Another path to Kingman model for the balance between mutation and selection.

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Abstract

We give a treatment of Kingman model for the balance between selection and mutation [4] showcasing new proofs based on the recent developments around the model, in particular by Yuan [7]; the main interest of these proofs, apart from being conceptually simpler, is to be more robust to changes in the model, as we exemplify in a companion paper [1].

The interplay between selection and mutation is a major issue ([2], chapter 6.2) in population genetics: mutations have the potential to increase genetic diversity, while selection tends to concentrate the distribution of the genotypes around the fittest ones. Kingman model [4] is a toy model designed to study the balance between the two effects, and a landmark property of this model is the condensation effect, when a positive fraction of the population concentrates at the highest possible fitness. The appeal of Kingman model is its great tractability that allows one to precisely describe the analytical conditions under which this phenomenon to occur.

Precisely, we are given a probability measure $q(dx)$ on the unit interval $[0,1]$, the mutation measure, and a real number $\beta \in (0,1)$ quantifying the strength of the mutation with respect to selection. For $p_0(dx)$ another probability measure on the unit interval (assuming that $q(dx)$ and $p_0(dx)$ are both distinct from δ_0), we define inductively a sequence of probability measures $(p_n(dx))_{n\geqslant 0}$ on the unit interval by :

$$p_{n+1}(dx) = \beta q(dx) + (1-\beta)\frac{xp_n(dx)}{w_n} \text{ and } w_n = \int_0^1 xp_n(dx) \tag{1}$$

Informally, $p_n(dx)$ should be thought of as the infinitesimal fraction of the population at time $n \geqslant 0$ with fitness $x \in [0,1]$. Selection biases the fitness distribution towards its highest values, whereas mutation drives back the fitness distribution towards a fixed, unaltered distribution $q(dx)$. The first summand describes the effect of mutation (a simple resampling), while the second describes that of selection (that acts through size biasing), and the relative strength of these effects is quantified by β. The point of this model is that condensation may arise, namely

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the creation of an atom at the top fitness while nor $p_0(dx)$ neither $q(dx)$ display such an atom, and that the criterion to decide for condensation is explicit.

These lecture notes are based on the seminal article by Kingman [4], and a series of papers by Yuan [5, 6, 7] on a more general version of the model focusing on the effect of a random parameter β). The impetus of that work has been the realisation that the arguments developed by Yuan [7] allow for a clean and conceptually straightforward derivation of Kingman's theorem, without the need of complex analysis and of the notion of completely monotone sequence, which were the key technical (and almost magical) tools in the original proof by Kingman. The elements showcased in this work are buried deep down in the work [7] and the aim of this work is to promote this set of ideas by exemplifying their robustness in a companion paper [1].

Also and this is perhaps the main novelty in this article, we try to dissociate the question of the convergence of the sequence $(p_n(dx))$ conditionally on the convergence of the sequence of the fitnesses (w_n) from that of the convergence of the sequence (w_n), the two questions being tackled by quite distinct methods and treatments. The convergence of the sequence (w_n) is done using some simple but clever monotonicity property hidden in the model and uncovered by Yuan [6], while the conditional convergence of $(p_n(dx))$ in total variation then follows by standard quantitative bounds. By comparison, the convergence of $(w_n)_n$ in the paper by Kingman was obtained separately in the two cases in Kingman : as the consequence of explicit computations on generating functions in the case without condensation, and using subtle properties of completely monotone sequences recently discovered again and independently in [3]) in the case with condensation.

1 The invariant measures

Technically, we think there is an interest to offer a quick derivation of the invariant measures, since they capture a lot of the complexity of the model. Also, this gives a very gentle starter for our exposition. We shall say that a probability measure $\pi(dx)$ is invariant if it is a fixed point of equation (1), namely:

$$\pi(dx) = \beta q(dx) + (1-\beta)\frac{x\pi(dx)}{w} \text{ and } w = \int_0^1 x\pi(dx) \tag{2}$$

Notice that, with respect to (1), the only parameters of an invariant measure are β and $q(dx)$. We set $\eta_q \in [0,1]$ the max of the support of the measure $q(dx)$, that is $\eta_q = \sup\{x \in [0,1] : q([x,1]) \neq 0\}$.

Theorem 1. *Set*

$$x_0 = \inf\left\{x' \in [\eta_q, \infty) : \int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{x'}} \leqslant 1\right\}. \tag{3}$$

There is the following alternative:

(i) If $x_0 \leqslant 1$, the set of invariant probability measures is

$$\left(\frac{\beta q(dx)}{1-\frac{x}{x_1}} + \pi_1 \delta_{x_1}\right)_{x_1 \in [x_0,1]},$$

with $\pi_1 = 1 - \int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{x_1}}$.

(ii) If $x_0 > 1$, the measure $\frac{\beta q(dx)}{1-\frac{x}{x_0}}$ is the unique invariant probability measure.

Remark. 1. For definiteness, in case $q(dx)$ has an atom at η_q we take the convention that $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_q}} = \infty$. Then the map

$$x' \mapsto \int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{x'}}$$

is continuous decreasing on $[\eta_q, \infty)$, hence the set of which we take the infimum in the RHS of (3) is a non-empty closed interval of the form $[x_0, \infty)$;

2. In case $x_0 \geqslant 1$, there is a unique invariant probability measure; this is in particular the case when $\eta_q = 1$.

3. It is natural to distinguish the following cases.

   (a) In case $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_q}} > 1$, we have $x_0 > \eta_q$; this is the case in particular when $q(dx)$ has an atom at η_q.

   (b) In case $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_q}} = 1$, we have $x_0 = \eta_q$ and the invariant measure associated with $x_1 = x_0$ has no atom at x_0.

   (c) In case $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_q}} < 1$, we have $x_0 = \eta_q$ again but the invariant measure associated with $x_1 = x_0$ has an atom at x_0.

4. π_1 is the atom at x_1 of the invariant measure indexed by x_1 in case (i) : indeed the only case that would prevent this from being true is when $x_1 = \eta_q$ and $q(dx)$ has an atom at η_q, but in such a case, $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_q}} = \infty$ hence $x_1 \geqslant x_0 > \eta_q$, a contradiction.

5. We have $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{x_0}} = 1$ iff $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_q}} \geqslant 1$, and in this case the invariant probability measure associated with $x_1 = x_0$ in case (i) has no atom at x_0.

6. In case $\eta_q < x_0 < 1$, the max of the support of the measure associated with x_1 is η_q if $x_1 = x_0$ (because $\pi_1 = 0$) and x_1 if $x_1 > x_0$ (because then $\pi_1 \neq 0$); as a consequence, the max of the support describes the set

$$\{\eta_q\} \cup (x_0, 1]$$

as x_1 describes $[x_0, x_1]$, which displays a "hole".

