Modules with integrable connections on analytic and complex varieties I : Regular connections and regular singular points in dimension 1.

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Reference : P. Deligne, Equations différentielles à points singuliers réguliers

N.M. Katz, An overview of Deligne's work on Hilbert's 21^{st} problem

B. Malgrange, Sur les points singuliers des equations differentielles - (Ens. Math).

C. Sabbah, Deformations isomonodromiques

Let X (connected) complex analytic function, $\mathcal{O}_X$ the sheaf of analytic functions.

(Some will work for $\mathcal{C}^\infty$, $\mathcal{C}^\omega$, algebraic varieties)

1. Vector bundles with connection

(E,∇), E vector bundle of rank e over X

$\mathcal{E}$ the sheaf of analytic sections of E, locally $\mathcal{E}|_U \simeq \mathcal{O}_{X|U}^{\oplus e}$

so $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module.

(This is an equivalence of categories...)

$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ morphism of sheaves of $\underline{\mathbb{C}}_X$-modules.

Leibniz rule : s, f are local sections of $\mathcal{E}$, $\mathcal{O}_X$

$$\nabla(sf) = \nabla(s)\cdot f + s \otimes df.$$

Alternatively, we can describe it as, for $U \hookrightarrow X$, $\theta \in \Theta(U)$

$\theta : \mathcal{O}_{X|U} \longrightarrow \mathcal{O}_{X|U}$ derivation

$\cdot\ ev_\theta : \Omega^1_{X|U} \longrightarrow \mathcal{O}_{X|U}$ "evaluation map"

$$\begin{tikzcd}
\nabla_\theta : \mathcal{E}|_U \arrow[rr] \arrow[dr, "\nabla|_U"'] & & \mathcal{E}|_U \\
& (\mathcal{E}\otimes\Omega^1_X)_U \arrow[ur, "id\otimes ev_\theta"'] &
\end{tikzcd}$$

$$\begin{aligned}
\Theta_X &\longmapsto \mathcal{H}om_{\mathbb{C}_X}(\mathcal{E},\mathcal{E}) \\
\theta &\longmapsto \nabla_\theta
\end{aligned}$$

$$\nabla_\theta(s\cdot f) = \nabla_\theta s\cdot f + s\cdot\theta(f).$$

Local description $U \hookrightarrow X$

frame: $\underline{s} = (s_1, \dots, s_e) : \mathcal{O}_X^{\oplus e}|_U \longrightarrow \mathcal{E}|_U$

$$(f_\alpha)_{1\le\alpha\le e} \longmapsto \sum_{\alpha=1}^{e} s_\alpha f_\alpha$$

$$\nabla \underline{s} = (\nabla s_1, \dots, \nabla s_e) = \underline{s}\,\Omega$$

$$\Omega = (\omega_{\alpha\beta})_{1\le\alpha,\beta\le e} \in M_e(\Omega^1_X(U)) \qquad \Big(\nabla s_\beta = \sum_{\alpha=1}^{e} s_\alpha \omega_{\alpha\beta}\Big)$$

$$s = \underline{s}\cdot\underline{f} = \sum_{\alpha=1}^{e} s_\alpha f_\alpha$$

$\otimes.$ "Christoffel's symbol"

$$\nabla s = \sum_{1\le\alpha\le e} (\nabla s_\alpha \cdot f_\alpha + s_\alpha \otimes df_\alpha) = \underline{s}\,\Omega\cdot\underline{f} + \underline{s}\,d\underline{f}$$
$$= \underline{s}\,(d\underline{f} + \Omega\underline{f})$$

Recall: We may extend ∇ to ∇^i, $i \geq 0$

$$\nabla^i : \mathcal{E}\otimes_{\mathcal{O}_X}\Omega^i_X \longrightarrow \mathcal{E}\otimes_{\mathcal{O}_X}\Omega^i_X \quad \text{morphism of } \mathbb{C}_X\text{-modules}$$

Characterization: $\nabla^0 = \nabla$

wedge.

$$\nabla^i(\underset{\mathcal{E}}{s}\cdot\underset{\Omega^i_X}{\varphi}) = \nabla^i(s)\cdot\varphi + s\cdot d\varphi$$

Existence: local formula.

