Asymptotic gaps in the Gaussian fractions

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Abstract

We are interested in the asymptotic *distance to nearest neighbour* (or *gap*) statistic of Gaussian fractions $\frac{p}{q}$, with $p, q \in \mathbb{Z}[i]$ and $0 < |q| \leqslant N$, as $N \to \infty$. We use the homogeneous dynamical approach of J. Marklof [Mar13] in order to derive the existence of a probability measure describing this asymptotic gap statistic.

Introduction

The *Gaussian fractions with height at most* $N > 0$ are defined as the points of the square torus in

$$\mathcal{G}_N = \left\{ \frac{p}{q} \mod \mathbb{Z}[i] \;:\; p, q \in \mathbb{Z}[i],\ 0 < |q| \leqslant N \right\} \subset \mathbb{C}/\mathbb{Z}[i].$$

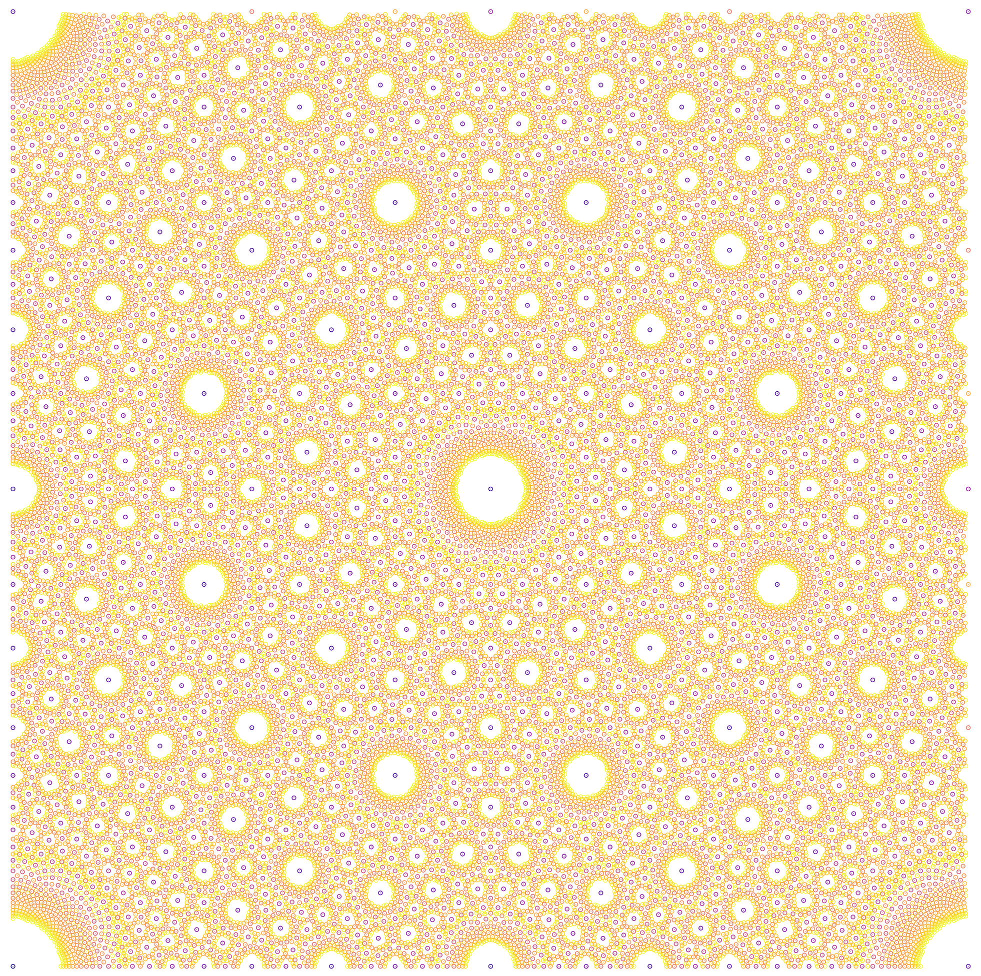


Figure 1: The set $\mathcal{G}_{20}$ with colours of points depending on their height.