Lemma 1. *Let $\pi(dx)$ be an invariant measure. Then the mean $w = intx\pi(dx)$ of $\pi(dx)$ satisfies: $w \geqslant (1-\beta)\eta_q$.*

Proof 1. Assume by contradiction that $(1-\beta)\eta_q > w$ then choose $\varepsilon > 0$ small enough such that $(1-\beta)(\eta_q - \varepsilon) > w$. Then we observe that if $\pi(dx)$ solves (2), then the support of $\pi(dx)$ contains that of $q(dx)$. This implies the strict inequality:

$$\int_{\eta_q-\varepsilon}^{\eta_q} \frac{1-\beta}{w} x\pi(dx) > \int_{\eta_q-\varepsilon}^{\eta_q} \pi(dx)$$

Then we integrate equation (2), $\pi(dx) = \beta q(dx) + \frac{1-\beta}{w} x\pi(dx)$, on $[\eta_q - \varepsilon, \eta_q]$ and find that $0 > \int_{\eta_q-\varepsilon}^{\eta_q} \beta q(dx)$, which is absurd. □

Proof 2. By induction, one can expand $\pi(dx)$ a solution of (2) for any integer $n \geqslant 0$ as follows:

$$\pi(dx) = \sum_{k=0}^{n} \left(\frac{1-\beta}{w}\right)^k x^k \beta q(dx) + \left(\frac{1-\beta}{w}\right)^{n+1} x^{n+1}\pi(dx) \tag{4}$$

Assume now by contradiction that $w < (1-\beta)\eta_q$ and choose $\varepsilon > 0$ small enough such that $w < (1-\beta)(\eta_q - \varepsilon)$. Then one may lower bound the n-th term in the sum on the RHS as follows:

$$\int_{\eta_q-\varepsilon}^{\eta_q} \left(\frac{1-\beta}{w}\right)^n x^n \beta q(dx) \geqslant \left(\frac{(1-\beta)(\eta_q-\varepsilon)}{w}\right)^n \beta\, q([\eta_q-\varepsilon, \eta_q]) \to \infty, \text{ as } n \to \infty$$

which is absurd since $\pi(dx)$ is a probability measure and all term in the decomposition (4) are non-negative measures. □

Proof of Theorem 1. First, on $[0, \eta_q)$, using that $w \geqslant (1-\beta)\eta_q$, we can divide and write :

$$\pi_{|[0,\eta_q)}(dx) = \frac{\beta q(dx)}{1 - \frac{1-\beta}{w} x}$$

If furthermore $w > (1-\beta)\eta_q$, then we have the reinforcement $\pi_{|[0,\eta_q]}(dx) = \frac{\beta q(dx)}{1-\frac{1-\beta}{w}x}$. Second, on $(\eta_q, 1]$, the equation for the invariant measure $\pi(dx)$ simplifies to:

$$\pi_{|(\eta_q,1]}(dx) = \left(\frac{1-\beta}{w}\right) x\pi_{|(\eta_q,1]}(dx),$$

and a moment of thought gives that this equation has $\lambda\delta_{w/(1-\beta)}$, $0 \leqslant \lambda \leqslant 1$, as a unique solution on the set of sub-probability measures. Altogether,

1. if $(1-\beta)\eta_q < w$, setting $x_1 = \frac{w}{1-\beta}$, the invariant measure has to take the form:

$$\frac{\beta q(dx)}{1 - \frac{x}{x_1}} + \pi_0 \delta_{x_1}$$

   which is indeed an invariant measure for every $x_1 \in (\eta_q, 1]$ such that $\int \frac{\beta q(dx)}{1-x/x_1} \leqslant 1$, setting then $\pi_0 = 1 - \int \frac{\beta q(dx)}{1-x/x_1}$: this the case $x_0 > \eta_q$.

2. if $(1-\beta)\eta_q = w$, the only possibility is

$$\frac{\beta q(dx)}{1-\frac{x}{\eta_q}} + \pi_0 \delta_{\eta_q},$$

which is indeed an invariant measure iff $\int \frac{\beta q(dx)}{1-x/\eta_q} \leqslant 1$, setting then $\pi_0 = 1 - \int \frac{\beta q(dx)}{1-x/\eta_q}$: this is the case $x_0 = \eta_q$.

□

2 Convergence of the sequence (p_n)

2.1 Kingman's theorem

We now turn to our main topic, namely the convergence of the fitness distributions in Kingman model. It will be assumed throughout that $\eta_0 = \sup\{x \in [0,1] : p_0([x,1]) \neq 0\}$ the max of the support of $p_0(dx)$ satisfies the property : $\eta_0 \geqslant \eta_q$. This is no loss of generality since one may start the recursion with $p_1(dx)$ instead of $p_0(dx)$, without changing the convergence issues. Recall the mean fitness w_n is defined by $w_n = \int x p_n(dx)$.

Theorem 2 (Kingman, 1987)**.** *Set*

$$y_0 = \inf\left\{x' \in [\eta_0, \infty) : \int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{x'}} \leqslant 1\right\}, \tag{5}$$

The sequence of probability measures $(p_n(dx))_n$ defined in (1) converges in total variation on $[0,\xi]$ for each $\xi < \eta_0$, and weakly on $[0,\eta_0]$, to :

$$\pi(dx) := \frac{\beta q(dx)}{1-\frac{x}{y_0}} + \pi_0 \delta_{\eta_0}, \tag{6}$$

where $\pi_0 := 1 - \int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{y_0}}$. The convergence also holds in total variation on $[0,\eta_0]$ in case $\pi_0 = 0$. Furthermore, the sequence of mean fitnesses $(w_n)_{n\geqslant 0}$ converges to $(1-\beta)y_0$.

Remark.

1. Observe that $\pi_0 \neq 0$ iff $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} < 1$, and in such a case, $y_0 = \eta_0$. Also, $y_0 = \eta_0$ iff $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} \leqslant 1$.

2. In case $\pi_0 \neq 0$ but $p_0(\{\eta_0\}) = q(\{\eta_0\}) = 0$, and in this case only, see Proposition 4, the limiting distribution is not absolutely continuous with respect to $q(dx)$ (in particular, the convergence does not hold in total variation); this phenomenon where an atom builds up at the max of the support of $p_0(dx)$ is sometimes coined "condensation".

3. Of course, π_0 coincides with $\pi(\{\eta_0\})$ when $\eta_q < \eta_0$. However, there exists cases where π_0 does not coincide with $\pi(\{\eta_0\})$: for instance, if $\eta_q = \eta_0$ and q has an atom at η_q, then $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} = \infty$ and $\pi_0 = 0$ by the previous remark, whereas, from (6), $\pi(\{\eta_0\}) \geqslant \beta\eta_q/(1-\eta_q/y_0) > 0$.

4. (Domain of attraction) Let $p_0(dx)$ be a given probability measure, with limit probability measure $\pi(dx)$; what is the set of probability measures $p'_0(dx)$ (with max of the support η'_0 such that the limit $\pi'(dx)$ concides with $\pi(dx)$? The simple answer depends as follows on π_0:

   (a) if $\pi_0 = 0$, then the set is $\{p'_0(dx) : \eta'_0 \leqslant y_0 \wedge 1\}$.

   (b) if $\pi_0 \neq 0$, then the set is $\{p'_0(dx) : \eta'_0 = \eta_0\}$.

Of course, Theorem 1 follows from Theorem 2: it is easy, using the continuity of the operator $p_0(dx) \mapsto p_1(dx)$, to check that any limit probability measure is invariant, and that the set of limiting measures coincides with the set of invariant probability measures as given by Theorem 1. We nonetheless chose to present Theorem 1 first because it allows to grasp the intuition of the model quickly, and because it admits a simple and autonomous proof.

2.2 Outline of the proof of Theorem 2

The general line of the proof is that, conditionally given the convergence of the sequence of mean fitnesses $(w_n)_n$, the convergence of the sequence $(p_n(dx))$ of fitness distribution is easy. This section details the general startegy of the proofs, that are presented hereafter in Section 2.4. We start as Kingman did by expanding $p_n(dx)$.

