 \leq i \leq n}$ of differential operators of order $\leq m$ satisfy[2]

$$[D_i, z_j] = [D_j, z_i]$$

then they come from an operator of degree at most one more, i.e. there exists $\widetilde{D}$ of order $\leq m+1$ such that

$$[\widetilde{D}, z_i] = D_i.$$

We look for $\widetilde{D}$ in the form

$$\widetilde{D} = \sum_{|\alpha| \leq m+1} A_\alpha(z) \partial^\alpha_z.$$

The following equality of operators hold

$$[\partial^\alpha_z, z_i] = \alpha_i \partial^{\alpha - e_i}_z,$$

where $e_i = (0, ..., \overset{i}{1}, ..., 0)$, and hence

$$[\widetilde{D}, z_i] = \sum_{|\alpha| \leq m+1} \alpha_i A_\alpha(z) \partial^{\alpha - e_i}_z$$

now this is equal to $D_i = \sum_{|\alpha| \leq m} a^i_\alpha(z) \partial^\alpha_z = \sum_\alpha a^i_{\alpha - e_i} \partial^{\alpha - e_i}_z$ if and only if for all α we have

$$\alpha_i A_\alpha(z) = a^i_{\alpha - e_i}(z).$$

So for each α such that $|\alpha| > 0$ i.e. such that there exists $i \in \{1, ..., n\}$ such that $\alpha_i \neq 0$ we set

$$A_\alpha(z) = \frac{1}{\alpha_i} a^i_{\alpha - e_i}(z).$$

To show this doesn't depend on i we need to show that for each $|\alpha| \leq m$ such that $\alpha_i \neq 0$ and $\alpha_j \neq 0$ we have

$$\alpha_j a^i_{\alpha - e_i} = \alpha_i a^j_{\alpha - e_j}$$

but this follows from the equality

$$[D_i, z_j] = [D_j, z_i].$$

[2]This condition looks "Fourier dual" to a Schwarzian integrability condition if $m = 1$, is there a bigger picture ?

In fact, if $i \neq j$ we have

$$[D_i, z_j] = \sum_{\alpha} \alpha_j a^i_\alpha(z) \partial_z^{\alpha - e_j} \overset{\alpha' = \alpha + e_i}{=} \sum_{\alpha'} \alpha'_j a^i_{\alpha' - e_i} \partial_z^{\alpha' - e_j - e_i},$$
$$[D_j, z_i] = \sum_{\alpha} \alpha_i a^j_\alpha(z) \partial_z^{\alpha - e_i} \overset{\alpha' = \alpha + e_j}{=} \sum_{\alpha'} \alpha'_i a^j_{\alpha' - e_j} \partial_z^{\alpha' - e_i - e_j},$$

and that yields the desired equality looking at the term corresponding to a given α' such that $\alpha'_i \neq 0$ and $\alpha'_j \neq 0$ because for such an α', we have

$$\alpha' - e_i - e_j \geq 0.$$

□

□

□

Lemma 4. *Let $D \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}\{z_1, ..., z_n\})$ such that for all $i \in \{1, ..., n\}$ we have*

$$[D, z_i] = 0,$$

then for all $f \in \mathbb{C}\{z_1, ..., z_n\}$,

$$[D, f] = 0.$$

Hence $D \in \mathcal{D}^{\leq 0}(\mathbb{C}\{z_1, ..., z_n\}) = \mathbb{C}\{z_1, ..., z_n\}$.

Proof. Here's an indication on how to prove it :

Any element of $f \in \mathbb{C}\{z_1, ..., z_n\}$ can be written (in a highly non unique way when $n \geq 2$)

$$f = f(0) + z_1 f_1 + ... + z_n f_n$$

so

$$[D, f] = [D, f_1] z_1 + ... + [D, f_n] z_n$$

and iterating there is a problem... because it should map into

$$\bigcap_{m > 0} (z_1, ..., z_n)^m = 0$$

so this is the zero map. □

Proposition 6. *Let U be an open set of X and $D \in \mathcal{D}_X(U)$ then for all $x \in X$ there exists a unique $m_x \in \mathbb{N}$ called the order of D at x such that*

$$D_x \in (\mathcal{D}_X^{\leq m_x})_x \setminus (\mathcal{D}_X^{\leq m_x - 1})_x.$$

The function

$$x \in U \longmapsto m_x \in \mathbb{N}$$

is locally constant and hence constant if U is connected. Consequently for any connected open U, we have the equality of $\mathcal{O}_X(U)$-algebras

$$\mathcal{D}_X(U) = \bigcup_{m \geq 0} \mathcal{D}_X^{\leq m}(U) = \mathcal{D}_X^{pre}(U).$$

Moreover for all choice of coordinates on a connected open neighborhood V of x say

$$z : V \longrightarrow \mathbb{C}^n$$

there exists uniquely determined functions $a_\alpha \in \mathcal{O}_{\mathbb{C}^n}(z(V))$ such that in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_V, \mathcal{O}_V)$ *we have*

$$D_{|V} = \sum_{|\alpha| \leq m_x} a_\alpha(z) \partial_z^\alpha.$$

Proof. The existence of such an m_x is clear by definition. Fix coordinates on some connected open neighborhood V of x. By assumption there exists an open W^x, which we may choose connected and inside of V, such that

$$D_{|W} \in \mathcal{D}_X^{\leq m_x}(W^x) - \mathcal{D}_X^{\leq m_x - 1}(W^x).$$

