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Complements in analysis and algebraic geometry

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Algebra		Analysis
$\mathbb{C}[X_1, \dots, X_m]$	$\mathbb{C}\{z\} \subset \mathbb{C}[[z]]$ DVR	$\mathcal{C}(I)$,
$\mathbb{C}[[X_1, \dots, X_m]]$		$\mathcal{C}^\infty(M)$
quotients ...	$U \overset{\circ}{\subset} \mathbb{C}$, $U \neq \emptyset$	
Noetherian	$\mathcal{O}^{an}(K)$ / $\mathcal{O}^{an}(U)$	Non-noetherian
noetherian factorial	$\rightarrow \mathbb{C}\{z_1, \dots, z_m\}$	complex analysis

$f \in \mathcal{O}^{an}(U)$

$f_m := \dfrac{f}{(z-a_1)\dots(z-a_m)}$ st f vanishes at the a_i at order 1

$f_m \mathcal{O}^{an}(U) \subset \dots$

$K \subset \mathbb{C}$ cpct connected $\neq \emptyset$

$\mathcal{O}^{an}(K) \simeq \varinjlim_{\varepsilon \to 0} \mathcal{O}^{an}(K_\varepsilon)$

$K_\varepsilon := K + \mathring{D}(0,\varepsilon)$

$\mathcal{O}^{an}(\bar{D}(0,1)) = \{\varphi \in \mathbb{C}[[z]] \mid R_{cv}\varphi > 1\}$

$\varphi \in \mathcal{O}^{an}(K) \setminus \{0\}$, $\deg \operatorname{div} \varphi = \sum_{p \in K} v_p(\varphi)$

Step 0

$U \overset{\circ}{\subset} \mathbb{C}^m$

$\mathcal{O}^{an}(U) := \{\varphi \in \mathcal{C}^1(U, \mathbb{C}) \mid D\varphi \text{ is } \mathbb{C}\text{-linear}\}$

$U \supset \prod_{i=1}^m \bar{D}(0, r_i) \ni z$ $\qquad \varphi = \sum_{I \in \mathbb{N}^m} a_I z^I$

$\mathcal{O}^{an}_{\mathbb{C}^m, 0} \subseteq \mathbb{C}[[z_1, \dots, z_m]]$ ← complet of $\mathcal{O}^{an}_{\mathbb{C}^m, 0}$

$\mathcal{O}^{an}_{\mathbb{C}^m,0} = \mathbb{C}\{z_1, \dots, z_m\}$

> X smooth alg var $/\mathbb{C}$
>
> $x \in X(\mathbb{C})$
>
> $\mathcal{O}_{X,x} \subset \mathcal{O}^{an}_{X,x} \subset \hat{\mathcal{O}}_{X,x}$
>
> Atiyah-McDonald chap 10 ex 5

Step 1

local th in several variables, Weierstraß, division / prep ...

Step 2 (Levi, Hartogs)

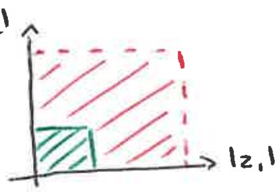
$U \overset{\circ}{\subset} \mathbb{C}$ $\quad H^1(U, \mathcal{O}^{an}) = 0$ $\quad$ Mittag-Leffler

$0 \to \mathcal{O}^{an} \to \mathcal{M}^{an} \to \mathcal{P} \to 0$

$H^0(U, \mathcal{M}^{an}) \to H^0(U, \mathcal{P}) \to 0$ $\qquad H^1(U, \mathcal{O}^{an}) = 0$

$H = \Delta(1,1) = \mathring{D}(0,1)^2 \setminus \bar{D}(0,r)^2 \subset \mathbb{C}^2$

$0 < r < 1$



"Hartogs cauldron" $\quad \mathcal{O}^{an}(\mathring{D}(0,1)^2) \xrightarrow{\sim} \mathcal{O}^{an}(H)$

$\varphi \longmapsto \varphi_{|H}$

$\mathring{C}(r,1) = \{z \in \mathbb{C} \mid r < |z| < 1\}$

$\underset{w}{U} \times \underset{z}{\mathring{C}(r,1)}$ $\quad \sum_{m \in \mathbb{Z}} a_m z^m$, $a_m \in \mathcal{O}^{an}(U)$ $\qquad \sum_i a_m z^m$ $R_{cv} \geq 1$

$\sum_{m \leq 0} a_{-m} w^m$ $\quad R_{cv} \geq \frac{1}{r}$

$\sum_{(p,q) \in \mathbb{Z}^2} a_{p,q} z_1^p z_2^q \xrightarrow{\text{can verify}} H^1(U, \mathcal{O}^{an}) = 0$

$'' 0$ si $(p,q) \notin \mathbb{Z}^2_{\geq 0}$

Weierstraß division

$m=1$ $\quad \mathbb{C}\{z\} \xrightarrow{v} \mathbb{N} \cup \{+\infty\}$

$\varphi \in \mathbb{C}\{z\} \setminus \{0\} \qquad k := v(\varphi)$

$$W_\varphi : \mathbb{C}\{z\} \oplus \mathbb{C}[z]_{<k} \xrightarrow{\sim} \mathbb{C}\{z\}$$
$$(q, r) \longmapsto \varphi q + r$$

variants: $\bar{D} = \bar{D}(0,1)$

$\mathcal{O}^{an}(\bar{D})$

$B(\bar{D}) := \{ \varphi \in \mathcal{C}(\bar{D}, \mathbb{C}) \mid \varphi_{|\mathring{D}} \in \mathcal{O}^{an}(\mathring{D}) \}$

