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Arc spaces and vertex algebras

– Monograph –

February 1, 2023

Springer Nature

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Part I
Vertex algebras: definitions and examples

Write an introduction.

Chapter 1
Jet schemes and arc spaces

This chapter is devoted to the study of jet schemes and arc schemes/spaces associated with a scheme X defined over the field of complex numbers.

Roughly speaking, an arc on a scheme X is a formal path on X, that is, a morphism from the formal disc $D = \operatorname{Spec} \mathbb{C}[[t]]$ to X. Jets on X are obtained by truncation of such paths at a finite order.

The study of singularities via the space of arcs was initiated by Nash [163]. He conjectured a tight relationship between the geometry of the arc space and the singularities of X, see Ishii and Kollár [105]. More precisely, he suggested that the study of the images by the truncation morphisms of the space of arcs should give information about the fibers over the singular points in a resolution of singularities of X. The work of Mustaţă [161] supports these predictions; for example, rational singularities of a locally complete intersection variety can be detected by the irreducibility of all its jet schemes. The space of arcs also plays a key role in motivic integration, as the domain over which functions are integrated. We refer the reader to the the recent book by Chambert-Loir, Nicaise and Sebag [46], and the references given there, for more about this topic.

It turns out that arc spaces are also of great importance in the theory of vertex algebras. One of the main reasons is that the structure sheaf of the arc scheme over a scheme X has the structure of a sheaf of commutative vertex algebras ([36, 77]), see Chap. 2. Moreover, any vertex algebra is canonical filtered, and the associated graded space is a quotient of the space of the functions on the arc space of the associated scheme of the vertex algebra, see Chap. 4 for more details. The space of the functions on an arc space will be thus the most important example of commutative vertex algebras.

The chapter is structured as follows. Section 1.1 is about the jet construction of differential algebras. Section 1.2, Section 1.3, and Section 1.4 concerns first properties and examples related to arc schemes. We study in Section 1.5 geometrical properties of arc spaces. In the context of vertex algebras one needs also to consider the *loop space* $\mathscr{L}X$ of an affine scheme X. This is the topic of Section 1.6. Arc spaces of group schemes acting on a scheme is discussed in Section 1.7.

Throughout this chapter, the ground field will be the field $\mathbb{C}$ of complex numbers. We shall work with the Zariski topology, and by *variety* we mean a reduced and separated scheme of finite type over $\mathbb{C}$.

1.1 Jet construction of differential algebras

Definition 1.1 In this book, a *differential algebra* is a commutative $\mathbb{C}$-algebra A equipped with a derivation ∂, that is, a homomorphism of vector spaces $\partial\colon A \to A$ satisfying the Leibniz product rule $\partial(ab) = \partial(a)b + a\partial(b)$ for every $a, b \in A$.

A *differential algebra homomorphism* $f\colon A \to A'$ between two differential algebras (A, ∂) and (A', ∂') is a $\mathbb{C}$-algebra homomorphism which commutes with the derivations, that is, $\partial'(f(a)) = f(\partial(a))$ for every $a \in A$.

Lemma 1.1 *For any finitely generated unital commutative algebra R, there exists an unique (up to an isomorphism) differential algebra $\mathscr{J}_\infty R$ such that*

$$\mathrm{Hom}_{Dif.\,Alg}(\mathscr{J}_\infty R, A) \cong \mathrm{Hom}_{Alg}(R, A) \tag{1.1}$$

for any differential algebra A. More precisely, the differential algebra $\mathscr{J}_\infty R$ satisfies the following universal property: we have an algebra morphism $j\colon R \to \mathscr{J}_\infty R$ such that for any algebra morphism $f\colon R \to A$ from R to a differential algebra (A, ∂), there is a unique differential algebra morphism $\tilde{f}\colon \mathscr{J}_\infty R \to A$ such that $\tilde{f} \circ j = f$.

Proof The uniqueness of $\mathscr{J}_\infty R$ follows from Yoneda's lemma.

Let us show the existence. First, let $R = \mathbb{C}[x_1, \dots, x_N]$. We define $\mathscr{J}_\infty R$ to be the polynomial ring $\mathbb{C}[\partial^j x_i\colon i = 1, \dots, N,\ j \geqslant 0]$ with infinitely many variables $\partial^j x_i$, $i = 1, \dots, N$, $j \geqslant 0$, with the differential

$$\partial\colon \partial^j x_i \longmapsto \partial^{j+1} x_i, \tag{1.2}$$

We have the embedding

$$j\colon R \hookrightarrow \mathscr{J}_\infty R, \quad x_i \longmapsto \partial^0 x_i, \tag{1.3}$$

and $\mathscr{J}_\infty R$ is generated by R as a differential algebra. From now, we identify x_i with $\partial^0 x_i$. It is clear that $(\mathscr{J}_\infty R, \partial)$ satisfies the desired property.

Next, let R be general. We may assume that

$$R = \mathbb{C}[x_1, \dots, x_N]/\langle f_1, f_2, \cdots, f_r\rangle$$

with $f_i \in \mathbb{C}[x_1, \dots, x_N]$. We define

$$\mathscr{J}_\infty R = \mathbb{C}[\partial^j x_i\colon i = 1, \dots, N,\ j \geqslant 0]/\langle \partial^j f_i\colon i = 1, \dots, r,\ j \geqslant 0\rangle, \tag{1.4}$$

where f_i is considered as an element of $\mathbb{C}[\partial^j x_i\colon i = 1, \dots, N,\ j \geqslant 0]$ by the embedding j. Since $\langle \partial^j f_i\colon i = 1, \dots, r,\ j \geqslant 0\rangle$ is a differential ideal, $\mathscr{J}_\infty R$ is naturally a differential algebra with the derivation ∂. Because $\langle \partial^j f_i : i = 1, \dots, r,\ j \geqslant 0 \rangle$ is the smallest differential ideal of $\mathbb{C}[\partial^j x_i : i = 1, \dots, N,\ j \geqslant 0]$ containing $\langle f_1, f_2, \cdots, f_r \rangle$, $(\mathscr{J}_\infty R, \partial)$ satisfies the required property.

From the above construction, one can also prove the uniqueness more explicitly. Assume that there is another differential algebra (B, δ) satisfying the universal property. First, since the identity is a differential algebra morphism from B to B, we get an algebra morphism j_B from R to B. Applying the universal property to B and $\mathscr{J}_\infty R$, we get algebra morphisms $f : B \to \mathscr{J}_\infty R$ and $g : \mathscr{J}_\infty R \to B$ such that $f \circ j_B = j$ and $g \circ j = j_B$. By the uniqueness, of liftings we get that $f \circ g = id_{\mathscr{J}_\infty R}$ and $g \circ f = id_B$. □

The differential algebra $\mathscr{J}_\infty R$ is called the *jet algebra* of R.

By the proof of Lemma 1.1 we have the embedding $j : R \to \mathscr{J}_\infty R$ given by the correspondence (1.3). In particular, R can be regarded as a subalgebra of $\mathscr{J}_\infty R$ and the isomorphism (1.1) is given by restriction.

Observe that the correspondence $R \mapsto \mathscr{J}_\infty R$ is functorial. If $f : R \to R'$ is an algebra homomorphism, then we naturally obtain a morphism $\mathscr{J}_\infty f : \mathscr{J}_\infty R \to \mathscr{J}_\infty R'$ making the following diagram commutative:

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "j"'] & R' \arrow[d, "j"] \\ \mathscr{J}_\infty R \arrow[r, "\mathscr{J}_\infty f"'] & \mathscr{J}_\infty R' \end{tikzcd}$$

Lemma 1.2 *Let R_1 and R_2 be finitely generated unital commutative algebras. Then*

$$\mathscr{J}_\infty (R_1 \otimes R_2) \cong \mathscr{J}_\infty R_1 \otimes \mathscr{J}_\infty R_2$$

as differential algebras, where the differential of $\mathscr{J}_\infty R_1 \otimes \mathscr{J}_\infty R_2$ is given by $\Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial$.

Proof For any differential algebra A, we have

$$\begin{aligned} \mathrm{Hom}_{Alg}(R_1 \otimes R_2, A) &\cong \mathrm{Hom}_{Alg}(R_1, A) \otimes \mathrm{Hom}_{Alg}(R_2, A) \\ &\cong \mathrm{Hom}_{Dif.\ Alg}(\mathscr{J}_\infty R_1, A) \otimes \mathrm{Hom}_{Dif.\ Alg}(\mathscr{J}_\infty R_2, A) \\ &\cong \mathrm{Hom}_{Dif.\ Alg}(\mathscr{J}_\infty R_1 \otimes \mathscr{J}_\infty R_2, A). \end{aligned}$$

Corollary 1.1 *Let A be a finitely generated commutative Hopf algebra with counit $\epsilon : A \to \mathbb{C}$, coproduct $\Delta : A \to A \otimes A$ and antipode $S : A \to A$. Then $\mathscr{J}_\infty A$ is a commutative Hopf algebra with counit $\mathscr{J}_\infty \epsilon$, coproduct $\mathscr{J}_\infty \Delta$ and antipode $\mathscr{J}_\infty S$. Moreover, if M is a comodule over A with comodule map $\mu : M \to A \otimes M$, then $\mathscr{J}_\infty M$ is a comodule over $\mathscr{J}_\infty A$ with comodule map $\mathscr{J}_\infty \mu$.*

Proof Note that $\mathscr{J}_\infty \mathbb{C} = \mathbb{C}$. Hence $\mathscr{J}_\infty \epsilon$ defines an algebra homomorphism $\mathscr{J}_\infty A \to \mathbb{C}$. It is straightforward to check the assertion using Lemma 1.2. □

For any $\mathbb{C}$-algebra A, we set

$$A[[z]] = \left\{ \sum_{n\geqslant 0} a_n z^n : a_n \in A \right\}, \tag{1.5}$$

which is naturally an algebra. Note that $A \otimes \mathbb{C}[[z]] \subsetneq A[[z]]$ in general.

As an algebra, $\mathscr{J}_\infty R$ has the following characterization.

Proposition 1.1 *For a finitely generated unital commutative $\mathbb{C}$-algebra R, $\mathscr{J}_\infty R$ is the unique (up to isomorphisms) unital commutative $\mathbb{C}$-algebra such that*

$$\mathrm{Hom}_{Alg}(\mathscr{J}_\infty R, A) \cong \mathrm{Hom}_{Alg}(R, A[[z]])$$

for any unital commutative $\mathbb{C}$-algebra A.

Proof The uniqueness follows from Yoneda's lemma.

For $f \in \mathscr{J}_\infty R$, we set

$$e^{z\partial} f = \sum_{n\geqslant 0} \frac{1}{n!} (\partial^n f) z^n \in (\mathscr{J}_\infty R)[[z]].$$

Next, for $\alpha \in \mathrm{Hom}_{Alg}(\mathscr{J}_\infty R, A)$, we define $\Phi(\alpha) \in \mathrm{Hom}_{Alg}(R, A[[z]])$ by

$$\Phi(\alpha)(f) = \alpha(e^{z\partial} f) = \sum_{n\geqslant 0} \alpha(\partial^n f/n!) z^n, \quad f \in R.$$

Since $e^{z\partial}(fg) = e^{z\partial}(f) e^{z\partial}(g)$, $\Phi(\alpha)$ is an algebra homomorphism. Hence, Φ defines a map $\Phi\colon \mathrm{Hom}_{Alg}(\mathscr{J}_\infty R, A) \to \mathrm{Hom}_{Alg}(R, A[[z]])$.

Conversely, define map $\Psi\colon \mathrm{Hom}_{Alg}(R, A[[z]]) \to \mathrm{Hom}_{Alg}(\mathscr{J}_\infty R, A)$ by

$$\Psi(\beta)(\partial^n f) = \lim_{z\to 0} \partial_z^n(\beta(f)), \quad f \in R,\ n \in \mathbb{Z}_{\geqslant 0}.$$

It is easy to see that Ψ is well-defined and we have $\Phi \circ \Psi = \Psi \circ \Phi = \mathrm{id}$. This completes the proof. □

We have

$$\mathscr{J}_\infty R = \varinjlim_m \mathscr{J}_m R, \tag{1.6}$$

where $\mathscr{J}_m R$ is the subalgebra of $\mathscr{J}_\infty R$ generated by $\partial^j x_i$ with $i = 1, \dots, N$, $j = 0, \dots, m$ in the presentation (1.4). The inductive limit is taken with respect to the natural inclusions $\mathscr{J}_n R \hookrightarrow \mathscr{J}_m R$, for $n \leqslant m$.

The proof of the following assertion is similar to that of Proposition 1.1 and is left to the reader.

Proposition 1.2 *Let R be a finitely generated unital commutative $\mathbb{C}$-algebra. For $m \geqslant 0$, $\mathscr{J}_m R$ is the unique (up to isomorphisms) commutative $\mathbb{C}$-algebra such that*

$$\mathrm{Hom}_{Alg}(\mathcal{J}_m R, A) \cong \mathrm{Hom}_{Alg}(R, A[z]/(z^{m+1}))$$

for any unital commutative $\mathbb{C}$-algebra A.

Exercise 1.1 Let K be a field of characteristic $p > 0$. Let us call a unital commutative K-algebra A a *differential algebra* if it is equipped with linear maps

$$\partial^{[n]} : A \to A, \quad n \geqslant 0,$$

(that corresponds to the divided power differential $\partial^n/n!$) such that

$$\partial^{[n]}(ab) = \sum_{j=0}^{n} \partial^{[j]}(a)\partial^{[n-j]}(b), \quad a, b \in A.$$

Show that statements of Lemma 1.1, Proposition 1.1 and Proposition 1.2 hold by replacing $\mathbb{C}$-algebra by K-algebra and defining $\mathcal{J}_m R$ to be the subalgebra generated by $\partial^{[j]}x_i$ with $i = 1, \ldots, N$, $j = 0, \ldots, m$.

1.2 Arc spaces for affine schemes

Let Sch be the category of schemes over $\mathbb{C}$.

Let X be an affine scheme of finite type, that is, $X = \operatorname{Spec} R$ for some finitely generated unital commutative $\mathbb{C}$-algebra R. Define

$$\mathcal{J}_\infty X := \operatorname{Spec}(\mathcal{J}_\infty R). \tag{1.7}$$

Then by Proposition 1.1,

$$\begin{aligned}\{\mathbb{C}\text{-points of } \mathcal{J}_\infty X\} = \mathrm{Hom}_{Sch}(\operatorname{Spec}\mathbb{C}, \mathcal{J}_\infty X) &= \mathrm{Hom}_{Alg}(\mathcal{J}_\infty R, \mathbb{C}) \\ &\cong \mathrm{Hom}_{Alg}(R, \mathbb{C}[[z]]) = \mathrm{Hom}_{Sch}(D, X),\end{aligned}$$

where D is the (formal) disc defined by

$$D = \operatorname{Spec}\mathbb{C}[[z]].$$

A morphism $\gamma\colon D \to X$ is called an *arc* of X. The scheme $\mathcal{J}_\infty X$, whose $\mathbb{C}$-points are arcs of X, is called the *arc space* of X. Note that $\mathcal{J}_\infty X$ is a scheme of infinite type in general.

By Proposition 1.1, we have

$$\mathrm{Hom}_{Sch}(\operatorname{Spec} A, \mathcal{J}_\infty X) \cong \mathrm{Hom}_{Sch}(\operatorname{Spec} A[[z]], X) \tag{1.8}$$

for any commutative $\mathbb{C}$-algebra A, and the arc space $\mathcal{J}_\infty X$ is characterized as the unique scheme satisfying this property (see e.g. [66, VI.1]).

We also define for $m \geqslant 0$

$$\mathscr{J}_m X = \mathrm{Spec}(\mathscr{J}_m R). \tag{1.9}$$

By the similar argument using Proposition 1.2, we find that the $\mathbb{C}$-points of $\mathscr{J}_m X$ are the *m-jets* of X, that is, the morphisms

$$\mathrm{Spec}(\mathbb{C}[z]/(z^{m+1})) \longrightarrow X.$$

The scheme $\mathscr{J}_m X$ is called the *m-th jet scheme* of X. It is a scheme of finite type.