With our choice of coordinates and thanks to the previous theorem, up to maybe shrinking W^x, there exists $a_\alpha^{W^x} \in \mathcal{O}_{\mathbb{C}^n}(z(W^x))$ such that

$$D_{|W^x} = \sum_{\alpha \leq m_x} a_\alpha^{W^x}(z) \partial_z^\alpha.$$

The same holds for any point $y \in V$, namely there exists a connected open $W^y \subset V$ containing y such that there exists $a_\alpha^{W^x} \in \mathcal{O}_{\mathbb{C}^n}(z(W^y))$ such that

$$D_{|W^y} = \sum_{\alpha \leq m_y} a_\alpha^{W^y}(z) \partial_z^\alpha.$$

Now for all $x, y \in V$ such that

$$W^x \cap W^y \neq \emptyset,$$

which happens for instance when $y \in W^x$, we have the equality in $\mathcal{D}_X(W^x \cap W^y)$

$$D_{|W^x \cap W^y} = D_{|W^y \cap W^x}$$

that is in particular

$$\sum_{\alpha \leq m_x} a_\alpha^{W^x}(z)\big|_{W^x \cap W^y} \partial_z^\alpha = \sum_{\alpha \leq m_y} a_\alpha^{W^y}(z)\big|_{W^y \cap W^x} \partial_z^\alpha.$$

so using unicity of the writing we get that for all α :

$$a_\alpha^{W^x}(z)\big|_{W^x \cap W^y} = a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}$$

In particular for all α such that $|\alpha| > m_x$ the previous equality becomes

$$0 = a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}.$$

Now by analytic continuation because W^y is connected and $W^y \cap W^x \neq \emptyset$ we get that $a_\alpha^{W^y}(z) = 0$ and hence $m_y \leq m_x$ so by symmetry $m_x = m_y$ and hence the fact that the order is constant on W^x and hence locally constant. We have an open covering

$$V = \bigcup_{x \in V} W^x$$

and because V is connected we have $m_x = m_y = m$ for all $x, y \in V$. For all α such that $|\alpha| \leq m$ we have by unicity again

$$a_\alpha^{W^x}(z)\big|_{W^x \cap W^y} = a_\alpha^{W^y}(z)\big|_{W^y \cap W^x}.$$

So for each α they glue to a certain holomorphic function $a_\alpha \in \mathcal{O}_X(V)$ such that for all $x \in V$ the following equality holds in $\mathcal{O}_X(W^x)$

$$a_\alpha(z)\big|_{W^x} = a_\alpha^{W^x}(z).$$

Now it is clear from the fact that $\mathcal{D}_X$ is a sheaf that

$$D_{|V} = \sum_{|\alpha| \leq m} a_\alpha(z)\partial_z^\alpha.$$

$\square$

Remark 7. *We note that this proves the expected fact that for all $n \geq 0$ the associative $\mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n)$-algebra of global differential operators on $\mathbb{C}^n$*

$$\mathcal{D}_{\mathbb{C}^n}(\mathbb{C}^n)$$

consists of all the elements of the form

$$\sum_\alpha a_\alpha(z)\partial_z^\alpha.$$

In particular for each $m \in \mathbb{N}$, the sheaf of $\mathcal{O}_X$-modules $\mathcal{D}_X^{\leq m}$ is a locally free of finite rank i.e. corresponds to a finite dimensional vector bundle. Explicitely for each connected coordinate chart U in coordinates $z_1, ..., z_n$ we have an isomorphism of $\mathcal{O}_U$-modules

$$\mathcal{D}_U^{\leq m} \simeq \bigoplus_{|\alpha| \leq m} \mathcal{O}_U \partial_z^\alpha.$$

In fact for every (possibly non connected) coordinate chart U in coordinates $z_1, ..., z_n$, we have

$$\mathcal{D}_U \simeq \bigoplus_\alpha \mathcal{O}_U \partial_z^\alpha$$

where the right hand side is the appropriately defined (=sheafified) infinite direct sum of sheaves. On every connected open set $V \subset U$, analytic continuation ensures that

$$\mathcal{D}_X(V) = \bigoplus_{\alpha} \mathcal{O}_X(V)\partial_z^{\alpha}.$$

So maybe one motivation for saying everything in terms of sheaves is to have a flexible setting that allows one to speak of $\mathcal{D}_X$, an infinite dimensional vector bundle, without every thinking of its transition maps as "infinite dimensional" matrices.

We can now describe the sheaf $\mathrm{Gr}(\mathcal{D}_X)$.

Proposition 7. *We have the equality of sheaves of $\mathcal{O}_X$-modules*

$$Gr^0(\mathcal{D}_X) = \mathcal{O}_X, Gr^1(\mathcal{D}_X) = \Theta_X.$$

That defines a map of sheaves of graded commutative $\mathcal{O}_X$-algebras

$$Sym_{\mathcal{O}_X}(\Theta_X) \longrightarrow Gr(\mathcal{D}_X).$$

which is an isomorphism. In particular, for all $m \geq 0$ we have an isomorphism of sheaves of $\mathcal{O}_X$-modules

$$Sym^m_{\mathcal{O}_X}(\Theta_X) \longrightarrow Gr^m(\mathcal{D}_X).$$

Sketch of proof. The construction of the map is clear by playing around with universal properties, to see this is an isomorphism one can work locally and then it is clear in local coordinates. □

Remark 8. *Give the interpretation in terms of the cotangent bundle. Sym in the notation stands for symmetric, funnily enough, it stands for (principal) symbol at the same time, so it's a really good notation.*