The cardinality of $\mathcal{G}_N$ can be estimated by Mertens' formula, as $N \to \infty$,

$$\operatorname{card} \mathcal{G}_N \sim c\, N^4 \quad \text{where} \quad c = \frac{\pi}{2\,\zeta_{\mathbb{Q}(i)}(2)}.$$

Let dx be the Haar probability measure on the square torus $\mathbb{C}/\mathbb{Z}[i]$ and let Δ_r denote the Dirac mass at any point r. Then the Gaussian fractions equidistribute.

> **Theorem** (S. Cosentino, 1999). *We have the vague convergence, as* $N \to \infty$,
>
> $$\frac{1}{\operatorname{card} \mathcal{G}_N} \sum_{r \in \mathcal{G}_N} \Delta_r \overset{*}{\rightharpoonup} dx.$$

This means that, as $N \to \infty$, the set $\mathcal{G}_N$ becomes denser and denser in a uniform way in $\mathbb{C}/\mathbb{Z}[i]$.

Nevertheless, the Gaussian fractions do not look as uniformly distributed in the square torus as the evenly spaced Riemann fractions $\mathcal{R}_N = \{\frac{p}{N} \mod \mathbb{Z}[i] \,:\, p \in \mathbb{Z}[i]\}$. An explanation lays in the asymptotic study of gaps in the sequence $(\mathcal{G}_N)_{N \in \mathbb{N}^*}$.

Let us define the probability measures

$$\mu_N = \frac{1}{\operatorname{card} \mathcal{G}_N} \sum_{r \in \mathcal{G}_N} \Delta_{N^2 d(r, \mathcal{G}_N \smallsetminus \{r\})}.$$

That is, μ_N is the uniform probability measure on the multiset of scaled gaps

$$\{\{\, N^2 d(r, \mathcal{G}_N \smallsetminus \{r\}) \;:\; r \in \mathcal{G}_N \,\}\}.$$

The scaling factor N^2 here has been chosen with comparison to the average area a_N of any partition of $\mathcal{G}_N$ (e.g. its Voronoï cells) since $\sqrt{a_N} = \frac{1}{\sqrt{\operatorname{card} \mathcal{G}_N}} \sim \frac{1}{\sqrt{c}\, N^2}$.

> **Theorem** (R. Sayous, 2024). *There exists a probability measure* μ *on* $\mathbb{R}$ *such that, as* $N \to \infty$,
>
> $$\mu_N \overset{*}{\rightharpoonup} \mu.$$
>
> *Moreover,* μ *has support* $[1, +\infty[$ *and has a density with respect to the Lebesgue measure.*

