Griffiths heights of pencils of hypersurfaces and geometric invariant theory

Thomas Mordant

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1. Preliminary: Variations of Hodge structures, Griffiths heights, and Deligne extensions

2. Griffiths heights of pencils of projective hypersurfaces and alternating sums of Griffiths heights

3. Pencils of projective hypersurfaces and geometric invariant theory

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I. Preliminary: Variations of Hodge structures, Griffiths heights, and Deligne extensions

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If n is a non-negative integer, an (integral) variation of Hodge structures of weight n on an analytic manifold S is the data $\mathbb{V} = (V_{\mathbb{Z}}, \mathcal{F}^{\bullet})$, where $V_{\mathbb{Z}}$ is an integral local system, namely a locally trivial sheaf in free $\mathbb{Z}$-modules over S, and

$$\mathcal{F}^{\bullet} : \mathcal{V} = \mathcal{F}^0 \supset ... \supset \mathcal{F}^n \supset \mathcal{F}^{n+1} = 0$$

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1. For every pair (p, q) of non-negative integers such that $p + q = n$:

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In particular, for every pair (p, q) of non-negative integers such that $p + q = n$, we can define an $\mathbb{R}$-analytic vector sub-bundle of $\mathcal{V}$ by $\mathcal{H}^{p,q} := \mathcal{F}^p \cap u(\mathcal{F}^q)$, and we get the ($\mathbb{R}$-analytic) Hodge decomposition of the vector bundle $\mathcal{V}$:

$$\mathcal{V} = \bigoplus_{p,q \geq 0,} \mathcal{H}^{p,q}.$$

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To such a variation of Hodge structures, or more generally to a vector bundle $\mathcal{V}$ endowed with a filtration by vector sub-bundles $\mathcal{F}^{\bullet \geq 0}$, can be attached its Griffiths line bundle:

$$\mathcal{GK}_S(\mathcal{V}, \mathcal{F}^\bullet) := \bigotimes_{i=1}^{n} \det \mathcal{F}^i \simeq \bigotimes_{r=0}^{n} (\det \mathcal{F}^r / \mathcal{F}^{r+1})^{\otimes r}.$$

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Let $\mathbb{V}$ be a variation of Hodge structures of weight n. A polarization is a bilinear form Q on $V_{\mathbb{Z}}$ which is symmetric (resp. antisymmetric) if n is even (resp. odd), such that, denoting also by Q the induced form on $\mathcal{V}$, for every p, the orthogonal of $\mathcal{F}^p$ is $\mathcal{F}^{n-p+1}$, and for every (p, q) with $p + q = n$, and for every non-zero v in $\mathcal{H}^{p,q}$, $Q(i^{p-q}v, u(v))$ is a positive real number.

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Theorem (Griffiths, 1970): *The Griffiths height of a polarized variation of Hodge structures is non-negative, and vanishes if and only if for every p, the sub-bundle $\mathcal{F}^p$ of $\mathcal{V}$ is flat relatively to the connection ∇.*

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FACT: *If T_x is the local monodromy of the vector bundle with connection $(\mathcal{V}, \nabla)_{|\mathring{C}}$ on a neigborhood of x where $\mathcal{V}$ is trivialized, the following equality holds, up to conjugation:*

$$T_x = e^{-2i\pi \mathrm{Res}_x(\nabla)}.$$

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THEOREM (Deligne): *For every choice of logarithms* $(\log(\alpha_{x,j}))_{x,j}$, *there is a unique (up to isomorphism) vector bundle with logarithmic connection* $(\overline{\mathcal{V}}, \overline{\nabla})$ *on* C, *whose restriction to* $\mathring{C}$ *is isomorphic to* $(\mathcal{V}, \nabla)$, *such that for every* x *in* Δ, *the eigenvalues of the residue* $\mathrm{Res}_x(\overline{\nabla})$ *are* $(-\frac{1}{2i\pi}\log(\alpha_{x,j}))_j$.

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Example: if for every x, T_x is unipotent, namely $T_x - \mathrm{Id}$ is nilpotent, then it is natural to choose $\log(\alpha_{x,j}) := 0$ for every x, j, so that the residues are nilpotent. This defines the canonical Deligne extension.