By Proposition 1.2, the m-th jet scheme $\mathscr{J}_m X$ is characterized as the unique scheme satisfying

$$\mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}\, A, \mathscr{J}_m X) \cong \mathrm{Hom}_{\mathcal{S}ch}(\mathrm{Spec}\, A[z]/(z^{m+1}), X) \tag{1.10}$$

for any commutative $\mathbb{C}$-algebra A.

Denote by π_m the canonical morphism

$$\pi_m \colon \mathscr{J}_m(X) \longrightarrow \mathscr{J}_0(X) \cong X$$

induced by the projection

$$\mathbb{C}[z]/(z^{m+1}) \twoheadrightarrow \mathbb{C}[z]/(z) \cong \mathbb{C}.$$

More generally, we have truncation morphisms:

$$\pi_{m,n} \colon \mathscr{J}_m(X) \longrightarrow \mathscr{J}_n(X), \qquad m \geqslant n,$$

induced by the projection

$$\mathbb{C}[z]/(z^{m+1}) \twoheadrightarrow \mathbb{C}[z]/(z^{n+1}).$$

Namely, $\pi_{m,n}$ is the affine scheme morphism whose comorphism is the embedding of $\mathbb{C}$-algebras $\mathscr{J}_n R \hookrightarrow \mathscr{J}_m R$.

By (1.6), we have

$$\mathscr{J}_\infty X = \varprojlim_m \mathscr{J}_m X \tag{1.11}$$

in the category of affine schemes. For $m \in \mathbb{Z}_{\geqslant 0}$, we have the canonical truncation morphism

$$\pi_{\infty,m} \colon \mathscr{J}_\infty(X) \longrightarrow \mathscr{J}_m(X)$$

whose comorphism is the the embedding of $\mathbb{C}$-algebras $\mathscr{J}_m R \hookrightarrow \mathscr{J}_\infty R$.

The canonical injection $\mathbb{C} \hookrightarrow \mathbb{C}[z]/(z^{m+1})$ induces a morphism $\iota_m \colon X \to \mathscr{J}_m(X)$ whose comorphism is the canonical projection of $\mathbb{C}$-algebras $\mathscr{J}_m R \twoheadrightarrow R$. Since $\pi_m \circ \iota_m = \mathrm{id}_X$, we get that π_m is surjective and ι_m is injective.

Example 1.1 Let us consider a concrete example. Let

$$X = \mathrm{Spec}\, \mathbb{C}[x^1, x^2, x^3]/((x^1)^2 + x^2 x^3) \subset \mathbb{A}^3.$$

Set

$$x^j_{(-i-1)} := \frac{1}{i!}\partial^i x^j, \qquad j = 1,2,3,\ i \geqslant 0.$$

Identifying x^1 with $x^1_{(-1)}$, x^2 with $x^2_{(-1)}$ and x^3 with $x^3_{(-1)}$, the equations of the embedding of $\mathscr{J}_\infty(X)$ in $\mathscr{J}_\infty(\mathbb{A}^3)$ are given by the vanishing of the coefficients of the polynomial

$$(x^1_{(-1)}+x^1_{(-2)}z+x^1_{(-3)}z^2+\cdots)^2+(x^2_{(-1)}+x^2_{(-2)}z+x^2_{(-3)}z^2+\cdots)(x^3_{(-1)}+x^3_{(-2)}z+x^3_{(-3)}z^2+\cdots)$$

in $\mathbb{C}[[z]]$ or, equivalently, by the following equations:

$$\begin{cases} (x^1_{(-1)})^2 + x^2_{(-1)}x^3_{(-1)} & = 0 \\ 2x^1_{(-1)}x^1_{(-2)} + x^2_{(-1)}x^3_{(-2)} + x^3_{(-1)}x^2_{(-2)} & = 0 \\ 2x^1_{(-1)}x^1_{(-3)} + 2(x^1_{(-2)})^2 + x^2_{(-1)}x^3_{(-3)} + x^2_{(-2)}x^3_{(-2)} + x^3_{(-1)}x^2_{(-3)} & = 0 \\ \vdots & \vdots \end{cases}.$$

The truncation morphism $\pi_{\infty,m}\colon \mathscr{J}_\infty(X) \to \mathscr{J}_m(X)$ is given by forgetting the coordinates $x^j_{(-i-1)}$, for $i > m$.

1.3 Arc spaces for general schemes

The result of this section is not used for the rest of the book.

Theorem 1.1 (Greenberg [96, 97]) *Let X be a scheme of finite type.*

i). For any $m \in \mathbb{Z}_{\geqslant 0}$ there exists a unique scheme $\mathscr{J}_m X$ such that

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}\,A, \mathscr{J}_m X) \cong \mathrm{Hom}_{Sch}(\mathrm{Spec}\,A[z]/(z^{m+1}), X) \tag{1.12}$$

for any commutative $\mathbb{C}$-algebra A. Equivalently,

$$\mathrm{Hom}_{Sch}(Z, \mathscr{J}_m X) \cong \mathrm{Hom}_{Sch}(Z \times_{\mathrm{Spec}\,\mathbb{C}} \mathbb{C}[z]/(z^{m+1}), X) \tag{1.13}$$

for any scheme Z.

ii). there exists a unique scheme $\mathscr{J}_\infty X$ such that

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}\,A, \mathscr{J}_\infty X) \cong \mathrm{Hom}_{Sch}(\mathrm{Spec}\,A[[z]], X) \tag{1.14}$$

for any commutative $\mathbb{C}$-algebra A. Equivalently,

$$\mathrm{Hom}_{Sch}(Z, \mathscr{J}_\infty X) \cong \mathrm{Hom}_{Sch}(Z \widehat{\times}_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,\mathbb{C}[[z]], X) \tag{1.15}$$

for any scheme Z, where $Z\widehat{\times}_{\mathrm{Spec}\,\mathbb{C}} \mathrm{Spec}\,\mathbb{C}[[z]]$ is the formal completion of $Z \times \mathrm{Spec}\,\mathbb{C}[[z]]$ with respect to $Z \times \{0\}$.

Thus, the $\mathbb{C}$-points of $\mathscr{J}_m(X)$ are the $\mathbb{C}[z]/(z^{m+1})$-points of X, and the the $\mathbb{C}$-points of $\mathscr{J}_\infty(X)$ are the $\mathbb{C}[[z]]$-points of X. From Theorem 1.1, we have for example that $\mathscr{J}_0(X) \simeq X$ and that $\mathscr{J}_1(X) \simeq TX$, where TX denotes the total tangent bundle of X.

We have a canonical projection $\pi_{m,n}\colon \mathscr{J}_m(X) \to \mathscr{J}_n(X)$ for $m \geqslant n$. It is defined at the level of the functor of points using (1.12): the induced map

$$\mathrm{Hom}_{Sch}(\mathrm{Spec}\, A[z]/(z^{m+1}), X) \longrightarrow \mathrm{Hom}_{Sch}(\mathrm{Spec}\, A[z]/(z^{n+1}), X)$$

is induced from the truncation morphism $A[z]/(z^{m+1}) \to A[z]/(z^{n+1})$. Similarly, we have a canonical projection $\pi_{\infty,m}\colon \mathscr{J}_\infty(X) \to \mathscr{J}_m(X)$ for $m \in \mathbb{Z}_{\geqslant 0}$.

For an arbitrary scheme X of finite type, the following lemma allows to describe the jet schemes $\mathscr{J}_m X$, for $m \in \mathbb{Z}_{\geqslant 0}$, and the arc scheme $\mathscr{J}_\infty X$ from the affine case.

Lemma 1.3 ([64]) *Given any $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ and any open subset U of X, $\mathscr{J}_m(U) = \pi_m^{-1}(U)$.*

Proof Assume first $m \in \mathbb{Z}_{\geqslant 0}$. Let A be a $\mathbb{C}$-algebra and

$$j_m\colon \mathrm{Spec}\, A \to \mathrm{Spec}\, A[z]/(z^{m+1})$$

be the morphism induced by truncation. An A-valued point of $\mathscr{J}_m(X)$ is a morphism of schemes $\gamma\colon \mathrm{Spec}\, A[z]/(z^{m+1}) \to X$. Such a morphism is an A-valued point of $\pi_m^{-1}(U)$ if and only if $\gamma \circ j_m$ factors through U. Clearly, if γ is an A-valued point of $\mathscr{J}_m(U)$, that is, the image of γ lies in U, then γ is an A-valued point of $\pi_m^{-1}(U)$.

Conversely, assume that $\gamma\colon \mathrm{Spec}\, A[z]/(z^{m+1}) \to X$ is an A-valued point of $\pi_m^{-1}(U)$. Then $\gamma \circ j_m$ factors through U. Note that the set of prime ideals of $A[z]/(z^{m+1}) = A \otimes \mathbb{C}[z]/(z^{m+1})$ is in one-to-one correspondence with the set of prime ideals of A since $\mathrm{Spec}\, \mathbb{C}[z]/(z^{m+1})$ contains a unique element. Hence, γ induces a map from $\mathrm{Spec}\, A[z]/(z^{m+1})$ to U (just between sets). Because U is open in X, we have $\mathscr{O}_U \cong \mathscr{O}_X|_U$. Hence the map induced from the morphism of schemes γ is automatically a morphism, too. So γ induces a morphism $\mathrm{Spec}\, A[z]/(z^{m+1}) \to U$, that is, an A-valued point of $\mathscr{J}_m(U)$.

For $m = \infty$, the statement is obtained by taking the projective limit since $\pi_{\infty,0}^{-1}(U) = \varprojlim_m \pi_m^{-1}(U)$ and $\mathscr{J}_\infty(U) = \varprojlim_m \mathscr{J}_m(U)$. □

It follows from the lemma that for an arbitrary scheme X of finite type with an affine open covering $\{U_i\}_{i \in I}$, its jet scheme $\mathscr{J}_m(X)$ is obtained by glueing the jet schemes $\mathscr{J}_m(U_i)$ (see [64, 104]). Over an affine open subset $U_i \subset X$, the space of arcs is described by

$$\left(\pi_{\infty,*}\mathscr{O}_{\mathscr{J}_\infty X}\right)(U_i) = \mathscr{O}_{\mathscr{J}_\infty X}(\pi_{\infty,0}^{-1}(U_i)) = \varinjlim \mathscr{O}_{\mathscr{J}_m X}(\pi_m^{-1}(U_i)) = \mathscr{O}_{\mathscr{J}_\infty X}(\mathscr{J}_\infty U_i),$$

where $\pi_{\infty,*}\mathscr{O}_{\mathscr{J}_\infty X}$ denotes the pushforward sheaf of $\mathscr{O}_{\mathscr{J}_\infty X}$ induced by

$$\pi_{\infty,0}\colon \mathscr{J}_\infty \to X.$$

In particular, the structure sheaf $(\pi_{\infty,0})_*\mathscr{O}_{\mathscr{J}_\infty(X)}$ is a sheaf of differential algebras on X.

1.4 Functorial properties

The map from a scheme to its jet schemes, or its arc space, is functorial. If $f\colon X \to Y$ is a morphism of schemes, then we naturally obtain a morphism $\mathscr{J}_m f\colon \mathscr{J}_m(X) \to \mathscr{J}_m(Y)$ making the following diagram commutative,

$$\begin{tikzcd} \mathscr{J}_m(X) \arrow[r, "\mathscr{J}_m f"] \arrow[d, "\pi_m"'] & \mathscr{J}_m(Y) \arrow[d, "\pi_m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

In terms of arcs, it means that $\mathscr{J}_m f(\alpha) = f \circ \alpha$ for $\alpha \in \mathscr{J}_m(X)$. This also holds for $m = \infty$.

In addition, we have the following results.

Lemma 1.4 **([64])** *Let $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. For every schemes X, Y, we have a canonical isomorphism*

$$\mathscr{J}_m(X \times Y) \simeq \mathscr{J}_m(X) \times \mathscr{J}_m(Y).$$

For X, Y affines, and $m = \infty$, the lemma is just a reformulation of Lemma 1.2.

Proof Assume first that $m \in \mathbb{Z}_{\geqslant 0}$. Then for any affine scheme Z in Sch,

$$\begin{aligned} \operatorname{Hom}(Z, \mathscr{J}_m(X \times Y)) &\cong \operatorname{Hom}(Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1}), X \times Y) \\ &\cong \operatorname{Hom}(Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1}), X) \times \operatorname{Hom}(Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1}), Y) \\ &\cong \operatorname{Hom}(Z, \mathscr{J}_m(X)) \times \operatorname{Hom}(Z, \mathscr{J}_m(Y)) \\ &\cong \operatorname{Hom}(Z, \mathscr{J}_m(X) \times \mathscr{J}_m(Y)), \end{aligned}$$

whence the statement in this case. For $m = \infty$, just replace $\mathbb{C}[z]/(z^{m+1})$ with $\mathbb{C}[[z]]$ and take the completion $Z \widehat{\times}_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}[[z]]$ instead of $Z \times_{\operatorname{Spec}_{\mathbb{C}}} \mathbb{C}[z]/(z^{m+1})$. $\square$

Let $f\colon X \to Y$ be a morphism between affine schemes $X, Y \in Sch$. Recall that f is called *formally smooth* (resp. *unramified*, *étale*) if for every $\mathbb{C}$-algebra A, every nilpotent ideal J of A, and every commutative square,

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "\varphi_0"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "\psi"'] \arrow[ur, dashed, "\varphi"] & Y, \end{tikzcd}$$

where $B = A/J$, there exists one (resp. at most one, one unique) diagonal arrow φ, called the *lifting*, making the two triangles commutatives, [98].

Since f is of finite type (the schemes X, Y are of finite type), the morphism f is formally smooth if and only if it is smooth. For the relation between formal and standard smoothness we refer for instance to [152, Chap. 10, Section 28].

Lemma 1.5 ([64],[46, Proposition 3.7.1 and 3.7.4]) *If $f\colon X \to Y$ is a smooth surjective morphism between affine schemes $X, Y \in Sch$, then for every $m \in \mathbb{Z}_{\geqslant 0}$, $\mathscr{J}_m f$ is also smooth and surjective. Moreover, $\mathscr{J}_\infty f$ is formally smooth and surjective.*

Recall [178, proposition 4.8] that a morphism of schemes $f\colon X \to Y$ is surjective if and only if for any field K and any $y \in Y(K)$ there is a field extension L/K and $x \in X(L)$ whose image by $X(L) \to Y(L)$ is the image of y under $Y(K) \to Y(L)$

Proof We prove at the same time the statements for $\mathscr{J}_m f$ and $\mathscr{J}_\infty f$. In the latter case, set $m = \infty$ and for every $\mathbb{C}$-algebra A, read $A[z]/(z^{m+1})$ as $A[[z]]$.

Let us first prove the surjectivity. Let K be a field. Given a K-valued point $\mathscr{J}_m Y(K)$ is the same as giving a morphism $\psi\colon \operatorname{Spec} K[z]/(z^{m+1}) \to Y$. Denoting by $\iota_K\colon \operatorname{Spec} K \to \operatorname{Spec} K[z]/(z^{m+1})$ the natural closed immersion, the composition map $\psi \circ \iota_K\colon \operatorname{Spec} K \to Y$ yields a K-valued point y. Since f is surjective, there is a field extension L/K and $x \in X(L)$ whose image by $X(L) \to Y(L)$ is the image of y under $Y(K) \to Y(L)$. Hence we get an L-valued point $\varphi_0\colon \operatorname{Spec} L \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[rr, "\varphi_0"] \arrow[d, "\iota_L"'] & & X \arrow[d, "f"] \\
\operatorname{Spec} L[z]/(z^{m+1}) \arrow[r, "\mu"'] & \operatorname{Spec} K[z]/(z^{m+1}) \arrow[r, "\psi"'] & Y,
\end{tikzcd}$$

where $\mu\colon \operatorname{Spec} L[z]/(z^{m+1}) \to \operatorname{Spec} K[z]/(z^{m+1})$ is the natural morphism induced from $K \hookrightarrow L$. Assume first that $m < \infty$. Then the ideal of $\ker \iota_L^*$ is generated by (z), hence it is nilpotent in $L[z]/(z^{m+1})$. The morphism f being formally smooth, there exists a morphism $\varphi\colon \operatorname{Spec} L[z]/(z^{m+1}) \to X$, making the two triangles commutative:

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r, "\varphi_0"] \arrow[d, "\iota_L"'] & X \arrow[d, "f"] \\
\operatorname{Spec} L[z]/(z^{m+1}) \arrow[r, "\psi\circ\mu"'] \arrow[ur, dashed, "\varphi"] & Y,
\end{tikzcd}$$

that is, $\mathscr{J}_m f(\varphi) = \psi'$, with $\psi' := \psi \circ \mu$. This proves the surjectivity of $\mathscr{J}_m f$.

