On cases where Litt's game is fair

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Abstract

A fair coin is flipped n times, and two finite sequences of heads and tails (words) A and B of the same length are given. Each time the word A appears in the sequence of coin flips, Alice gets a point, and each time the word B appears, Bob gets a point. Who is more likely to win? This puzzle is a slight extension of Litt's game [4] that recently set Twitter abuzz. We show that Litt's game is fair for any value of n and any two words that have the same auto-correlation structure by building up a bijection that exchanges Bob and Alice scores; the fact that the inter-correlations do not come into play in this case may come up as a surprise.

1 Introduction and main results

In [4], Litt came up with the following puzzle: a fair coin is tossed n times, Alice gets a point each time the sequence HH appears, while Bob scores a point each time the sequence HT appears (and these sequences may be overlapping). The game may result in a win for Alice, a win for Bob or a tie (the possibilities being exclusive). Who is more likely to win, Alice or Bob ? The perhaps surprising answer (if one judges by the Twitter poll) is that:

1. for any size $n \geqslant 3$, Bob is more likely to win than Alice,
2. for large n, it holds:

$$2 \cdot \big(\mathbb{P}_n(\text{Bob wins}) - \mathbb{P}_n(\text{Alice wins})\big) \sim \mathbb{P}_n(\text{Tie}) \sim \frac{1}{\sqrt{\pi n}}.$$

Several elements of proofs for the first fact quickly emerged on Twitter [8, 9, 11, 12], then more formal proofs of both facts appeared on the arXiv [1, 10]. Nica [5] recorded a YouTube video to introduce the problem to a wider audience. Zeilberger online journal [13] maintains a list of

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contributions and questions on the problem; among the proofs given, we would like to advertise an early probabilistic proof: in [8], Ramesh builds a "fair majorant" for the score difference of Alice and Bob that consists in a delayed simple random walk; the score difference of Alice and Bob is a deterministic function of this walk, at distance at most 1 below it, see right after the bibliography for more details. A rigorous proof of both facts 1 and 2 is then at an easy reach. Another by-product of his approach is that giving Alice an initial advantage of only one point reverses the statement of the first fact in a strong sense: for any size $n \geqslant 0$, Alice then has probability $>1/2$ of winning; in particular, Alice is more likely to win than Bob.

Considering a contest between HT and TH is also possible, but arguably less interesting: flipping the sequence of tosses, or reading the sequence in the reverse order exchanges Alice and Bob points, which results in a fair game. If you found this second contest boring, beware: the purpose of this short note is a vast generalization of it - the symmetry being slightly more hidden though...

For $\ell \geqslant 0$, a *word* of length ℓ is a sequence $A = (a_1, \ldots, a_\ell)$ in $\{0,1\}^\ell$. For $X_n := (\varepsilon_k)_{1 \leqslant k \leqslant n}$ a finite sequence in $\{0,1\}^n$, we denote by

$$N_A(X_n) := |\{\ell \leqslant k \leqslant n, (\varepsilon_{k-\ell+1}, \ldots, \varepsilon_k) = (a_1, \ldots, a_\ell)\}|$$

the number of occurrences of the word A in the sequence X_n. Assume now that X_n is an i.i.d. sequence with distribution $\mathcal{B}(1/2)$ the Bernoulli distribution with parameter $1/2$, and let A and B be two words of size ℓ. In the generalized Litt's game, Alice wins if $N_A(X_n) > N_B(X_n)$, Bob wins if $N_B(X_n) > N_A(X_n)$, and this is a tie otherwise. A key quantity encoding the intersections of A and B is the correlation of A and B, which may be represented as a subset of integers or simply as a number (the base 2 expansion of the subset):

$$[A|B] = \sum_{k \in \mathrm{Cor}(A,B)} 2^k \quad \text{where} \quad \mathrm{Cor}(A,B) = \{1 \leqslant k \leqslant \ell - 1,\ (a_{\ell-k+1}, \ldots, a_\ell) = (b_1, \ldots, b_k)\}.$$

To be more specific, we shall call *inter-correlation* the correlation of two distinct words (beware the order matters), and *auto-correlation* the correlation of a word with itself. Our main result in this note is that Litt's game is fair for words A and B with the same auto-correlation, regardless of their inter-correlation.

