03/12/24

last time Riemann-Hilbert on proj. alg. curves C

$x_0 \in C$ connect. smooth proj. curve $/\mathbb{C}$

$$MIC(C) \xrightarrow{\sim} \mathrm{Rep}\,\pi_1(C, x_0)$$

(finite dim rep)

Today: over quasiprojective alg. curves

$\mathring{C} = C \setminus \Sigma$, Σ finite, C smooth proj. curves

any vect. bundle on $\mathring{C}$ extend to a vect. bundle over C

alg vector bundles $/\mathring{C}$ + morphisms of $\mathcal{O}_{\mathring{C}}$-modules (E, ∇) $\simeq$ alg vect. bundle $/C$ + morphisms of $\mathcal{O}_{\mathring{C}}$ modules of their ext to $\mathring{C}$ $(\overline{E}, \overline{\nabla})$, $\nabla : \overline{\xi} \longrightarrow \overline{\xi} \otimes \Omega^1_C(*\Sigma)$

GAGA $\wr\wr$

analytic vect. bundles on C^{an} + morphisms: analytic morphism on $\mathring{C}^{an}$ + meromorphic at Σ $(\overline{E}^a, \nabla^a)$

$\nabla^a : \overline{E}^a \longrightarrow \overline{E}^a \otimes \Omega^1_{C^{an}}(*\Sigma)$ analytic

ex: $C = \mathbb{P}^1_{\mathbb{C}}$, $\Sigma = \{\infty\}$

$\mathring{C} = \mathbb{P}^1_{\mathbb{C}} \setminus \{\infty\} = \mathbb{A}^1_{\mathbb{C}}$

$C^{an} = \mathbb{C}$

$(\mathcal{O}_{\mathring{C}}, d - dz)$ $\quad d\varphi = \varphi dz$

$\nabla = d - dz$

$(\mathcal{O}^{an}_{\mathring{C}})^{\nabla = 0} = \underline{\underline{\mathbb{C}}}_{\mathring{C}^{an}} \exp$

near ∞, $w = \frac{1}{z}$

$\nabla = d + \frac{1}{w^2} dw$, $\exp z = \exp\left(\frac{1}{w}\right)$

essential sing. at 0

trivial monodromy but not alg. isom over $\mathring{C}$ ($\mathbb{A}^1$) to $(\mathcal{O}_{\mathring{C}}, d)$

two non isom. objects in $MIC(\mathbb{A}^1)$ are isom in $MIC(\mathbb{A}^{1\,an})$

Too many objects in $MIC(\mathring{C})$, needs to "select" good ones, which avoid "essential sing." for flat sections.

Regular singular points in dim 1 $\quad$ "local analytic theory"

$0 \in D = \mathring{D}(0, r)$, $r > 0$ $\quad \mathcal{O} = \mathcal{O}^{an}_D$

$\dot{D} = D \setminus \{0\}$

∇ on $\mathcal{O}_{D|\dot{D}}^{\oplus e}$ + meromorphic at 0

$\nabla = d + A\,dz$, $A \in M_e\big(\mathcal{O}(\dot{D}) \cap \mathcal{M}(D)\big)$

Laurentz serie $\frac{1}{z}\mathbb{C}[1/z] \oplus \mathbb{C}[[z]]$ (R. of. cv $\geq r$)

$g \in GL_e\big(\mathcal{O}(\dot{D}) \cap \mathcal{M}(D)\big)$

$$g^{-1}\nabla(g\cdot) = d + \left(g^{-1}Ag + g^{-1}\frac{dg}{dz}\right)dz$$

$T :=$ monodromy of $(\mathcal{O}_{\dot{D}}^{\oplus e}, \nabla)$ (at some $z_0 \in \dot{D}$, well def. conjugacy class in $GL_e(\mathbb{C})$)

We know $MIC(\dot{D}) \xrightarrow{\sim} Rep\,\pi_1(\dot{D})$, $\pi_1(\dot{D}) = \mathbb{Z}$