Lemma 2 (Expanded form of p_n). *We have for every integer $n \geqslant 1$:*

$$p_n(dx) = \sum_{0 \leqslant k \leqslant n-1} \frac{(1-\beta)^k}{w_{n-1} \dots w_{n-k}} x^k \beta q(dx) + \frac{(1-\beta)^n}{w_{n-1} \dots w_0} x^n p_0(dx). \tag{7}$$

We take the convention that the product over an empty set is 1, so that the term associated with $k = 0$ in the sum on the RHS is $\beta q(dx)$. The proof, by induction, is left to the reader. We deduce the following a-priori bound. Recall our standing assumption that $\eta_0 \geqslant \eta_q$.

Corollary 1. *Assume $w_n \to w$. Then $w \geqslant (1-\beta)\eta_0$.*

Proof. The proof follows from inspection of the second term of the sum in the RHS of (7), proceeding as in proof 2 of Lemma 1. □

The next step is to prove, starting from (7), the following key Proposition.

Proposition 1. *Assume $w_n \to w$.*

- *In the general case where $w \geqslant (1-\beta)\eta_0$, the sequence $(p_n(dx))_{n \geqslant 0}$ converges towards the sub-probability measure $\frac{\beta q(dx)}{1-\frac{1-\beta}{w}x}$ in total variation on $[0, \xi]$ for every $\xi < \eta_0$, hence also weakly on $[0, 1]$, towards the probability measure*

$$\frac{\beta q(dx)}{1 - \frac{1-\beta}{w}x} + \left(1 - \int_0^{\eta_q} \frac{\beta q(dx)}{1 - \frac{1-\beta}{w}x}\right) \delta_{\eta_0}. \tag{8}$$

- *If furthermore $w > (1-\beta)\eta_0$, the sequence $(p_n(dx))_{n\geqslant 0}$ converges towards :*

$$\frac{\beta q(dx)}{1-\frac{1-\beta}{w}x} \tag{9}$$

  in total variation on $[0,\eta_0]$ this time, in particular, $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{1-\beta}{w}x} = 1$.

The proof of Proposition 1 (given in the next section) is technical but quite standard; intuitively it justifies one can pass to the limit "under the sum" in (7). In a nutshell, the key idea to get bounds in total variation is to put some space between $(1-\beta)x/w$ and 1 for every x in the support of $p_0(dx)$ (which includes the support of $q(dx)$ under our assumption). In general however, we only have $(1-\beta)\eta_0/w \leqslant 1$, and in the equality case, we will need to localize to $[0,\xi]$, where $(1-\beta)\xi/w < 1$, which explains that Proposition 1 takes a different form in the two regimes. Theorem 2 then readily follows, had we proved the sequence (w_n) converge, as we now show.

Proof of Theorem 2 conditionally on the convergence of the sequence (w_n). Assume that $w_n \to w$. Based on Proposition 1, we only have to identify the limiting probability measures (8) or (9) with the ones given in the statement of Theorem 2. We shall distinguish two cases:

- If $\frac{1-\beta}{w} = \frac{1}{\eta_0}$, the measure $\frac{\beta q(dx)}{1-\frac{1-\beta}{w}x}$ is only a sub-probability measure in general, that is:

$$\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} \leqslant 1,$$

  which implies that y_0 defined by (5) satisfies $y_0 = \eta_0$ hence $\frac{1}{y_0} = \frac{1-\beta}{w}$ as expected.

- If $\frac{1-\beta}{w} < \frac{1}{\eta_0}$, the measure $\frac{\beta q(dx)}{1-\frac{1-\beta}{w}x}$ is a probability measure, which implies by (strict) monotonicity

$$\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} > 1$$

  hence $\frac{1-\beta}{w} = \frac{1}{y_0}$ again holds.

Also, in both cases, we have that the limit w of $(w_n)_{n\geqslant 0}$ satisfies $\frac{1-\beta}{w} = \frac{1}{y_0}$. Last, from Lemma 8, in case $w = (1-\beta)\eta_0$ and $\int_0^{\eta_q} \frac{\beta q(dx)}{1-\frac{1-\beta}{w}x} = 1$, the convergence stated in Proposition 1 also holds in total variation on $[0,\eta_0]$. □

We now turn to the meat of the proof, namely the convergence of the sequence of mean fitnesses $(w_n)_{n\geqslant 0}$, building on Yuan [7]. We start with the easy cases, then show how to build on them to catch the remaining and more difficult cases.

Lemma 3. *It holds*

- *$(p_n^{(\delta_\eta)}(A))_{n\geqslant 0}$ is non-decreasing for every $A \in [0,\eta)$, and $(p_n^{(\delta_\eta)}(\{\eta\}))_{n\geqslant 0}$ is non-increasing.*

- *In particular,* $(w_n^{(\delta_\eta)})_{n\geqslant 0}$ *is a non-increasing sequence.*
- *In particular,* $(p_n^{(\delta_\eta)})_{n\geqslant 0}$ *converges in total variation, and if* $\pi^{(\delta_\eta)}$ *denotes its limit, it holds:*

$$\|p_n^{(\delta_\eta)} - \pi^{(\delta_\eta)}\|_{TV,[0,\eta]} = p_n^{(\delta_\eta)}(\{\eta\}) - \pi^{(\delta_\eta)}(\{\eta\}) \to 0.$$

Relying on this special case, we now come back to the general case of an arbitrary initial measure $p_0(dx)$, whose support has maximal element η_0. The following is the key to the novel approach pioneered by Yuan. Let us set

$$W_n^{(\delta_{\eta_0})} = \prod_{k=0}^{n-1} w_k^{(\delta_{\eta_0})}.$$

Lemma 4. *Let* $p_0(dx)$ *be a probability measure and* η_0 *be the max of its support. Assume that* $\lim_n((1-\beta)\eta_0)^n/W_n^{(\delta_{\eta_0})} = 0$.

- *It holds*
$$\|p_n - p_n^{(\delta_{\eta_0})}\|_{TV,[0,\eta_0]} \leqslant \frac{((1-\beta)\eta_0)^n}{W_n^{(\delta_{\eta_0})}}$$
- *The sequence of probability measures* $(p_n)_{n\geqslant 0}$ *converges in total variation towards* $\pi^{(\delta_\eta)}$.

In the remaining cases, we have no direct access to convergence in total variation, but we can still compare the fitnesses. We denote by $w^{(\delta_{\eta_0})}$ the limit of the non-increasing sequence $(w_n^{(\delta_{\eta_0})})_{n\geqslant 0}$.

Proposition 2. *Let* $p_0(dx)$ *be a probability measure and* η_0 *be the max of its support. The sequence* $(w_n)_{n\geqslant 0}$ *converges towards* $w^{(\delta_{\eta_0})}$.

In case $p_0 = \delta_{\eta_0}$, the proposition is proved in Lemma 3. If $p_0 \neq \delta_{\eta_0}$, the idea is to compare p_n with $p_n^{(\delta_{\eta_0})}$. Since the sequence $w_n^{(\delta_{\eta_0})}$ is non-increasing with limit $\geqslant (1-\beta)\eta_0$, by point 2 of Lemma 3 and Corollary 1, we get that $w_n^{(\delta_{\eta_0})} \geqslant (1-\beta)\eta_0$ holds for each n, hence the sequence $((1-\beta)\eta_0)^n/W_n^{(\delta_{\eta_0})}$ is non-increasing and has a limit; we shall then distinguish according to the following cases[1]:

$A-$ $\lim_n((1-\beta)\eta_0)^n/W_n^{(\delta_{\eta_0})} = 0$.