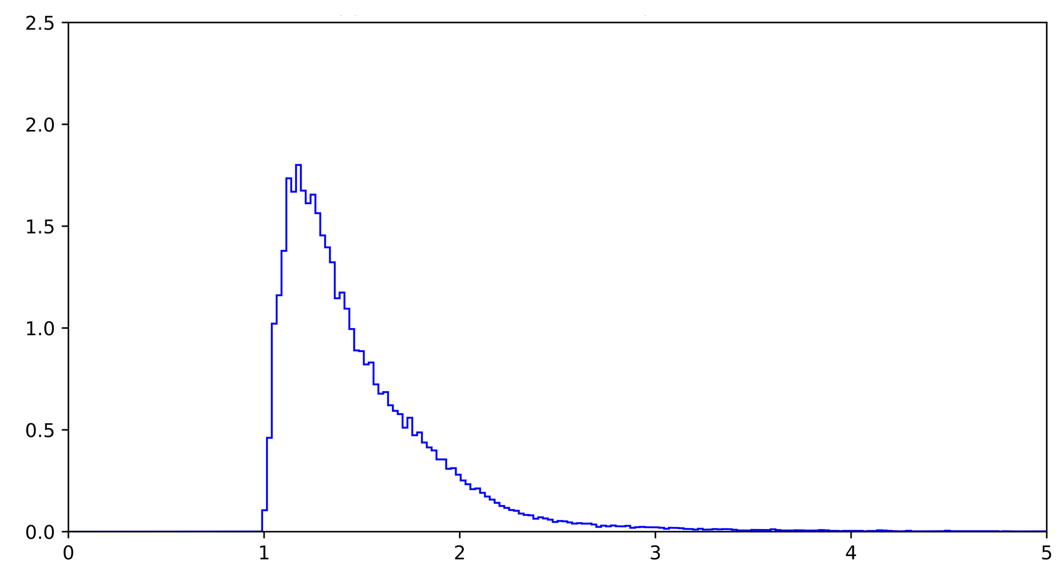


Figure 2: A numerical approximation of the density of μ, using points of $\mathcal{G}_{50}$.

In comparison, all scaled gaps in the Riemann fractions $\mathcal{R}_N$ are equal to $N\, d(\frac{p}{N}, \mathcal{R}_N \smallsetminus \{\frac{p}{N}\}) = 1$.

An integral formula and a tail estimate

For $z \in \mathbb{C}$ and $r, R \geqslant 0$, let $A(z_0, r, R) = \{z \in \mathbb{C} \,:\, r \leqslant |z - z_0| \leqslant R\}$ denotes the closed annulus in $\mathbb{C}$ centred at z with radii r and R. We have the following integral formula for the cumulative distribution function of μ.

> **Theorem** (R. Sayous, 2024). *For every* $\delta > 0$, *we have*
>
> $$\mu([0, \delta]) = 2 \int_{s=0}^{+\infty} dx \Big(\bigcup_{\substack{p, q \in \mathbb{Z}[i] \\ p \neq 0}} A\Big(\frac{q}{p}, \frac{e^s}{\delta}, \frac{e^{\frac{s}{2}}}{|p|}\Big) \mod \mathbb{Z}[i] \Big)\, e^{-2s}\, ds.$$

From this integral formula and a series of geometric arguments, we can derive the tail estimate, as $\delta \to +\infty$,

> $$\mu(\,]\delta, +\infty[\,) = \frac{1}{\delta^4} + \mathrm{O}\Big(\frac{1}{\delta^5}\Big).$$