$\underline{s}$ as above, Ω, $\nabla^i(\underline{s}\cdot\underset{(\Omega^i_X)^{\oplus e}}{\underline{\varphi}}) = \underbrace{\underline{s}\cdot(d\underline{\varphi} + \Omega\cancel{d}\underline{\varphi})}_{(\Omega_X^{i+1})^e}$

Fact: $\nabla^2 = \nabla^1\circ\nabla^0 : \mathcal{E} \longrightarrow \mathcal{E}\otimes\Omega^2_X$ is $\mathcal{O}_X$-linear

$\nabla^2 s = R s$ ("R section of $\mathrm{End}(E)\otimes\Omega^2_X$ ~ R curvature tensor form).

Indeed:

$$\nabla^2(\underline{s}\cdot\underline{f}) = \nabla^1(\underline{s}(d\underline{f} + \Omega\underline{f}))$$
$$= \underline{s}\,[d(d f + \Omega\underline{f}) + \Omega(d f + \Omega\underline{f})]$$
$$= \underline{s}\,[d\Omega f - \Omega df + \Omega d\underline{f} + \Omega^2 f]$$
$$= \underline{s}\,R\,\underline{f}$$

where $R = d\Omega + \Omega^2$.
$\in M_e(\Omega^1_X(U))$.

$$R(\theta_1,\theta_2)s = ([\nabla_{\theta_1},\nabla_{\theta_2}]-\nabla_{[\theta_1,\theta_2]})s$$

1.b | ~~Modules~~ Vector bundles with connections form a tensor category

Direct sum $\quad (E_1,\nabla_1),(E_2,\nabla_2) \rightsquigarrow (E_1\oplus E_2, \nabla_1\oplus\nabla_2)$

Internal Hom $\quad \mathrm{Hom}(E_1,E_2) \longleftrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}_1,\mathcal{E}_2).$

$$\mathrm{Hom}(E_1,E_2)_x = \mathrm{Hom}_{\mathbb{C}}(E_{1,x},E_{2,x})$$

There exists a unique connection ∇ on $\mathrm{Hom}(E_1,E_2)$ such that:

$$\nabla_{2,\theta}(\varphi(s)) = \nabla_\theta\varphi(s)+\varphi(\nabla_{1,\theta}s).$$

section of Hom $\qquad$ section of E_1

Special case: $(\mathcal{E}_2,\nabla)=(\mathcal{O}_X,d)$

$\rightsquigarrow (E_1^\vee,\nabla_1^\vee)\qquad \theta(\xi\cdot s) = \nabla_\theta^\vee\xi\cdot s+\xi\,\nabla_{1,\theta}(s).$

$\qquad\qquad\qquad\qquad\qquad$ (4) $\quad E_1^\vee \quad E_1$

Tensor product

$$E_1\otimes E_2 \longleftrightarrow \mathcal{E}_1\otimes_{\mathcal{O}_X}\mathcal{E}_2$$

$$\nabla_\theta^\otimes(s_1\otimes s_2) = \nabla_{1,\theta}s_1\otimes s_2 + s_1\otimes\nabla_{2,\theta}s_2$$

On a local frame of E_i $(i=1,2)$ $\underline{s}_i$:

$g \in GL_e(G_x)$

(E,∇)	$\underline{\underline{s}}$	Ω	R
(E,∇)	$s' \cdot s \cdot g$	$\Omega' = g^{-1}\Omega g + g^{-1}dg$	
$E_1 \oplus E_2$	$(\underline{\underline{s_1}}, \underline{\underline{s_2}})$	$\begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$	$\begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}$
(G_X, d)	1	0	0
$(E^\vee, \nabla^\vee)$	$\underline{\underline{s}}^*$	$-{}^t\Omega.$	$-{}^tR$
$(E_1 \otimes E_2, \nabla^\otimes)$	$\underline{\underline{s_1}} \otimes \underline{\underline{s_2}}$	$\Omega_1 \otimes I_{e_2} + I_{e_2} \otimes \Omega_1$	$R_1 \otimes I_{e_2} + I_{e_1} \otimes R_2$

N.B: Connexion on E is a tensor over $\Gamma(X, \mathrm{End}(E) \otimes \Omega^1_X)$.