$B-$ $\lim_n((1-\beta)\eta_0)^n/W_n^{(\delta_{\eta_0})} > 0$ and $p_0(\{\eta_0\}) \neq 0$.

$C-$ $\lim_n((1-\beta)\eta_0)^n/W_n^{(\delta_{\eta_0})} > 0$ and $p_0(\{\eta_0\}) = 0$.

[1]for the existence of the limit :

2.3 Proof of Proposition 2

We introduce two partial orders between probability measures, the second one (introduced by Yuan in [6]) being taylor-made to fit Kingman's recursion.

Definition 1. *For two probability measures $p(dx)$ and $q(dx)$ on $[0,1]$, and $\eta \in (0,1]$, we write:*

- $p \preccurlyeq q$ *if* $p([0,x]) \geqslant q([0,x])$ *for any* $x \in [0,1]$ *(*standard stochastic order*)*
- $p \leqslant_{\eta-} q$ *if* $p(A) \leqslant q(A)$ *for any* $A \subset [0,\eta)$ *Borel.*

Let us collect in a Lemma a few facts :

Lemma 5. *(i)* $p \leqslant_{\eta-} q$ *iff for any non-negative* $h : [0,1] \to \mathbb{R}^+$ *such that* $h(x) = 0$ *for* $x \geqslant \eta$, $\int_0^1 h(x)p(dx) \leqslant \int_0^1 h(x)q(dx)$.

(ii) $p \preccurlyeq q$ *iff for any non-negative non-decreasing* $h : [0,1] \to \mathbb{R}^+$, $\int_0^1 h(x)p(dx) \leqslant \int_0^1 h(x)q(dx)$.

(iii) $p \leqslant_{1-} q$ *implies* $q \preccurlyeq p$.

(iv) $p \preccurlyeq q$ *implies* $\int_0^1 xp(dx) \leqslant \int_0^1 xq(dx)$,

(v) $p \leqslant_{\eta_q-} q$ *implies* $\int_0^1 xp(dx) \geqslant \int_0^1 xq(dx)$.

Notice the inequality is reversed in fact (v) with respect to fact (ii). Fact (v) perhaps requires a justification: applying (i) with $h(x) = (1 - \frac{x}{\eta_q})\mathbb{1}_{x \leqslant \eta_q}$, we find $\int_0^{\eta_q}(1 - \frac{x}{\eta_q})p(dx) \leqslant \int_0^{\eta_q}(1 - \frac{x}{\eta_q})q(dx)$ hence

$$\int_0^{\eta_q} xq(dx) \leqslant \int_0^{\eta_q} xp(dx) + \eta_q \cdot p((\eta_q, 1]) \leqslant \int_0^1 xp(dx),$$

using Markov inequality at the last inequality.

Lemma 6. *Let p_0 and p'_0 be two probability measures, and let η'_0 be the max of the support of p'_0; if $p_0 \leqslant_{\eta'_0-} p'_0$, then the probability measures p_1 and p'_1 obtained from p_0 and p'_0 after one step of Kingman recursion satisfy:*

$$p_1 \leqslant_{\eta'_0-} p'_1. \tag{10}$$

Proof of Lemma 6. We apply fact (ii) with $h(x) = x\mathbb{1}_A(x)$, and use the inequality on the mean, fact (v), to get the inequality:

$$p_1(A) - p'_1(A) = (1-\beta)\left(\frac{\int_A xp_0(dx)}{\int_0^\eta xp_0(dx)} - \frac{\int_A xp'_0(dx)}{\int_0^\eta xp'_0(dx)}\right) \leqslant 0.$$

□

Lemma 6 has two direct corollaries. We denote by $(p_n^{(\delta_{\eta_0})}(dx))_{n\geqslant 0}$ the fitness distribution sequence started at δ_{η_0}, and by $w_n^{(\delta_{\eta_0})} = \int xp_n^{(\delta_{\eta_0})}(dx)$ its fitness.

Corollary 2. *Let $p_0(dx)$ be a probability measure on $[0,1]$ and η_0 be the max of its support ($\eta_0 \geqslant \eta_q$). For each integer $n \geqslant 0$, it holds,*

$$p_n^{(\delta_{\eta_0})} \leqslant_{\eta_0 -} p_n, \text{ hence } w_n^{(\delta_{\eta_0})} \geqslant w_n.$$

Corollary 3. *Let $\eta_0 \in (0,1]$ be such that $\eta_0 \geqslant \eta_q$. For each integer $n \geqslant 0$, it holds*

$$p_n^{(\delta_{\eta_0})} \leqslant_{\eta_0 -} p_{n+1}^{(\delta_{\eta_0})}, \text{ hence } w_n^{(\delta_{\eta_0})} \geqslant w_{n+1}^{(\delta_{\eta_0})}.$$

The proof of both corollaries is by induction and follows from Lemma 6, noting for the initialisation that the assumption $p_0 \leqslant_{\eta_0 -} p_1$ is satisfied since $p_0(A) = \delta_{\eta_0}(A) = 0$ for any $A \subset [0, \eta_0)$, and noting for the induction that η_0 is the max of the support of the measures p_n and $p_{n+1}^{(\delta_{\eta_0})}$. The fact that $(w_n^{\delta_{\eta_0}})_n$ is non-increasing then follows from fact (v).

Proof of Lemma 3. For the bound in total variation, observe that for any $A \subset [0, \eta)$, $p_n^{(\delta_\eta)}(A) \leqslant p_{n+1}^{(\delta_\eta)}(A)$ hence $p_n^{(\delta_\eta)}(A) \leqslant p^{(\delta_\eta)}(A)$, and a moment of thought then gives :

$$\|p_n^{(\delta_\eta)} - \pi^{(\delta_\eta)}\|_{TV,[0,\eta]} = \sup_{A \subset [0,\eta]} \left(p_n^{(\delta_\eta)}(A) - \pi^{(\delta_\eta)}(A) \right) = p_n^{(\delta_\eta)}(\{\eta\}) - \pi^{(\delta_\eta)}(\{\eta\}).$$

□

Proof of Lemma 4. Let p_0 have max of its support η_0, we wish to compare p_n started from p_0 and $p_n^{(\delta_{\eta_0})}$ We decompose as before :

$$p_n(dx) = q_n(dx) + p_{0,n}(dx)$$
$$p_n^{(\delta_{\eta_0})}(dx) = q_n^{(\delta_{\eta_0})}(dx) + p_{0,n}^{(\delta_{\eta_0})}(dx)$$

