Internal operations on $\mathcal{D}$-modules – (Ref: P. Schneiders) [illegible]

X $\mathbb{C}$-manifold, Θ sheaf of holomorphic vector fields, subsheaf of $\underline{\mathrm{End}}_{\mathbb{C}_X}(\mathcal{O}_X)$.

1) $h \in \mathcal{O}_X \rightsquigarrow h\cdot \in \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{O}_X)$

2) $\theta \in \Theta_X \rightsquigarrow \theta = \sum a_i \partial_{x_i}$, $h \in \mathcal{O}_X$, $L_\theta h := \sum a_i \partial_i h$.

Rmk: $\mathcal{O}_X, \Theta_X \subseteq \underline{\mathrm{End}}_{\mathbb{C}}(\mathcal{O}_X)$, $\mathcal{D} = \langle \mathcal{O}_X, \Theta_X \rangle$ sheaf of diff. op.

Prop: Let $\mathcal{M}$ be an $\mathcal{O}_X$-module, $\alpha : \Theta_X \longrightarrow \underline{\mathrm{End}}_{\mathbb{C}}(\mathcal{M})$, $\mathbb{C}$-linear

s.t. a) $[\alpha(\theta), h\cdot] = L_\theta h\cdot$ (resp. $-L_\theta h$)

b) $[\alpha(\theta), \alpha(\psi)] = \alpha([\theta, \psi])$ $\psi, \theta \in \Theta_X$. (resp. $\alpha[\psi,\theta]$)

c) $(h \cdot \alpha(\theta)) = \alpha(h\theta)$ (resp. $\alpha(\theta) \circ (h\cdot) = \alpha(h\cdot\theta)$)

then $\mathcal{M}$ is a left $\mathcal{D}_X$-module. (right $\mathcal{D}$-module)

Example: · $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module, $L_\theta h =: \theta \cdot h$.

($\dim X = n$) · $\Omega_X = \Lambda^n \Omega^1_X$, locally free of rank 1, then Ω_X is a right $\mathcal{D}$-module.

$\omega \in \Omega_X$, $\theta \in \Theta$

$$\omega \cdot \theta := - L_\theta \omega. \quad \text{(Definition)}$$

In coordinates $\theta = \sum \theta^i \partial_{z_i}$ $\omega(z) = a(z)\, dz_1 \wedge \dots \wedge dz_n$.

$$L_\theta \omega = \sum_{i=1}^n \partial_{z_i}(\theta^i a)\, dz_1 \wedge \dots \wedge dz_n$$

locally $\theta = \theta^i \partial_{z_i}$ ... , $\exists\, \varphi_t : U \longrightarrow U$ 1-parameter family of biholo, $t \in B_\varepsilon(0) \subset \mathbb{C}$

$p \in X$, $U \ni p$ small neighborhood. s.t. $\varphi_0 = id_U$.

$$\forall p \in U, \quad (\partial_t \varphi_t(p))|_{t=0} = \theta(p).$$

$$L_\theta \omega := \partial_t (\varphi_t^* \omega)|_{t=0}$$

Rmk: Related to the adjoint of differential operators.

If $\omega \in$ top. forms
P differential operator.
$f \in$ Function.

$$\int \omega(Pf) = \int (\omega P) f$$

$({}^tP\omega)$

Internal operations

${}_{\mathcal{D}}\mathrm{Mod}$ = category of left $\mathcal{D}$-modules.

$\mathcal{M}, \mathcal{N} \in {}_{\mathcal{D}}\mathrm{Mod}$, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is an $\mathcal{O}_X$-module.

Lemma: $\forall \theta \in \Theta$, $m \otimes n \in \mathcal{M} \otimes \mathcal{N}$

$$\theta(m \otimes n) = \theta(m) \otimes n + m \otimes \theta(n)$$

then $\mathcal{M} \otimes \mathcal{N}$ is a left $\mathcal{D}$-module.

CHECK this is well defined.

It works...