Assume now that $m = \infty$. Since $\mathscr{J}_\infty X$ is the projective limit of the $\mathscr{J}_m X$, in order to show the surjectivity of $\mathscr{J}_\infty f$ it is enough to check the compatibility of $\mathscr{J}_m f$ with the truncation morphisms $\pi_{m,n}\colon \mathscr{J}_m X \to \mathscr{J}_n X$. The foregoing shows that for every n, there exists a morphism $\varphi_n\colon \operatorname{Spec} L[z]/(z^{n+1}) \to X$ such that $\psi' \circ \iota_n = f \circ \varphi_n$, that is, $\pi_{\infty,n}(\psi) = \mathscr{J}_n f(\varphi_n)$:

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r, "\varphi_0"] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\
\operatorname{Spec} L[z]/(z^{n+1}) \arrow[r] \arrow[ur, dashed, "\varphi_n"] \arrow[d, "\iota_n"'] & Y \\
\operatorname{Spec} L[[z]], \arrow[ur, dashed, "\psi'"'] &
\end{tikzcd}$$

where $\iota_n\colon \operatorname{Spec} L[z]/(z^{n+1}) \to \operatorname{Spec} L[[z]]$ is the canonical closed immersion. Moreover, $\pi_{m,n}(\varphi_m) = \varphi_n$ for every $m \geqslant n$. Therefore, the family $(\varphi_n)_n$ defines an L-valued point of $\varprojlim \mathscr{J}_n(X)$, hence, an L-arc φ of X. This proves the surjectivity of $\mathscr{J}_\infty f$.

We now show that $\mathscr{J}_m(X)$ is formally smooth, for $m \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. As before, we prove together the statement for $\mathscr{J}_m f$ and $\mathscr{J}_\infty f$. Notice that for $m < \infty$, $\mathscr{J}_m f$ is of finite type because f is so. Hence $\mathscr{J}_m f$ will be smooth if formally smooth.

Let A be a $\mathbb{C}$-algebra, J a nilpotent ideal of A and set $B = A/J$. Let $\psi\colon \operatorname{Spec} A[z]/(z^{m+1}) \to Y$ be an A-valued point of $\mathscr{J}_m Y$ and $\varphi_0\colon \operatorname{Spec} B[z]/(z^{m+1}) \to X$ a B-valued point of $\mathscr{J}_m X$ such that $f \circ \varphi_0 = \psi \circ j$,

$$\begin{tikzcd}
\operatorname{Spec} B[z]/(z^{m+1}) \arrow[r, "\varphi_0"] \arrow[d, "j"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A[z]/(z^{m+1}) \arrow[r, "\psi"'] & Y,
\end{tikzcd}$$

where $j\colon \operatorname{Spec} B[z]/(z^{m+1}) \to \operatorname{Spec} A[z]/(z^{m+1})$ is the canonical closed immersion. The ideal of $\ker j^*$ is generated by $J[z]/(z^{m+1})$, hence it is nilpotent. Since f is formally smooth, there exists a morphism $\varphi\colon \operatorname{Spec} A[z]/(z^{m+1}) \to X$ making the two triangles commutatives:

$$\begin{tikzcd}
\operatorname{Spec} B[z]/(z^{m+1}) \arrow[r, "\varphi_0"] \arrow[d, "j"'] & X \arrow[d, "f"] \\
\operatorname{Spec} A[z]/(z^{m+1}) \arrow[r, "\psi"'] \arrow[ur, dashed, "\varphi"] & Y.
\end{tikzcd}$$

This shows that the morphism $\mathscr{J}_m f$ is formally smooth. Indeed, by definition, an A-valued point of $\mathscr{J}_m Y$ (respectively, a B-valued point of $\mathscr{J}_m X$) is an $A[z]/(z^{m+1})$-valued point of Y (respectively, $B[z]/(z^{m+1})$-valued point of X). $\square$

Remark 1.1 Similarly, one can show that if $f\colon X \to Y$ is a formally étale morphism of affine schemes, then the canonical morphism $\mathscr{J}_m(X) \to \mathscr{J}_m(Y) \times_Y X$ induced by $\mathscr{J}_m(f)$ and $\pi_m\colon \mathscr{J}_m(Y) \to Y$ is an isomorphism. Hence Lemma 1.3 also follows from this fact applied to the open immersion $U \hookrightarrow X$ since $\pi_m^{-1}(U) \cong \mathscr{J}_m(X) \times_X U$.

1.5 Geometric properties of arc spaces

It is known that the geometry of the jet schemes $\mathscr{J}_m(X)$, for $m \geqslant 1$, is closely linked to that of X. More precisely, we can transport some geometrical properties from $\mathscr{J}_m(X)$ to X.

The following proposition gives examples of such phenomena:

Proposition 1.3 *Let $m \in \mathbb{Z}_{\geqslant 0}$, and let X be an affine scheme of finite type. If $\mathscr{J}_m(X)$ is smooth (respectively, irreducible, reduced, normal, locally a complete intersection) for some m, then so is X.*

For smoothness, the converse is true, even with "every m" instead of "for some m". In fact, for smooth varieties, we have the following more precise statement, [64, Corollary 2.11].

Proposition 1.4 *If X is a smooth variety of dimension N, then the truncation morphism $\pi_{m,p}$, for $p \in \{0, \dots, m\}$, is a Zariski locally trivial projection with fiber isomorphic to $\mathbb{A}^{(m-p)N}$. In particular, $\mathscr{J}_m(X)$ is a smooth variety of dimension $(m+1)N$.*

Proof Around every point in X we can find an open subset U and an étale morphism $U \to \mathbb{A}^N$. Using Remark 1.1 the assertion reduced to the case where X is the affine space $\mathbb{A}^N$, in which case the statement is clear by Sections 1.1 and 1.2. ◻

For the other properties stated in Proposition 1.3, the converse is not true in general. We refer for instance to [104, §3] for counter-examples. See also [160] for counter-examples in the setting of nilpotent orbit closures in a simple Lie algebra.

The following lemma gives a necessary and sufficient condition for the converse of Proposition 1.3 to hold for irreducibility.

Lemma 1.6 *Assume that X is an irreducible reduced affine scheme of finite type over $\mathbb{C}$, and let $m \in \mathbb{Z}_{\geqslant 0}$. Then the Zariski closure of $\pi_{X,m}^{-1}(X_{\mathrm{reg}})$ is an irreducible component of $\mathscr{J}_m(X)$, and $\mathscr{J}_m(X)$ is irreducible if and only if $\pi_{X,m}^{-1}(X_{\mathrm{sing}})$ is contained in the Zariski closure of $\pi_{X,m}^{-1}(X_{\mathrm{reg}})$. Here, X_{reg} stands for the smooth part of X, and X_{sing} for its complement in X.*

Proof Since X_{reg} is smooth and irreducible, the Zariski closure $\overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$ of $\pi_{X,m}^{-1}(X_{\mathrm{reg}})$ is an irreducible closed subset of $\mathscr{J}_m(X)$ of dimension $(m+1)\dim X$ by Proposition 1.4. Then the lemma easily follows from the fact that we have the decomposition

$$\mathscr{J}_m(X) = \pi_{X,m}^{-1}(X_{\mathrm{sing}}) \cup \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$$

of closed subsets, and that $\pi_{X,m}^{-1}(X_{\mathrm{sing}}) \not\supset \overline{\pi_{X,m}^{-1}(X_{\mathrm{reg}})}$. ◻

There are also subtle connections between the geometry of $\mathscr{J}_m(X)$, for $m \geqslant 1$, and the singularities of X. In particular, by results of Mustaţă, we have:

Theorem 1.2 ([161]) *Let X be an irreducible affine variety over $\mathbb{C}$.*

i). If X is a complete intersection, then $\mathscr{J}_m(X)$ is irreducible for every $m \geqslant 1$ if and only if X has rational singularities.
ii). If X is a complete intersection and if $\mathscr{J}_m(X)$ is irreducible for some $m \geqslant 1$, then $\mathscr{J}_m(X)$ is also reduced.

We have seen that jet schemes and arc spaces share several functorial properties. For topological properties, they behave rather differently. The main reason is that $\mathbb{C}[[z]]$ is a domain, contrary to $\mathbb{C}[z]/(z^{m+1})$. Thereby, although $\mathscr{J}_\infty(X)$ is not of finite type in general, its geometric properties are somehow simpler than those of the finite jet schemes $\mathscr{J}_m(X)$.

Let us now turn to topological properties of the arc spaces.

Lemma 1.7 *The natural morphism $X_{\mathrm{red}} \to X$ induces an isomorphism*

$$\mathscr{J}_\infty X_{\mathrm{red}} \xrightarrow{\simeq} \mathscr{J}_\infty X$$

of topological spaces. Here X_{red} stands for the reduced scheme associated with X.

Proof We may assume that $X = \operatorname{Spec} R$, with R a ring. An arc α of X corresponds to a ring homomorphism $\alpha^*\colon R \to \mathbb{C}[[z]]$. Since $\mathbb{C}[[z]]$ is an integral domain, it decomposes as $\alpha^*\colon R \to R/\sqrt{0} \to \mathbb{C}[[z]]$. Thus, α is an arc of X_{red}. □

Note that Lemma 1.7 is false for the schemes $\mathscr{J}_m(X)$.

! Warning

If $\mathscr{J}_\infty X$ is reduced, then X is reduced, but $\mathscr{J}_\infty X$ no need to be reduced if X is reduced.

The following example was discovered by Julien Sebag [172]: let X be the hypersurface of $\mathbb{A}^2$ defined by equation $x^3 - y^2 = 0$. Then X is reduced, and one can verify that $3y_{(-1)}x_{(-2)} - 2x_{(-1)}y_{(-2)}$ is a nilpotent element of $\mathbb{C}[\mathscr{J}_\infty X]$, where $x_{(-1)}$ and $y_{(-1)}$ identify with x and y, respectively, and $x_{(-2)} = \partial x_{(-1)}$, $y_{(-2)} = \partial y_{(-1)}$.

Mustaţă's result (Theorem 1.2) furnishes a converse to the above "warning" in the case where X is a locally complete intersection with rational singularities:

Theorem 1.3 ([161]) *If X is a locally complete intersection with rational singularities, then $\mathscr{J}_\infty$ is reduced (and irreducible).*

If X is a point (as topological space), then $\mathscr{J}_\infty(X)$ is also a point (as topological space), because $\operatorname{Hom}(D, X) = \operatorname{Hom}(\mathbb{C}, \mathbb{C}[[z]])$ consists of only one element. Thus, Lemma 1.7 implies the following.

Corollary 1.2 *If X is zero-dimensional, then $\mathscr{J}_\infty(X)$ is also zero-dimensional.*

In contrast to jet schemes, the irreducibility property is preserved for the space of arcs.

Theorem 1.4 (Kolchin) *The arc scheme $\mathscr{J}_\infty(X)$ is irreducible if and only if X is irreducible.*

Proof Since $\mathscr{J}_\infty X \cong \mathscr{J}_\infty X_{\mathrm{red}}$ as topological spaces, we may assume that X is reduced. Assume first that X is smooth. In this case, the result is easy. Indeed, by Proposition 1.4, the jet schemes $\mathscr{J}_m X$ are smooth for any m and the canonical projections $\mathscr{J}_\infty X \to \mathscr{J}_m X$ are all surjective. Therefore $\mathscr{J}_\infty X = \varprojlim_m \mathscr{J}_m X$ with the projective limit topology is irreducible, too.

Consider now the general case. We argue by induction on $d = \dim X$, the $d = 0$ case being trivial. By Hironaka's Theorem, there is a resolution of singularities $f\colon X' \to X$. In particular, f is a proper morphism and X' is smooth. Suppose that Z is a proper closed subset of X such that f is an isomorphism over $U = X \setminus Z$.

We claim that

$$\mathscr{J}_\infty(X) = \mathscr{J}_\infty(Z) \cup \operatorname{Im}(\mathscr{J}_\infty f). \tag{1.16}$$

Indeed, since f is proper, the Valuative Criterion for properness implies that an arc $\gamma\colon \operatorname{Spec}\mathbb{C}[[z]] \to X$ lies in the image of $\mathscr{J}_\infty(f)$ if and only if the induced morphism $\overline{\gamma}\colon \operatorname{Spec}\mathbb{C}((z)) \to X$ can be lifted to X' (moreover, if the lifting of $\overline{\gamma}$ is unique, then the lifting of γ is also unique). On the other hand, γ does not lie in $\mathscr{J}_\infty(Z)$ if and only if $\overline{\gamma}$ factors through $U \hookrightarrow X$. In this case, the lifting of $\overline{\gamma}$ exists and is unique since f is an isomorphism over U. This proves (1.16).

The smooth case implies that $\mathscr{J}_\infty(X')$ is irreducible and so is $\operatorname{Im}(\mathscr{J}_\infty f)$. Hence by (1.16) it only remains to prove that $\mathscr{J}_\infty(Z)$ is contained in the closure of $\operatorname{Im}(\mathscr{J}_\infty f)$.

Consider the irreducible decomposition $Z = Z_1 \cup \ldots \cup Z_r$, inducing by $\mathscr{J}_\infty(Z) = \mathscr{J}_\infty(Z_1) \cup \ldots \cup \mathscr{J}_\infty(Z_r)$. Since f is surjective and proper, for any i, there is an irreducible component Z_i' of $f^{-1}(Z_i)$ such that the induced map $Z_i' \to Z_i$ is surjective. By the Generic Smoothness Theorem ([134, Corollary 10.7]), one can find open subsets U_i' and U_i in Z_i' and Z_i, respectively, such that induced morphisms $g_i\colon U_i' \to U_i$ are smooth and surjective. In particular, we get

$$\mathscr{J}_\infty(U_i) = \operatorname{Im}(\mathscr{J}_\infty(g_i)) \subseteq \operatorname{Im}(\mathscr{J}_\infty f).$$

On the other hand, by induction, every $\mathscr{J}_\infty(Z_i)$ are irreducible. Since $\mathscr{J}_\infty(U_i)$ is a nonempty open subset of $\mathscr{J}_\infty(Z_i)$, it follows that

$$\mathscr{J}_\infty(Z_i) \subseteq \overline{\operatorname{Im}(\mathscr{J}_\infty f)}$$

for every i. This completes the proof of the theorem. □

This result is classically referred to as the Kolchin irreducibility theorem, and is an analogue for arc schemes of a theorem in differential algebra [130, IV.17, Prop. 10].

As a consequence of Kolchin's Irreducibility Theorem, if $X_1, \ldots, X_r$ are the irreducible components of X, then $\mathscr{J}_\infty(X_1), \ldots, \mathscr{J}_\infty(X_r)$ are the irreducible components of $\mathscr{J}_\infty X$.

Lemma 1.8 *Let Y be an irreducible affine scheme, and let $f\colon X \to Y$ be a morphism that restricts to a bijection between some open subsets $U \subset X$ and $V \subset Y$. Then $\mathscr{J}_\infty f\colon \mathscr{J}_\infty(X) \to \mathscr{J}_\infty(Y)$ is dominant.*

Proof The map $\mathscr{J}_\infty f$ restricts to the isomorphism $\mathscr{J}_\infty(U) \xrightarrow{\simeq} \mathscr{J}_\infty(V)$, and the open subset $\mathscr{J}_\infty(V)$ is dense in $\mathscr{J}_\infty(Y)$ since $\mathscr{J}_\infty(Y)$ is irreducible. □

Remark 1.2 Note that all results of Sections 1.4 and 1.5 hold for any scheme of finite type (not necessarily affine).

1.6 Loop spaces

In the context of vertex algebras one needs also to consider the *loop space* $\mathscr{L}X$ of an affine scheme X. One of the reasons is that an $\mathscr{O}(\mathscr{J}_\infty X)$-module as a vertex algebra is the same as a *smooth module* over the topological ring $\mathscr{O}(\mathscr{L}X)$ (see Section 2.14).

Assume that $X = \operatorname{Spec} R$ is an affine scheme of finite type over $\mathbb{C}$.