2 Differential operators on the projective line

Let $\mathbb{P}^1_{\mathbb{C}} = \mathbb{P}^1$ be the projective line over the complex number. It is covered by two charts

$$\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}_z,$$
$$\mathbb{P}^1 \setminus \{0\} = \mathbb{C}_w,$$

and on the intersection, i.e. on $\mathbb{P}^1 \setminus \{0, \infty\}$ the change of coordinate is given by

$$z \in \mathbb{C}_z^* \longrightarrow \frac{1}{w} \in \mathbb{C}_w^*.$$

In the previous section we defined the sheaf $\mathcal{D}_{\mathbb{P}^1}$ of differential operators on $\mathbb{P}^1$. Here we want to describe the global differential operators on the projective line

$$\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1),$$

as a $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$-algebra.

First recall that $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C}$ because any holomorphic function $\mathbb{P}^1 \longrightarrow \mathbb{C}$ needs to be bounded and hence constant. So really what we are describing is a (a priori non-commutative) $\mathbb{C}$-algebra.

Because $\mathbb{P}^1$ is connected, the order of a global differential operator is well defined, that is, as $\mathbb{C}$-algebras we have

$$\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1) = \bigcup_{m\geq 0} \mathcal{D}^{\leq m}_{\mathbb{P}^1}(\mathbb{P}^1).$$

We have the following equality of $\mathbb{C}$-vector spaces

$$\mathcal{D}^{\leq 1}_{\mathbb{P}^1}(\mathbb{P}^1) = \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \oplus \Theta_{\mathbb{P}^1}(\mathbb{P}^1) = \mathbb{C} \oplus \Theta_{\mathbb{P}^1}(\mathbb{P}^1).$$

where $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$ is the Lie algebra of global vector fields on $\mathbb{P}^1$. Let us first compute $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$.

Lemma 5. *The Lie algebra $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$ is 3-dimensional with basis f, h, e satisfying*

$$\begin{aligned}
[h, f] &= -2f,\\
[e, f] &= h,\\
[h, e] &= 2e.
\end{aligned}$$

That is, there is an isomorphism of Lie algebra

$$\Theta_{\mathbb{P}^1}(\mathbb{P}^1) \simeq \mathfrak{sl}_2.$$

Proof. Let $\theta \in \Theta_{\mathbb{P}^1}(\mathbb{P}^1)$ then by definition,

$$\begin{aligned}
\theta_{|\mathbb{C}_z} &= a(z)\partial_z,\\
\theta_{|\mathbb{C}_w} &= b(w)\partial_w,
\end{aligned}$$

for some $a \in \mathcal{O}_{\mathbb{C}_z}(\mathbb{C}_z)$ and $b \in \mathcal{O}_{\mathbb{C}_w}(\mathbb{C}_w)$. Now because θ is globally defined it means that the vector fields coincide on the intersection i.e. on they agree on

$$\mathbb{C}_z^* \simeq \mathbb{P}^1 \setminus \{0, \infty\} \simeq \mathbb{C}_w^*.$$

In the coordinate w that means we have, as vector fields on $\mathbb{C}_w^*$ the equality

$$a(\frac{1}{w})(-w^2)\partial w = b(w)\partial w.$$

That is, for all $w \in \mathbb{C}_w^*$ we have

$$b(w) = -w^2 a(\frac{1}{w})$$

which implies that f, and g are both polynomials of order at most 2. That is we can write

$$\begin{aligned}
a(z) &= a_0 + a_1 z + a_2 z^2,\ a_0, a_1, a_2 \in \mathbb{C},\\
b(w) &= b_0 + b_1 w + b_2 w^2,\ b_0, b_1, b_2 \in \mathbb{C},
\end{aligned}$$

and the equality $b(w) = -w^2 a(\frac{1}{w})$ implies

$$b_0 + b_1 w + b_2 w^2 = -w^2 a_0 + -af_1 - a_2$$

i.e.

$$\begin{aligned} b_0 &= -a_2 \\ b_1 &= -a_1 \\ b_2 &= -a_0. \end{aligned}$$

So a $\mathbb{C}$-basis of the Lie algebra of global vector fields is for instance, symbolically given by the vector fields

$$\begin{aligned} e &= -\partial_z = w^2 \partial_w, \\ h &= -2z\partial_z = 2w\partial_w, \\ f &= z^2 \partial_z = -\partial_w. \end{aligned}$$

Then one easily sees that (by computing on each chart) that they satisfy the commutation relation of $\mathfrak{sl}_2$. □

Remark 9. *Another way to see it is to say that SL_2 acts on $\mathbb{P}^1$ and so by differentiating the action $\mathfrak{sl}_2$ lies in $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$.*

So far we have described the global differential operators of order ≤ 1, we have that

$$\mathcal{D}_{\mathbb{P}^1}^{\leq 1}(\mathbb{P}^1) = \mathbb{C} \oplus \mathfrak{sl}_2.$$