Banach alg of an. fct. on $\mathring{D}$

$\varphi \in \mathcal{O}^{an}(\bar{D})$ or $B(\bar{D})$ $+ \varphi_{|\partial D}$ does not vanish

$$W_\varphi : \mathcal{O}^{an}(\bar{D}) \oplus \mathbb{C}[z]_{<n} \xrightarrow{\sim} \mathcal{O}^{an}(\bar{D})$$
$$W_\varphi : B(\bar{D}) \oplus \mathbb{C}[z]_{<n} \xrightarrow{\sim} B(\bar{D})$$

$\varphi \mapsto W_\varphi \in \mathrm{Is}(B(\bar{D}) \oplus \mathbb{C}[z]_{<n}, B(\bar{D}))$ analytic in φ

$\not\#$

$U \hookrightarrow \mathbb{C}^{m-1}$ connected

$f \in \mathcal{O}^{an}(U \times \bar{D}) \qquad f(w, z_m)$

assume $f(U \times \partial \bar{D}) \subset \mathbb{C}^\times$

$w \in U, \quad f(w, \cdot) \in \mathcal{O}^{an}(\bar{D})$

$W_{f(w,\cdot)} \in \mathrm{Is}(\mathcal{O}(\bar{D}) \oplus \mathbb{C}[z]_{<k}, \mathcal{O}(\bar{D}))$

$\mathbb{C}$-an in $w \in U$

$g \in \mathcal{O}(U \times \bar{D}), \quad W^{-1}_{f(w,\cdot)} g(w, u) = (q_w, r_w)$ am-dep of w

$$\left.\begin{aligned} r(w, z_m) &:= r_w(z_m) \\ q(w, z_m) &:= q_w(z_m) \end{aligned}\right\} \Rightarrow g(w, z_m) = f(w, z_m) q(w, z_m) + r(w, z_m)$$

$\to$ Th [Weierstraß division] : $g = fq + r$

local version: $f \in \mathbb{C}\{w, z_m\}$, $w = (w_1, \dots, w_{m-1})$

$f(0, z_m) \not\equiv 0 \qquad k = v(f(0, \cdot))$

Any $g \in \mathbb{C}\{w, z_m\}$ may uniquely written $g = fq + r$, $q \in \mathbb{C}\{w, z_m\}$, $r \in \mathbb{C}\{w\}[z_m]_{<k}$

Cor: (Weierstraß prep.)

f and k as above

$\exists u \in \mathbb{C}\{w, z_m\}^\times, \; u(0,0) \neq 0$

st $\quad uf(w, z_m) = z_m^k + \sum_{i=0}^{k-1} a_i(w) z_m^i \qquad a_i(0) = 0, \; a_i \in \mathbb{C}\{w\}$

démo : $g = z_m^k$ $\qquad n = -\sum_i a_i(w) z^i$

$$\underbrace{z_m^k - n}_{W.P} = f g$$

$w = 0$ $\qquad z_m^k + \sum_i^{k-1} a_i(0) z^i = f(0, z_m)\, g(0, z_m) \quad \in \mathbb{C}\{z_m\}$

$\Rightarrow a_i(0) = 0$

$f(0,0) \in \mathbb{C}^\times$ $\qquad\square$

$I_m \subset I_{m+1} \subset \dots \subset \mathcal{O}_m = \mathbb{C}\{z_1, \dots, z_m\}$

may assume $f \in I_0 \setminus \{0\}$, $f(0, \cdot) \neq 0$

$R_f : \mathbb{C}\{z_1, \dots, z_m\} \longrightarrow \mathbb{C}\{z_1, \dots, z_{m-1}\}^{\oplus k} = \mathcal{O}_{m-1}$

"rest of div by f"

$I_{k+1} = R_f(I_k) + f\mathcal{O}_m$ $\qquad R_f(I_k) \subset R_f(I_{k+1}) \subseteq \dots$

Noetherianity

Coherence in analytic geometry

X top sp., $\mathcal{A}$ sheaf of rings over X

$\mathcal{F}$ sheaf of $\mathcal{A}$-modules

$\mathcal{F}$ is said to be locally finitely generated if $\forall x_0 \in X$, $\exists\, \Omega$ open nbgh of x_0 in X

(LFG) and $F_1, \dots, F_q \in \mathcal{F}(\Omega)$ st $\forall x \in \Omega$, $\mathcal{F}_x$ is generated by $F_{1,x}, \dots, F_{q,x}$ as $\mathcal{A}_x$-module

NB: when this holds, if $\mathcal{F}_{x_0} = \{0\}$, then $\exists\, U \ni x_0$ open nbgh of $x_0 \in X$ st $\mathcal{F}_{|U} = 0$

does not behave well

Def: $\mathcal{F}$ is coherent $\Leftrightarrow$ $\mathcal{F}$ is LFG

$(\overset{?}{\Rightarrow})$ $\forall U \subset_{op} X$, $F_1, \dots, F_q \in \mathcal{F}(U)$, $R(F_1, \dots, F_q)$ is LFG

$R(F_1, \dots, F_q)_x := \ker (F_{1,x}, \dots, F_{q,x}) : \mathcal{A}_x^{\oplus q} \longrightarrow \mathcal{F}_x$

$R(F_1, \dots, F_q) := \ker (F_1, \dots, F_q) : \mathcal{A}_{|U} \longrightarrow \mathcal{F}_{|U}$