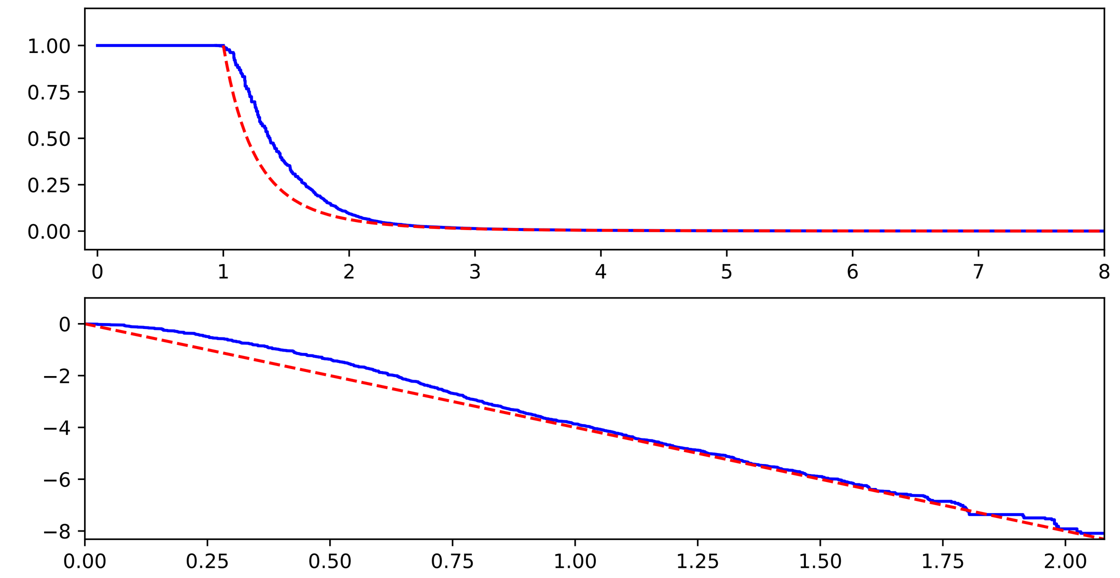


Figure 3: At the top, the empirical tail distribution $\delta \to \mu_N(\,]\delta, +\infty[\,)$ (in blue) with $N = 30$ and the graph of $\delta \mapsto \frac{1}{\delta^4}$ (in red). At the bottom, a logarithmic version of the top graph: we illustrate the tail estimate of μ by comparing the functions $\ell \mapsto \ln(\mu_N(\,]e^\ell, +\infty[\,))$ and $\ell \mapsto -4\ell$.

Idea of the proof: a link with dynamic on the hyperbolic space $\mathbb{H}^3$

Let us use the upper half-space model of the 3-dimensional (real) hyperbolic space $\mathbb{H}^3$. The positive isometries of $\mathbb{H}^3$ are described by the action of the group $G = \mathrm{PSL}_2(\mathbb{C})$ by homographies on the boundary at infinity $\partial_\infty \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. In particular, (a reparametrisation of) the geodesic flow from ∞ to 0 is given by the Cartan subgroup of G

$$A = \{a(t) \,:\, t > 0\} \text{ where } a(t) = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & t \end{bmatrix}.$$

Translations on each horizontal plane (i.e. on each horosphere centred at ∞) are given by the abelian group

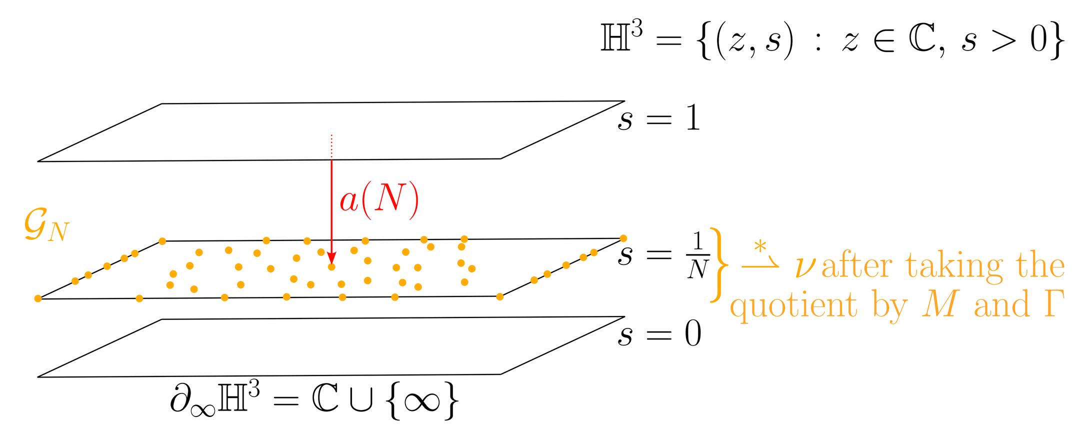
$$H = \{h(z) \,:\, z \in \mathbb{C}\} \text{ where } h(z) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}.$$

And rotations with axis $]0, \infty[$ give the compact group

$$M = \{m(\theta) \,:\, \theta \in \mathbb{R}\} \text{ where } m(\theta) = \begin{bmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{bmatrix}.$$

Let also $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$ denote the Bianchi group of $\mathbb{Q}(i)$. The following theorem is a result of equidistribution of Gaussian fractions set on an horosphere following the geodesic flow towards the boundary at infinity in $\Gamma \backslash \mathbb{H}^3$.