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In general, there are two natural choices: taking the only logarithms whose imaginary parts are in $[0, 2\pi[$ (resp. $]-2\pi, 0]$). The real parts of the eigenvalues of the residues are then in $]-1, 0]$ (resp. $[0, 1[$). This defines the upper (resp. lower) Deligne extension $(\overline{\mathcal{V}}_+, \overline{\nabla}_+)$ (resp. $(\overline{\mathcal{V}}_-, \overline{\nabla}_-)$).

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Let $(\mathcal{V}, \nabla)$ be a vector bundle with connection on $\mathring{C}$, and for every x in Δ, T_x the local monodromy of $(\mathcal{V}, \nabla)$ at x and $(\alpha_{x,j})_j$ its eigenvalues.

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Observe that if for every x, the monodromy T_x is quasi-unipotent with order a multiple of $r \geq 1$, namely if $T_x^r - \mathrm{Id}$ is nilpotent, then the eigenvalues of the residues of the upper (resp. lower) Deligne extension can be written as $-\frac{k}{r}$ (resp. $\frac{k}{r}$) where k is an integer such that $0 \leq k < r$.

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FACT (Griffiths, Schmid): *The sub-bundles $\mathcal{F}^p$ of $\mathcal{V}$ on $\mathring{C}$ can be extended into sub-bundles $\overline{\mathcal{F}^p}_{\pm}$ of the Deligne extension $\overline{\mathcal{V}}_{\pm}$.*

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Therefore we can extend the filtration $\mathcal{F}^\bullet$ of $\mathcal{V}$ into a filtration $\overline{\mathcal{F}}^\bullet_\pm$ of the Deligne extension $\overline{\mathcal{V}}_\pm$, and define the upper and lower Griffiths-Kato heights:

$$\mathrm{ht}_{GK,\pm}(\mathbb{V}) := \mathrm{ht}_{GK}(\overline{\mathcal{V}}_\pm, \overline{\mathcal{F}}^\bullet_\pm).$$

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THEOREM (Peters, 1984): *If the variation of Hodge structures $\mathbb{V}$ is polarized, then the height $\mathrm{ht}_{GK,+}(\mathbb{V})$ is non-negative, and it vanishes if and only if the $\mathcal{F}^p$ are flat for the connection ∇ and if the local monodromy is unipotent.*

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Let C' be a connected smooth projective complex curve and $\sigma : C' \to C$ a finite morphism such that for every x' in C' with image x in Δ, the morphism σ has an expression in local coordinates of the form:

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$$\mathrm{ht}_{GK,stab}(\mathbb{V}) := \frac{1}{\deg(\sigma)} \mathrm{ht}_{GK,+}(\sigma^*\mathbb{V}).$$

It does not depend on the choice of C' and σ.

The height $\mathrm{ht}_{GK,stab}(\mathbb{V})$

FACT (Borel, Schmid): *For every x in Δ, the local monodromy T_x at x of the vector bundle with connection $(\mathcal{V}, \nabla)$ defined by the variation of Hodge structures $\mathbb{V}$ is quasi-unipotent.*

For every x in Δ, let $r_x \geq 1$ be an integer such that $T_x^{r_x} - \mathrm{Id}$ is nilpotent.

Let C' be a connected smooth projective complex curve and $\sigma : C' \to C$ a finite morphism such that for every x' in C' with image x in Δ, the morphism σ has an expression in local coordinates of the form:

$$\sigma : t' \mapsto t'^{s_{x'}},$$

where $s_{x'} \geq 1$ is a multiple of r_x. In particular, the local monodromy of $(\sigma^*\mathcal{V}, \sigma^*\nabla)$ at every point of $\sigma^{-1}(\Delta)$ is unipotent. We define the stable Griffiths-Kato height of $\mathbb{V}$:

$$\mathrm{ht}_{GK,stab}(\mathbb{V}) := \frac{1}{\deg(\sigma)} \mathrm{ht}_{GK,+}(\sigma^*\mathbb{V}).$$

It does not depend on the choice of C' and σ.