It is harder to deal with higher order differential operators by looking explicitely at the change of coordinates. In fact, fix an operator of order 2 on each chart

$$\begin{aligned} D_z &= a_0(z) + a_1(z)\partial_z + a_2(z)\partial_z^2, \\ D_w &= b_0(w) + b_1(w)\partial_w + b_2(w)\partial_w^2, \end{aligned}$$

with holomorphic coefficients. They glue if and only if they coincide on the intersection after change of coordinates, that is, if and only if

$$\begin{aligned} & a_0(\frac{1}{w}) + a_1(\frac{1}{w})(-w^2)\partial_w + a_2(\frac{1}{w})(-w^2)\partial_w(-w^2)\partial_w \\ &\quad = a_0(\frac{1}{w}) - w^2 a_1(\frac{1}{w})\partial_w - w^2 a_2(\frac{1}{w})(-w^2)\partial_w\partial_w - w^2 a_2(\frac{1}{w})(-2w)\partial_w \\ &\quad = a_0(\frac{1}{w}) - w^2 a_1(\frac{1}{w})\partial_w - w^2 a_2(\frac{1}{w})(-w^2)\partial_w\partial_w - w^2 a_2(\frac{1}{w})(-2w)\partial_w \\ &\qquad\qquad = b_0(w) + b_1(w)\partial_w + b_2(w)\partial_w^2. \end{aligned}$$

That is to say if and only if for all $w \in \mathbb{C}_w^*$ we have

$$b_0(w) = a_0(\frac{1}{w}), \tag{1}$$

$$b_1(w) = -w^2 a_1(\frac{1}{w}) + 2w^3 a_2(\frac{1}{w}), \tag{2}$$

$$b_2(w) = w^4 a_2(\frac{1}{w}). \tag{3}$$

This is still manageable, and maybe this is explicit enough for many purposes. Notice that the top coefficients a_2 and b_2 behave "nicely" i.e. the glueing conditions look familiar. It is a general feature of the principal symbol of a differential operator, the condition (3) is nothing by the glueing condition of the tensor product of the tangent sheaf with itself, namely this is nothing but us explicitely looking at the sheaf equality

$$\mathrm{Gr}^2(\mathcal{D}_{\mathbb{P}^1}) = \mathrm{Sym}^2(\Theta_X) = \Theta_X \otimes \Theta_X,$$

that we already proved !

Hopefully this serves as a motivation to show that the tools we introduced before are there to help us make **explicit computations**. The additional algebraic input will be the representation theory of the Lie algebra $\mathfrak{sl}_2$.

Let's begin, because $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ is an associative algebra, there exists a morphism of associative algebras

$$\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1),$$

where $\mathcal{U}(\mathfrak{sl}_2)$ is the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. This is defined via generators and relations as the quotient of the tensor algebra

$$\mathcal{T}(\mathfrak{sl}_2) = \bigoplus_{n \geq 0} (\mathfrak{sl}_2)^{\otimes n} = \mathbb{C} \oplus \mathfrak{sl}_2 \oplus (\mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathfrak{sl}_2) \oplus ...$$

obtained by imposing the defining relations of $\mathfrak{sl}_2$ i.e. asking that

$$h \otimes f - f \otimes h = -2f,$$
$$e \otimes f - f \otimes e = h,$$
$$h \otimes e - e \otimes h = 2e.$$

The image of φ inside of $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ is the $\mathbb{C}$-subalgebra generated by $\mathfrak{sl}_2$ i.e. the subalgebra of global differential operators generated by global vector fields.

By definition, $\mathcal{T}(\mathfrak{sl}_2)$ is a filtered algebra, so we can define a filtration on its quotient $\mathcal{U}(\mathfrak{sl}_2)$ by mean of that filtration. Precisely we define for all $m \in \mathbb{Z}$

$$\mathcal{U}(\mathfrak{sl}_2)^{\leq m} = \pi(\mathcal{T}^{\leq m}(\mathfrak{sl}_2))$$

where $\pi : \mathcal{T}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2)$ is the quotient map. Recall the following Theorem in the case of the Lie algebra $\mathfrak{sl}_2$, it explicitely describes the graded algebra associated with the previously described filtration

Theorem 2 (Poincaré-Birkhoff-Witt). *The following family of elements of* $\mathcal{U}(\mathfrak{sl}_2)$
$$(f^\alpha h^\beta e^\gamma)_{\alpha,\beta,\gamma\in\mathbb{N}}$$
form a $\mathbb{C}$*-basis of* $\mathcal{U}(\mathfrak{sl}_2)$*. More precisely, for each* $m \in \mathbb{N}$*, the family of elements of* $\mathcal{U}(\mathfrak{sl}_2)$
$$(f^\alpha h^\beta e^\gamma)_{\alpha+\beta+\gamma\leq m}$$
forms a basis of $\mathcal{U}^{\leq m}(\mathfrak{sl}_2)$*. The associated graded algebra of* $\mathcal{U}(\mathfrak{sl}_2)$ *is isomorphic to the symmetric algebra on the generators (of degree one)* f, h *and* e*. That is to say there exists an isomorphism of graded* $\mathbb{C}$*-algebras*
$$Sym(\mathfrak{sl}_2) = \mathbb{C}[f,h,e] \simeq Gr(\mathcal{U}(\mathfrak{sl}_2)).$$

The map of associative algebras $\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ previously defined is compatible with the filtrations on both side.

We are now in following general setting, we have a filtered map φ between two filtered associative algebras. And to check if it is an isomorphism (between non-commutative objects), it is enough to check if it is an isomorphism at the level of the associated graded algebras (that so happen to be commutative here).