Theorem 1. *Let A, B two words of length ℓ such that $[A|A] = [B|B]$. Let $X_n = (\varepsilon_i)_{1 \leqslant i \leqslant n}$ be an i.i.d. sequence of $\mathcal{B}(1/2)$ random variables. Then for each $n \geqslant 1$, $(N_A(X_n), N_B(X_n))$ and $(N_B(X_n), N_A(X_n))$ have the same distribution. In particular, for any $n \geqslant 1$*

$$\mathbb{P}_n(\text{Bob wins}) = \mathbb{P}_n(\text{Alice wins}).$$

Let us illustrate this result with an example. Choose $A = \text{HHTHTH}$ and $B = \text{HTTTHH}$. These words have the same auto-correlation 2 hence Litt's game between A and B is fair despite the lack of symmetry between these words.

It is then possible to estimate the quantity in the last equality, or equivalently the probability of a tie $\mathbb{P}_n(\text{Tie})$, using a Local Central Limit theorem for sums of weakly dependent random variables.

We do not venture into this for the following reason: in a recent breakthrough announced online [6], Nica (together with Janson) pointed to a generic method based on Edgeworth expansions to tackle the case of words A and B with possibly distinct auto-correlations; the sketch of proof, together with some basic moment computations, hints to the following (yet still conjectural at the moment of writing) asymptotic estimates: for $A \neq B$, as n gets large, the following asymptotics:

$$\mathbb{P}_n(\text{Bob wins}) - \mathbb{P}_n(\text{Alice wins}) = \frac{[A|A]-[B|B]}{\sqrt{2^\ell + [A|A] + [B|B] - [A|B] - [B|A]}} \cdot \frac{1}{2\sqrt{\pi n}} + o\Big(\frac{1}{\sqrt{n}}\Big),$$

$$\mathbb{P}_n(\text{Tie}) = \frac{2^\ell}{\sqrt{2^\ell + [A|A] + [B|B] - [A|B] - [B|A]}} \cdot \frac{1}{2\sqrt{\pi n}} + o\Big(\frac{1}{\sqrt{n}}\Big)$$

hold as soon as the denominators on both RHS are non null (the four[1] pair of words giving a null denominator being associated with degenerated cases). In case $[A|A] = [B|B]$, Theorem 1 refines on the first formula by stating that the quantity is, in fact, identically null for all n.

A key feature of the first formula is that for $\mathbb{P}_n(\text{Bob wins}) - \mathbb{P}_n(\text{Alice wins})$ has the sign of $[A|A] - [B|B]$. In light of this observation together with some extensive numerical computations, we are lead to state the following conjecture for a fixed length n of the underlying word. We use Δ to denote the symmetric difference of two sets.

Conjecture 1. *For each*[2] $n \geqslant 2\ell - \max\{\mathrm{Cor}(A)\Delta\mathrm{Cor}(B)\}$,

$$\mathbb{P}_n(\text{Bob wins}) - \mathbb{P}_n(\text{Alice wins}) \textit{ has the sign of } [A|A] - [B|B],$$

meaning these quantities are either both positive, null, or negative.

Our main result, Theorem 1, answers the conjecture in case the two words have the same auto-correlation, $[A|A] = [B|B]$. This case would also be, according to our conjecture, the only case where Litt's game is fair. Last, the inequality $|[A|A] - [B|B]| \leqslant 2^\ell - 2$ holds for every pair of words A,B, hence the probability of a tie is asymptotically always larger than the absolute value of the difference of the win probabilities for Alice and Bob.

We should also point that the topic of pattern matching and overlaps has been the subject of many investigations, starting with the non-transitive *Penney Ante game* named after Penney [7] (and famously solved by Conway) in which one is asked to compute the probability that a word is the first to appear in between two words in a sequence of fair coins, see [2, 3] and the references therein.

2 Proof of Theorem 1

The auto-correlation and inter-correlation of two words are quantities that appear naturally when we look at the probability of one word appearing before another in a sequence of coin flips. The formal definition, which we repeat below, is the following.

[1] $(\mathrm{TH}^{\ell-1}, \mathrm{H}^{\ell-1}\mathrm{T})$ and its siblings, for which $N_A(X_n) - N_B(X_n)$ belongs to $\{-1, 0, 1\}$.