Applying Corollary 2, we get $w_n \leqslant w_n^{(\delta_{\eta_0})}$ implies $q_n(A) \geqslant q_n^{(\delta_{\eta_0})}(A)$ for any $A \in [0, \eta_0]$ (we stress the interval $[0, \eta_0]$ is closed here), hence :

$$p_n(A) = q_n(A) + p_{0,n}(A) \geqslant q_n^{(\delta_{\eta_0})}(A) = p_n^{(\delta_{\eta_0})}(A) - \frac{((1-\beta)\eta_0)^n}{W_n^{(\delta_{\eta_0})}} \mathbb{1}_A(\eta_0)$$

which implies :

$$\left\| p_n - p_n^{(\delta_{\eta_0})} \right\|_{TV,[0,\eta_0]} = \sup_{A \subset [0,\eta_0]} \left(p_n(A) - p_n^{(\delta_{\eta_0})}(A) \right) \leqslant \frac{((1-\beta)\eta_0)^n}{W_n^{(\delta_{\eta_0})}} \to 0$$

Recalling from Lemma 3 that $\left\| p_n^{(\delta_{\eta_0})} - \pi^{(\delta_{\eta_0})} \right\|_{TV,[0,\eta_0]} \to 0$ we deduce that $(p_n(dx)_{n \geqslant 0}$ converges to $\pi^{(\delta_{\eta_0})}(dx)$ in total variation, which implies in particular, $w_n \to w^{(0)}$. □

Proof of Proposition 2, case A. This is a consequence of Lemma 4 since convergence in total variation of the sequence $(p_n(dx))_n$ towards $p^{(\delta_\eta)}(dx)$ implies convergence of the fitness sequence $(w_n)_n$ towards $w^{(\delta_\eta)}$. □

Proof of Proposition 2, case B. Observe that the atom of $p_n(dx)$ at η_0 has mass :

$$p_n(\{\eta_0\}) = \sum_{k=0}^{n-1}(1-\beta)^k \frac{W_{n-k}}{W_n}\eta_0^k \beta q(\{\eta_0\}) + \frac{(1-\beta)^n}{W_n}\eta_0^n p_0(\{\eta_0\}).$$

Let $\gamma > 0$ be arbitrary, and set $K_n := \text{Card}\left\{0 \leqslant k \leqslant n : w_k \leqslant (1-\gamma)w_k^{(\delta_{\eta_0})}\right\}$. Applying Corollary 2, we have $w_k^{(\delta_{\eta_0})} \geqslant w_k$ for $k = 0, \dots, n-1$, which implies $W_n^{(\delta_{\eta_0})} \geqslant W_n$. Combining these two elements, we get the lower bound :

$$1 \geqslant p_n(\{\eta_0\}) \geqslant \frac{((1-\beta)\eta_0)^n}{W_n} p_0(\{\eta_0\}) \geqslant \frac{1}{(1-\gamma)^{K_n}} \frac{((1-\beta)\eta_0)^n}{W_n^{(\delta_{\eta_0})}} p_0(\{\eta_0\}),$$

hence

$$\liminf (1-\gamma)^{K_n} \geqslant p_0(\{\eta_0\}) \lim_n \frac{(1-\beta)\eta_0)^n}{W_n^{(\delta_{\eta_0})}} > 0$$

which ensures that the sequence $(K_n)_n = (K_n(\gamma))_n$ remains bounded. This is valid for every $\gamma > 0$, therefore the limit $w^{(\delta_{\eta_0})}$ of $(w_n^{(\delta_{\eta_0})})_n$ satisfies $w^{(\delta_{\eta_0})} \leqslant \liminf w_n$. Since we already knew $\limsup w_n \leqslant \limsup w_n^{(\delta_{\eta_0})} = w^{(\delta_{\eta_0})}$, we deduce that $\lim w_n$ exists and is equal to $w^{(\delta_{\eta_0})}$.

□

Definition 2. *Let $p(dx)$ be a probability measure on $[0,1]$ and $\rho \in [0,1]$, let us call "rabot" at ρ of $p(dx)$ the probability measure $R_\rho(p)$ given by :*

$$R_\rho(p)(dx) = p(dx)\mathbb{1}_{x<\rho} + p([\rho,1])\delta_\rho.$$

Let us now single out another corollary of Lemma 6. Choose $\eta_0' < \eta_0$ and define a sequence $p_n'(dx)$ by using for initial distribution the rabot at ρ of $p_0(dx)$, and for mutation measure the rabot at ρ of $q(dx)$, precisely:

$$p_0'(dx) = R_{\eta_0'}(p_0)(dx)$$
$$p_{n+1}'(dx) = \beta R_{\eta_0'}(q)(dx) + (1-\beta)\frac{xp_n'(dx)}{w_n'}$$

Corollary 4. *It holds that*

$$p_n \leqslant_{\eta_0'-} p_n', \ \textit{for } n \geqslant 0.$$

Proof. We prove the lemma by induction. It holds for $n = 0$, since $p_0 \leqslant_{\eta_0'-} R_{\eta_0'}(p_0)$ (indeed, both measures coincide on $[0, \eta_0')$). Then assume that $p_n \leqslant_{\eta_0'-} p_n'$, observe that p_n' is supported on $[0,\eta_0']$, apply Lemma 6 to get $p_{n+1} \leqslant_{\eta_0'-} \theta_1(p_n')$, where $\theta_1(p_n')$ is the probability measure obtained by one step of Kingman's recursion using p_n' as starting measure. But $\theta_1(p_n')$ coincides with p_{n+1}' on $[0,\eta_0')$ (in fact, we have $R_{\eta_0'}(\theta_1(p_n')) = p_{n+1}'$), hence $p_{n+1} \leqslant_{\eta_0'-} p_{n+1}'$. □

Proof of Proposition 2, case C. The assumption $\lim_n((1-\beta)\eta_0)^n/W_n^{(\delta_{\eta_0})} > 0$ implies $w^{(\delta_{\eta_0})} = (1-\beta)\eta_0$. Using Corollaries 2 and 4 now, we have:

$$w_n' \leqslant w_n \leqslant w_n^{(\delta_{\eta_0})}.$$

Since $R_{\eta_0'}(p_0)$ has an atom at η_0', the sequence (p_n') now falls in one of the two cases A and B of Proposition 2 already discussed, hence the sequence $(w_n')_n$ converges. Furthermore, its limit $w' = w'(\eta_0')$ satisfies $(1-\beta)\eta_0' \leqslant w'$ by Lemma 1. Therefore:

$$(1-\beta)\eta_0' \leqslant \liminf w_n \leqslant \limsup w_n \leqslant (1-\beta)\eta_0,$$

and η_0' being arbitrary we conclude that $\lim w_n = (1-\beta)\eta_0 = w^{(\delta_{\eta_0})}$, as expected. □