Hence, before applying this general idea, we first need to describe the associated graded of the associative algebra
$$\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1) = \bigcup_{m\geq 0} \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1).$$

For each $m \in \mathbb{N}$, by the very definition of the sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules $\mathrm{Gr}^{m+1}\mathcal{D}_{\mathbb{P}^1}$, we have a short exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules
$$0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m} \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1} \longrightarrow \mathrm{Gr}^{m+1}\mathcal{D}_{\mathbb{P}^1} \longrightarrow 0,$$
for $m = 0$ this is an exact sequence that we have already encountered
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq 1} \longrightarrow \Theta_{\mathbb{P}^1} \longrightarrow 0.$$
In general, we have explained that for all $m \geq 0$ we have the equality of sheaves of $\mathcal{O}_X$-modules
$$\mathrm{Gr}^m(\mathcal{D}_{\mathbb{P}^1}) = \mathrm{Sym}^m(\Theta_{\mathbb{P}^1}).$$

Lemma 6. *Let* $m \geq 0$*, then we have the equalities of sheaves of* $\mathcal{O}_X$*-modules*
$$Sym^n(\Theta_{\mathbb{P}^1}) = \Theta_{\mathbb{P}^1}^{\otimes m}.$$
In particular we have,
$$\dim_{\mathbb{C}}(\Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1)) = 2m + 1.$$

Proof. Because Θ_X is a locally free sheaf of rank 1 i.e. a line bundle, its symmetric and tensor powers are equal and are again locally free sheaves of rank r. This is not any deeper than the fact that

$$S^m_R(R) = R^{\otimes m} = R \otimes_R ... \otimes_R R = R$$

for any ring R. So we are left to describe what line bundle is $\Theta_X^{\otimes m}$. Let's be very explicit, clearly when restricting to say the open U_z, the multiplication of $\mathcal{O}_{U_z}$ yields an isomorphism between the presheaf on U_z

$$U \longmapsto \Theta_{U_z}^{\otimes m} \simeq \mathcal{O}_{U_z}^{\otimes m}$$

and the sheaf $\mathcal{O}_{U_z}$. This says that we don't need to sheafify our definition of (iterated) tensor product when looking at open sets contained in U_z or U_w. Now because $\Theta_{\mathbb{P}^1}^{\otimes m}$ is a sheaf, to give a section of $\mathbb{P}^1$ is the same as giving sections on U_z and U_w that transform accordingly under change of coordinates. A section on U_z is, because by the previous remark we don't need to sheafify,

$$f_1(z)\partial_z \otimes f_2(z)\partial_z \otimes ... \otimes f_m(z)\partial_z = f_1(z)...f_m(z)(\partial_z \otimes ... \otimes \partial_z) \simeq f_1(z)...f_m(z)$$

where $f_1, ..., f_m \in \mathcal{O}_{U_z}(U_z)$. For each $i \in \{1, ..., m\}$, the vector field $f_i(z)\partial_z$ written in coordinate w becomes $-w^2 f_i(\frac{1}{w})\partial_w$.

Summing up we are looking for holomorphic functions $f_1, ..., f_m \in \mathcal{O}_{U_z}(U_z)$ (resp. $g_1, ..., g_m \in \mathcal{O}_{U_w}(U_w)$) that satisfy for each $w \in \mathbb{C}^*$ the equality

$$(-1)^m w^{2m} f_1(\frac{1}{w})...f_m(\frac{1}{w}) = g_1(w)...g_m(w).$$

Now it is trivial to note that when $f_1, ..., f_m$ describes $\mathcal{O}_{U_z}(U_z)$ then so does their product F, and the same goes with $g_1, ..., g_m$ by calling the product G. So global sections of $\Theta_{\mathbb{P}^1}^{\otimes m}$ are in bijection with holomorphic functions F and G satisfying for all $w \in \mathbb{C}^*$ the equality

$$(-1)^m w^{2m} F(\frac{1}{w}) = G(w),$$

i.e. polynomials of degree at most $2m$. Clearly this is a vector space of degree $2m + 1$. $\square$

Remark 10. *For those who know about the family of sheaves $\mathcal{O}_{\mathbb{P}^1}(m)$ on $\mathbb{P}^1$ for $m \in \mathbb{Z}$, it is clear that $\Theta_{\mathbb{P}^1} = \mathcal{O}(2)$ and we are of course just iterating the fact that*

$$\mathcal{O}_{\mathbb{P}^1}(m) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(m') = \mathcal{O}_{\mathbb{P}^1}(m + m').$$

Seeing tensor products of sheaves for the first time can be confusing. In particular these sheaves give an example as to why we need to sheafify the definition of tensor products. For instance the global sections of $\Theta_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} \Theta_{\mathbb{P}^1}$ is a vector space of dimension 5 as we explained. But it is not equal to the "naive guess" (i.e the value of the presheaf) which is equal to $\Theta_{\mathbb{P}^1}(\mathbb{P}^1) \otimes_{\mathbb{C}} \Theta_{\mathbb{P}^1}(\mathbb{P}^1)$, a vector space of dimension $3 * 3 = 9$! Looking at the proof, it really boils down to *the following fact, when multiplying two polynomials f_1, f_2 of degree ≤ 2 in the variable z, one always gets a polynomial F of degree ≤ 4. But a polynomial of degree ≤ 4 may not be written uniquely in that form*

$$z^2 = z \cdot z = z^2 \cdot 1 = 1 \cdot z^2.$$

It is good for these kind of issues to keep in mind the example of the tensor product of Ω^1 (the sheaf of differential 1-forms = the cotangent bundle = $\mathcal{O}(-2)$) and Θ_X (the sheaf of vector fields = the tangent bundle = $\mathcal{O}(2)$) which gives $\mathcal{O}_X$ although there is no globally defined holomorphic 1-form on $\mathbb{P}^1$.