Proposition 1.5 *i). There exists a unique, up to isomorphism, ind-scheme $\mathscr{L}X$ which is the inductive limit of affine schemes $\mathscr{L}_nX$ of infinite type such that for any commutative $\mathbb{C}$-algebra A,*

$$\operatorname{Hom}_{Alg}(\mathscr{O}(X), A((z))) \cong \operatorname{Hom}_{Alg}(\mathscr{O}(\mathscr{L}X), A),$$

where $A((z)) = \varinjlim_n z^{-n}A[[z]]$.
ii). If X is smooth, then $\mathscr{L}X$ is formally smooth.

For a commutative $\mathbb{C}$-algebra A, by

$$\operatorname{Hom}_{Alg}(\mathscr{O}(\mathscr{L}X), A)$$

we always mean the set of *continuous morphisms*, that is, the morphisms from $\mathscr{O}(\mathscr{L}X)$ to A which factorize through one of the quotients of the projective limit,

$$\rho_{\infty,n}\colon \mathscr{O}(\mathscr{L}X) \twoheadrightarrow \mathscr{O}(\mathscr{L}_nX).$$

Proof The unicity of the ind-scheme $\mathscr{L}X$ follows from Yoneda's lemma.

(i) Assume first that $X = \mathbb{A}^N = \operatorname{Spec} \mathbb{C}[x^i]_{i=1,\dots,N}$. Then set

$$\mathscr{L}X = \varinjlim_n \operatorname{Spec} \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n},$$

where the coordinate $x^i_{(-j-1)}$ is defined by sending a morphism $\gamma\colon \mathbb{C}[x^i]_i \to \mathbb{C}((z))$ giving by $\gamma(x^i) = \sum_{j\gg -\infty} \gamma^i_{(-j-1)} z^j$ to the scalar $\gamma^i_{(-j-1)}$. We have

$$\mathscr{O}(\mathscr{L}X) = \varprojlim_n \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}, \tag{1.17}$$

with respect to the surjective homomorphisms

$$\rho_{m,n}\colon \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -m} \twoheadrightarrow \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}, \qquad m \geqslant n.$$

A continuous algebra morphism from $\mathscr{O}(\mathscr{L}X)$ to a $\mathbb{C}$-algebra A is the same as a morphism from $\mathscr{O}(\mathscr{L}X)$ to A which factorizes through one the quotient morphisms

$$\rho_{\infty,n}\colon \mathscr{O}(\mathscr{L}X) \twoheadrightarrow \mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}.$$

Hence we get that for every commutative $\mathbb{C}$-algebra A,

$$\begin{aligned}\mathrm{Hom}_{Alg}(\mathscr{O}(\mathscr{L}X), A) &\cong \varinjlim_n \mathrm{Hom}_{Alg}(\mathbb{C}[x^i_{(-j-1)}]_{i,j\geqslant -n}, A)\\ &\cong \mathrm{Hom}_{Alg}(\mathscr{O}(X), \varinjlim_n z^{-n}A[[z]])\\ &\cong \mathrm{Hom}_{Alg}(\mathscr{O}(X), A((z))),\end{aligned}$$

and $\mathscr{L}X$ satisfies the required condition.

Suppose now that $X = \operatorname{Spec} R$ if an affine subscheme of $\mathbb{A}^N$ defined by equations $f_1, \dots, f_r$. Any polynomial $f \in \mathbb{C}[x^i]_{i=1,\dots,N}$ induces a morphism of ind-schemes $\tilde{f}\colon \mathscr{L}\mathbb{A}^N \to \mathscr{L}\mathbb{A}$ via base extension. Hence one may realize the loop space $\mathscr{L}X$ as the sub-ind-scheme of $\mathscr{L}\mathbb{A}^N$ defined by the equations $\tilde{f}_1, \dots, \tilde{f}_r$. More concretely, replacing x^i by $x^i(z) = \sum_{j\geqslant -n} x^i_{(-j-1)} z^j$ in the equations f_k, we get, for each m, a system of equations in $\mathbb{C}[x^i_{(-j-1)}\colon i = 1, \dots, N,\ j \geqslant -m]$ which defines a subscheme in $\mathscr{L}\mathbb{A}^N$. Our desired ind-scheme $\mathscr{L}X$ is the inductive limit of these schemes as $n \to \infty$.

(ii) Assume that $X = \operatorname{Spec} R$ is smooth. We need to prove that for any surjection of $\mathbb{C}$-algebras $B \to A$ whose kernel J satisfies $I^n = 0$ for some n, the map of sets $\mathrm{Hom}_{Alg}(R, B((z))) \to \mathrm{Hom}_{alg}(R, A((z)))$ is surjective. But the kernel of $B((z)) \to A((z))$ is $J((z))$ which is also nilpotent of order n. So the smoothness of R implies that any morphism $R \to A((z))$ can be lifted to a morphism $R \to B((z))$. □

Since X is separated, the valuative criterion for separated morphisms gives an inclusion of the arc space $\mathscr{J}_\infty X$ into the loop space $\mathscr{L}X$. One could extend the definition to any scheme, and the inclusion

$$\mathscr{J}_\infty X \subset \mathscr{L}X$$

would still hold. The valuative criterion for properness guarantees that this inclusion is a bijection if and only if X is proper. In fact, if X is proper, let's say projective, then there is no difference between $A[[z]]$-points and $A((z))$-points of X. However, the category of $\mathscr{O}(\mathscr{J}X)$ is different than the category of $\mathscr{O}(\mathscr{L}X)$. In this book, we only need to consider the case of affine schemes. We refer the reader to [121] for an appropriate construction in a more general setting.

1.7 Arc spaces of group schemes acting on an algebraic variety

A *proalgebraic group* is an inverse limit of algebraic groups. As a consequence of Lemma 1.4 we get the following result.

Lemma 1.9 *Let $m \in \mathbb{Z}_{\geqslant 0}$ (respectively, $m = \infty$). If G is a group scheme over $\mathbb{C}$, then $\mathscr{J}_m(G)$ is also a group scheme (respectively, a proalgebraic group scheme) over $\mathbb{C}$. Moreover, if G acts on X, then $\mathscr{J}_m(G)$ acts on $\mathscr{J}_m(X)$.*

Proof According to Lemma 1.4, the multiplication morphism $\mu\colon G \times G \to G$ induces a morphism $\mathscr{J}_m\mu\colon \mathscr{J}_m(G \times G) \cong \mathscr{J}_m G \times \mathscr{J}_m G \to \mathscr{J}_m G$ for any m. Moreover, since the jet of a point is a point, the restrictions to $\{e\} \times G \to G$ and $G \times \{e\} \to G$ of μ induces morphisms $\{e\} \times \mathscr{J}_m G \to \mathscr{J}_m G$ and $\mathscr{J}_m G \times \{e\} \to \mathscr{J}_m G$ and, so, the neutral element e of G is still a neutral element for the operation $\mathscr{J}_m\mu$. From this, it is easy to verify that the operation $\mathscr{J}_m\mu$ gives to $\mathscr{J}_m G$ a group scheme (respectively, a proalgebraic group scheme) structure.

Suppose now that G acts on X. The above group scheme (respectively, a proalgebraic group scheme) structure shows that $\mathscr{J}_m G$ acts on $\mathscr{J}_m X$ using the map $\mathscr{J}_m(G \times X) \cong \mathscr{J}_m G \times \mathscr{J}_m X \to \mathscr{J}_m X$ induces from the action map $G \times X \to X$. □

Remark 1.3 As a consequence of Lemma 1.9, if G is an affine group scheme over $\mathbb{C}$ acting on an affine scheme X, the action comorphism,

$$\mathscr{O}(\mathscr{J}_\infty X) \to \mathscr{O}(\mathscr{J}_\infty G) \otimes \mathscr{O}(\mathscr{J}_\infty X),$$

is a morphism of differential algebras.

Example 1.2 Let G be an linear algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$. By Lemma 1.9, $\mathscr{J}_\infty G$ is an affine proalgebraic group, whose $\mathbb{C}$-points are the $\mathbb{C}[[t]]$-points of G. We denote by $G[[t]]$ the set of $\mathbb{C}$-points of G. We have

$$\mathrm{Lie}(\mathscr{J}_\infty G) = \mathscr{J}_\infty \mathfrak{g} = \mathfrak{g}[[t]], \quad \mathrm{Lie}(\mathscr{J}_r G) = \mathscr{J}_r \mathfrak{g} = \mathfrak{g}[t]/(t^{r+1}),$$

with Lie bracket:

$$[xt^m, yt^n] = [x, y]t^{m+n}, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z}_{\geqslant 0}. \tag{1.18}$$

Indeed, by definition, for $r \in \mathbb{Z}_{\geqslant 0} \sqcup \{\infty\}$, $\mathrm{Lie}(\mathscr{J}_r G)$ is the Lie algebra of the left invariant vector fields on $\mathscr{J}_r G$, that is,

$$\mathrm{Lie}(\mathscr{J}_r G) = \{D \in \mathrm{Der}(\mathscr{O}(\mathscr{J}_r G))\colon \Delta \circ D = (1 \otimes D) \circ \Delta\}$$

(see Appendix B, Section B.3), where $\Delta\colon \mathscr{O}(\mathscr{J}_r G) \to \mathscr{O}(\mathscr{J}_r G) \otimes \mathscr{O}(\mathscr{J}_r G)$ is the coproduct induced by the coproduct of $\mathscr{O}(G)$, see Corollary 1.1. Note that if $\Delta(f) = \sum_i u_i \otimes v_i$ for $f \in \mathscr{O}(G)$, we have

$$\Delta(f_{(-n-1)}) = \sum_i \sum_{k=0}^{n} (u_i)_{(k-n-1)} \otimes (v_i)_{(-k-1)}, \tag{1.19}$$

where $f_{(-n-1)} = \partial^n f/n!$. This is clear for $r = \infty$ since Δ is the homomorphism of differential algebras, and the coproduct of $\mathscr{O}(\mathscr{J}_r G)$ is obtained by restricting the coproduct of $\mathscr{O}(\mathscr{J}_\infty G)$ to $\mathscr{O}(\mathscr{J}_r G)$. Let $r \in \mathbb{Z}_{\geqslant 0}$ and consider the Lie algebra homomorphism $\phi\colon \mathfrak{g}[t]/(t^{r+1}) \to \mathrm{Der}(\mathscr{O}(\mathscr{J}_r G))$ defined by

$$\phi(xt^m) f_{(-n-1)} = (x_L f)_{(m-n-1)}, \tag{1.20}$$

where x_L is the left invariant vector field on G corresponding to $x \in \mathfrak{g}$ and we have put $f_{(n)} = 0$ for $n \geqslant 0$. We find from (1.19) that the image of ϕ is contained in $\mathrm{Lie}(\mathscr{J}_r G)$, and thus, we have the Lie algebra homomorphism

$$\psi\colon \mathfrak{g}[t]/(t^{r+1}) \longrightarrow \mathrm{Lie}(\mathscr{J}_r G).$$

The map ψ is injective since $\mathfrak{g} = \mathrm{Lie}(G)$, and therefore, ϕ must be isomorphism since $\dim \mathfrak{g}[t]/(t^{r+1}) = \dim \mathrm{Lie}(\mathscr{J}_r G)$. As this is true for all $m \geqslant 0$, we find that $\mathfrak{g}[[t]] \cong \mathrm{Lie}(\mathscr{J}_\infty G)$, and that the action of $\mathfrak{g}[[t]]$ on $\mathscr{O}(\mathscr{J}_\infty G)$ as left invariant vector fields is given by the formula (1.20).

By Lemma 1.9, note that the adjoint action of G on $\mathfrak{g}$ induces an action of $\mathscr{J}_\infty(G)$ on $\mathscr{J}_\infty(\mathfrak{g})$, and the coadjoint action of G on $\mathfrak{g}^*$ induces an action of $\mathscr{J}_\infty(G)$ on $\mathscr{J}_\infty(\mathfrak{g}^*)$.

We conclude this section with an application of Theorem 1.3 to the nilpotent cone $\mathscr{N}$ of a simple Lie algebra $\mathfrak{g}$.

Example 1.3 Assume that $\mathfrak{g}$ is simple, and let $\mathscr{N}$ be the *nilpotent cone* of $\mathfrak{g}$ that is, the set of nilpotent elements of $\mathfrak{g}$. (The reader is referred to Appendix A for basics on semisimple Lie algebras, and to Appendix D for properties of the nilpotent cone and nilpotent elements.). It is well-known that $\mathscr{N}$ is the reduced scheme of $\mathfrak{g}$ defined by the equations $p_1, \dots, p_r$, where $p_1, \dots, p_r$ are homogeneous generators of $\mathscr{O}(\mathfrak{g})^G$. Hence, $\mathscr{J}_\infty(\mathscr{N})$ is the subscheme of $\mathfrak{g}[[t]]$ defined by the equations $\partial^j p_i$, $i = 1 \dots, r$ and $j \geqslant 0$.

Furthermore, according to Kostant [133], the nilpotent cone is a complete intersection, which is irreducible and reduced. Moreover, it was proved by Hesselink [101] that it has rational (hence canonical) singularities. Using Mustaţă's result (Theorem 1.3), it was shown that Eisenbud-Frenkel [65] that[1]:

$$\mathscr{J}_\infty(\mathfrak{g}/\!/G) \cong \mathscr{J}_\infty\mathfrak{g}/\!/\mathscr{J}_\infty G,$$

where $\mathfrak{g}/\!/G = \mathrm{Spec}\,\mathscr{O}(\mathfrak{g})^G$ and $\mathscr{J}_\infty\mathfrak{g}/\!/\mathscr{J}_\infty G = \mathrm{Spec}\,\mathscr{O}([\mathscr{J}_\infty\mathfrak{g})^{\mathscr{J}_\infty G}$. In other words, the invariant ring $\mathscr{O}(\mathscr{J}_\infty\mathfrak{g})^{\mathscr{J}_\infty G}$ is the polynomial ring

$$\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}/\!/G)) = \mathbb{C}[\partial^j p_i : i = 1, \dots, r, j \geqslant 0],$$

since $\mathscr{O}(\mathfrak{g}/\!/G) = \mathbb{C}[p_1, \dots, p_r]$. In particular,

[1] This result was obtained independently by Raïs-Tauvel [169] and Beilinson-Drinfeld [37] by other methods.

$$\mathscr{J}_\infty(\mathscr{N}) = \operatorname{Spec} \mathscr{O}(\mathscr{J}_\infty\mathfrak{g})/\mathscr{O}(\mathscr{J}_\infty\mathfrak{g})_+^{\mathscr{J}_\infty G},$$

where $\mathscr{O}(\mathscr{J}_\infty\mathfrak{g})_+^{\mathscr{J}_\infty G}$ is the augmentation ideal of $\mathscr{O}(\mathscr{J}_\infty\mathfrak{g})^{J_\infty G}$.

Chapter 2
Operator product expansion and vertex algebras

In this chapter, we collect the basic definitions and standard properties of vertex algebras (see Section 2.7). We give several equivalent caracterisation of the *locality axiom*, which is the most important axiom of a vertex algebra, and derive from this the Borcherds identities (see Section 2.3 and 2.8). The easiest examples of vertex algebras are the commutatives vertex algebras which are discussed in Section 2.9. First interesting examples of non-commutative vertex algebras are given in the next chapter (Chap. 3). Other important examples of non-commutative vertex algebras will occur in the rest of this book.

The best general references for this chapter are [77, 112].

2.1 Notation

For R a $\mathbb{C}$-algebra and $n \in \mathbb{Z}_{>0}$, we denote by $R[[z_1^{\pm}, \dots, z_n^{\pm}]]$ the vector space of all R-valued *formal power series* (or *formal Laurent series*) in the variables $z_1, \dots, z_n$, that is, the elements of the form

$$\sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} a_{i_1,\dots,i_n} z_1^{i_1} \dots z_n^{i_n}, \tag{2.1}$$

where each $a_{i_1,\dots,i_n}$ is in R. If $a \in R[[z_1^{\pm}, \dots, z_n^{\pm}]]$ and $b \in R[[w_1^{\pm}, \dots, w_m^{\pm}]]$, $m, n > 0$, then the product ab is well-defined in $R[[z_1^{\pm}, \dots, z_n^{\pm}, w_1^{\pm}, \dots, w_m^{\pm}]]$. But if a, b are two elements of $R[[z_1^{\pm}, \dots, z_n^{\pm}]]$, then their product does not make sense in general since the coefficient in a given z_i^j, for $i = 1, \dots, n$ and $j \in \mathbb{Z}$, of the product may be an infinite sum. However, the product of $a \in R[[z_1^{\pm}, \dots, z_n^{\pm}]]$ by a *Laurent polynomial*, that is, a series as in (2.1) such that $a_{i_1,\dots,i_n} = 0$ for all but finitely many n-tuples $i_1, \dots, i_n$, is well-defined.