The three heights satisfy the following inequalities:

$$\mathrm{ht}_{GK,-}(\mathbb{V}) \leq \mathrm{ht}_{GK,stab}(\mathbb{V}) \leq \mathrm{ht}_{GK,+}(\mathbb{V}),$$

and equalities hold if the local monodromy of $(\mathcal{V}, \nabla)$ at every point of Δ is unipotent.

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Theorem (Picard-Lefschetz): *For every x in Δ, if N is even, the local monodromy at x of the variation of Hodge structures $\mathbb{H}^{N-1}(H_{\mathring{C}}/\mathring{C})$ is unipotent; and if N is odd, its only eigenvalues are 1 and -1 and the multiplicity of -1 is the cardinal $|\Sigma_x|$.*

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Example: ordinary double points and Picard-Lefschetz theorem II

Theorem (Eriksson, Freixas, Mourougane): *If N is odd, the eigenvalue -1 "only appears in $\mathcal{F}^{(N-1)/2}/\mathcal{F}^{(N-1)/2+1}$".*

Consequently, if N is even:

$$\mathrm{ht}_{GK,+}(\mathbb{H}^{N-1}(H_{\mathring{C}}/\mathring{C})) = \mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}(H_{\mathring{C}}/\mathring{C})) = \mathrm{ht}_{GK,-}(\mathbb{H}^{N-1}(H_{\mathring{C}}/\mathring{C}))$$

and if N is odd:

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II. Griffiths heights of pencils of projective hypersurfaces and alternating sums of Griffiths heights

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The intersection-theoretic height of H is defined by the rational number:

$$\begin{aligned}\mathrm{ht}_{int}(H/C) &:= \int_{\mathbb{P}(E)} c_1(\mathcal{O}_E(1))^N \cap [H] + dN\mu(E)\\ &= (-1)^N (N+1)^{-N} \int_{\mathbb{P}(E)} c_1(\omega_{\mathbb{P}(E)/C})^N \cap [H].\end{aligned}$$

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It depends only on H as a subscheme of the projective bundle $\mathbb{P} := \mathbb{P}(E)$, and not on the actual choice of a vector bundle E such that $\mathbb{P} \simeq \mathbb{P}(E)$.

PPH Theorem

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Let C be a connected smooth projective complex curve with generic point η, E a vector bundle of rank $N+1$ on C, and $H \subset \mathbb{P}(E)$ an horizontal hypersurface of relative degree d, smooth over $\mathbb{C}$, such that $\pi_{|H} : H \to C$ has a finite set Σ of critical points, all of which are non-degenerate.

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$$|\Sigma| = (N+1)(d-1)^N \operatorname{ht}_{int}(H/C)$$

and for $\varepsilon \in \{+, -, stab\}$, we have:

$$\operatorname{ht}_{GK,\varepsilon}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_\varepsilon(d,N) \operatorname{ht}_{int}(H/C),$$

where, when N is odd:

$$F_{stab}(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N - d^2 - 2dN - 2) + 2(d^2-1)\right],$$

$$F_\pm(d,N) := F_{stab}(d,N) \pm \frac{(N+1)(N-1)(d-1)^N}{4},$$

and when N is even:

$$F_\pm(d,N) = F_{stab}(d,N) := \frac{N+1}{24d^2}\left[(d-1)^N(d^2N + 2d^2 - 2dN - 2) - 2(d^2-1)\right].$$

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We have $\mathrm{ht}_{int}(H/C) \geq 0$ if $d \geq 2$, and $F_{stab}(d,N) \geq 0$ and $F_+(d,N) \geq 0$ as predicted by Peters' theorem.

But $F_-(d,N)$ can be negative!

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Their work leads to consider, when the smooth fibers of the morphism g are Calabi-Yau manifolds, the alternating sum:

$$\sum_{k=0}^{2n}(-1)^{k-1}\,\mathrm{ht}_{GK,-}(\mathbb{H}^k(Y_\eta/C_\eta)) = \sum_{0\leq p\leq n}(-1)^{p-1}p\,\deg\det R^\bullet g_*\omega^p_{Y/C}.$$

In this case, it is the degree of the BCOV line bundle, whose metric properties are studied by Eriksson, Freixas and Mourougane.