$\theta(mf \otimes n)$
$\theta(m \otimes fn)$

Pf:

$$\alpha : \Theta \longrightarrow \underline{\mathrm{End}}_{\mathbb{C}}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})$$

" $[\alpha(\theta), h\cdot] m \otimes n = (L_\theta h \cdot)$ "

$LHS := \alpha(\theta)(hm \otimes n) - h(\alpha(\theta)(m \otimes n))$
$= (\theta hm) \otimes n + hm \otimes \theta n - h[\theta m \otimes n + m \otimes \theta n]$
$= h\theta m \otimes n + L_\theta h m \otimes n + hm \otimes \theta n - h\theta m \otimes n - hm \otimes \theta n$
$= RHS.$

.....

Lemma: $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is a left $\mathcal{D}_X$-module with $(\theta f)(m) = f(\theta m) + \theta f(m)$.

RIGHT. $\mathrm{Mod}_{\mathcal{D}} \ni \mathcal{M}, \mathcal{N}$

$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$, $(m \otimes n)\theta = (m \otimes n\theta) + (m\theta) \otimes n$.

$\mathcal{M}od^{r}_{\mathcal{D}}$

${}_{\mathcal{D}}\mathcal{M}od \ni \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ $(\theta f)(m) = f(m)\theta - f(m\theta)$.

MIXED: $\mathcal{M} \in \mathcal{M}od_{\mathcal{D}}$, $\mathcal{N} \in {}_{\mathcal{D}}\mathcal{M}od$

$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ with $(m \otimes n)\theta = (m\theta) \otimes n - m \otimes \theta n$ is RIGHT $\mathcal{D}$-module

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{M})$ $(f\theta)(n) = f(\theta n) - f(n)\theta$ is RIGHT $\mathcal{D}$-module.

Properties:

$$\begin{aligned} {}_{\mathcal{D}}\mathcal{M}od &\longrightarrow \mathcal{M}od_{\mathcal{D}} \\ \mathcal{M} &\longmapsto \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \mathcal{H}om_{\mathcal{O}_X}(\Omega, \mathcal{N}) &\longleftarrow\!\shortmid \mathcal{N} \end{aligned}$$

Proof: Claim $\mathcal{M} \xrightarrow{\varphi} \mathcal{H}om_{\mathcal{O}}(\Omega, \Omega \otimes \mathcal{M})$ is an iso of left $\mathcal{D}$-module.

$m \longmapsto f_m : \omega \mapsto \omega \otimes m$.

it is clear that φ is an iso of $\mathcal{O}_X$-modules

$\theta \in \Theta$, $(\theta \cdot f_m)(\omega) = f_m(\omega\theta) - \theta f_m(\omega)$

$= \omega\theta \otimes m - \theta(\omega \otimes m)$

$= (\omega\theta) \otimes m - ((\omega\theta) \otimes m - \omega \otimes \theta m)$

$= \omega \otimes \theta m = f_{\theta m}(\omega)$

So $f_{\theta m} = \theta f_m$.

Adjoint operator

$\mathcal{M} \in {}_{\mathcal{D}}\mathrm{Mod}$, $\Omega \otimes \mathcal{M}$ is right module.

Proposition: $V \overset{\text{open}}{\subset} X$, $\omega \in \Omega(V)$ generator.

For any $P \in \mathcal{D}_X(V)$ there exists a unique $\widetilde{P}^{\omega} \in \mathcal{D}_X(V)$ such that

$\forall R \in \mathcal{O}_X(V)$, $\quad (\widetilde{P}^{\omega} R)\cdot\omega = (R\omega)\cdot P$

Moreover $P, Q \in \mathcal{D}_X(V)$ then $\quad \widetilde{(Q \circ P)}^{\omega} = \widetilde{P}^{\omega} \circ \widetilde{Q}^{\omega} \quad (\ast\ast)$

Pf: Locally $P = \sum_{\alpha} a_{\alpha}(z)\partial_z^{\alpha}$, $\omega = a\, dz_1 \wedge \dots \wedge dz_n$, a holo never vanishing.