Given a formal power series in one variable $a(z) = \sum\limits_{n \in \mathbb{Z}} a_n z^n \in R[[z^{\pm}]]$, we define its *residue* at $z = 0$ as:

$$\operatorname{Res}_{z=0} a(z) := a_{-1}.$$

If $R = \mathbb{C}$ and if $a(z)$ is the Laurent series of a meromorphic function defined on a punctured disc at 0, having pole only at 0, then

$$\mathrm{Res}_{z=0}\, a(z) = \frac{1}{2\pi\sqrt{-1}} \int_\gamma a(z)dz,$$

where the integral is taken over any closed curve γ winding once around 0.

2.2 Formal delta function

Define the *formal delta-function* by

$$\delta(z-w) = \frac{1}{z}\sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \in \mathbb{C}[[z^{\pm}, w^{\pm}]].$$

We have

$$\delta(z-w) = \tau_{z,w}\left(\frac{1}{z-w}\right) - \tau_{w,z}\left(\frac{1}{z-w}\right), \tag{2.2}$$

where the two maps $\tau_{z,w}$ and $\tau_{w,z}$ are the embeddings of algebras defined by:

$$\begin{aligned}
\tau_{z,w} &: \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] \longrightarrow \mathbb{C}((z))((w)), \quad \frac{1}{z-w} \longmapsto \frac{1}{z}\sum_{n\geqslant 0}\left(\frac{w}{z}\right)^n,\\
\tau_{w,z} &: \mathbb{C}[z, w, z^{-1}, w^{-1}, \frac{1}{z-w}] \longrightarrow \mathbb{C}((w))((z)), \quad \frac{1}{z-w} \longmapsto -\frac{1}{z}\sum_{n>0}\left(\frac{z}{w}\right)^n.
\end{aligned}$$

Thus the map $\tau_{z,w}(f)$ is the expansion of f in $|z| > |w|$ and $\tau_{w,z}(f)$ is the expansion of f in $|w| > |z|$.

Lemma 2.1 *For any $\mathbb{C}$-algebra R and any $f \in R[z, z^{-1}]$, where $R[z, z^{-1}]$ is the set of all Laurent polynomials in the variable z with coefficients in R, we have*

$$f(z)\delta(z-w) = f(w)\delta(z-w). \tag{2.3}$$

Proof Note that $f(z) - f(w)$ is divisible by $z - w$. We have

$$\begin{aligned}
(z-w)\delta(z-w) &= (z-w)\left(\tau_{z,w}\left(\frac{1}{(z-w)}\right) - \tau_{w,z}\left(\frac{1}{(z-w)}\right)\right)\\
&= \tau_{z,w}(1) - \tau_{w,z}(1) = 0,
\end{aligned}$$

whence the assertion. □

Remark 2.1 In fact, for any formal series $f \in R[[z, z^{-1}]]$, the multiplication $f(z)\delta(z-w)$ makes sense and the equality (2.3) holds.

Both homomorphisms $\tau_{z,w}$ and $\tau_{w,z}$ commute with ∂_w and ∂_z. Therefore, it follows in the same way as above that

$$(z-w)^{n+1}\frac{1}{n!}\partial_w^n\delta(z-w)=0 \tag{2.4}$$

for $n \geqslant 0$.

Lemma 2.2 *For $m, n \geqslant 0$ we have*

$$\mathrm{Res}_{z=0}\left((z-w)^m\frac{1}{n!}\partial_w^n\delta(z-w)\right)=\delta_{m,n}$$

Proof Observe that

$$(z-w)^m\frac{1}{n!}\partial_w^n\delta(z-w)=\tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right)-\tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right).$$

The meromorphic function $f(z)=\frac{1}{(z-w)^{n-m+1}}$, for fixed $w\in\mathbb{C}$, has poles contained in $\{w,0,\infty\}$. It admits the following Laurent series expansions:

$$f(z)=\begin{cases}\tau_{z,w}\left(\dfrac{1}{(z-w)^{n-m+1}}\right)=\sum_{m\in\mathbb{Z}}a_m(w)z^m \text{ if } |z|>|w|\\[2ex] \tau_{w,z}\left(\dfrac{1}{(z-w)^{n-m+1}}\right)=\sum_{n\in\mathbb{Z}}b_n(w)z^n \text{ if } |w|>|z|,\end{cases}$$

with $a_m(w), b_n(w)\in\mathbb{C}[[w^{\pm}]]$,

Now we have

$$\mathrm{Res}_{z=0}\left(\tau_{z,w}\left(\frac{1}{(z-w)^{n-m+1}}\right)\right)=a_{-1}(w)=\frac{1}{2\pi\sqrt{-1}}\int_{C_{1,w}}\frac{dz}{(z-w)^{n-m+1}}, \tag{2.5}$$

where $C_{1,w}$ is the contour described in Figure 2.1 (a circle centred at 0 with radius $r_1>|w|=r$), while

$$\mathrm{Res}_{z=0}\left(\tau_{w,z}\left(\frac{1}{(z-w)^{n-m+1}}\right)\right)=b_{-1}(w)=\frac{1}{2\pi\sqrt{-1}}\int_{C_{2,w}}\frac{dz}{(z-w)^{n-m+1}}, \tag{2.6}$$

where $C_{2,w}$ is the contour described in Figure 2.2 (a circle centred at 0 with radius $r_2<|w|=r$).

Clearly, by the residue theorem applied to the meromorphic function $f(z)$ defined on the domain $\mathbb{C}\setminus\{w,0\}$, we get

$$(2.5)-(2.6)=\frac{1}{2\pi\sqrt{-1}}\int_{C_w}\frac{dz}{(z-w)^{n-m+1}}=\delta_{m,n},$$

where C_w is the contour described in Figure 2.3 (a circle with center w and radius $<\min(r-r_2, r_1-r)$). This concludes the proof.

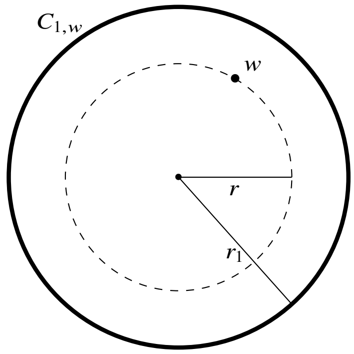

Fig. 2.1 The contour $C_{1,w}$

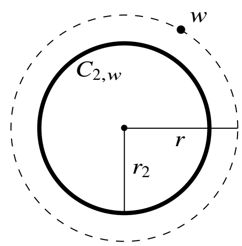

Fig. 2.2 The contour $C_{2,w}$

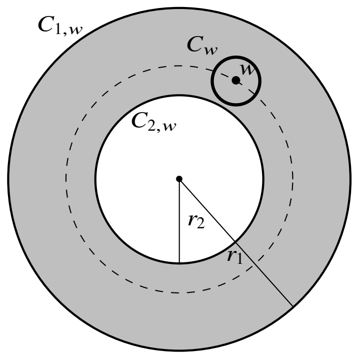

Fig. 2.3 The contour C_w

2.3 Locality and Operator product expansion (OPE)

Let V be a vector space over $\mathbb{C}$. We denote by $(\operatorname{End} V)[[z, z^{-1}]]$ the set of all formal Laurent series in the variable z with coefficients in the space $\operatorname{End} V$. We call elements $a(z)$ of $(\operatorname{End} V)[[z, z^{-1}]]$ a *series* on V. For a series $a(z)$ on V, we set

$$a_{(n)} = \operatorname{Res}_{z=0} a(z) z^n$$

so that the expansion of $a(z)$ is

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}. \tag{2.7}$$

The coefficient $a_{(n)}$ is called a *Fourier mode* of $a(z)$. We write

$$a(z)b = \sum_{n\in\mathbb{Z}} a_{(n)}bz^{-n-1}$$

for $b \in V$.

Definition 2.1 A series $a(z) \in (\operatorname{End} V)[[z, z^{-1}]]$ is called a *field* on V if for any $b \in V$, $a(z)b \in V((z))$, that is, for any $b \in V$, $a_{(n)}b = 0$ for large enough n.

In the sequel, the space of all fields on V will be denoted by $\mathscr{F}ields(V)$.

For $a(z), b(z) \in \mathscr{F}ields(V)$, the product $a(z)b(z)$ does not make sense in general. However, the *normally ordered product*

$$:a(z)b(z): = a(z)_+b(z) + b(z)a(z)_-,$$

where

$$a(z)_+ = \sum_{n<0} a_{(n)}z^{-n-1}, \qquad a(z)_- = \sum_{n\geqslant 0} a_{(n)}z^{-n-1},$$

does make sense and belongs to $\mathscr{F}ields(V)$. However, the normally ordered product is neither commutative nor associative. By definition, $:a(z)b(z)c(z):$ stands for $:a(z):b(z)c(z)::$.

Although $a(z)b(w)$ makes sense, we also consider the following normally ordered product in $\operatorname{End}(V)[[z^{\pm}, w^{\pm}]]$:

$$:a(z)b(w): = a(z)_+b(w) + b(w)a(z)_-.$$

Note that $:a(z)b(w):v \in V[[z, w]][z^{-1}, w^{-1}]$, while $a(z)b(w)v \in V((z))((w))$, for $a(z), b(z) \in \mathscr{F}ields(V)$, $v \in V$.

Definition 2.2 We say two fields $a(z), b(z)$ on V are *mutually local* if

$$(z-w)^N [a(z), b(w)] = 0$$

in $(\operatorname{End} V)[[z^{\pm}, w^{\pm}]]$ for a sufficiently large N.

We note that a field $a(z)$ needs not be local to itself.

Proposition 2.1 ([112], see also [156]) *Fix two fields $a(z), b(z)$ on a vector space V. The following assertions are equivalent:*

i). $a(z)$ and $b(z)$ are mutually local, that is, $(z-w)^N [a(z), b(w)] = 0$ for some $N \in \mathbb{Z}_{\geqslant 0}$ in $(\operatorname{End} V)[[z^{\pm}, w^{\pm}]]$;

ii). There exist $c_0(w), c_1(w), \ldots, c_{N-1}(w) \in \mathscr{F}ields(V)$ such that

$$[a(z), b(w)] = \sum_{n=0}^{N-1} c_n(w)\frac{1}{n!}\partial_w^n \delta(z-w).$$

in $(\operatorname{End} V)[[z^{\pm 1}, w^{\pm 1}]]$;

iii). There exist $c_0(w), c_1(w), \dots, c_{N-1}(w) \in \mathscr{F}ields(V)$ *such that*

$$a(z)b(w) = \sum_{n=0}^{N-1} c_n(w)\tau_{z,w}\left(\frac{1}{(z-w)^{n+1}}\right) + {}^{\circ}_{\circ}a(z)b(w){}^{\circ}_{\circ}$$

and

$$b(w)a(z) = \sum_{n=0}^{N-1} c_n(w)\tau_{w,z}\left(\frac{1}{(z-w)^{n+1}}\right) + {}^{\circ}_{\circ}a(z)b(w){}^{\circ}_{\circ}$$

in $(\operatorname{End} V)[[z^{\pm 1}, w^{\pm 1}]]$.

Proof The direction (iii) $\Rightarrow$ (ii) is obvious and (ii) $\Rightarrow$ (i) follows from (2.4). We shall show that (i) $\Rightarrow$ (iii). We have

$$\begin{aligned} a(z)b(w) - {}^{\circ}_{\circ}a(z)b(w){}^{\circ}_{\circ} &= [a(z)_-, b(w)], \\ b(w)a(z) - {}^{\circ}_{\circ}a(z)b(w){}^{\circ}_{\circ} &= [b(w), a(z)_+]. \end{aligned}$$

By the locality assumption,

$$(z-w)^N[a(z)_-, b(w)] = (z-w)^N[b(w), a(z)_+]. \tag{2.8}$$

Observe that the left-hand-side of (2.8) does not have terms greater than $N-1$ in z whereas the right does not have terms of negative degree in z. Hence, they are polynomials of degree at most $N-1$ in z. It follows that there exists $c_j(w) \in (\operatorname{End} V)[[w, w^{-1}]]$, $j = 0, \dots, N-1$, such that

$$(z-w)^N[a(z)_-, b(w)] = \sum_{j=0}^{N-1} c_j(w)(z-w)^{N-j-1}.$$

For $v \in V$, the element $[a(z)_-, b(w)]v = (a(z)b(w) - {}^{\circ}_{\circ}a(z)b(w){}^{\circ}_{\circ})v$ belongs to $V((z))((w))$, which is a vector space over $\mathbb{C}((z))((w))$. We have

$$\begin{aligned} [a(z)_-, b(w)]v &= \tau_{z,w}\left(\frac{1}{(z-w)^N}\right)(z-w)^N[a(z)_-, b(w)]v \\ &= \tau_{z,w}\left(\frac{1}{(z-w)^N}\right)\sum_{j=0}^{N-1}(z-w)^{N-j-1}c_j(w)v \\ &= \sum_{j=0}^{N-1}\tau_{z,w}\left(\frac{1}{(z-w)^{j+1}}\right)c_j(w)v. \end{aligned}$$

Since $v \in V$ is an arbitrary, we have obtained the first formula of (iii). The second formula is similarly shown. Finally, we need to show that each $c_j(w)$ is a field. Since we have shown that $[a(z), b(w)] = \sum_{j=0}^{N-1} c_j(w)\frac{1}{j!}\partial_w^j\delta(z-w)$, it follows from Lemma 2.2 that

$$c_j(w) = \mathrm{Res}_{z=0}((z-w)^j [a(z), b(w)]). \tag{2.9}$$

As both $a(z)$ and $b(w)$ are fields, $c_j(w)$ is a field as well. □

By abuse of notation we often just write

$$a(z)b(w) \sim \sum_{n=0}^{N-1} \frac{c_n(w)}{(z-w)^{n+1}} \tag{2.10}$$

for the relations of Proposition 2.1 (iii).

Definition 2.3 Formula (2.10) is called the *operator product expansion* (OPE) of $a(z)$ and $b(w)$.

Proposition 2.2 *The OPE* (2.10)*, or the relations of Proposition 2.1 (iii), is equivalent to the relation,*

$$[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} (c_j)_{(m+n-j)} \quad (m, n \in \mathbb{Z})$$

for $a, b \in V$*,* $m, n \in \mathbb{Z}$*, in* $\mathrm{End}\, V$.

In the above formulas, the notation $\binom{m}{j}$ for $j \geqslant 0$ and $m \in \mathbb{Z}$ means

$$\binom{m}{j} = \frac{m(m-1) \times \cdots \times (m-j+1)}{j(j-1) \times \cdots \times 1},$$

with the convention $\binom{m}{0} = 1$.

Proof We only show that (2.10) implies that the above relation. The other direction is easy to see. We have

$$[a_{(m)}, b_{(n)}] = \mathrm{Res}_{w=0} \left(w^n \, \mathrm{Res}_{z=0}(z^m [a(z), b(w)]) \right).$$

As in the same manner as in the proof of Lemma 2.2, we get

$$\mathrm{Res}_{z=0}(z^m [a(z), b(w)]) = \sum_{j=0}^{N-1} \mathrm{Res}_{z=w} \left(\frac{z^m}{(z-w)^{j+1}} \right) c_j(w) = \sum_{j=0}^{N-1} \binom{m}{j} c_j(w) w^{m-j}.$$

This completes the proof. □

Proposition 2.2 says that the right-hand-side of the OPE encodes all the brackets between all the coefficients of mutually local fields $a(z)$ and $b(z)$.