2.4 Proof of Proposition 1

Proof of Proposition 1, first item, local convergence in total variation on $[0,\eta_0)$*, and* $w \geqslant (1-\beta)\eta_0$*.* Let $\xi < \eta_0$. We want to compare:

$$\begin{aligned} p_n(dx) &= \sum_{k=0}^{n-1} \frac{(1-\beta)^k}{w_{n-1}\dots w_{n-k}} x^k \beta q(dx) + \frac{(1-\beta)^n}{w_{n-1}\dots w_0} x^n p_0(dx) &&=: \sum_{k=0}^{n-1} q_{n,k}(dx) + p_{0,n}(dx) \\ \pi(dx) &:= \sum_{k\geqslant 0} \left(\frac{1-\beta}{w}\right)^k x^k \beta q(dx) &&=: \sum_{k\geqslant 0} q_{\infty,k}(dx) \end{aligned}$$

in the sense we wish to establish the convergence in total variation of $p_n(dx)$ towards $\mu(dx)$ on $[0,\xi]$. We point out that $\pi(dx)$ is a sub-probability measure by Fatou lemma. First,

$$\|q_{n,k}(dx) - q_{\infty,k}(dx)\|_{TV,[0,\xi]} = \left|\frac{(1-\beta)^k}{w_{n-1}\dots w_{n-k}} - \left(\frac{1-\beta}{w}\right)^k\right| \int_0^\xi x^k \beta q(dx) = \left|\frac{w^k}{w_{n-1}\dots w_{n-k}} - 1\right| q_{\infty,k}([0,\xi]),$$

hence, for any k_0:

$$\begin{aligned} \sum_{k=0}^{k_0-1} \|q_{n,k}(dx) - q_{\infty,k}(dx)\|_{TV,[0,\xi]} &\leqslant \sum_{k=0}^{k_0-1} q_{\infty,k}([0,\xi]) \left|\frac{w^k}{w_{n-1}\dots w_{n-k}} - 1\right| \\ &\leqslant \max_{0\leqslant k<k_0} \left|\frac{w^k}{w_{n-1}\dots w_{n-k}} - 1\right| \end{aligned} \tag{11}$$

since $\sum_{k=0}^{\infty} q_{\infty,k}([0,\xi]) \leqslant 1$, and k_0 being fixed, the last term goes to 0 as $n \to \infty$, using our assumption that $w_n \to w$.

Second,

$$\begin{aligned} \sum_{k\geqslant k_0} q_{\infty,k}([0,\xi]) &= \int_0^\xi \sum_{k\geqslant k_0} \left(\frac{1-\beta}{w}\right)^k x^k \beta q(dx) \\ &\leqslant \left(\frac{(1-\beta)\xi}{w}\right)^{k_0} \sum_{k\geqslant 0} q_{\infty,k}([0,\xi]) \\ &\leqslant \left(\frac{(1-\beta)\xi}{w}\right)^{k_0} \end{aligned} \tag{12}$$

using that $\pi(dx)$ is a subprobability measure at the last line; also we point out that $\frac{(1-\beta)\xi}{w}$ is such that $\frac{(1-\beta)\xi}{w} < \frac{(1-\beta)\eta_0}{w} \leqslant 1$ by assumption.

Third, choosing $\delta > 0$ small enough such that $\frac{(1-\beta)\xi}{(1-\delta)w} < 1$, and then n_0 large enough such that $w_n \geqslant (1-\delta)w$ for $n \geqslant n_0$, we find, for n such that $n \geqslant k_0 + n_0$:

$$\begin{aligned}\sum_{k=k_0}^{n-1} q_{n,k}([0,\xi]) &= \int_0^{\xi} \sum_{k=k_0}^{n-1} \frac{(1-\beta)^k}{w_{n-1}\dots w_{n-k}} x^k \beta q(dx) \\ &\leqslant \left(\frac{(1-\beta)\xi}{(1-\delta)w}\right)^{k_0} \sum_{k=0}^{n-k_0-1} q_{n-k_0,k}([0,\xi]) \\ &\leqslant \left(\frac{(1-\beta)\xi}{(1-\delta)w}\right)^{k_0} \end{aligned} \tag{13}$$

Fourth, the term implying p_0 is dealt with similarly, and we find, for $n \geqslant k_0 + n_0$:

$$\begin{aligned} p_{0,n}([0,\xi]) &= \int_0^{\xi} \frac{(1-\beta)^n}{w_{n-1}\dots w_0} x^n p_0(dx) \\ &\leqslant \left(\frac{(1-\beta)\xi}{(1-\delta)w}\right)^{k_0} \int_0^{\xi} \frac{(1-\beta)^{n-k_0}}{w_{n-k_0-1}\dots w_0} x^{n-k_0} p_0(dx) \\ &= \left(\frac{(1-\beta)\xi}{(1-\delta)w}\right)^{k_0} p_{0,n-k_0}([0,\xi]) \\ &\leqslant \left(\frac{(1-\beta)\xi}{(1-\delta)w}\right)^{k_0} \end{aligned} \tag{14}$$

The sum of the four terms in (11),(12), (13) and (14) now give an upper bound for $\|p_n - \pi\|_{TV,[0,\xi]}$. We choose the parameters in this order : we choose k_0 large enough such that the expression in (13),(14) (hence (12)) is small and then n large enough, such that $n \geqslant n_0 + k_0$ and (11) is small.

□

Proof of Proposition 1, first item, weak convergence on $[0,\eta_0]$*, case* $w \geqslant (1-\beta)\eta_0$. This is abstract non-sense : the set of probability measures on $[0,\eta_0]$ is compact for the topology of weak convergence; now take any subsequence of $(p_n(dx))_{n\geqslant 0}$; it admits a converging subsubsequence, which should agree with the limit in total variation on $[0,\eta_0)$, that is with $\frac{\beta q(dx)}{1-\frac{x}{y_0}}$, and be supported on $[0,\eta_0]$: the only possibility left is then given by (8) on $[0,\eta_0]$. The set of accumulation points of $(p_n(dx))_{n\geqslant 0}$ therefore consists in the singleton given by (8): in other words, the sequence $(p_n)_{n\geqslant 0}$ has weak limit given by (8) on $[0,\eta_0]$. □

Proof of Proposition 1, second item, convergence in total variation on $[0,\eta_0]$*, and* $w > (1-\beta)\eta_0$*.* The proof is similar, even easier, and consists in replacing ξ by the larger quantity η_0 in those bounds involving ξ; precisely, (11) is unaffected, then choosing δ such that $\frac{(1-\beta)\eta_0}{(1-\delta)w} < 1$ (and n_0

as before), we get the following substitutes for (12) (13) (14):

$$\left\| \sum_{k \geqslant k_0} q_{\infty,k}(dx) \right\|_{TV,[0,\eta_0]} \leqslant \left(\frac{(1-\beta)\eta_0}{w} \right)^{k_0}$$

$$\left\| \sum_{k=k_0}^{n-1} q_{n,k}(dx) \right\|_{TV,[0,\eta_0]} \leqslant \left(\frac{(1-\beta)\eta_0}{(1-\delta)w} \right)^{k_0}$$

$$\left\| p_{0,n}(dx) \right\|_{TV,[0,\eta_0]} \leqslant \left(\frac{(1-\beta)\eta_0}{(1-\delta)w} \right)^{k_0},$$

□

3 Miscellaneous

We first look at a monotonicity property satisfied by $(p_n^{(\delta_0)}(dx))_{n\geqslant 0}$; if the size-bias of $\delta_0(dx)$ is not formally defined, it is reasonable to define it to be $\delta_0(dx)$ which leads to define $p_1^{(\delta_0)} = \beta q + (1-\beta)p_0$, while the rest of the terms of the sequence $(p_n^{(\delta_0)}(dx))_{n\geqslant 0}$ are defined unambiguously. With this definition, one can then state the:

Proposition 3. *The sequence $(p_n^{(\delta_0)}(dx))_{n\geqslant 0}$ is non-decreasing for the stochastic order, in particular the sequence $(w_n^{(\delta_0)})$ is non-decreasing.*

This contrasts with the facts that $(p_n^{(\delta_1)}(dx))_{n\geqslant 0}$ is non-increasing for the stochastic order, and the sequence $(w_n^{(\delta_1)})$ is non-increasing. A key tool will be the following (standard) lemma on preservation of stochastic order.