$$(R\omega)P = (Ra\, dz_1 \wedge \dots \wedge dz_n)P = \frac{1}{a}\sum_{\alpha}(-\partial_z)^{\alpha}(a_{\alpha}Ra)\, a\, dz_1 \wedge \dots \wedge dz_n.$$

$$= \frac{1}{a}\sum_{\alpha}(-\partial_z)^{\alpha}(a_{\alpha}Ra)\omega.$$

$$\widetilde{P}^{\omega} := \frac{1}{a}\sum_{\alpha}(-\partial_z)^{\alpha}(a_{\alpha}a\,\cdot).$$

$$(\widetilde{Q}^{\omega} \circ \widetilde{P}^{\omega})(R)\omega = \widetilde{Q}^{\omega}(\underbrace{\widetilde{P}^{\omega}R}_{=S})\omega = (S\omega)Q = (\widetilde{P}^{\omega}R\omega)Q = (R\omega P)Q$$

$$= R\omega(P \circ Q).$$

Proposition: $\mathcal{M} \in {}_{\mathcal{D}}\mathrm{Mod}$, there is an explicit description of the $\mathcal{D}$-module structure on $\Omega\otimes\mathcal{M}$

i.e. $\quad (\omega \otimes m)P = \omega \otimes \widetilde{P}^{\omega} m$

Pf: Since $(\ast\ast)$ we can assume $P = \partial_{z_i}$, locally $\omega = a\, dz_1 \wedge \dots \wedge dz_n$.

$$(\omega \otimes m)\partial_{z_i} = (\omega\partial_{z_i}) \otimes m - \omega \otimes \partial_{z_i} m$$

$$= -(\partial_{z_i}a)dz_1 \wedge \dots \wedge dz_n \otimes m - a\, dz_1 \wedge \dots \wedge dz_n \otimes \partial_{z_i} m.$$

$$= a\, dz_1 \wedge \dots \wedge dz_n \otimes \left(-\partial_{z_i}(am)\right) = \omega \otimes \widetilde{\partial_{z_i}}(m) \quad \blacksquare$$

The De Rham system

$U \overset{open}{\subseteq} X \qquad \mathcal{D}^n|_U \longrightarrow \mathcal{D}|_U \longrightarrow \mathcal{O}_X \qquad$ exact $(*)$.

$$Q \longmapsto Q\cdot 1.$$

$$(P_i) \longmapsto \sum P_i \partial_{z_i}$$

- $\mathcal{O}_X$ is the $\mathcal{D}$-module associated with the system of diff eq. $\begin{cases} \partial_{z_1} u = 0 \\ \vdots \\ \partial_{z_n} u = 0 \end{cases}$ $\forall i = 1, \dots, n$.

- Is there a resolution in free $\mathcal{D}_X$-modules of $\mathcal{O}_X$?

KOSZUL COMPLEX: A ring, $M \in \mathrm{Mod}_A$, $e_1, \dots, e_p \in \mathbb{Z}^p$, $M^k = M \otimes \bigwedge^k \mathbb{Z}^p$

$$m \otimes e_{i_1 \dots i_k} \in M^k, \quad i_1 < \dots < i_k, \qquad m \otimes e_{i_1 \dots i_k} = m \otimes e_{i_1} \wedge \dots \wedge e_{i_k}$$

Def: Positive Koszul complex

$$d : M^k \longrightarrow M^{k+1}$$
$$m \otimes e_{i_1 \dots i_k} \longmapsto \sum \varphi_j(m) \otimes e_j \wedge e_{i_1} \dots e_{i_k}$$

$\Phi = (\varphi_1, \dots, \varphi_p)$ commuting endomorphism of M.

Rmk: $d \circ d = 0$.

Notation: $K^\bullet(\Phi, M)$

Def: Negative Koszul complex $K_\bullet(\Phi, M)$: $\delta_k : M^k \longrightarrow M^{k-1}$

$$m \otimes e_{i_1 \dots i_k} \longmapsto \sum_{j=1}^{k} (-1)^{j+1} \varphi_{i_j}(m) \otimes e_{i_1} \wedge \dots \widehat{e_{i_j}} \wedge \dots e_{i_k}.$$

RMK: $\delta \circ \delta$.