[2] for smaller values of n, the game is fair.

Definition 2. *Let A, B two words of length ℓ. We define the indices of inter-correlation of A and B by*

$$\mathrm{Cor}(A,B) = \{1 \leq k \leq \ell-1, (a_{\ell-k+1},\dots,a_\ell) = (b_1,\dots,b_k)\}.$$

We write $\mathrm{Cor}(A)$ *to denote* $\mathrm{Cor}(A,A)$*. We define the inter-correlation* $[A|B]$ *of* (A,B) *by*

$$[A|B] = \sum_{k \in \mathrm{Cor}(A,B)} 2^k$$

and the auto-correlation of A as $[A|A]$.

Let us make some remarks about this definition

- The number $[A|B]$ is not in general equal to $[B|A]$.
- For all A, B of length ℓ, $[A|B] \in [\![0; 2^\ell - 2]\!]$.
- For every words A, B, C, D of length ℓ, $[A|B] = [C|D]$ if and only if $\mathrm{Cor}(A,B) = \mathrm{Cor}(C,D)$.

Fix any two words A and B with length ℓ and the same auto-correlation. To prove that $(N_A(X_n), N_B(X_n))$ has the same law as $(N_B(X_n), N_A(X_n))$, we prove the existence of a bijection ϕ from $\{0,1\}^n$ to $\{0,1\}^n$ such that, for any sequence $X_n \in \{0,1\}^n$, $(N_A(X_n), N_B(X_n)) = (N_B(\phi(X_n)), N_A(\phi(X_n)))$.

We introduce some additional notation. If C, D are two words, CD will be the concatenation of C and D. Besides, for two words C, D of length ℓ and $m \in \mathrm{Cor}(C,D)$, we denote $C^m D$ the word of length $2\ell - m$ beginning by C and ending by D. For example, if $C = \underline{100101}$, $D = \overline{010111}$, we have $C^2 D = \underline{1001\overline{\mathbf{01}}}\overline{0111}$. We extend this notation to k words $C_1, \dots, C_k$ of length ℓ and $m_1, \dots, m_k \geqslant 1$ in the obvious way: if $m_i \in \mathrm{Cor}(C_i, C_{i+1})$ for each i, we define the word of length $k\ell - \sum m_i$

$$Y = {C_1}^{m_1} {C_2}^{m_2} {C_3}^{m_3} \dots {C_{k-1}}^{m_{k-1}} C_k. \tag{1}$$

Definition 3. *Let A, B be two words of length ℓ. We call* overlap *of A and B a word Y of the form* (1) *where $C_1, \dots, C_k \in \{A, B\}^k$. We denote by $\mathcal{E}(A,B)$ the set of all* overlaps *of A and B.*

Note that do not accept $m_i = 0$ in (1). In particular, the concatenation $Y = AB$ of A and B may not be in $\mathcal{E}(A,B)$. We notice also that, for $Y \in \mathcal{E}(A,B)$, the expression of Y in the form given by (1) may be not unique. We call the maximal decomposition of Y the one such that, if $Y = {C_1}^{m_1} {C_2}^{m_2} {C_3}^{m_3} \dots {C_{k-1}}^{m_{k-1}} C_k$, we have

$$N_A(Y) = |\{1 \leq i \leq k, C_i = A\}| \qquad N_B(Y) = |\{1 \leq i \leq k, C_i = B\}|.$$

We have the following decomposition of a word.

Definition 4. *Let Y a word. There exists a unique way to write Y in the form*

$$Y = X_0 E_1 X_1 E_2 \dots X_{k-1} E_k X_k \tag{2}$$

such that

- *the words $E_1, \dots, E_k$ belong to $\mathcal{E}(A, B)$.*
- *For all $i \in [\![0, k]\!]$, neither A nor B appears in X_i (word X_i may be empty).*
- $N_A(Y) = \sum_{i=1}^k N_A(E_i)$ *and* $N_B(Y) = \sum_{i=1}^k N_B(E_i)$.