2.4 Example

Let $\mathcal{B}$ be the unital associative algebra generated by elements b_n, for $n \in \mathbb{Z}$, with relations

$$[b_m, b_n] = m\delta_{m+n,0}, \quad m, n \in \mathbb{Z}.$$

A $\mathcal{B}$-module M is called *smooth* if for each $m \in M$ there exits an integer N such that $b_n m = 0$ for $n > N$. If M is a smooth $\mathcal{B}$-module,

$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}$$

is a field on M. We have

$$[b(z), b(w)] = \sum_{m,n \in \mathbb{Z}} [b_m, b_n] z^{-m-1} w^{-n-1} = \sum_{m \in \mathbb{Z}} m z^{-m-1} w^{m-1} = \partial_w \delta(z - w).$$

Hence, $b(z)$ is local to itself and

$$b(z)b(w) \sim \frac{1}{(z-w)^2}.$$

2.5 n-th product of fields

Let $a(z), b(z)$ be mutually local fields on V, so that $(z-w)^N [a(z), b(w)] = 0$ for some N. For $n \geqslant 0$, define the field $a(z)_{(n)} b(z)$ by

$$a(z)_{(n)} b(z) := \mathrm{Res}_{w=0}((w-z)^n [a(w), b(z)]).$$

Then, the OPE of $a(z)$ and $b(z)$ is expressed as

$$a(z)b(w) \sim \sum_{j \geqslant 0} \frac{a(w)_{(j)} b(w)}{(z-w)^{j+1}}, \tag{2.11}$$

see (2.9). (Note that $a(z)_{(j)} b(z) = 0$ for $j \geqslant N$.)

In fact, the field $a(z)_{(n)} b(z)$ makes sense for all $n \in \mathbb{Z}$, where we undestand $\mathrm{Res}_{w=0}((w-z)^n [a(w), b(z)])$ as

$$\mathrm{Res}_{w=0} \left(\tau_{w,z}((w-z)^n) a(w) b(z) \right) - \mathrm{Res}_{w=0} \left(\tau_{z,w}((w-z)^n) b(z) a(w) \right).$$

Explicitely, we have

$$a(z)_{(n)} b(z) = \sum_{k \in \mathbb{Z}} \left(\sum_{i \geqslant 0} \binom{n}{i} \left(a_{(n-i)} b_{(k+i)} - (-1)^n b_{(n+k-i)} a_{(i)} \right) \right) z^{-k-1}. \tag{2.12}$$

Definition 2.4 The field $a(z)_{(n)}b(z)$ is called the *n-th product* of $a(z)$ and $b(z)$.

Note that

$$a(z)_{(-1)}b(z) = \,{}^{\circ}_{\circ}\, a(z)b(z) \,{}^{\circ}_{\circ}\,, \tag{2.13}$$

$$a(z)_{(-n)}\,\mathrm{id}_V = \begin{cases} \frac{1}{(n-1)!}\partial_z^{(n-1)}a(z) & \text{if } n > 0, \\ 0 & \text{if } n \leqslant 0. \end{cases} \tag{2.14}$$

The last formula follows from the fact that

$$a(z)_{(-n)}\,\mathrm{id}_V = \mathrm{Res}_{w=z}\left(\frac{a(w)}{(w-z)^n}\right)$$

(recall the proof of Lemma 2.2), where

$$\mathrm{Res}_{w=z}\left(\frac{a(w)}{(w-z)^n}\right) := \sum_{n\in\mathbb{Z}} a_{(n)}\,\mathrm{Res}_{w=z}\left(\frac{w^{-n-1}}{(w-z)^n}\right).$$

Similarly, we have

$$(\mathrm{id}_V)_{(n)}a(z) = \delta_{n,-1}a(z). \tag{2.15}$$

Also, we have

$$\partial_z(a(z)_{(n)}b(z)) = (\partial_z a(z))_{(n)}\,b(z) + a(z)_{(n)}\,(\partial_z b(z))\,. \tag{2.16}$$

This is clear from the fact that $\mathrm{Res}_{z=0}\,\partial_z(\ldots) = 0$.

The n-th product is not associative. By definition, $a(z)_{(m)}b(z)_{(n)}c(z)$ stands for $a(z)_{(m)}(b(z)_{(n)}c(z))$ as in the case of the normally ordered product.

Lemma 2.3 ([141]) *If $a(z)$, $b(z)$, $c(z)$ are three mutually local fields on a vector space V, then the fields $a(z)_{(n)}b(z)$ and $c(z)$ are also mutually local for all $n \in \mathbb{Z}$.*

Lemma 2.3 is usually referred to as *Dong's Lemma*.

Proof By assumption there exists $N \geqslant 0$ such that

$$(z-w)^N a(z)b(w) = (z-w)^N b(w)a(z), \tag{2.17}$$

$$(z-u)^N a(z)c(u) = (z-u)^N c(u)a(z), \tag{2.18}$$

$$(w-u)^N b(w)c(u) = (w-u)^N c(u)b(w). \tag{2.19}$$

We may assume that $N + n \geqslant 0$. We claim that

$$\begin{aligned} &(w-u)^{4N}\left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right)c(u) \\ &= (w-u)^{4N}c(u)\left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right). \end{aligned} \tag{2.20}$$

Indeed, we have

$$(w-u)^{4N} = (w-u)^N \sum_{s=0}^{3N} \binom{3N}{s} (z-u)^s (w-z)^{3N-s}.$$

If $0 \leqslant s \leqslant N$, $(w-z)^{3N-s}\tau_{z,w}((z-w)^n) = (-1)^{3N-s}\tau_{z,w}((z-w)^{3N-s+n})$ and $3N-s+n \geqslant N$. Thus, $(w-z)^{3N-s}(\tau_{z,w}((z-w)^n)a(z)b(w)-\tau_{w,z}((z-w)^n)b(w)a(z)) = 0$ by (2.17), and so the left-hand-side of (2.20) is equal to

$$\sum_{s=N+1}^{3N} (w-u)^N (z-u)^s (w-z)^{3N-s} \left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right) c(u).$$

Similarly, the right-hand-side of (2.20) is equal to

$$\sum_{s=N+1}^{3N} (w-u)^N (z-u)^s (w-z)^{3N-s} c(u) \left(\tau_{z,w}((z-w)^n)a(z)b(w) - \tau_{w,z}((z-w)^n)b(w)a(z)\right).$$

But these two are equal thanks to (2.18) and (2.19).

The assertion follows by taking $\mathrm{Res}_{z=0}$ of both sides of (2.20). □

The following assertion should be compared with Proposition 2.2.

Proposition 2.3 *Let $a(z)$, $b(z)$, $c(z)$ be three mutually local fields on a vector space V. Then,*

$$a(z)_{(m)}b(z)_{(n)}c(z) - b(z)_{(n)}a(z)_{(m)}c(z) = \sum_{j\geqslant 0} \binom{m}{j} (a(z)_{(j)}b(z))_{(m+n-j)}c(z) \tag{2.21}$$

for $m, n \in \mathbb{Z}$.

Proof The left-hand-side is equal to the sum of the following two terms:

$$\begin{aligned} &\mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{w,z}((w-z)^m)\tau_{u,z}((u-z)^n)a(w)b(u)c(z)\right) \\ &-\mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{w,z}((w-z)^m)\tau_{u,z}((u-z)^n)b(u)a(w)c(z)\right), \end{aligned} \tag{2.22}$$

$$\begin{aligned} &-\mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{z,w}((w-z)^m)\tau_{z,u}((u-z)^n)c(z)a(w)b(u)\right) \\ &+\mathrm{Res}_{w=0}\,\mathrm{Res}_{u=0}\left(\tau_{z,w}((w-z)^m)\tau_{z,u}((u-z)^n)c(z)b(u)a(w)\right). \end{aligned} \tag{2.23}$$

By using the formula

$$(w-z)^m = \sum_{j\geqslant 0} \binom{m}{j} (w-u)^j (u-z)^{m-j},$$

and the fact that

$$\tau_{w,z}((w-u)^j(u-z)^{m-j}) = \begin{cases} \tau_{w,u}((w-u)^j)\tau_{u,z}((u-z)^{m-j}) & \text{in } \mathbb{C}((w))((u))((z)), \\ \tau_{u,w}((w-u)^j)\tau_{u,z}((u-z)^{m-j}) & \text{in } \mathbb{C}((u))((w))((z)), \end{cases}$$

we find that (2.22) is equal to

$$\sum_{j\geqslant 0}\binom{m}{j}\operatorname{Res}_{u=0}\left(\tau_{u,z}((u-z)^{m+n-j})\left(a(u)_{(j)}b(u)\right)c(z)\right).$$

Similarly, we find (2.23) is equal to

$$-\sum_{j\geqslant 0}\binom{m}{j}\operatorname{Res}_{u=0}\left(\tau_{z,u}((u-z)^{m+n-j})\left(a(u)_{(j)}b(u)\right)c(z)\right).$$

This completes the proof. □

Exercise 2.1 Show that

$$a(z)_{(-n-1)}b(z) = \frac{1}{n!} :(\partial_z^n a(z))b(z): \quad \text{for } n \geqslant 0.$$

2.6 Wick formula and an example (continued from Section 2.4)

Wick's formula that we shall present below is very useful to compute OPE's between mutually local fields.

Recall that the normally ordered product of fields $a^1(z), \ldots, a^k(z)$ over a vector spaces V is defined inductively from right to left:

$$:a^1(z)\ldots a^k(z): \;=\; :a^1(z)\ldots :a^{k-1}(z)a^k(z): \ldots :.$$

It is a sum of 2^k terms of the form

$$a^{i_1}(z)_+ a^{i_2}(z)_+ \ldots a^{j_1}(z)_- a^{j_2}(z)_- \ldots, \tag{2.24}$$

where $i_1 < i_2 \cdots, j_1 > j_2 > \cdots$ is a permutation of the index set $\{1, \ldots, k\}$.

Remark 2.2 It is clear from (2.24) that if $[a^i(z)_\pm, a^j(z)_\pm] = 0$ for all i, j, then $:a^1(z)\ldots a^k(z): \;=\; :a^{i_1}(z)\ldots a^{i_k}(z):$ for any permutation $\{i_1, \ldots, i_k\}$. It follows that in this case the normally ordered products is commutative and associative.

We write $\langle a^i, a^j\rangle = [a^i(z)_-, a^j(w)]$ for the *contraction* of $a^i(z)$ and $a^j(w)$. As already observed in the proof of Proposition 2.1, we have

$$\langle a^i, a^j\rangle = a^i(z)a^j(w) - :a^i(z)a^j(w):$$

so that $\langle a^i, a^j\rangle$ represents the "singular part" of $a^i(z)a^j(w)$.

Theorem 2.1 (Wick's formula) *Let $a^1(z), \dots, a^m(z)$ and $b^1(z), \dots, b^n(z)$ be two collections of fields such that the following properties hold:*

i). $[\langle a^i, b^j\rangle, a^k(z)_\pm] = 0$ and $[\langle a^i, b^j\rangle, b^k(z)_\pm] = 0$ for all i, j, k;
ii). $[a^i(z)_\pm, b^j(w)_\pm] = 0$ for all i and j.

Then one has the following OPE:

$$
\begin{aligned}
&{:}a^1(z)\dots a^m(z){:}\,{:}b^1(w)\dots b^n(w){:} \qquad (2.25)\\
&= \sum_{s=0}^{\min(m,n)} \sum_{\substack{i_1<\dots<i_s\\ j_1\neq\dots\neq j_s}} \Big(\langle a^{i_1}, b^{j_1}\rangle \cdots \langle a^{i_s}, b^{j_s}\rangle\, {:}a^1(z)\dots a^m(z) b^1(w)\dots b^n(w){:}_{(i_1,\dots,i_s;j_1,\dots,j_s)}\Big),
\end{aligned}
$$

where the subscript $(i_1, \dots, i_s; j_1, \dots, j_s)$ means that the fields $a^{i_1}(z)\dots a^{i_s}(z) b^{j_1}(w)\dots b^{j_s}(w)$ are removed.

Proof The typical term of the left-hand-side of (2.25) is

$$
\left(a^{j_1}(z)_+ a^{j_2}(z)_+ \dots a^{i_1}(z)_- a^{i_2}(z)_- \dots\right)\left(b^{k_1}(w)_+ b^{k_2}(w)_+ \dots b^{l_1}(w)_- b^{l_2}(w)_- \dots .\right)
$$

Then we have to move the $a^i(z)_-$ across the $b^j(w)_+$ in order to bring this product to the normally ordered product as in (2.24). Due to the condition (ii) of the theorem, we have

$$
a^i(z)_- b^j(w)_+ = b^j(w)_+ a^i(z)_- + \langle a^i, b^j\rangle .
$$

But due to condition (i), the contractions commute with all fields $b^k(w)_\pm$, hence can be moved to the left. This proves the theorem. □

Using Proposition 2.3 for $n = -1$ we obtain the *non-commutative Wick formula* :

$$
\begin{aligned}
&a(z)_{(m)}\, {:}b(z)c(z){:} \qquad (2.26)\\
&= {:}\left(a(z)_{(m)} b(z)\right) c(z){:} + {:}b(z)\left(a(z)_{(m)} c(z)\right){:} + \sum_{j=0}^{m-1} \binom{m}{j} \left(a(z)_{(j)} b(z)\right)_{(m-1-j)} c(z).
\end{aligned}
$$

Formulas (2.25) and (2.26) allow to compute OPE of arbitrary normally ordered product of pairwise local fields from the knowledge of the OPE of these fields if they form a closed system under n-th products for $n \in \mathbb{Z}_{\geqslant 0}$.

Remark 2.3 There is a Mathematica package [177] which provides a computer program for these OPE calculations.

Remark 2.4 It is not difficult to adapt Wick's formulas (2.25) and (2.26) in the case where V is a superspace, see [112, Section 3.3].

Keep the notation of Section 2.4. For $\alpha \in \mathbb{C}$, set

$$
L(z) = \frac{1}{2}{:}b(z)^2{:} + \alpha \partial_z b(z).
$$

By Lemma 2.3, $L(z)$ is local to $b(z)$ and itself. To compute the OPE's between them, we use Wick's formula.

Exercise 2.2 Using Wick's formula, show that

$$
\begin{aligned}
&{:}b(z)^2{:}\,{:}b(w)^2{:} \sim \frac{2}{(z-w)^4} + \frac{4}{(z-w)^2}{:}b(w)^2{:} + \frac{4}{(z-w)}{:}(\partial_w b(w))b(w){:},\\
&\partial_z b(z)({:}b(w)^2{:}) \sim -\frac{4}{(z-w)^3}b(w),\\
&({:}b(z)^2{:})\partial_w b(w) \sim \frac{4}{(z-w)^3}b(w) + \frac{4}{(z-w)^2}\partial_w b(w) + \frac{2}{(z-w)}\partial_w^2 b(w),\\
&\partial_z b(z)\partial_w b(w) \sim -\frac{6}{(z-w)^4},\\
&L(z)b(w) \sim -\frac{2\alpha}{(z-w)^3} + \frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{(z-w)},\\
&L(z)L(w) \sim \frac{(1-12\alpha^2)/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}.
\end{aligned}
\tag{2.27}
$$

2.7 Definition of vertex algebras

Definition 2.5 A *vertex algebra* is a vector space V equipped with the following data:

- (*the vacuum vector*) a vector $|0\rangle \in V$,
- (*the vertex operator*) a linear map

$$Y\colon V \to \mathscr{F}ields(V), \quad a \mapsto Y(a,z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

- (*the translation operator*) a linear map $T\colon V \to V$.

These data are subject to the following axioms:

- (*the vacuum axiom*) $Y(|0\rangle, z) = \mathrm{Id}_V$. Furthermore, for all $a \in V$,

  $$Y(a,z)|0\rangle \in V[[z]]$$

  and $\lim_{z\to 0} Y(a,z)|0\rangle = a$. In other words, $a_{(n)}|0\rangle = 0$ for $n \geqslant 0$ and $a_{(-1)}|0\rangle = a$,
- (*the translation axiom*) we have $T|0\rangle = 0$ and for any $a \in V$,

  $$[T, Y(a,z)] = \partial_z Y(a,z),$$

- (*the locality axiom*) for all $a, b \in V$, the fields $Y(a,z)$ and $Y(b,w)$ are mutually local, that is,

  $$(z-w)^N [Y(a,z), Y(b,w)] = 0 \tag{2.28}$$

for some $N = N_{a,b} \in \mathbb{Z}_{\geqslant 0}$.