Lemma 7. *Let $r_1 \preccurlyeq r_2 \preccurlyeq \ldots \preccurlyeq r_k$ be an ordered sequence of probability measures on the set of real numbers (for the stochastic order $\preccurlyeq$), and $\alpha_1, \ldots, \alpha_k$ and $\alpha'_1, \ldots, \alpha'_k$ be two finite sequences of non-negative real numbers summing to 1, such that*

$$\sum_{i=1}^{j} \alpha_i \leqslant \sum_{i=1}^{j} \alpha'_i, \text{ for each } 0 \leqslant j \leqslant k-1. \tag{15}$$

Then $\sum_{i=1}^k \alpha_i r_i$ and $\sum_{i=1}^k \alpha'_i r_i$ are two probability measures satisfying: $\sum_{i=1}^k \alpha_i r_i \succcurlyeq \sum_{i=1}^k \alpha'_i r_i$.

Intuitively, the first ponderation by the sequence $(\alpha_i)_i$ gives more weight to the largest elements among $(r_i)_i$, hence results in a stochastically larger probability measure.

Proof of Lemma 7. A moment of thought reveals that the result follows from the following claim:

"Let $\alpha_1, \ldots, \alpha_k$ and $\alpha'_1, \ldots, \alpha'_k$ be two finite sequences of non-negative real numbers such that $\sum_{i=1}^j \alpha_i \leqslant \sum_{i=1}^j \alpha'_i$ for each $j \in \{1, \ldots, k\}$, and let $\beta_1 \geqslant \ldots \geqslant \beta_k \geqslant 0$ be a finite sequence of of non-negative real numbers. It then holds:

$$\sum_{i=1}^{k} \alpha_i \beta_i \leqslant \sum_{i=1}^{k} \alpha'_i \beta_i."$$

The claim is proven by induction on k. Indeed to prove $\sum_{i=1}^{k}\alpha_i\beta_i \leqslant \sum_{i=1}^{k}\alpha'_i\beta_i$, it is enough to have $\sum_{i=1}^{k-1}\alpha_i(\beta_i-\beta_k) \leqslant \sum_{i=1}^{k-1}\alpha'_i(\beta_i-\beta_k)$, but this is a consequence of the claim at "step" $k-1$.

□

Proof of Proposition 3. We prove that $(w_n)_n^{(\delta_0)}$ is non-decreasing. First we observe that $w_0 = 0 \leqslant \beta\int q(dx) = w_1$, which initialises the induction. Next we assume that $w_0 \leqslant w_1 \leqslant \ldots \leqslant w_n$ and prove that this implies $w_0 \leqslant w_1 \leqslant \ldots \leqslant w_{n+1}$ (strong induction). For this we use the expanded form (7) of p_n (the last term of which vanishes because $p_0 = \delta_0$) to define a set of non-negative real numbers $(\alpha_k^{(n)})_{0\leqslant k\leqslant n-1}$ as follows :

$$p_n(dx) = \sum_{k=0}^{n-1}\frac{(1-\beta)^k}{w_{n-1}\ldots w_{n-k}}x^k\beta q(dx) =: \sum_{k=0}^{n-1}\alpha_k^{(n)}\frac{x^k q(dx)}{\int x^k q(dx)}$$

On the one hand, $w_0 \leqslant w_1 \leqslant \ldots \leqslant w_n$ now implies: $\alpha_k^{(n+1)} \leqslant \alpha_k^{(n)}$, $k = 0,\ldots,n-1$, which in particular gives $\sum_{k=0}^{j}\alpha_k^{(n+1)} \leqslant \sum_{k=0}^{j}\alpha_k^{(n)}$, $j = 0,\ldots,n-1$. On the other hand, we claim that, for each $k \geqslant 0$,

$$\frac{x^k q(dx)}{\int x^k q(dx)} \preccurlyeq \frac{x^{k+1}q(dx)}{\int x^{k+1}q(dx)} \tag{16}$$

Indeed for any pair of functions f, g monotone, integrable, whose product fg is again integrable, with respect to a probability measure μ on the Borel sets of $\mathbb{R}$, we have the following standard correlation inequality : $\int fgd\mu \geqslant \int fd\mu\int gd\mu$ [2], apply that formula with $\mu(dx) = x^k q(dx)/\int x^k q(dx)$ and $f(x) = \mathbb{1}_{x\geqslant y}$ and $g(x) = x$ to get

$$\int_y^1 x^{k+1}q(dx)\int_0^1 x^k q(dx) \geqslant \int_y^1 x^k q(dx)\int_0^1 x^{k+1}q(dx),$$

which gives (16).

We are now in position to apply Lemma 7 to get $p_n \preccurlyeq p_{n+1}$ which in turn implies the desired inequality $w_n \leqslant w_{n+1}$. □

Let us point out the following monotonicity property in the decomposition of $p_n^{(\delta_\eta)}(\{\eta\})$ starting from δ_η:

$$p_n^{(\delta_\eta)}(\{\eta\}) = \left(\sum_{k=0}^{n-1}\frac{(1-\beta)^k}{w_{n-1}^{(\delta_\eta)}\ldots w_{n-k}^{(\delta_\eta)}}\eta^k\right)\beta q(\{\eta\}) + \frac{(1-\beta)^n}{w_{n-1}^{(\delta_\eta)}\ldots w_0^{(\delta_\eta)}}\eta^n,$$

the LHS is non-increasing in n, while on the RHS, the first term is non-decreasing in n and the second term is non-increasing in n : this is a consequence of the fact that $(w_n^{(\delta_\eta)})_n$ is non-increasing with limit $w^{(\delta_\eta)} \geqslant (1-\beta)\eta$. One may then distinguish the two cases :

- $q(\{\eta\}) = 0$ in which case $p_n^{(\delta_\eta)}(\{\eta\}) = \frac{(1-\beta)^n}{w_{n-1}^{(\delta_\eta)}\ldots w_0^{(\delta_\eta)}}\eta^n$ is non-increasing in n (the limit may be null or not).

[2] this is an application of Fubini considering $\int_{\mathbb{R}^2}(f(x)-f(y))(g(x)-g(y))\mu(dx)\mu(dy) \geqslant 0$

- $q(\{\eta\}) > 0$, in which case $(w_n^{(\delta_\eta)})_n$ converges towards $w^{(\delta_\eta)} > (1-\beta)\eta$, hence the second term on the RHS vanishes, and $p_n^{(\delta_\eta)}(\{\eta\}) \to \frac{\beta q(\{\eta\})}{1-(1-\beta)\eta/w^{(\delta_\eta)}} > 0$.

Let us recall the definition of π_0, from Theorem 2, $\pi_0 = 1 - \int_0^{\eta_0} \frac{\beta q(dx)}{1-\frac{x}{y_0}}$, and also recall that

$$\pi_0 > 0 \text{ iff } \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} < 1.$$

We have seen that $(p_n(dx))_{n\geqslant 0}$ converges in total variation on $[0,\eta_0]$ if π_0 is null, we now give the converse statement :

Proposition 4. *The sequence $(p_n(dx))_{n\geqslant 0}$ converges in total variation on $[0,\eta_0]$ towards the probability measure $\pi(dx)$ given in Theorem 2 iff*

$$\pi_0 = 0 \text{ or } p_0(\{\eta_0\}) \neq 0$$

The main ingredient of the proof will be the following result focusing on initial distributions $p_0(dx)$ such that $p_0(\{\eta_0\}) \neq 0$.