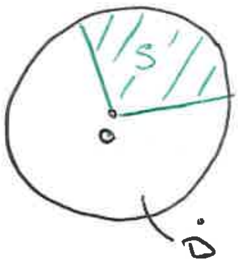
$T = \exp(2\pi i B)$, $B \in M_e(\mathbb{C})$

$$(\mathcal{O}_{\dot{D}}^{\oplus e}, \nabla) \xleftarrow[\sim]{\tilde{R}} \left(\mathcal{O}_{\dot{D}}^{\oplus e}, d - B\frac{dz}{z}\right) \text{ over } \dot{D}$$

$\tilde{R} \in GL_e(\mathcal{O}(\dot{D}))$ unique up to right mult. by a cst. mat. in $GL_e(\mathbb{C})$ with T

$$(d + A\,dz)\tilde{R} = \tilde{R}\left(d - B\frac{dz}{z}\right)$$

$$\tilde{R}' + A\tilde{R} = -\tilde{R}B\frac{1}{z}$$



S sector in $\dot{D}$, choose $\log z$ on S

flat section of $\left(\mathcal{O}_{\dot{D}}^{\oplus e}, d - B\frac{dz}{z}\right)$

are the $\underline{\varphi} = \exp(B\log z)\,\underline{\varphi}_0 \in \mathbb{C}^e$

$= $ " $z^B \underline{\varphi}_0$ "

flat sections of $(\mathcal{O}_{\dot{D}}^{\oplus e}, \nabla)$ over S:

$\tilde{R}(z)\,z^B\,\underline{\varphi}_0$ $\qquad R_S(z) = \tilde{R}(z)\,z^B$

"Resolvant over S" $R_S(z, z_0) := R_S(z)\,R_S(z_0)^{-1}$

The construction of $\tilde{R}$ and R_S are compatible with "gauge change"

$(\nabla \mapsto g^{-1}\nabla(g\cdot))$

Th: (TFCAE)

(1) $\exists \varepsilon \in (0,1)$, $\exists g \in GL_e(\mathcal{O}(\dot{D}_\varepsilon) \cap \mathcal{M}(D_\varepsilon))$, $\exists C \in M_e(\mathbb{C})$,

$$g^{-1}\nabla(g\cdot) = d + C\frac{dz}{z} \qquad T = \exp(-2\pi i C)$$

(2) $\exists \varepsilon$ ___, $\exists g$ ___, $\exists \tilde{C} \in M_e(\mathcal{O}(D_\varepsilon))$,

$$g^{-1}\nabla(g\cdot) = d + \tilde{C}(z)\frac{dz}{z}$$

$\exp(-2\pi i \tilde{C}(0))$ same eig. on T

(3) $\forall S$ sector of $\dot{D}_1$ + choice of log on S

$\exists N > 0$, $\|R_S(z)\| = O(|z|^{-N})$ $\quad z \in S,\ |z| \to 0$

(4) $\tilde{R}$ is meromorphic at 0 $\quad \tilde{R} \in GL_e(\mathcal{O}(\dot{D}) \cap \mathcal{M}(D))$

When this holds, we say that 0 is a regular singular point of $(\mathcal{O}, \nabla)$

proof: $\tilde{R}(z) z^B \underline{\varphi_0}$ $\qquad R_S(z) = \tilde{R}(z) z^B$

(1) ⇒ (2)

(3') $\exists \tilde{N} > 0$, $\|\tilde{R}(z)\| = O(|z|^{-\tilde{N}})$ $\quad z \in \dot{D},\ |z| \to 0$

(3) ⇔ (3') ⇔ (4)

(4) ⇒ (1) $\quad g = \tilde{R}$, $C = -B$

(2) ⇒ (3) $\qquad A(z) = \frac{\tilde{C}(z)}{z} \qquad \|A(z)\| \leq \frac{\tilde{N}}{|z|} \qquad |z| \leq r_0 < 1$

$$\frac{d R_S(z)}{dz} = -A(z) R_S(z)$$

$$\left\| \frac{dR_S}{dz}(z) \right\| \leq \frac{\tilde{N}}{|z|} \|R_S(z)\|$$

Lemma: $\varphi : (0,1) \to \mathbb{C}^e \setminus \{0\}$ C^1

$$\|\varphi'(t)\| \leq \frac{C}{t}\|\varphi(t)\|$$

$0 < t \leq t_0 \qquad \|\varphi(t)\| \leq \left(\frac{t}{t_0}\right)^{-C} \|\varphi(t_0)\|$