Category of vector bundles with connections over X

$$\mathrm{Hom}((E_1,\nabla_1),(E_2,\nabla_2)) := \{\varphi \in \mathrm{Hom}_{G_X}(E_1,E_2) \mid \forall \theta, \ \nabla_{2,\theta} \circ \varphi = \varphi \circ \nabla_{1,\theta}\}$$

$$\mathrm{Hom}((E_1,\nabla_1),(E_2,\nabla_2)) = \Gamma(X, \mathrm{Hom}(E_1,E_2))^{\nabla=0}.$$

(E,∇) $\quad \Gamma(X,E)^{\nabla=0} = \ker(\Gamma(X,E) \to \Gamma(X, E \otimes \Omega^1_X))$

"flat sections of (E,∇)"

locally $\quad \varphi \underline{\underline{s_1}} = \underline{\underline{s_2}} A, \quad A \in \Pi_{e_2,e_1}(G_X(U))$

$$\varphi(\underline{\underline{s}}\, \underline{f}) = \underline{\underline{s_2}} A \underline{f}.$$

$$\nabla_2(\varphi \underline{\underline{s_1}}) = \nabla_2(\underline{\underline{s_2}} A) = \underline{\underline{s_2}}(dA + \Omega_2 A)$$

$$\varphi(\nabla_1 \underline{\underline{s}}_1) = \varphi(\underline{\underline{s}}_1 \Omega_1) = \underline{\underline{s}}_2 A \Omega_1$$

Flatness condition $\Leftrightarrow$ $dA = A\Omega_1 - \Omega_2 A \in \Gamma_{e_2,e_1}(\Omega^1_X(U))$

1.c Inverse image of vector bundles with connections

$$f : Y \longrightarrow X \qquad \mathbb{C}\text{-analytic}$$

$$E, \mathcal{E}, \nabla$$

$F := f^* E$

$\mathcal{F} = f^{-1}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_Y$

$$\nabla^F : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_Y \simeq f^{-1}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_X} \Omega^1_X.$$

$$\text{"}{}^t Df\text{"} : f^{-1}\Omega^1_X \longrightarrow \Omega^1_Y$$

$$\text{"}I \otimes {}^t Df\text{"} : f^{-1}(\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X) \to f^{-1}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_X} \Omega^1_Y \simeq \mathcal{F} \otimes_{\mathcal{O}_Y} \Omega^1_Y.$$

$$\nabla^F(\underset{\in \mathcal{E}}{f^{-1}s} \otimes \underset{\in \mathcal{O}_Y}{\lambda}) := (I \otimes {}^t Df)(f^{-1} \underbrace{\nabla^E s}_{\mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X})\lambda + f^{-1}s \otimes d\lambda.$$

Well defined.

In a local frame: $U \hookrightarrow X$ $\underline{\underline{s}}$ frame of E over U,

$\rightsquigarrow f^{-1}\underline{\underline{s}}$ frame of F over $f^{-1}(U)$.

$$\nabla^E \underline{\underline{s}} = \underline{\underline{s}} \cdot \Omega^E$$

$$\rightsquigarrow \nabla^F(f^{-1}\underline{\underline{s}}) = f^{-1}s \cdot \underbrace{f^* \Omega^E}_{\substack{\text{connexion for } f^{-1}\underline{\underline{s}} \\ \Gamma_e(\Omega^1_X(f^{-1}(U)))}} \qquad \Omega^F = f^* \Omega^E.$$

R^E ; $\mathrm{End}(E) \otimes \Omega$

$R^F = f^* R^E \qquad \mathrm{End}(F) \otimes \Omega^2_Y$

(locally $R^F = d\Omega^F + (\Omega^F)^2$)

$= f^*(d\Omega^E + \Omega^{E\,2})$

<u>NB</u>: Everything holds in the algebraic setting.
Pseudo functors is defined from.

$$\left\{ \begin{matrix} \text{smooth analytic} \\ \text{complex manifold} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Category of} \\ \text{Groupoids} \end{matrix} \right\}$$

"prestack"
in alg. setting],
Stack over
smooth scheme in the
étale cohomology..

$$X \rightsquigarrow \left\{ \begin{matrix} \text{Vector bundle with connection over } X \\ + \text{ isom of } \ldots\ldots \end{matrix} \right\}$$

$\underline{\underline{2}}$ MIC and Local systems of finite dim. $\mathbb{C}$-vector spaces.