Proposition: If φ_1 is injective and $\varphi_j : \dfrac{M}{\langle \mathrm{Im}(\varphi_1) + \dots + \mathrm{Im}(\varphi_{j-1}) \rangle}$ is injective then,

$$H_\bullet(K_\bullet(\Phi, M)) = \begin{cases} 0 & \text{if } k \neq 0. \\ \dfrac{M}{\mathrm{Im}\varphi_1 + \dots + \mathrm{Im}\varphi_p} & \text{if } k = 0. \end{cases}$$

FACT: There exists $w^k : M^k \xrightarrow[\text{A-lin.}]{} M^{p-k}$ such that

$$w^k(m \otimes e_{i_1 \dots i_k}) = s\, m \otimes e_{j_1 \dots j_{p-k}}$$

where $s = \mathrm{sign}\begin{pmatrix} i_1 \dots i_k\, j_1 \dots j_{p-k} \\ 1 \longrightarrow p \end{pmatrix}$ and $\{i_1, \dots, i_k\} \cup \{j_1, \dots, j_{p-k}\} = \{1, \dots, p\}$

Moreover $d^{p-k} \circ w^k = (-1)^{k-1} w^{k-1} \circ \delta^k$

$(-1)^{\frac{k(k-1)}{2}} w^k : M^k \longrightarrow M^{p-k}$ gives an isomorphism of complexes.

$$K_\bullet(\Phi, M) \xrightarrow{\sim} K^\bullet(\Phi, M)[p].$$

In particular $\quad H_k(\Phi, \Pi) = H^{p-k}(\Phi, \Pi)$.

Universal Spencer and De Rham complexes.

Consider

Def: (Spencer complex) $\Theta^p := \Lambda^p \Theta$

$$SP_p := \mathcal{D} \otimes_{\mathcal{O}_X} \Theta^p$$

$$\delta^p : \underset{\overset{\parallel}{\mathcal{D} \otimes \Theta^p}}{SP_p} \longrightarrow SP_{p-1}$$

$$Q \otimes (\theta_1 \wedge \dots \wedge \theta_p) \longmapsto \sum_{i=1}^{p} (-1)^{i-1} (Q\theta_i) \otimes \theta_1 \wedge \dots \wedge \hat{\theta}_i \wedge \dots \wedge \theta_p$$

$$+ \sum_{i<j} (-1)^{i+j} Q \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \dots \wedge \hat{\theta}_i \wedge \dots \wedge \hat{\theta}_j \wedge \dots \wedge \theta_p.$$

Exercise $\delta \circ \delta = 0$.

Proposition: ~~[illegible]~~

The $\mathcal{D}_X$-linear morphism $\quad \begin{array}{ccc} \mathcal{D}_X & \longrightarrow & \mathcal{O}_X \\ Q & \longmapsto & Q \cdot 1 \end{array}$

induces a quasi-iso. $\quad SP_\bullet \simeq \mathcal{O}_X$.

Pf: $U \subseteq X$ coordinate chart

$$\delta(Q \otimes \partial_{z_{i_1}} \wedge \dots \wedge \partial_{z_{i_p}}) = \sum_{i=1}^{p} (-1)^{i-1} (Q\partial_{z_i}) \otimes \partial_{z_1} \wedge \dots \wedge \hat{\partial}_{z_i} \wedge \dots \wedge \partial_{z_p}.$$

Hence $Sp_\bullet|_U = K((\cdot\partial_{z_1}, \dots, \cdot\partial_{z_n}), \mathcal{D}|_U)$.

- ∂_{z_1} is injective on $\mathcal{D}|_U$.
- ∂_{z_j} $\circlearrowright$ $\mathcal{D} / (\mathcal{D}\partial_{z_1} + \dots + \mathcal{D}\partial_{z_{j-1}})$ injective.

Use the order filtration? directly.

We get $H_k(Sp|_U) = \begin{cases} 0 & k \neq 0 \\ \mathcal{D}|_U / (\mathcal{D}_U\partial_{z_1} + \dots + \mathcal{D}_U\partial_{z_n}) = \mathcal{O}_U \end{cases}$ $\blacksquare$