Let V, W be vertex algebras. The tensor product $V \otimes W$ is a vertex algebra with the vacuum vector $|0\rangle \otimes |0\rangle$, the translation operator $T \otimes 1 + 1 \otimes T$, and the vertex operator $Y(a \otimes b, z) = Y(a,z) \otimes Y(b,z)$. A *vertex algebra homomorphism* from V to W is a linear map $\phi : V \to W$ such that $\phi(|0\rangle) = |0\rangle$, $\phi(Ta) = T\phi(a)$, and $\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b)$ for all $a, b \in V$, $n \in V$.

2.8 Goddard's uniqueness theorem and Borcherds identities

Theorem 2.2 (Goddard's uniqueness theorem) *Let V be a vertex algebra, and $A(z)$ a field on V. Suppose there exists a vector $a \in V$ such that*

$$A(z)|0\rangle = Y(a,z)|0\rangle$$

and $A(z)$ is local with $Y(b,z)$ for all $b \in V$. Then $A(z) = Y(a,z)$.

Proof Let $c \in V$. By the hypothesis and the locality axiom, we obtain that for N large enough, the following equalities hold in $V[[z^{\pm 1}, w^{\pm 1}]]$:

$$\begin{aligned}(z-w)^N A(z)Y(c,w)|0\rangle &= (z-w)^N Y(c,w)A(z)|0\rangle \\ &= (z-w)^N Y(c,w)Y(a,z)|0\rangle = (z-w)^N Y(a,z)Y(c,w)|0\rangle.\end{aligned}$$

Using the vacuum axiom, we deduce evaluating at $w = 0$ the above equalities, that for every $c \in V$,

$$A(z)c = Y(a,z)c,$$

that is, $A(z) = Y(a,z)$ as expected. □

Remark 2.5 We notice that in the proof of Goddard's uniqueness theorem, only the vacuum and the locality axioms are used, but not the translation axiom.

Corollary 2.1 *For $a \in V$, we have $Y(Ta,z) = \partial_z Y(a,z)$.*

Proof First, we have

$$Y(a,z)|0\rangle = e^{zT}a = \sum_{n \geqslant 0} \frac{1}{n!}(T^n a)z^n. \tag{2.29}$$

Indeed, $\partial_z Y(a,z)|0\rangle = [T, Y(a,z))]|0\rangle = TY(a,z)|0\rangle$. Using this repeatedly, we obtain $\partial_z^n Y(a,z)|0\rangle = T^n Y(a,z)|0\rangle$ since $(\partial_z Y(a,z))|0\rangle = \partial_z(Y(a,z)|0\rangle)$ which can checked directly. In particular, we have

$$\lim_{z \to 0} \partial_z^n Y(a,z)|0\rangle = T^n a,$$

which proves (2.29). Therefore, $\partial_z Y(a,z)|0\rangle = \partial_z(e^{zT}a) = e^{zT}(Ta) = Y(Ta,z)|0\rangle$ and the assertion follows from Theorem 2.2. □

By Corollary 2.1 and its proof, we have

$$(Ta)_{(n)} = -na_{(n-1)}$$

for all $a \in V, n \in \mathbb{Z}$, and

$$a_{(-n-1)}|0\rangle = \frac{1}{n!}T^n a, \qquad \text{for all} \qquad n > 0. \tag{2.30}$$

Proposition 2.4 (skew-symmetry) *Let V be a vertex algebra. Then the identity*

$$Y(a,z)b = e^{zT}Y(b,-z)a.$$

holds in $V((z))$.

Proof By (2.29) and locality,

$$\begin{aligned}(z-w)^N Y(a,z)e^{wT}b = (z-w)^N Y(a,z)Y(b,w)|0\rangle &= (z-w)^N Y(b,w)Y(a,z)|0\rangle \\ &= (z-w)^N Y(b,w)e^{zT}a\end{aligned}$$

for a sufficiently large N. Now we have

$$e^{zT}Y(b,w)e^{-zT} = \sum_{n\geqslant 0}\frac{1}{n!}\operatorname{ad}(zT)^n(Y(b,w)) = \sum_{n\geqslant 0}\frac{z^n}{n!}\partial_w^n Y(b,w) = Y(b,z+w)$$

in $(\operatorname{End} V)[[z^{\pm}, w^{\pm}]]$, where by $(z+w)^{-1}$ we understand its expansion $\tau_{z,w}(1/(z+w))$. (The formal variable version of the Taylor formula.) Hence,

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N e^{zT}Y(b,w-z)a,$$

where by $(w-z)^{-1}$ we understand its expansion $\tau_{z,w}(1/(w-z))$. Since there is no negative power of w on the left-hand-side, we can set $w = 0$ on both sides to get the desired formula. □

Lemma 2.4 *Let V be a vertex alegbra, $a, b \in V$, $n \in \mathbb{Z}$. Then,*

$$Y(a_{(n)}b,z) = Y(a,z)_{(n)}Y(b,z).$$

Proof By Lemma 2.3, the field $Y(a,z)_{(n)}Y(b,z)$ is mutually local to all $Y(v,z)$, $v \in V$. Hence it is sufficient to show that $Y(a_{(n)}b,z)|0\rangle = Y(a,z)_{(n)}Y(b,z)|0\rangle$ by Theorem 2.2.

We have $Y(a,z)_{(n)}Y(b,z)|0\rangle \in V[[z]]$ and

$$\lim_{z\to 0} Y(a,z)_{(n)}Y(b,z)|0\rangle = a_{(n)}b_{(-1)}|0\rangle = a_{(n)}b,$$

see (2.12). Also, we have

$$(Ta)_{(n)}b = -na_{(n-1)}b + a_{(n)}(Tb),$$

while (2.16), we have,

$$\partial_z \left(Y(a,z)_{(n)}Y(b,z)\right) = -nY(a,z)_{(n-1)}Y(b,z) + Y(a,z)_{(n)}(\partial_z Y(b,z)).$$

Hence, we obtain

$$\lim_{z\to 0} Y(T^k a_{(n)}b,z)|0\rangle = \lim_{z\to 0} \partial_z^k Y(a,z)_{(n)}Y(b,z)|0\rangle \tag{2.31}$$

inductively for all $k \geqslant 0$. By Corollary 2.1, this is equivalent to the required formula $Y(a_{(n)}b,z)|0\rangle = Y(a,z)_{(n)}Y(b,z)|0\rangle$. □

Theorem 2.3 (Borcherds identities) *Let V be a vertex algebra, $a, b \in V$. We have*

$$[a_{(m)}, b_{(n)}] = \sum_{i\geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}, \tag{2.32}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}), \tag{2.33}$$

for $m, n \in \mathbb{Z}$.

Proof By (2.11) and lemma 2.4, we have

$$Y(a,z)Y(b,w) \sim \sum_{i\geqslant 0} \frac{Y(a_{(i)}b,w)}{(z-w)^{i+1}}.$$

Hence, (2.32) follows from Proposition 2.1. As for (2.33), it is equivalent to the statement of Lemma 2.4 and formula (2.12). □

The relations (2.32) and (2.33) are called *Borcherds identities*.

Remark 2.6 The two identifies (2.32) and (2.33) are equivalent to the following single identity, for $p, q, r \in \mathbb{Z}$:

$$\sum_{i\geqslant 0} \binom{p}{i} (a_{(r+i)}b)_{(p+q-i)} = \sum_{i\geqslant 0} (-1)^i \binom{r}{i} \left(a_{(p+r-i)}b_{(q+i)} - (-1)^r b_{(q+r-i)}a_{(p+i)}\right), \tag{2.34}$$

which is equivalent to the *Jacobi identity* in [139], see [156]. Note that (2.34) is also equivalent to the following identity:

$$\begin{aligned} &\mathrm{Res}_{z-w}\, Y(Y(a,z-w)b,w)\tau_{w,z-w}F(z,w) \\ &\quad = \mathrm{Res}_z\, Y(a,z)Y(b,w)\tau_{z,w}F(z,w) - \mathrm{Res}_z\, Y(b,w)Y(a,z)\tau_{w,z}F(z,w), \end{aligned} \tag{2.35}$$

where $F(z,w) = z^p w^q (z-w)^r$.

Remark 2.7 It is easy to adapt the definition of a vertex algebra to the supercase. To be more specific, if $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a superspace, then the data and axioms shoud be modified as follows: if $a \in V_{\bar{i}}$, then all Fourier modes of $Y(a,z)$ should be endomorphisms of V of parity $\bar{i}$, $|0\rangle$ should be an element of $V_{\bar{0}}$, T should have even parity and the locality axiom should be:

$$(z-w)^N Y(a,z)Y(b,w) = (-1)^{|a||b|}(z-w)^N Y(b,w)Y(a,z)$$

for N sufficiently large, where $|a|$ denotes the parity of $a \in V$. The Borcherds identities have to be understood in the supercase as follows:

$$[a_{(m)}, b_{(n)}] = a_{(m)}b_{(n)} - (-1)^{|a||b|}b_{(n)}a_{(m)} = \sum_{i\geqslant 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}, \tag{2.36}$$

$$(a_{(m)}b)_{(n)} = \sum_{j\geqslant 0}(-1)^j\binom{m}{j}\left(a_{(m-j)}b_{(n+j)} - (-1)^{|a||b|}(-1)^m b_{(m+n-j)}a_{(j)}\right). \tag{2.37}$$

2.9 Commutative vertex algebras

A vertex algebra V is called *commutative* if all vertex operators $Y(a,z)$, $a \in V$, commute each other (i.e., we have $N_{a,b} = 0$ in the locality axiom (2.28)). This condition is equivalent to that

$$[a_{(m)}, b_{(n)}] = 0 \quad \text{for all} \quad a, b \in V,\ m, n \in \mathbb{Z}.$$

This condition is also equivalent to that $a_{(n)}b = 0$ for all $n \geqslant 0$, $a, b \in V$, that is, $Y(a,z) \in \operatorname{End} V[[z]]$ for all $a \in V$. Indeed, if $Y(a,z) \in \operatorname{End} V[[z]]$ for all $a \in V$ then V is commutative by (2.32). Conversely, if V is commutative, then $a_{(n)}b = a_{(n)}b_{(-1)}|0\rangle = b_{(-1)}a_{(n)}|0\rangle = 0$ for $n \geqslant 0$.

Suppose that V is commutative. Then, the relation (2.33) for $m = n = -1$ simplifies to $(a_{(-1)}b)_{(-1)} = a_{(-1)}b_{(-1)}$, that is,

$$(a_{(-1)}b)_{(-1)}c = a_{(-1)}(b_{(-1)}c).$$

for all $a, b, c \in V$. It follows that a commutative vertex algebra has a structure of a unital commutative algebra with the product:

$$a \cdot b = a_{(-1)}b,$$

where the unit is given by the vacuum vector $|0\rangle$. The translation operator T of V acts on V as a derivation with respect to this product:

$$T(a \cdot b) = (Ta)\cdot b + a\cdot(Tb).$$

Therefore a commutative vertex algebra has the structure of a differential algebra, see Definition 1.1.

The converse holds according to the following exercice.

Exercise 2.3 Show that a differential algebra R with a derivation ∂ carries a canonical commutative vertex algebra structure such that the vacuum vector is the unit, and

$$Y(a,z)b = \left(e^{z\partial}a\right)b = \sum_{n\geqslant 0}\frac{z^n}{n!}(\partial^n a)b \quad \text{for all} \quad a,b\in R.$$

This correspondence gives the following result.

Theorem 2.4 ([39]) *The category of commutative vertex algebras is the same as that of differential algebras.*

Example 2.1 If $X = \operatorname{Spec} R$ is an affine scheme, then $(\mathscr{O}(\mathscr{J}_\infty R), T)$ is a differential algebra (see Section 1.1 and Section 1.2,) hence a commutative vertex algebra by Theorem 2.4, where $T = \partial$ is the derivation defined by (1.2). More generally, $(\pi_\infty)_*\mathscr{O}_{\mathscr{J}_\infty X}$ is a sheaf of commutative vertex algebras on a scheme X.

2.10 Vertex subalgebra, commutant and center

Definition 2.6 A subspace W of a vertex algebra V is called a *vertex subalgebra* if $|0\rangle \in W$, $TW\subset W$, and $a_{(n)}b\in W$ for all $a,n\in W$, $n\in\mathbb{Z}$.

Let W be a vertex subalgebra of V. We set

$$\operatorname{Com}(W,V) = \{v\in V\colon\, [w_{(m)}, v_{(n)}] = 0 \text{ for all } w\in W,\ m,n\in\mathbb{Z}\}. \tag{2.38}$$

Then,

$$\operatorname{Com}(W,V) = \{v\in V\colon\, w_{(n)}v = 0 \text{ for all } w\in W, n\geqslant 0\}. \tag{2.39}$$

Indeed, if $w_{(n)}v = 0$ for all $w\in W$, $n\geqslant 0$, then $v\in\operatorname{Com}(W,V)$ by (2.32). Conversely, if $v\in\operatorname{Com}(W,V)$ then $w_{(n)}v = w_{(n)}v_{(-1)}|0\rangle = v_{(-1)}w_{(n)}|0\rangle = 0$ for $n\geqslant 0$. It is straightforward to see that $\operatorname{Com}(W,V)$ is a vertex subalgebra of V. $\operatorname{Com}(W,V)$ is called the *commutant* of W in V, or the *coset* of V by W.

The same line of arguments shows that we also have

$$\operatorname{Com}(W,V) = \{v\in V\colon\, v_{(n)}w = 0 \text{ for all } w\in W, n\geqslant 0\}. \tag{2.40}$$

Vertex sualgebras W_1, W_2 of V are said to form a *dual pair* of $W_1 = \operatorname{Com}(W_2,V)$ and $W_2 = \operatorname{Com}(W_1,V)$.

The commutant $\operatorname{Com}(V,V)$ of V in V is called the *center* of V is denoted also by $Z(V)$.

2.11 Vertex algebra of local fields and reconstruction theorem

Theorem 2.5 ([141]) *Let M be a vector space, and let $\mathcal{V}$ be a subspace of $\mathscr{F}ields(M)$ that satisfies the following properties.*

i). $a(z)$ and $b(z)$ are mutually local for all $a(z), b(z) \in \mathcal{V}$,
ii). $\mathrm{id}_M \in \mathcal{V}$,
iii). $a(z)_{(n)}b(z) \in \mathcal{V}$ for all $a(z), b(z) \in \mathcal{V}$, $n \in \mathbb{Z}$.

Then $\mathcal{V}$ has the structure of a vertex algebra with the vacuum vector id_M, the translation operator ∂_z, and

$$Y(a(z), \xi) = \sum_{n\in\mathbb{Z}} a(z)_{(n)}\xi^{-n-1},$$

where $a(z)_{(n)}$ denotes the linear map $b(z) \mapsto a(z)_{(n)}b(z)$ on $\mathcal{V}$.

Proof By (2.14) for $n = 2$, $\mathcal{V}$ is stable under the translation operator $T = \partial_z$. The vaccum axiom is satisfied by (2.14) and (2.15). By definition and (2.16), we have $[\partial_z, Y(a(z), \xi)] = Y(\partial_z a(z), \xi)$. Since $\mathrm{Res}_{w=0}\, \partial_w((w-z)^n[a(w), b(z)]) = 0$, we get that $(\partial_z a(z))_{(n)} = -na(z)_{(n-1)}$, and hence the translation axiom holds. The locality axiom holds by Proposition 2.3, in view of Proposition 2.1. $\square$

Let $\mathcal{S}$ be a set of pairwise mutually local fields on a vector space M. Denote by $\langle\mathcal{S}\rangle_M$ the subspace of $\mathscr{F}ields(M)$ spanned by the fields constructed by successive application of the n-th products to the fields in $\mathcal{S}$ as well as the identify field id_M. By Lemma 2.3 and Theorem 2.5, $\langle\mathcal{S}\rangle_M$ has a structure of a vertex algebra. The vertex algebra $\langle\mathcal{S}\rangle_M$ is called the *vertex algebra of the local fields* generated by $\mathcal{S}$.

Lemma 2.5 (State-field correspondence) *Let V be a vertex algebra, $\mathcal{S} = \{Y(a, z) : a \in V\} \subset \mathscr{F}ields(V)$. Then the linear map*

$$V \to \langle\mathcal{S}\rangle_V, \quad a \mapsto Y(a, z), \tag{2.41}$$

is an isomorphism of vertex algebras.