We call pattern *of the word Y with respect to A and B the words $(E_1, \dots, E_k)$ which appears in (2) and write $\boldsymbol{Patt}_{A,B}(Y) = (E_1, E_2, \dots, E_k)$.*

Proposition 1. *Let A, B be two words with the same auto-correlation. Let $\phi : \mathcal{E}(A, B) \to \mathcal{E}(A, B)$ define in the following way. If $Y := {C_1}^{m_1}{C_2}^{m_2} \dots {C_{k-1}}^{m_{k-1}}C_k$ with $C_i \in \{A, B\}$, we set*

$$\phi(Y) = {\bar{C}_k}^{m_{k-1}}{\bar{C}_{k-1}}^{m_{k-2}} \dots {\bar{C}_2}^{m_1}\bar{C}_1$$

where $\bar{C}_i = A$ if $C_i = B$ and $\bar{C}_i = B$ if $C_i = A$. Then ϕ is well-defined, it is independent of the decomposition chosen for Y, it is an involution and $\phi(Y)$ has the same length as Y. Moreover, we have $(N_A(Y), N_B(Y)) = (N_B(\phi(Y)), N_A(\phi(Y)))$.

Proof. We first prove that $\phi(Y)$ is well defined *i.e.* if $m_i \in \mathrm{Cor}(C_i, C_{i+1})$, then $m_i \in \mathrm{Cor}(\bar{C}_{i+1}, \bar{C}_i)$. Recall that we assume that A, B have the same auto-correlation, *i.e.* $\mathrm{Cor}(A) = \mathrm{Cor}(B)$. We have two cases:

- Either $C_i = C_{i+1}$, for example $C_i = A$. Then $\bar{C}_i = \bar{C}_{i+1} = B$. Hence, we have $\mathrm{Cor}(C_i, C_{i+1}) = \mathrm{Cor}(A) = \mathrm{Cor}(B) = \mathrm{Cor}(\bar{C}_{i+1}, \bar{C}_i)$.
- Or $C_i \neq C_{i+1}$, for example $(C_i, C_{i+1}) = (A, B)$. Then $(\bar{C}_{i+1}, \bar{C}_i)$ is also equal to (A, B) and so $\mathrm{Cor}(C_i, C_{i+1}) = \mathrm{Cor}(\bar{C}_{i+1}, \bar{C}_i)$.

Note that ϕ does not depend of the decomposition chosen for Y. To justify this claim, it is enough to consider the case of words Y with two decompositions $Y = {C_1}^{m_1}{C_2}^{m_2}C_3$ and $Y = {C_1}^{m}C_3$. Note that $|Y| = 2\ell - m = 3\ell - m_1 - m_2$. Applying ϕ with the first decomposition we get $\phi(Y) = {\bar{C}_3}^{m_2}{\bar{C}_2}^{m_1}\bar{C}_1$ and since $|\phi(Y)| = |Y| < 2\ell$, necessarily, $\bar{C}_3$ and $\bar{C}_1$ overlap in the writing of $\phi(Y)$ and thus we also have $\phi(Y) = {\bar{C}_3}^{m}\bar{C}_1$. The fact that ϕ is an involution is clear.

We write now Y with its maximal decomposition. By construction, we directly get that $N_A(\phi(Y)) \geqslant N_B(Y)$ and $N_B(\phi(Y)) \geqslant N_A(Y)$. We claim that, by maximality of the decomposition, there is equality in these two inequalities. Indeed, consider the case of words Y whose maximal decomposition consist in two words: ${C_1}^{m}C_2$. Among those words Y, only the words where $C_1 = C_2$ have to be considered. If $Y_0 = {C_1}^{m}C_1$, it holds $\phi(Y_0) = {\bar{C}_1}^{m}\bar{C}_1$. Now, assume that $\phi(Y_0) = {\bar{C}_1}^{m_1}{C_3}^{m_2}\bar{C}_1$ for some $C_3 \in \{A, B\}$. The map ϕ being independent of the decomposition, we get $Y_0 = \phi(\phi(Y_0)) = {C_1}^{m_1}{\bar{C}_3}^{m_2}C_1$, which has a distinct count of the $\bar{C}_3$-word; this negates the fact that $Y_0 = {C_1}^{m}C_1$ is the maximal decomposition in the first place. □