> **Theorem** (J. Parkkonen, F. Paulin, 2024). *There exists a probability measure* ν *on* $M \backslash G / \Gamma$ *(explicitly computed in [PP24, Cor. 4.2]) such that we have the vague convergence, as* $N \to \infty$,
>
> $$\nu_N = \frac{1}{\operatorname{card} \mathcal{G}_N} \sum_{r \in \mathcal{G}_N} \Delta_{M a(\frac{1}{N}) h(-r) \Gamma} \overset{*}{\rightharpoonup} \nu.$$



The points $M a(\frac{1}{N}) h(-r) \Gamma$ with $r \in \mathcal{G}_N$ may give us information about the gaps in $\mathcal{G}_N$. In order to see that, first notice that, modulo the four invertible elements $\mathbb{Z}[i]^\times = \{\pm 1, \pm i\}$, the data of irreducible fractions $\frac{p}{q}$ in $\mathcal{G}_N$ is equivalent to the data of vectors (p, q) in $\mathbb{Z}[i]^2$ with coprime coordinate p and q. Then, we compute

$$m(-\theta)\, a\Big(\frac{1}{N}\Big)\, h(-z) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} (\frac{p}{q} - z)\, q\, N\, e^{\frac{i\theta}{2}} \\ \frac{q}{N}\, e^{-\frac{i\theta}{2}} \end{pmatrix}.$$

By taking the modulus of each coordinate, this formula allows to code in a subset $\mathcal{C}(\delta)$ of $M \backslash G / \Gamma$ the two conditions that the denominator satisfies $|q| \leqslant N$ and that the scaled gap around z is greater than some parameter $\delta > 0$. This yields the formula, for every $\delta > 0$,

$$\mu_N(\,]\delta, +\infty[\,) = \nu_N(\mathcal{C}(\delta)).$$

Then it remains:

- to prove that the convergence $\nu_N(\mathcal{C}(\delta)) \to \nu(\mathcal{C}(\delta))$, by showing that $\nu(\partial\, \mathcal{C}(\delta)) = 0$ and that the noncompactness of $\mathcal{C}(\delta)$ is not an issue here,
- to compute $\nu(\mathcal{C}(\delta))$ and checks that it is a tail distribution function of δ.

Both steps are detailed in [Say24] and extended to any quadratic number field.

A look at the Eisenstein fractions

In the case of the Eisenstein fractions, i.e. elements of the quadratic field $\mathbb{Q}(i\sqrt{3})$, the study conducted in [Say24] focuses on the asymptotic gap statistics in the following sequence of point clouds in the elliptic curve $\mathbb{C}/\mathbb{Z}[e^{i\frac{2\pi}{6}}]$.

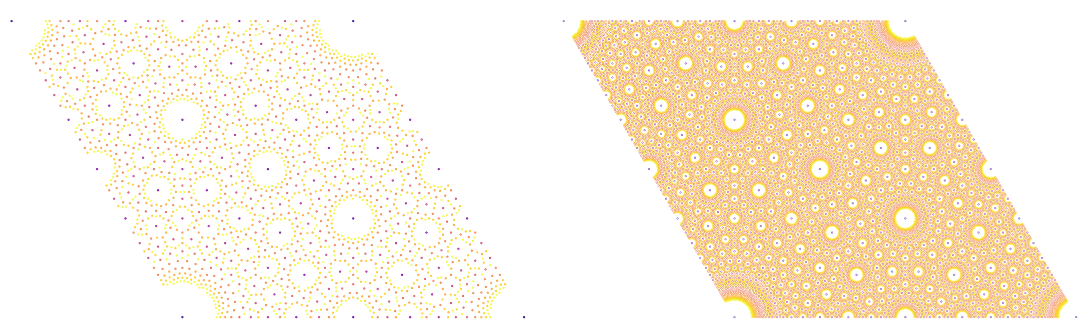


Figure 4: The Eisenstein fractions of height at most 10 (left) and 20 (right).

References

[Mar13] J. Marklof, "Fine-Scale Statistics for the Multidimensional Farey Sequence", in "Limit theorems in probability statistics and number theory", Springer Proc. Math. Stat. **42** (2013).

[PP24] J. Parkkonen and F. Paulin, "Joint partial equidistribution of Farey rays in negatively curved manifolds and trees", *Ergodic Theory Dynam. Systems* **44** (2024), pp. 2700–2736.

[Say24] R. Sayous, *Gaps in the complex Farey sequence of an imaginary quadratic field*, 2024, arXiv: 2407.04380.