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This alternating sum remains relevant for the study of pencils $g: Y \to C$ of arbitrary projective varieties!

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If D is a divisor with normal crossings, we define the vector bundle of logarithmic differential forms $\Omega^1_Y(\log D)$: on a neighborhood of a point where D is defined by the equation $(y_1^{m_1}...y_k^{m_k} = 0)$, this bundle is generated by the family:

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If g is a morphism from Y to a smooth curve C, which is smooth over the complement of a finite subset Δ, and such that $Y_\Delta := g^{-1}(\Delta)$ is a divisor with normal crossings, we define a vector bundle of rank $n-1$ on Y by:

$$\omega^1_{Y/C} := \Omega^1_Y(\log Y_\Delta)/g^* \Omega^1_C(\log \Delta).$$

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Then we have:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \int_Y \rho_{N-1}(\omega^{1\vee}_{Y/C}) \frac{\mathrm{Td}([T_g])}{\mathrm{Td}(\omega^{1\vee}_{Y/C})}$$

where $[T_g] := [T_{Y/\mathbb{C}}] - g^*[T_{C/\mathbb{C}}] \in K^0(Y)$, *and where* Td *is the Todd genus and* ρ_{N-1} *is the characteristic class:*

$$\rho_{N-1} := c_{N-2} - \frac{N-1}{2} c_{N-1} + \frac{1}{12} c_1 c_{N-1}.$$

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- the Grothendieck-Riemann-Roch theorem:

$$\deg_C \det R^\bullet g_* \mathcal{F} = \int_Y \mathrm{ch}(\mathcal{F}) \,\mathrm{Td}([T_g]);$$

“Ingredients” for the proof of the main theorem

- Steenbrink's theory: identification of the lower Deligne extension of $\mathbb{H}^n(Y_\eta/C_\eta)$ (resp. of $\mathrm{gr}_F^p \mathbb{H}^{p+q}(Y_\eta/C_\eta)$) with $R^n g_*(\omega^\bullet_{Y/C})$ (resp. $R^q g_*(\omega^p_{Y/C})$), where $\omega^p_{Y/C} := \Lambda^p \omega^1_{Y/C}$, and $\omega^\bullet_{Y/C}$ denotes the logarithmic variant of the relative de Rham complex; establishes the equality:

$$\sum_{k=0}^{2n} (-1)^{k-1}\, \mathrm{ht}_{GK,-}(\mathbb{H}^k(Y_\eta/C_\eta)) = \sum_{0\le p\le n} (-1)^{p-1} p \deg\det R^\bullet g_* \omega^p_{Y/C};$$

- the Grothendieck-Riemann-Roch theorem:

$$\deg_C \det R^\bullet g_* \mathcal{F} = \int_Y \mathrm{ch}(\mathcal{F})\,\mathrm{Td}([T_g]);$$

- an equality of characteristic classes:

$$\sum_{p=0}^{N-1} (-1)^{p-1} p\, \mathrm{ch}(\Lambda^p E^\vee)\,\mathrm{Td}(E) = \rho_{N-1}(E) \mod \mathrm{CH}^{\ge N+1}$$

for every vector bundle E of rank $N-1$ (already in [BCOV]).

Main theorem II – Notation

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With the notation of the main theorem, let us write the divisor Y_Δ as follows:

$$Y_\Delta = \sum_{i \in I} m_i D_i,$$

where I is a finite set, equipped with a total order $\prec$, and where $(m_i)_i \in (\mathbb{N}^*)^I$ and the D_i are pairwise distinct non-singular connected divisors of Y.

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The set I can be written as the disjoint union:

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where, for every $x \in \Delta$,

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Finally let:

$$\mathring{D}_i := D_i \setminus D_i \cap D^2 \quad \text{and} \quad \mathring{D}_{ij} := D_{ij} \setminus D_{ij} \cap D^3.$$

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Under the hypotheses of the main theorem, the following equality holds:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,-}(\mathbb{H}^n(Y_\eta/C_\eta)) = \frac{1}{12}\int_Y c_1(\omega_{Y/C}^{1\vee}) c_{N-1}(\omega_{Y/C}^{1\vee}) + \sum_{x\in\Delta} \alpha_x,$$

where for every x in Δ, $\alpha_x \in \mathbb{Q}$ is defined by:

$$\alpha_x = \frac{N-1}{4}\sum_{i\in I_x}(m_i - 1)\chi_{\mathrm{top}}(\mathring{D}_i) + \frac{1}{12}\sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}} \left(3 - \frac{m_i}{m_j} - \frac{m_j}{m_i}\right)\chi_{\mathrm{top}}(\mathring{D}_{ij})$$

where χ_{top} denotes the topological Euler characteristic.