Hence we have an exact sequence of $\mathcal{O}_{\mathbb{P}^1}$-modules

$$0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m} \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1} \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m} \longrightarrow 0$$

looking at global sections yields the exact sequence of $\mathbb{C}$-vector spaces ($= \mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)-$modules)

$$0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1) \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}(\mathbb{P}^1) \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1).$$

There is no zero on the right, and this is not a mistake ! Taking global sections is a left exact operation, in a sense this is the reason why one needs to sheafify the definition of quotients (see Remark 5). The very definition of sheaf cohomology shows that there is a long exact sequence

$$\begin{aligned}
0 \longrightarrow{}& \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1) \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}(\mathbb{P}^1) \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1) \\
\longrightarrow{}& \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) \\
\longrightarrow{}& \mathrm{H}^2(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) \longrightarrow \mathrm{H}^2(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}) \longrightarrow \mathrm{H}^2(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) \\
\longrightarrow{}& \dots
\end{aligned}$$

For dimension reasons, the cohomology of the sheaves we consider vanishes starting from degree 2. That is we have an exact sequence of $\mathbb{C}$-vector spaces

$$\begin{aligned}
0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m}(\mathbb{P}^1) \longrightarrow \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}(\mathbb{P}^1) \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1)& \\
\longrightarrow \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m+1}) \longrightarrow \mathrm{H}^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) \longrightarrow 0.& \quad (*)
\end{aligned}$$

We can compute the first cohomology groups of the line bundles that appear.

Lemma 7. *We have*

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0,$$

and more generally for all $m \geq 0$, we have

$$H^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}^{\otimes m}) = 0.$$

Proof. Omitted. □

From there we can show

Proposition 8. *For all $m \geq 0$,*

$$H^1(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}^{\leq m}) = 0$$

and

$$Gr^m(\mathcal{D}_{\mathbb{P}^1})(\mathbb{P}^1) = \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1).$$

In particular

$$\dim_{\mathbb{C}}(Gr^m(\mathcal{D}_{\mathbb{P}^1})(\mathbb{P}^1)) = 2m+1.$$

Proof. It is clear by induction on m, the previous lemma and the long exact sequence (*). The computation of the dimension what done in Lemma 6. □

Let's come back to our problem, we want to check if

$$\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

is an isomorphism of filtered associative algebras. Passing to the graded objects we get a graded morphism

$$\mathrm{Gr}(\varphi) : \mathrm{Gr}(\mathcal{U}(\mathfrak{sl}_2)) \simeq \mathbb{C}[f,h,e] \longrightarrow \mathrm{Gr}(\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)) = \bigoplus_{m \geq 0} \Theta_{\mathbb{P}^1}^{\otimes m}(\mathbb{P}^1).$$

To check if it is an isomorphism we can check degree by degree

$$\begin{aligned}
&\mathrm{Gr}^0(\varphi) : \mathbb{C} \longrightarrow \mathbb{C},\\
&\mathrm{Gr}^1(\varphi) : f\mathbb{C} \oplus h\mathbb{C} \oplus e\mathbb{C} \longrightarrow \Theta_{\mathbb{P}^1}(\mathbb{P}^1),\\
&\mathrm{Gr}^2(\varphi) : f^2\mathbb{C} \oplus h^2\mathbb{C} \oplus e^2\mathbb{C} \oplus fe\mathbb{C} \oplus fh\mathbb{C} \oplus he\mathbb{C} \longrightarrow \Theta_{\mathbb{P}^1}^{\otimes 2}(\mathbb{P}^1).
\end{aligned}$$

But there clearly is a problem in degree 2 because the dimensions don't match, the dimension of the left hand side is 6 while on the right we computed it to be 5. This can mean two things, either we have been too optimistic and missed a relation in degree 2 in $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ and moding out by this relation will give the correct answer. Or that there are too much relation of order higher than one between the vector fields inside of $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ and it is not even true that $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$ is generated by $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)$ and $\Theta_{\mathbb{P}^1}(\mathbb{P}^1)$.

Question 4. *Can one formulate a sufficient condition on X so that $\mathcal{D}_X(X)$ is generated as an associative algebra by $\mathcal{O}_X(X)$ and $\Theta_X(X)$?*

Let's try and find a relation, it is natural from the perspective of Lie theory to look at the following element, known as the Casimir element

$$C = \frac{1}{2}h^2 + ef + fe \in \mathcal{U}(\mathfrak{sl}_2),$$

its importance lies in the fact that it generates the center of $\mathcal{U}(\mathfrak{sl}_2)$. It corresponds to the vector field , in the z-coordinate it corresponds to

$$
\begin{aligned}
C &= \frac{1}{2}(-2z\partial_z)(-2z\partial_z) - \partial_z z^2\partial_z + z^2\partial_z(-\partial_z) \\
&= 2z\partial_z z\partial_z - \partial_z z^2\partial_z - z^2\partial_z\partial_z \\
&= 2z^2\partial_z\partial_z + 2z\partial_z - z^2\partial_z\partial_z - 2z\partial_z - z^2\partial_z\partial_z \\
&= 0.
\end{aligned}
$$

and a similar computation shows it too in the w-coordinate.