Proposition 5. *Let $p_0(dx)$ be such that $p_0(\{\eta_0\}) \neq 0$ and $\pi_0 \neq 0$.*

- *It holds*
$$\frac{((1-\beta)\eta_0)^n}{W_n} p_0(\{\eta_0\}) \to \pi_0 := 1 - \int_0^{\eta_0} \frac{\beta q(dx)}{1-\frac{x}{y_0}},$$
for y_0 as defined in Theorem 2.

- *There is convergence of $(p_n(dx))_{n\geqslant 0}$ towards $p(dx)$ in total variation on $[0,\eta_0]$.*

Proof of Proposition 5, first item. We start again from the decomposition:

$$p_n(dx) = \left(\sum_{k=0}^{n-1} \frac{(1-\beta)^k}{w_{n-1}^{(\delta_\eta)}\dots w_{n-k}^{(\delta_\eta)}} x^k\right) \beta q(dx) + \frac{(1-\beta)^n}{w_{n-1}^{(\delta_\eta)}\dots w_0^{(\delta_\eta)}} x^n p_0(dx)$$

Using the characterization of weak convergence (ensured by Theorem 2) given in Portmanteau theorem, we get for the weak limit $p(dx)$ of $p_n(dx)$ the following lower bound at η_0:

$$\limsup_n \frac{((1-\beta)\eta_0)^n}{W_n} p_0(\{\eta_0\}) \leqslant \limsup_n p_n(\{\eta_0\}) \leqslant \pi(\{\eta_0\})$$

and the following upper bound:

$$\pi(\{\eta_0\}) \leqslant \pi((\eta_0-\varepsilon,\eta_0]) \leqslant \liminf p_n((\eta_0-\varepsilon,\eta_0])$$

Now we claim there is a finite constant C independent of ε and n, to be defined later, such that:

$$\liminf p_n((\eta_0-\varepsilon,\eta_0]) \leqslant C \int_{\eta_0-\varepsilon}^{\eta_0} \frac{\beta q(dx)}{1-\frac{x}{\eta_0}} + \liminf \frac{((1-\beta)\eta_0)^n}{W_n} p_0(\{\eta_0\})$$

If this holds true, since the quantity $\int_{\eta_0-\varepsilon}^{\eta_0} \frac{1}{1-\frac{x}{y_0}} \beta q(dx)$ may be rendered as small as we wish, choosing ε small enough, we deduce that the limit of the sequence $(\frac{((1-\beta)\eta_0)^n}{W_n} p_0(\{\eta_0\})_{n\geqslant 0}$ exists and equals $\pi(\{\eta_0\}) = \pi_0 = 1 - \int_0^{\eta_0} \frac{\beta}{1-\frac{x}{y_0}} q(dx)$, which finishes the proof.

We now check our claim. For short, let us set $v_n = \frac{((1-\beta)\eta_0)^n}{W_n}$, and similarly for $v_n^{(\delta_{\eta_0})}$. We start from the following bound on v_n, deduced from Corollary 2 :

$$v_n^{(\delta_{\eta_0})} \leqslant v_n \leqslant \frac{1}{p_0(\{\eta_0\})} v_n^{(\delta_{\eta_0})} \tag{17}$$

then we rewrite the canonical decomposition (7) in term of the sequence (v_n):

$$p_n(dx) = \left(\sum_{k=0}^{n-1} \frac{v_{n-k}}{v_n} \left(\frac{x}{\eta_0} \right)^k \right) \beta q(dx) + v_n \left(\frac{x}{\eta_0} \right)^n p_0(dx)$$

then bound $\frac{v_{n-k}}{v_n}$ by C in virtue of (17) to deduce that

$$\liminf p_n((\eta_0 - \varepsilon, \eta_0]) \leqslant C \int_{[\eta_0-\varepsilon,\eta_0]} \frac{\beta q(dx)}{1 - \frac{x}{\eta_0}} + \liminf \left(v_n \int_{[\eta_0-\varepsilon,\eta_0]} \left(\frac{x}{\eta_0} \right)^n p_0(dx) \right).$$

Now the sequence of functions $(\frac{x}{\eta_0})^n$ is dominated by 1, and converges pointwise to $\mathbb{1}_{\eta_0}$ hence Lebesgue dominated convergence theorem applies to give that the limit of $\int_{[\eta_0-\varepsilon,\eta_0]} (\frac{x}{\eta_0})^n p_0(dx)$ exists and is $p_0(\{\eta_0\})$, hence the last term on the RHS is the one given in our claim. □

We now state and prove Lemma 8, that is the last building block necessary to conclude our proofs.

Lemma 8. *Let $(\mu_n)_{n\geqslant 0}, \mu$ be probability measures on $[0, \eta]$, such that :*

- *for each $\xi < \eta$, (μ_n) converges in total variation on $[0, \xi]$ towards μ .*
- *$\mu_n(\{\eta\}) \to \mu(\{\eta\})$.*

Then $(\mu_n)_{n\geqslant 0}$ converges in total variation on $[0, \eta]$ towards μ.

Proof of Lemma 8. Let $\varepsilon > 0$. Since μ is a probability measure, there exists $\xi = \xi_\varepsilon < \eta$ such that: $\mu([0, \xi) \cup \{\eta\}) > 1 - \varepsilon$. Also, we deduce from the assumptions that there exists n_0 such that for $n \geqslant n_0$, $\mu_n([0, \xi) \cup \{\eta\}) > 1 - 2\varepsilon$. Then,

$$\sup_{A\subset[0,\eta]} |\mu_n(A) - \mu(A)| \leqslant \sup_{A\subset[0,\eta]} |\mu_n(A\cap[0, \xi]) - \mu(A\cap[0, \xi])| + |\mu_n(\{\eta\}) - \mu(\{\eta\})| + \mu_n((\xi, \eta)) + \mu((\xi, \eta))$$

By the first assumption, there exists n_1 such that for $n \geqslant n_1$, the first term on the RHS is $\leqslant \varepsilon$, and n_2 such that for $n \geqslant n_2$, the second term on the RHS is $\leqslant \varepsilon$ Taking $n \geqslant n_0 \vee n_1 \vee n_2$, the RHS is $\leqslant 4\varepsilon$, which gives our claim. □

Proof of Proposition 5, second item. Combine the first item of Proposition 5, Theorem 2 and Lemma 8. □

Proof of Proposition 4. The convergence under the assumption $\pi_0 = 0$ is Theorem 2, and under the assumption $p_0(\{\eta_0\}) \neq 0$ is Proposition 5. To get the converse, let us consider cases where $\pi_0 \neq 0$ and $p_0(\{\eta_0\}) = 0$. Then we claim that $q(\{\eta_0\}) = 0$. Otherwise indeed, we would need $\eta_q = \eta_0$, but $q(\{\eta_q\}) \neq 0$ implies $\int_0^{\eta_q} \beta q(dx)/(1 - x/\eta_q) = \infty$, hence $y_0 > \eta_0$, which entails $\pi_0 = 0$, a contradiction. Now, from $p_0(\{\eta_0\}) = q(\{\eta_0\}) = 0$ and (7), we have that $p_n(\{\eta_0\}) = 0$, whereas $\pi(\{\eta_0\}) = \pi_0 \neq 0$: hence there is no convergence in total variation on $[0, \eta_0]$. □

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