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$$= \frac{1}{12}\sum_{i\in I_x}\left[3(N-1)(m_i-1)\chi_{\mathrm{top}}(\mathring{D}_i) + \int_{D_i} c_1(\mathcal{N}_i)c_{N-2}(\Omega^{1\vee}_{D_i}(\log D_i\cap D^2))\right]$$

$$+\frac{1}{4}\sum_{\substack{(i,j)\in I_x^2,\\ i\prec j}}\chi_{\mathrm{top}}(\mathring{D}_{ij}) \in (1/12)\mathbb{Z},$$

where, for every i, $\mathcal{N}_i$ denotes the normal bundle of the divisor D_i in Y.

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If $E=\bigcup_{i\in I}E_i$ is a divisor with strict normal crossings in a smooth manifold X, we have:

$$c(\Omega^1_X(\log E)) = \sum_{J\subset I} i_{E_J*}c(\Omega^1_{E_J}),$$

where $i_{E_J}: E_J := \bigcap_{j\in J}E_j \hookrightarrow X$.

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Let C be a connected smooth projective complex curve with generic point η, H a N-dimensional connected smooth projective complex manifold, and:

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Then for $\varepsilon \in \{+,-\}$, the following equality holds:

$$\sum_{n=0}^{2(N-1)} (-1)^{n-1} \mathrm{ht}_{GK,\varepsilon}(\mathbb{H}^n(H_\eta/C_\eta)) = \frac{1}{12}\int_H c_1([\Omega^1_{H/C}]^\vee) c_{N-1}([\Omega^1_{H/C}]^\vee) + u_N^\varepsilon\,|\Sigma|,$$

where
$$(u_N^+, u_N^-) := \begin{cases} \big(-(7N-9)/24, (5N-3)/24\big) & \text{if } N \text{ is odd} \\ \big(N/24, N/24\big) & \text{if } N \text{ is even.} \end{cases}$$

“Ingredients” for the formula on pencils with ordinary double points

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- The construction of the blow-up

$$\nu : \widetilde{H} \longrightarrow H$$

of Σ in H and of the composition

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so that the following equality of divisors of $\widetilde{H}$ holds:

$$\widetilde{H}_{\Delta} = 2E + W,$$

where E and W are smooth divisors with normal intersection;

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$$\mathrm{ht}_{GK,+}(\mathbb{H}^n(H_\eta/C_\eta)) = \mathrm{ht}_{GK,-}(\mathbb{H}^n(H_\eta/C_\eta)) + \delta^{n,N-1}\eta_N \frac{N-1}{2}|\Sigma|,$$

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If moreover L is ample relatively to π, then for $\varepsilon \in \{+,-\}$, we have:

$$\begin{aligned} &\mathrm{ht}_{GK,\varepsilon}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) \\ &\quad = \mathrm{ht}_{GK}(\mathbb{H}^{N-1}(X/C)) + \mathrm{ht}_{GK}(\mathbb{H}^{N+1}(X/C)) - \mathrm{ht}_{GK}(\mathbb{H}^{N}(X/C)) \\ &\quad + \frac{1}{12}\int_X \left[(1-c_1(L))^{-1} c_1(\Omega^1_{X/C}) c(\Omega^1_{X/C}) - c_1(L) c_N(\Omega^1_{X/C})\right] + v_N^\varepsilon\,|\Sigma|, \end{aligned}$$

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Applications of the PH theorem: the PPH theorem

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When the Griffiths-Kato bundles $\mathcal{GK}_C(\mathbb{H}^n(X/C))$, $0 \leq n \leq 2N$ are trivial on C, the PH theorem gives the following formula:

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We can notably apply this formula when X is a projective bundle $\mathbb{P}(E)$ where E is a vector bundle of rank $N+1$ on C, π is the projection, and H is an horizontal hypersurface of relative degree d of $\mathbb{P}(E)$.