Remark 11. *This could have been expected because algebras of differential operators typically don't have central elements ? Can I prove it thought ?*

We have shown that the map of filtered associative algebras

$$\varphi : \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

factors through the quotient $\mathcal{U}(\mathfrak{sl}_2)_0$ of $\mathcal{U}(\mathfrak{sl}_2)$ by the two-sided ideal generated by the Casimir element. Hence we have defined a morphism of filtered associative algebras

$$\widetilde{\varphi} : \mathcal{U}(\mathfrak{sl}_2)_0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1).$$

The result is the following, the rest of this section is devoted to proving it.

Theorem 3. *The morphism of associative algebras*

$$\widetilde{\varphi} : \mathcal{U}(\mathfrak{sl}_2)_0 = \mathcal{U}(\mathfrak{sl}_2)/((\frac{1}{2}h^2 + ef + fe)\mathcal{U}(\mathfrak{sl}_2)) \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

is an isomorphism.

Both side carry an action of $\mathfrak{sl}_2$, on the left it comes from the adjoint action and on the right it comes from the inclusion

$$\mathfrak{sl}_2 = \Theta_{\mathbb{P}^1}(\mathbb{P}^1) \subset \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

by having $x \in \mathfrak{sl}_2$ acting as $[x, \cdot]$ on $\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$. It is clear from the fact that φ is a morphism of associative algebra that it is a morphism of $\mathfrak{sl}_2$-modules with respect to these structures. Moreover it is clear that $C\mathcal{U}(\mathfrak{sl}_2)$ is a $\mathfrak{sl}_2$-submodule of $\mathcal{U}(\mathfrak{sl}_2)$. For all $x \in \mathfrak{sl}_2$ and all $y \in \mathcal{U}(\mathfrak{sl}_2)$ we have

$$[x, Cy] = C[x, y]$$

because C is central. Hence $C\mathcal{U}(\mathfrak{sl}_2)$ is isomorphic to $\mathcal{U}(\mathfrak{sl}_2)$ as an $\mathfrak{sl}_2$-module.

First let's show that φ is surjective. For all $m \in \mathbb{N}$, the vector $e^m \in \mathcal{U}(\mathfrak{sl}_2)$ is clearly an highest weight vector of weight $2m$. It is mapped by φ to the differential operator $e^m \in \mathrm{Sym}^m(\Theta_{\mathbb{P}^1})(\mathbb{P}^1)$ which in the z-chart reads

$$(-1)^m\partial_z^m,$$

in particular it is nonzero. As such it is an highest weight vector of weight $2m$ so the $\mathfrak{sl}_2$-submodule generated by it is of dimension $2m+1$. For dimension reasons we see that it is the whole of $\mathrm{Sym}^n(\Theta_{\mathbb{P}^1})(\mathbb{P}^1)$, that proving the surjectivity of φ and hence of $\widetilde{\varphi}$.

Injectivity is more subtle. We have a surjective map between two semisimple $\mathfrak{sl}_2$-modules, so to conclude we have an isomorphism it is enough to explain why both side have the same decomposition in simple $\mathfrak{sl}_2$-modules. To show this, we'll make use of a certain notion of character, it measures the "size" of a graded vector space with finite dimensional graded pieces. To be more precise, there are two gradations in our setting, one is given by the weight with respect to the action of $\mathrm{ad}(h)$ and the other is given by the degree. On then enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$, the degree gradation is the usual grading of the polynomial algebra

$$\mathrm{Gr}(\mathcal{U}(\mathfrak{sl}_2)) = \mathbb{C}[f, h, e]$$

by requiring that $\deg(f) = \deg(h) = \deg(e) = 1$ (Recall that as $\mathbb{C}$-vector spaces, $\mathcal{U}(\mathfrak{sl}_2) \simeq \mathrm{Gr}(\mathcal{U}(\mathfrak{sl}_2)))$. On the right hand side, it is the gradation given by the order of the differential operator. For a $\mathbb{Z}$-bigraded $\mathbb{C}$-vector spaces $V = \bigoplus_{m,m' \in \mathbb{Z}} V_{m,m'}$ such that each $V_{m,m'}$ is of finite dimension, we define its character by

$$\mathrm{ch}(V) = \sum_{m,m'} \dim_{\mathbb{C}}(V_{m,m'}) x^m t^{m'} \in \mathbb{Z}[[x^{\pm 1}, t^{\pm 1}]].$$

So the variable x indicates the weight gradation and the variable t the degree gradation.

Remark 12. *Be careful that in what follows, whenever we take the character of a graded vector space, it is is assumed to have finite dimensional graded pieces with respect to our gradation.*

Proposition 9. *For any short exact sequence of $\mathbb{Z}^2$-graded modules*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we have

$$ch(B) = ch(A) + ch(C).$$

And for all pair A, B of $\mathbb{Z}^2$-graded vector spaces, we have

$$ch(A \otimes_{\mathbb{C}} B) = ch(A) ch(B).$$

For any family of graded vector spaces $(A_m)_{m \in \mathbb{N}}$, we have

$$ch(\bigoplus_{m \in \mathbb{N}} A_m) = \sum_{m \in \mathbb{N}} ch(A_m).$$

Proof. Omitted. □

For each $k \in \mathbb{N}$, we denote by $L(k)$ the simple representation of $\mathfrak{sl}_2$ of highest weight k. Its character with respect to the $\mathbb{Z}$-gradation given by the h-weights is given by

$$\mathrm{ch}(L(k)) = x^{-k} + x^{-k+2} + ... + x^{k-2} + x^k = \frac{x^{k+1} - x^{-k-1}}{x - x^{-1}}.$$