$\varphi(t) = R_S(tu)$, $|u| = 1$

□

⚠ invariant by "gauge change" by $g \in GL_e(\mathcal{O}(\dot{D}) \cap \mathcal{M}(D))$

But beware that if you allow $g \in GL_e(\mathcal{O}(\dot{D}))$ always "reg. sing"

For any (E,∇) in $MIC(\dot{D})$, there exists an analyt. fn. ξ of E over $\dot{D}$

$$\left(\mathcal{O}^{\oplus e}_{\dot{D}},\ d - B\frac{dz}{z}\right) \xrightarrow[\sim]{\xi} (E,\nabla)$$

Regular singular points in dim 1 : global algebraic theory

$$\underset{\text{finite}}{\Sigma_i} \subset C \supset \mathring{C} = C \setminus \Sigma_i \qquad (E,\nabla)\ MIC \text{ on } \mathring{C}$$

(C proj.)

$\bar{E}$ on C alg vect. bundle ext. E

$\alpha \in \Sigma_i$:

$$\begin{array}{ccc} \alpha \in U & \hookrightarrow & C^{an} \\ \downarrow \ \ \wr\downarrow z & & \\ 0 \in D & & \end{array} \qquad \begin{array}{c} U\setminus\{\alpha\} \\ \downarrow ? \\ \dot{D} \end{array}$$

$$\xi : \mathcal{O}_U^{an\oplus e} \xrightarrow[\sim]{an} \bar{E}^{an}_U$$

$\longrightarrow$ $\nabla \rightsquigarrow d + A$

$A \in M_e(\mathcal{O}^{an}(\dot{U}) \cap \mathcal{M}(U))$

$\quad \wr\wr$

$\mathcal{O}^{an}(\dot{D}) \cap \mathcal{M}(D)$

Prop: (1) $d+A$ on $\dot{D}$ has a regular sing. at 0

$\Leftrightarrow$ (2) $\exists \bar{E}'$ alg ext of E to $C_\alpha := \mathring{C} \cup \{\alpha\} = C \setminus (\Sigma_i \setminus \{\alpha\})$

st ∇ defines a morphism $\nabla : \bar{E}' \longrightarrow \bar{E}' \otimes \Omega^1_{C_\alpha}(\alpha)$

When this holds for every $\alpha \in \Sigma_i$, we say that (E,∇) has regular singular points, and construct an extension $\tilde{E}$ of E to C s.t. $\tilde{\nabla}$ exists

"logarithmic connection" $\tilde{\nabla} : \tilde{E} \longrightarrow \tilde{E} \otimes \Omega^1_C(\Sigma_i)$

$MIC_{reg}(\mathring{C})$ + comp. with $\oplus, \vee, \otimes$, Hom

Deligne's thm: (in dim 1)

$x_0 \in \mathring{C}$ connected

equivalence of categ.

$$MIC_{reg}(\mathring{C}) \xrightarrow{\text{"an"}} MIC(\mathring{C}^{an}) \longrightarrow Rep(\pi_1(\mathring{C}, x_0))$$

$$(E,\nabla) \longmapsto (E^{an}, \nabla^{an}) \longrightarrow \text{monodromy}$$

NB: $MIC(\mathring{C}) \xrightarrow{an} MIC(\mathring{C}^{an})$ is not an equivalence when $\Sigma_i \neq \emptyset$!

Idea of the proof:

1) "an" is fully faithfull

$\quad\uparrow$ internal hom

Obs: For any (E,∇) in $MIC_{reg}(\mathring{C})$

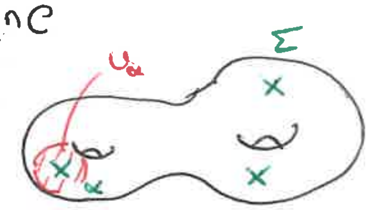
$$\Gamma(\mathring{C}^{an}, E^{an})^{\nabla^{an}=0} \xleftarrow[an]{\sim} \Gamma(\mathring{C}, E)^{\nabla=0}$$

a horizontal sect. of E^{an} on $\mathring{C}^{an}$ has moderate growth near Σ_i hence is "algebraic"