The function ϕ defined in the previous proposition can be extended to a function on patterns. For $(E_1, \dots, E_k) \in \mathcal{E}(A, B)^k$, we set

$$\phi(E_1, \dots, E_k) = (\phi(E_k), \dots, \phi(E_1))$$

which defines an involution on $\mathcal{E}(A,B)^k$. Given a pattern $M = (E_1,\dots,E_k) \in \mathcal{E}(A,B)^k$ and $n \geqslant 1$, we define $L_M(n)$ as the number of words of length n with pattern M:

$$L_M(n) = |\{\text{word } Y,\ |Y| = n \text{ and } \mathbf{Patt}_{A,B}(Y) = M\}|.$$

To prove Theorem 1, it is sufficient to show that, for any pattern M and any $n \geqslant 1$, we have

$$L_M(n) = L_{\phi(M)}(n). \tag{3}$$

Indeed, if Y is a word such that $\mathbf{Patt}_{A,B}(Y) = M$ and Z is a word such that $\mathbf{Patt}_{A,B}(Z) = \phi(M)$, we have

$$(N_A(Y), N_B(Y)) = (\sum_{i=1}^{k} N_A(E_i), \sum_{i=1}^{k} N_B(E_i)) = (\sum_{i=1}^{k} N_B(\phi(E_i)), \sum_{i=1}^{k} N_A(\phi(E_i))) = (N_B(Z), N_A(Z)).$$

In view of Equality (3), we conclude that

$$\begin{aligned} P\big((N_A(X_n), N_B(X_n)) = (a,b)\big) &= \frac{1}{2^n} \sum_{\substack{\text{pattern } M \text{ s.t.} \\ (N_A(M), N_B(M)) = (a,b)}} L_M(n) = \frac{1}{2^n} \sum_{\substack{\text{pattern } M \text{ s.t.} \\ (N_A(M), N_B(M)) = (a,b)}} L_{\phi(M)}(n) \\ &= P\big((N_B(X_n), N_A(X_n)) = (a,b)\big). \end{aligned}$$

In order to establish (3), we prove the more slightly more precise result:

Lemma 1. *For any pattern $M = (E_1,\dots,E_k)$, for any $I = (i_0,\dots,i_k)$, set*

$$L^M(I) = |\{Y = X_0 E_1 X_1 \dots E_k X_k : \boldsymbol{Patt}_{A,B}(Y) = M; \forall j,\ |X_j| = i_j\}|.$$

Then

$$|L^M(I)| = |L^{\phi(M)}(I')|$$

where $I' = (i_k,\dots,i_0)$.

Proof. Let us remark that we have

$$L^M(I) = L^{E_1}(i_0,0) \left(\prod_{j=1}^{k-1} L^{(E_j,E_{j+1})}(0,i_j,0) \right) L^{E_k}(0,i_k)$$

and

$$L^{\phi(M)}(I') = L^{\phi(E_k)}(i_k,0) \left(\prod_{j=1}^{k-1} L^{(\phi(E_{j+1}),\phi(E_j))}(0,i_j,0) \right) L^{\phi(E_1)}(0,i_0).$$

Moreover, the value of $L^{(E_j,E_{j+1})}(0,i_j,0)$ only depends on i_j, on the word ending E_j and on the word beginning E_{j+1}. For example, if E_j ends with an A and E_{j+1} starts with a B, we have

$$L^{(E_j,E_{j+1})}(0,i_j,0) = L^{(A,B)}(0,i_j,0).$$

Now, if E_j ends with an A and E_{j+1} starts with a B, then $\phi(E_{j+1})$ ends with an A and $\phi(E_j)$ starts with an B. Thus, in this case, we directly get

$$L^{(\phi(E_{j+1}),\phi(E_j))}(0,i_j,0) = L^{(E_j,E_{j+1})}(0,i_j,0) = L^{(A,B)}(0,i_j,0).$$

The situation is more involved when E_j ends with the same word than E_{j+1} starts with, let say the word A. Then indeed

$$L^{(\phi(E_{j+1}),\phi(E_j))}(0,i_j,0) = L^{(B,B)}(0,i_j,0) \text{ while } L^{(E_j,E_{j+1})}(0,i_j,0) = L^{(A,A)}(0,i_j,0).$$