<u>2.a</u> $\underset{\substack{\wr \\ \text{Module with integrable connection}}}{\text{MIC}} := (E, \nabla)$ module with connection such that curvature $R \in \Gamma(X, \mathcal{E}nd\, E \otimes \Omega^2_X)$.

$$\theta_1, \theta_2, \quad [\nabla_{\theta_1}, \nabla_{\theta_2}] = \nabla_{[\theta_1, \theta_2]}.$$

$\boxed{\text{In dimension } 1\text{, this is always satisfied.}}$

2.b: Local systems of finite dimensional $\mathbb{C}$-vector space.

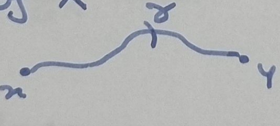
<u>Recall</u>: X "nice" connected topological space, e.g. manifold.

$x_0 \in X$, $\pi_1 = \pi_1(X, x_0)$

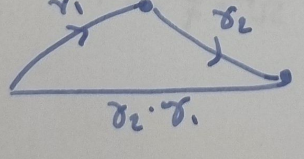
Fundamental groupoid:

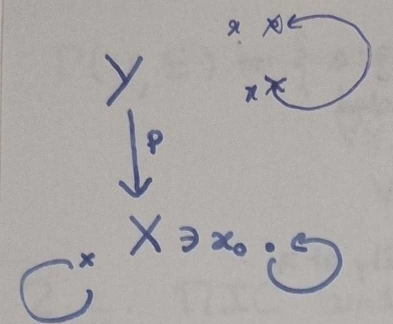
<u>objects</u>: points of X

<u>morphism</u>:



Morphism: $\gamma : [0,1] \longrightarrow X$
$\gamma(0) = x$
$\gamma(1) = y$.





topological covering.

Natural left action of $\pi_1(X,x_0)$ on $p^{-1}(x_0)$.

$\rightsquigarrow$ Equivalence of categories

$$\{\text{topological cov of } X\} \simeq \{\pi_1\text{- sets.}\}$$

connected $\iff$ transitive action.

Pointed top. covering X, $y_0 \in p^{-1}(x_0)$ $\iff$ Pointed π_1- set

$(\tilde{X},\tilde{x}_0)$ universal pointed covering $\iff$ $\pi_1 \ni e$ left action.

$$\iota : \mathrm{Aut}(\tilde{X}/X) \simeq \pi_1(X,x_0).$$
$$\alpha \longmapsto [\gamma].$$

$$\alpha^{-1}(\tilde{x}_0) = [\gamma]\cdot x_0.$$

Define: A local system of f.d. $\mathbb{C}$-vector space over X is a locally constant sheaf of ______.

locally $\mathbb{E} \simeq \underline{\mathbb{C}}_X^{\oplus e}$ $\quad$ $s: \underline{\mathbb{C}}_U^{\oplus e} \xrightarrow{\sim} \mathbb{E}|_U$ free

$\underline{\mathbb{C}}_X$- module.

$$\mathbb{E}_x \simeq \mathbb{C}^{\oplus e}$$

Facts:

① $x_0 \in \mathcal{U} \hookrightarrow X$ connected simply connected

$\exists$ unique iso

$$\underset{\underline{\underline{E}}_{x_0}}{\overset{E_{x_0}\otimes_{\mathbb{C}} \underline{\underline{\mathbb{C}}}_U}{\shortparallel}} \simeq \mathbb{E}|_U \quad \text{"equality at } x_0\text{"}.$$

② $f: Y \longrightarrow X$ C^0, $\mathbb{E}$ local system on X.

$\rightsquigarrow f^{-1}\mathbb{E}$ local system on Y

$\underline{\underline{s}}$ horizontal frame of $\mathbb{E}$ over $U \hookrightarrow X$.

$\Rightarrow f^{-1}\underline{\underline{s}}$ —— $f^{-1}\mathbb{E}$ —— $f^{-1}(U) \hookrightarrow Y$.

③ $\gamma: [0,1] \longrightarrow X$ continuous path.

$\gamma^{-1}\mathbb{E}$ is a trivial local system.

$$I_\gamma : \mathbb{E}_{\gamma(0)} \otimes_{\mathbb{C}} \mathbb{C}_{|[0,1]} \xrightarrow{\sim} \gamma^{-1}\mathbb{E}$$

$$\rightsquigarrow I_\gamma(1) : \mathbb{E}_{\gamma(0)} \xrightarrow{\sim} \mathbb{E}_{\gamma(1)}$$

This is compatible with composition and is invariant over homotopy.