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Using computations of characteristic classes, formula (1) applied in this situation implies the PPH theorem.

III. Pencils of projective hypersurfaces and geometric invariant theory

Motivation and reminders on geometric invariant theory

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Let us recall some notions of geometric invariant theory. Let k be an algebraically closed field, e and v two positive integers, and

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Thanks to geometric invariant theory, we get the open subset $\mathbb{P}^{v-1}_{k,ss}$ of $\mathbb{P}^{v-1}_k$ defined by the semistable points under the action of $\rho_{|\mathrm{SL}_{e,k}}$, and a morphism:

$$q : \mathbb{P}^{v-1}_{k,ss} \longrightarrow M(\rho) := \mathbb{P}^{v-1}_k // \mathrm{SL}_{e,k},$$

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Moreover $M(\rho)$ is projective and endowed with a natural ample $\mathbb{Q}$-line bundle L such that:

$$q^* L \simeq \mathcal{O}_{\mathbb{P}^{v-1}_k}(1)_{|\mathbb{P}^{v-1}_{k,ss}}.$$

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If moreover the representation ρ is homogeneous, namely maps $\mathbb{G}_m \mathrm{Id}_e$ into $\mathbb{G}_m \mathrm{Id}_v$, then the height ht_{GIT} is invariant under the action of $\mathrm{GL}_e(k(C))$ on $\mathbb{P}^{v-1}(k(C))$ through ρ.

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Let us assume that the base field is $k = \mathbb{C}$, fix N and d two positive integers, define $e := N + 1$ and $v := \binom{N+d}{N}$, and consider the homogeneous representation:

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We interpret $\mathbb{P}^{v-1}_{\mathbb{C}}$ as the space of hypersurfaces of degree d in $\mathbb{P}^N_{\mathbb{C}}$, and the action of ρ as the natural action of GL_{N+1} on homogeneous polynomials of degree d with $N + 1$ indeterminates.

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For every vector bundle E of rank $N+1$ on C whose generic fiber is trivialised, for every horizontal hypersurface H of degree d in $\mathbb{P}(E)$ whose generic fiber in $\mathbb{P}^N_{\mathbb{C}(C)}$ is defined by an homogeneous polynomial F which is semistable relatively to the action of $\rho_{|\mathrm{SL}_{N+1}}$, the following inequality holds:

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Moreover equality holds in (2) if and only if all the fibers H_x, $x \in C$, seen as hypersurfaces of degree d of the projective spaces $\mathbb{P}(E_x) \simeq \mathbb{P}^N_{\mathbb{C}}$, are semistable relatively to the action of $\rho_{|\mathrm{SL}_{N+1}}$.

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Applying the PPH theorem, one obtains that *for every smooth hypersurface H_η of degree d in $\mathbb{P}^N_{\mathbb{C}(C)}$ defined by an homogeneous form F in $\mathbb{C}(C)[X_0, \ldots, X_N]_d$, the following equality holds:*

$$\mathrm{ht}_{GK,stab}(\mathbb{H}^{N-1}(H_\eta/C_\eta)) = F_{stab}(d,N)\,\mathrm{ht}_{GIT}([F])$$

when $N \geq 2$, $d \geq 3$, and the hypersurface H_η of $\mathbb{P}^N_{\mathbb{C}(C)}$ admits for model an horizontal hypersurface $H \subset \mathbb{P}(E)$ in the projective bundle $\mathbb{P}(E)$ associated to a vector bundle E of rank $N+1$ on C, satisfying the hypotheses of the PPH theorem.

Conclusion: arithmetic analogues ?

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We consider an homogeneous polynomial $P \in \mathbb{Z}[X_0, ..., X_N]_d$, such that

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and we consider the smooth hypersurface

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- Use of the p-adic comparison theorems,
- Understanding of the asymptotics of the analytic torsions associated to families of smooth complex hypersurfaces when they acquire singularities (for which I intend to collaborate with Gerard Freixas).