We have explained before that as $\mathfrak{sl}_2$-modules we have

$$\mathrm{Sym}^m(\Theta_{\mathbb{P}^1})(\mathbb{P}^1) = L(2m),$$

so

$$\mathrm{ch}(\mathrm{Gr}(\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1))) = \mathrm{ch}(\bigoplus_{m\in\mathbb{N}} L(2m)) = \sum_{m\geq 0} \mathrm{ch}(L(2m)) = \sum_{m\geq 0} \frac{x^{2m+1} - x^{-2m-1}}{x - x^{-1}} t^m.$$

To compute the character of the enveloping algebra side, notice we have an exact sequence of $\mathfrak{sl}_2$-modules

$$0 \longrightarrow C\mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2) \longrightarrow \mathcal{U}(\mathfrak{sl}_2)_0 \longrightarrow 0.$$

It is an exact sequence of $\mathbb{Z}^2$-graded modules, so we have the equality

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = \mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)) - \mathrm{ch}(C\mathcal{U}(\mathfrak{sl}_2)).$$

As we noticed $\mathcal{U}(\mathfrak{sl}_2)$ and $C\mathcal{U}(\mathfrak{sl}_2)$ are isomorphic as $\mathfrak{sl}_2$-modules, and only the degree is shifted by this isomorphism, so we have

$$\mathrm{ch}(C\mathcal{U}(\mathfrak{sl}_2)) = t^2 \mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)).$$

That readily implies the equality

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = (1 - t^2)\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)).$$

Now it is clear from the equality of graded vector spaces

$$\mathcal{U}(\mathfrak{sl}_2) = \mathbb{C}[f, h, e] = \mathbb{C}[f] \otimes \mathbb{C}[h] \otimes \mathbb{C}[e]$$

that

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)) = \mathrm{ch}(\mathbb{C}[f])\mathrm{ch}(\mathbb{C}[h])\mathrm{ch}(\mathbb{C}[e]) = \frac{1}{1 - x^{-2}t} \cdot \frac{1}{1 - t} \cdot \frac{1}{1 - x^2 t}.$$

Putting everything together we have

$$\mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = \frac{1 - t^2}{(1 - x^{-2}t)(1 - t)(1 - x^2 t)}.$$

Now it is enough to check that

$$\frac{1 - t^2}{(1 - x^{-2}t)(1 - t)(1 - x^2 t)} = \mathrm{ch}(\mathcal{U}(\mathfrak{sl}_2)_0) = \mathrm{ch}(\mathrm{Gr}(\mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1))) = \sum_{m\geq 0} \frac{x^{2m+1} - x^{-2m-1}}{x - x^{-1}} t^m$$

to decide if

$$\widetilde{\varphi} : \mathcal{U}(\mathfrak{sl}_2)_0 \longrightarrow \mathcal{D}_{\mathbb{P}^1}(\mathbb{P}^1)$$

is an isomorphism of associative algebras.

It is indeed the case

Lemma 8. *The following equality*

$$\frac{1-t^2}{(1-x^{-2}t)(1-t)(1-x^2t)} = \sum_{m\geq 0} \frac{x^{2m+1}-x^{-2m-1}}{x-x^{-1}} t^m$$

holds in $\mathbb{Z}[[x^{\pm 1}, t]]$.

Proof. We have

$$\begin{aligned}
\sum_{m\geq 0} \frac{x^{2m+1}-x^{-2m-1}}{x-x^{-1}} t^m &= \frac{1}{x-x^{-1}} \left(x \cdot \sum_{m\geq 0} (x^2t)^m - x^{-1} \cdot \sum_{m\geq 0} (x^{-2}t)^m \right) \\
&= \frac{1}{x-x^{-1}} \left(x \cdot \frac{1}{1-x^2t} - x^{-1} \cdot \frac{1}{1-x^{-2}t} \right) \\
&= \frac{1}{x-x^{-1}} \left(\frac{x}{1-x^2t} - \frac{x^{-1}}{1-x^{-2}t} \right) \\
&= \frac{1}{x-x^{-1}} \left(\frac{x(1-x^{-2}t) - x^{-1}(1-x^2t)}{(1-x^2t)(1-x^{-2}t)} \right) \\
&= \frac{1}{x-x^{-1}} \left(\frac{x - x^{-1} - x^{-1}t + x^1 t}{(1-x^2t)(1-x^{-2}t)} \right) \\
&= \frac{x-x^{-1}}{x-x^{-1}} \frac{1+t}{(1-x^2t)(1-x^{-2}t)} \\
&= \frac{(1+t)(1-t)}{(1-x^2t)(1-x^{-2}t)(1-t)} = \frac{1-t^2}{(1-x^2t)(1-x^{-2}t)(1-t)}.
\end{aligned}$$

□

Remark 13. *Note that we have essentially proven the following equalities of* $\mathfrak{sl}_2$*-modules, that describe the decomposition of* $\mathcal{U}(\mathfrak{sl}_2)$ *and* $\mathcal{U}(\mathfrak{sl}_2)_0$ *in simple* $\mathfrak{sl}_2$*-modules under the adjoint action*

$$\begin{aligned}
\mathcal{U}(\mathfrak{sl}_2) &= \bigoplus_{k\geq 0} \mathbb{C}[\frac{1}{2}h^2 + ef + fe] \otimes L(2k), \\
\mathcal{U}(\mathfrak{sl}_2)_0 &= \bigoplus_{k\geq 0} L(2k).
\end{aligned}$$

Question 5. *How does that generalizes to a simple Lie algebra* $\mathfrak{g}$ *?*