Besides, if E_1 starts with A, then $\phi(E_1)$ ends with a B, and we have

$$L^{\phi(E_1)}(0,i_0) = L^{(B)}(0,i_0) \text{ while } L^{E_1}(i_0,0) = L^{(A)}(i_0,0),$$

and a similar assertion holds for $L^{E_k}(0,i_k)$ and $L^{\phi(E_k)}(i_k,0)$. Combining all these remarks, we see that Lemma 1 will be proved as soon as we establish the following equality: for each $i \geqslant 0$,

$$L^{(A,A)}(0,i,0) = L^{(B,B)}(0,i,0) \qquad \text{and} \qquad L^{(B)}(0,i) = L^{(A)}(i,0). \tag{4}$$

We prove (4) by induction on i. The proposition clearly holds for $i=0$. Assume it holds for $k \leq i-1$. Thus, for any pattern M, if $I=(i_0,\ldots,i_k)$ with $i_j < i$ for all j, we get

$$|L^M(I)| = |L^{\phi(M)}(I')|.$$

Let us now prove that $L^{(B)}(0,i) = L^{(A)}(i,0)$. Writing $|I| = \sum_{j=0}^k i_j$ if $I=(i_0,\ldots,i_k)$ and $|M| = \sum_{i=1}^k |E_i|$ for the length of the pattern $M=(E_1,\ldots,E_k)$, we find that

$$\begin{aligned} L^{(A)}(i,0) = \; &= |\{\text{words } XA : \; |X|=i \text{ and } \mathbf{Patt}_{A,B}(XA)=A\}| \\ &= 2^i - \sum_{k\geqslant 1} \sum_{\substack{M=(E_1,\ldots,E_k)\neq(A)\\ E_k \text{ ends with } A}} \sum_{\substack{I=(i_0,\ldots,i_{k-1},0)\\ |I|+|M|=i+|A|}} |L^M(I)| \\ &= 2^i - \sum_{k\geqslant 1} \sum_{\substack{M=(E_1,\ldots,E_k)\neq(A)\\ E_k \text{ ends with } A}} \sum_{\substack{I=(i_0,\ldots,i_{k-1},0)\\ |I|+|M|=i+|A|}} |L^{\phi(M)}(I')|. \end{aligned}$$

At this point, we use the recurrence assumption noticing that $|I| < i$ since $|M| > |A|$. Recalling that if M ends with A, then $\phi(M)$ starts with B, we see that the last line is also equal to

$$2^i - \sum_{k\geqslant 1} \sum_{\substack{M=(E_1,\ldots,E_k)\neq(B)\\ E_1 \text{ starts with } B}} \sum_{\substack{I=(0,i_1,\ldots,i_k)\\ |I|+|M|=i+|B|}} |L^M(I)| \; = \; L^{(B)}(0,i).$$

The equality $L^{(A,A)}(0,i,0) = L^{(B,B)}(0,i,0)$ is proved with a similar argument. $\square$

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For the ease of reference, we reproduce here[3] *verbatim* the tweets [8] with the permission of their author Sridhar Ramesh.

[3]"Consider a random walk in which one takes equally likely steps of one unit up or one unit down, but with different distributions of speeds. (E.g., maybe up steps take one hour, while down steps have probability 1/2 of taking 2 hours, 1/4 of taking 3 hours, 1/8 of 4 hours, etc). The time it takes to return to the origin is independent of whether the first step is up and last step is down or vice versa, as doing the same steps in reverse order has the same probability. Thus, for any fixed walk time, the last step away from the origin begun before the time limit is equally likely to be up or down. Thus, at the end when "the buzzer goes off", one is equally likely to be above or below the origin (possibly in the middle of an uncompleted step). Applied to our game, with HH as a step up in one unit of time and HT^{n}H as a step down over $n+1$ units of time, this says we are equally likely to end above the origin (Alice wins or we are in the middle of an HT^{n}H step which has tied the game) or below it (Bob wins). Since it is indeed possible to end in the middle of a game-tying HT^{n}H step (e.g., if the game consists of HHT followed by all T's), Alice is less likely to win than Bob. QED. The salient difference is that the "buzzer" can cut off HT^{n}H in the middle (after awarding Bob a game-tying point but before returning to H), which it cannot do for HH. The random walk framing perhaps allows some ready generalization to other interesting problems."