Applied to loops;

$\rightsquigarrow$ Monodromy representation $\pi_1(X, x_0) \longrightarrow GL(\mathbb{E}_{x_0})$.

In fact there is an equivalence of categories between

$$\left\{\begin{array}{c}\text{local system of} \\ \text{f. dim } \mathbb{C}\text{-vector spaces} \\ \text{on } X\end{array}\right\} \xrightarrow[\substack{\oplus, \otimes, \mathrm{Hom}, \\ \text{pullback.}}]{\sim} \left\{\begin{array}{c}\text{Finite dim rep} \\ \text{of } \pi_1(X, x_0)\end{array}\right\}.$$

Description of the quasi-inverse;

$$\begin{array}{l}\pi : \pi_1(X, x_0) \longrightarrow GL(V). \\ \quad\ \wr\uparrow \\ \ \mathrm{Aut}(\tilde{X}/X)\end{array}$$

$$\begin{array}{c}\tilde{X} \ni \tilde{x}_0 \\ {\scriptstyle \text{universal cover}}\ \downarrow \\ X\end{array} \qquad V \otimes_{\mathbb{C}} \underline{\underline{\mathbb{C}}}_{\tilde{X}} \circlearrowleft \mathrm{Aut}(\tilde{X}/X) \text{ equivariant}$$

descends to a local system $\mathbb{E}$ over x.

$\Gamma(U, \mathbb{E}) = \{ s : p^{-1}(U) \longrightarrow V$ locally constant

$\forall \gamma, \forall \tilde{x} \in \tilde{X} \qquad s(\gamma \cdot \tilde{x}) = \pi(\iota(\gamma))^{-1} s(\tilde{x}) \}$

$\forall \gamma \in \mathrm{Aut}(\tilde{X}/X)$

2.c. MIC and local systems

X $\mathbb{C}$-analytic, $\mathcal{C}^\infty$, $\mathcal{C}^\omega$ (NOT ALGEBRAIC)

Construction: $\mathbb{E}$ - local system over X

$(\mathcal{E}, \nabla) \rightsquigarrow$ MIC

$\mathcal{E} := \mathbb{E} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X \qquad \qquad \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X .$

$\nabla = \mathrm{Id}_{\mathbb{E}} \otimes d . \qquad \qquad \qquad \qquad \qquad \qquad \shortparallel$

$\nabla : \mathbb{E} \otimes_{\underline{\mathbb{C}}_X} \mathcal{O}_X \longrightarrow \mathbb{E} \otimes_{\underline{\mathbb{C}}} \Omega^1_X .$

functorial + compatible with tensor product & pullback.

Locally $(\mathcal{E}, \nabla) = (\mathcal{O}_X^{\oplus r}, \mathrm{Id} \otimes d)$

$\hookrightarrow \Omega = 0, \; R = 0.$

Observe: $\mathbb{E}_x$ "$=$" E_x.

$\underline{s}$ horizontal frame on U $\iff$ $\underline{s}$ frame of $\mathcal{E}$ s.t. $\nabla \underline{s} = 0$.

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \shortparallel$

$\qquad \qquad \qquad \qquad \qquad \qquad \quad \underline{s} \otimes 1 .$

Theorem: Integrability of integrable connections.

Theorem 1: The construction above is an equivalence of categories.

Theorem 2: E vector bundle over X, there is a bijection

$$\left\{ \begin{array}{l} \text{subsheaves of } \mathcal{E}, \text{ called } \mathbb{E} \\ \text{which are local systems} \\ \text{s.t.} \\ \mathbb{E} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X \xrightarrow{\sim} \mathcal{E} \end{array} \right\} \xrightarrow{\quad \sim \quad} \left\{ \begin{array}{c} \text{Integrable connections} \\ \nabla \text{ on } E \end{array} \right\}$$

$$\mathbb{E} \longmapsto \nabla = \mathrm{Id} \otimes d$$

$$\mathbb{E} = \mathcal{E}^{\nabla = 0} \longleftarrow\!\!\shortmid \nabla$$

Scholium:

If X is connected then there is an equivalence of categories

$$\left\{ \begin{matrix} \text{finite dims} \\ \text{rep of } \pi_1(X, x_0) \end{matrix} \right\} \xrightarrow{\sim} \Pi I C(X)$$