2) "am" is essentially surjective (E^a, ∇^a) in $MIC(\dot{C}^{am})$

$\alpha \in \Sigma$: $\quad \alpha \in U_\alpha \hookrightarrow C \qquad \dot{U}_\alpha = U_\alpha \setminus \{\alpha\} = U_\alpha \cap C$

$$\begin{array}{ccc} \alpha \in U_\alpha & & \dot{U}_\alpha \\ \downarrow \; \wr z_\alpha & & \downarrow \wr \\ 0 \in D & & \dot{D} \end{array}$$



may choose $\underline{s}_\alpha : \left(\mathcal{O}^{\oplus e}_{\dot{U}_\alpha}, d - B_\alpha \frac{dz_\alpha}{z_\alpha}\right) \simeq \left(E^a_{|\dot{U}_\alpha}, \nabla^a_{|\dot{U}_\alpha}\right)$

$\exp(2\pi B_\alpha)$

Define $\bar{E}^a$ on C^{am} by gluing E^a on $\dot{C}^{am}$ by means of the $\underline{s}_\alpha$
$\mathcal{O}^{\oplus e}_{|U_\alpha}$ on U_α

GAGA $\rightsquigarrow \bar{E}^a \simeq \bar{E}^{am}$ for some $\bar{E}$ alg vect. bundle over C

By ext. ∇ extend to $\bar{\nabla} : \bar{E}^a \longrightarrow \bar{E}^a \otimes \Omega^1_{C^{am}}(\Sigma)$

GAGA $\quad \bar{\nabla} : \bar{E} \longrightarrow \bar{E} \otimes \Omega^1_C(\Sigma)$

$(E, \nabla) := (\bar{E}_{\dot{C}}, \nabla)$ clearly has reg. sing.

Higher dimensional theory - Deligne

- purely analytic th already discussed
  ~~[illegible]~~ M complex analyt. manif., $x_0 \in U$
  connected

$$MIC(M) \xrightarrow{\sim} Rep\, \pi_1(M, x_0)$$

- local analytic th of regular singularities:

$$\Delta = D^{a+b} \hookleftarrow \dot{\Delta} = \dot{D}^a \times D^b = \Delta \setminus \underbrace{\{z_1 = \dots = z_a = 0\}}_{= D}$$

$(\mathcal{O}^{\oplus e}_{\dot{\Delta}}, d + A) \quad A \in M_e(\Omega^1_{am}(\dot{\Delta}) \cap \mathcal{H}(\Delta))$

$dA + A^2 = 0$

logarithm. different. $\Gamma^{am}(\Delta, \Omega^1_\Delta(\log D))$

$$\ni \quad \sum_{i=1}^{a} a_i \frac{dz_i}{z_i} + \sum_{i=a+1}^{a+b} a_i dz_i \qquad (a_i \text{ hol})$$

$\Omega^1_\Delta(\log D)^\vee \hookrightarrow \Theta_\Delta$

$\cup$ vector fields tgt to D

$\nabla = d + A$ has reg. sing along $D \Leftrightarrow \exists g \in GL_e(\mathcal{O}(\dot{\Delta}) \cap \mathcal{H}(\Delta))$, $g^{-1}\nabla(g \cdot) = d + \tilde{A}$

$\tilde{A} \in M_e(\Gamma^{am}(\Delta, \Omega^1_\Delta(\log D)))$

• global theory

X smooth complex connect. alg variety

$\downarrow$ Nagata

$\bar{X}$ proper complex alg. variety

$\bar{X}^{\sim} \xrightarrow{\nu} \bar{X}$

$\bar{X}^{\sim} \setminus X =: D$ divisor with SNC (locally in the analytic top $D \hookrightarrow \Delta$)

(E, ∇) alg vect. bundle on X with IC has regular singularities

$\Leftrightarrow$ - - - locally along D as in the above discuss / Δ

$\Leftrightarrow$ $\exists\, \bar{E}$ alg. vect. bundle on $\bar{X}^{\sim}$ extending E

st $\nabla : \bar{E} \to \bar{E} \otimes \Omega^1_{\bar{X}^{\sim}}(\log D)$

equivalence of categories:

$$MIC_{reg}(X) \xrightarrow{\text{“an”}} MIC(X^{an}) \longrightarrow Rep\, \pi_1(X, x_0)$$