Proof It is a vertex algebra homomorphism by Corollary 2.1 and Lemma 2.4. It is an isomorphism since we have the inverse map $Y(a, z) \mapsto \lim_{z\to 0} Y(a, z)|0\rangle$. $\square$

Theorem 2.6 (Reconstruction theorem [78]) *Let V be a vector space, $|0\rangle$ a nonzero vector, and T an endomorphism of V. Let I be a set and $\{a^i\}_{i\in I}$ be a collection of vectors in V. Suppose also that we have given fields*

$$a^i(z) = \sum_{n\in\mathbb{Z}} a^i_{(n)} z^{-n-1} \in (\mathrm{End}\, V)[[z, z^{-1}]], \quad i \in I,$$

such that the following conditions holds:

(1) *For all i, $a^i(z)|0\rangle \in a^i + zV[[z]]$,*

(2) *$T|0\rangle = 0$ and $[T, a^i(z)] = \partial_z a^i(z)$ for all i,*
(3) *all fields $a^i(z)$ are mutually local,*
(4) *V is spanned by the vectors*

$$a^{i_1}_{(n_1)} \dots a^{i_m}_{(n_m)}|0\rangle, \quad n_j < 0.$$

Then there exists a unique vertex algebra structure on V such that $Y(a^i, z) = a^i(z)$ for $i \in I$ and $|0\rangle$ is the vacuum vector.

Proof Let $\mathcal{V} = \langle a^i(z) \colon i \in I\rangle_V$, the vertex algebra of local fields on V generated by $a^i(z)$.

By the assumption (1), we deduce by induction and (2.12) that $a(z)|0\rangle \in V[[z]]$ for all $a(z) \in \mathcal{V}$. Moreover,

$$\lim_{z\to 0} a^i(z)_{(n)} b(z)|0\rangle = a^i_{(n)} b_{(-1)}|0\rangle \tag{2.42}$$

for $i \in I$, $n \in \mathbb{Z}$ and $b(z) \in \mathcal{V}$. Then by induction,

$$\lim_{z\to 0} a^{i_1}(z)_{(n_1)} \dots a^{i_r}(z)_{(n_r)} b(z)|0\rangle = a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)} b_{(-1)}|0\rangle \tag{2.43}$$

for $i_j \in I$, $n_j \in \mathbb{Z}$ and $b(z) \in \mathcal{V}$.

Since $\operatorname{ad} T$ and ∂_z act as derivations on the m-th product, we deduce by induction and the assumption (2) that

$$[T, a(z)] = \partial_z a(z)$$

for all $a(z) \in \mathcal{V}$. It follows that $\partial_z^n a(z)|0\rangle = \operatorname{ad}(T)^n(a(z))|0\rangle = T^n a(z)|0\rangle$ for all $n \geqslant 0$. By setting $z = 0$ on both sides, we get that $\lim_{z\to 0} \partial_z^n a(z)|0\rangle = T^n a_{(-1)}|0\rangle$, or equivalently,

$$a(z)|0\rangle = e^{zT} a_{(-1)}|0\rangle. \tag{2.44}$$

Consider the linear map

$$\mathcal{V} \to V, \quad a(z) \mapsto \lim_{z\to 0} a(z)|0\rangle = a_{(-1)}|0\rangle. \tag{2.45}$$

By the assumption (4) and (2.43) with $a(z) = \mathrm{id}_V$, this map is surjective. We claim that this map is injective as well. Indeed, if $a_{(-1)}|0\rangle = 0$, then $a(z)|0\rangle = 0$ by (2.44). It follows in the same manner as the proof of Theorem 2.2 that $a(z) = 0$.

It is now clear that there is a unique algebra structure on V which makes the linear isomorphism (2.45) a vertex algebra isomorphism. Namely, we set

$$Y(a_{(-1)}|0\rangle, z) = a(z)$$

for $a(z) \in \mathcal{V}$. □

A collection $\{a^i : i \in I\}$ of elements of a vertex algebra V is called *strong generators* of V if V is spanned by

$$a^{i_1}_{(-n_1)} \dots a^{i_s}_{(-n_s)}|0\rangle$$

with $s \geqslant 0$, $n_r \geqslant 1$ and $i_r \in I$. In view of Theorem 2.6, the structure of V is completely determined by the OPEs among $a^i(z)$, $i \in I$.

Remark 2.8 By Exercise 2.1, the vertex operator for V in Theorem 2.6 can be explicitly described as

$$\begin{aligned} &Y(a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \dots a^{i_r}_{(-n_r-1)}|0\rangle, z) \\ &= \frac{1}{n_1!n_2!\dots n_r!} {\circ \atop \circ} (\partial_z^{n_1} a^{i_1}(z))(\partial_z^{n_2} a^{i_2}(z)) \dots (\partial_z^{n_r} a^{i_r}(z)) {\circ \atop \circ} \end{aligned}$$

for $n_i \geqslant 1$.

2.12 Example (continued from Section 2.6)

Let

$$\pi = \mathbb{C}[b_{-1}, b_{-2}, \dots,].$$

Then π is a smooth $\mathcal{B}$-module on which b_n, $n \geqslant 0$, acts as $n\frac{\partial}{\partial b_{-n}}$, and b_{-n}, $n > 0$, acts as multiplication by b_{-n}. Define

$$T = \sum_{n>0} n b_{-n-1} \frac{\partial}{\partial b_{-n}} \in \operatorname{End} \pi.$$

Then $[T, b(z)] = \partial_z b(z)$ on π. It follows from Theorem 2.6 that there is a unique vertex algebra structure on π such that 1 is the vacuum vector and $Y(b_{-1}, z) = b(z)$.

Exercise 2.4 Let M be a smooth $\mathcal{B}$-module.

i). Show that the following correspondence gives the vertex algebra $\langle b(z)\rangle_M$ a $\mathcal{B}$-module structure:

$$\mathcal{B} \to \operatorname{End}(\langle b(z)\rangle_M) \quad b_n \mapsto b(z)_{(n)}$$

ii). Show that there is a surjective homomorphism $\pi \to \langle b(z)\rangle_M$ of vertex algerbas.

Exercise 2.5 i). Set $\omega = \frac{1}{2}b_{-1}^2 + \alpha b_{-2} \in \pi$, so that $L(z) = Y(\omega, z)$. Verify that the OPE (2.27) is equivalent to the following relations:

$$b_0\omega = 0, \quad b_1\omega = b_{-1}, \quad b_2\omega = 2\alpha.$$

ii). Show that $L_{-1} = T$ on π.

2.13 Vertex ideals, vertex algebra modules and quotient vertex algebras

A *representation* M of a vertex algebra V, or a V*-module* M, is a vertex algebra homomorphism from V to a vertex algebra of the local fields on M. We denote by $Y_M(a,z) = a^M(z) = \sum_{n\in\mathbb{Z}} a^M_{(n)} z^{-n-1}$ the image of $a \in V$ in $\mathscr{F}ields(M)$, or simply by $Y(a,z) = a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ if no confusion should occur.

The following assertion is clear.

Lemma 2.6 *A vector space M is a module over a vertex algebra V if and only if there exists a linear map $V \to \mathscr{F}ields(M)$, $a \mapsto Y_M(a,z)$, such that*

$$Y_M(|0\rangle, z) = \mathrm{id}_M, \tag{2.46}$$

$$[Y_M(a,z), Y_M(b,w)] = \sum_{j\geqslant 0} Y_M(a_{(j)}b, w)\frac{1}{j!}\partial_w^j \delta(z-w), \tag{2.47}$$

$$Y_M(a_{(n)}b, z) = Y_M(a,z)_{(n)} Y_M(b,z) \tag{2.48}$$

for all $a, b \in V$, $n \in \mathbb{Z}$,

A vertex algebra is a module over itself by (2.41), which is called the *adjoint representation*.

By definition, a subspace N of a V-module M is a *submodule* if $a_{(n)}N \subset N$ for all $a \in V$, $n \in \mathbb{Z}$. It is clear that the category V-Mod of V-modules is an abelian category.

A T-stable proper submodule of the adjoint representation is called an *ideal* of V. If $f : V \to V'$ is a vertex algebra homomorphism, ker f is an ideal of V. For an ideal I of V, the quotient V/I inherits the vertex algebra structure from V. Indeed, there are two ways to see this. One is to use Reconstruction Theorem (Theorem 2.6), since V/I is spanned by the images of $a_{(-1)}|0\rangle$, $a \in V$. The other one is to use the skew symmetry (Proposition 2.4), as it shows that $Y(a,z)b = e^{zT}Y(b,-z)a \in I$ for $a \in I$, $b \in V$.

The category V/I-Mod is a full subcategory of V-Mod consisting of objects M such that $Y_M(a,z) = 0$ for all $a \in I$.

Exercise 2.6 Show that the vertex algebra π is simple, that is, there is no non-trivial ideal of π. This implies that the vertex algebra $\langle b(z)\rangle_M$ of local fields on *any* non-trivial smooth $\mathscr{B}$-module M is isomorphic to π.

2.14 Loop spaces and commutative vertex algebras

Let V be a commutative vertex algebra, and let M be a V-module. Then,

$$[Y_M(a,z), Y_M(b,w)] = 0$$

for all $a, b \in V$ by (2.47). However, $Y_M(a, z)$ needs not be in $(\operatorname{End} M)[[z]]$. This implies that a V-module as a vertex algebra is *not* the same as a V-module as a differential algebra. In fact, we have the following assertion.

Theorem 2.7 *Let X be an affine scheme. Then the category of vertex $\mathscr{O}(\mathscr{J}_\infty X)$-modules is the same as the category of smooth $\mathscr{O}(\mathscr{L}X)$-modules.*

Here, by *smooth $\mathscr{O}(\mathscr{L}X)$-module* we mean an $\mathscr{O}(\mathscr{L}X)$-module M such that, in the notation of §1.6, $f_{(n)}.m = 0$ for sufficiently large n.

Proof First, let $X = \mathbb{A}^N = \operatorname{Spec} \mathbb{C}[x^1, \dots, x^N]$. Recall that

$$\mathscr{O}(\mathscr{L}X) = \varprojlim_r \mathbb{C}[x^i_{(n)} : i = 1, \dots, N,\ n \leqslant r]$$

see (1.17). Let M be a smooth $\mathscr{O}(\mathscr{L}X)$-module. Then

$$x^i(z) := \sum_{n \in \mathbb{Z}} x^i_{(n)} z^{-n-1}$$

is a field on M, since $x^i_{(n)}$ acts as zero for a sufficiently large n because M is smooth. Moreover, $x^i(z)$ and $x^j(z)$ are mutually local as they commute each other. Therefore, we have a well-defined vertex algebra homomorphism

$$\mathscr{O}(\mathscr{J}_\infty X) \to \langle x^i(z) : i = 1, \dots N \rangle_M \subset (\operatorname{End} M)[[z, z^{-1}]]$$

that sends $x^i \in \mathscr{O}(X) \subset \mathscr{O}(\mathscr{J}_\infty X)$ to $xi(z)$. Conversely, let M be a vertex $\mathscr{O}(\mathscr{J}_\infty X)$-module. Then the correspondence

$$\mathscr{O}(\mathscr{L}X) \to \operatorname{End}(M), \quad x^i_{(n)} \mapsto \operatorname{Res}_{z=0} z^n Y_M(x^i, z),$$

defines a smooth $\mathscr{O}(\mathscr{L}X)$-module structure on M. It is clear that this correspondence is compatible with the morphisms.

Next, let $X = \operatorname{Spec} R$ with

$$R = \mathbb{C}[x^1, x^2, \cdots, x^N]/(f_1, f_2, \cdots, f_r).$$

Then $\mathscr{O}(\mathscr{J}_\infty X) = \mathscr{O}(\mathscr{J}_\infty \mathbb{A}^N)/I$, where $I = \langle T^j f_i : i = 1, \dots, r, j \geqslant 0 \rangle$. Hence, $\mathscr{O}(\mathscr{J}_\infty X)$ -Mod is the full subcategory of $\mathscr{O}(\mathscr{J}_\infty \mathbb{A}^N)$ -Mod consisting of modules M such that

$$Y_M(f_i, z) = 0$$

for all $i = 1, \dots, r$. (Here we have used the fact that $Y_M(Ta, z) = \partial_z Y_M(a, z)$.) But under the above identification of $\mathscr{O}(\mathscr{J}_\infty \mathbb{A}^N)$ -Mod with the category of smooth $\mathscr{O}(\mathscr{L}\mathbb{A}^N)$-modules, this is nothing but the category of smooth $\mathscr{O}(\mathscr{L}X)$-modules. □

One of the advantages of vertex algebras to loop spaces is that one can avoid using completions, which can be sometimes tedious.

2.15 Conical vertex algebras

A *Hamiltonian* of a vertex algebra V is a semisimple operator H on V satisfying

$$[H, a_{(n)}] = -(n+1)a_{(n)} + (Ha)_{(n)} \tag{2.49}$$

for all $a \in V$, $n \in \mathbb{Z}$.

Definition 2.7 A vertex algebra equipped with a Hamiltonian H is called *graded*. In that case, set $V_\Delta = \{a \in V : Ha = \Delta a\}$ for $\Delta \in \mathbb{C}$, so that $V = \bigoplus_{\Delta \in \mathbb{C}} V_\Delta$. For $a \in V_\Delta$, Δ is called the *conformal weight* of a and it is denoted by Δ_a. We have

$$a_{(n)}b \in V_{\Delta_a + \Delta_b - n - 1} \tag{2.50}$$

for homogeneous elements $a, b \in V$. A graded vertex algebra is called *conical* if there exists a positive integer m such that $V = \bigoplus_{\Delta \in \frac{1}{m}\mathbb{Z}_{\geqslant 0}} V_\Delta$ and $V_0 = \mathbb{C}$.

We set

$$a_n = a_{(n+\Delta_a - 1)}$$

for $n \in -\Delta_a + \mathbb{Z}$, so that $a_n V_\Delta \subset V_{\Delta - n}$. Then we have

$$a(z) = \sum_{n \in -\Delta_a + \mathbb{Z}} a_n z^{-n-\Delta_a}, \tag{2.51}$$

which is more standard notation in physics than (2.7).

Any (proper) graded ideal of a conical vertex algebra V does not contain the vacuum vector $|0\rangle$, and hence, there is a unique simple graded quotient of V.

Let X be a conical affine scheme, that is, $X = \operatorname{Spec} R$ with a graded ring $R = \bigoplus_{\Delta \in \frac{1}{m}\mathbb{Z}_{\geqslant 0}} R_\Delta$ such that $R_0 = \mathbb{C}$, where m is some positive integer. Then the commutative vertex algebra $\mathscr{O}(\mathscr{J}_\infty X) = \mathscr{J}_\infty R$ is conical, where the Hamiltonian is defined by

$$[H, f_{(-n)}] = (\Delta + n - 1) f_{(-n)}, \quad f \in R_\Delta.$$

In particular the scheme $\mathscr{J}_\infty X$ is conical, and we have a contracting $\mathbb{C}^*$-action on $\mathscr{J}_\infty X$ corresponding to the comorphism $\mathscr{J}_\infty R \to \mathbb{C}[t, t^{-1}] \otimes \mathscr{J}_\infty R$, $f_{(-n)} \mapsto t^{\Delta + n - 1} \otimes f_{(-n)}$ ($f \in R_\Delta$).

Chapter 3
Examples of non-commutative vertex algebras

We present in this chapter important first examples of non-commutative vertex algebras: the Heisenberg vertex algebras (see Example 3.1), the universal affine vertex algebras (cf. Section 3.1) and the Virasoro vertex algebras (cf. Section 3.2). Heisenberg vertex algebras are particular cases of the affine one, and these three families of examples are all constructed from infinite-dimensional Lie algebras (affine Kac-Moody Lie algebras, Virasoro Lie algebras). We will see next chapter more sophisticated examples using BRST reduction.

By considering quotients of these examples of vertex algebras, that is, quotient by vertex ideals, we construct many other interesting families of vertex algebras.

3.1 Universal affine vertex algebras

Let $\mathfrak{a}$ be a Lie algebra endowed with a symmetric invariant bilinear form κ. Here, a bilinear form κ on $\mathfrak{a}$ is called invariant if $\kappa([x, y], z) = \kappa(x, [y, z]) = 0$ for $x, y, z \in \mathfrak{a}$. Let

$$\hat{\mathfrak{a}}_\kappa = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$$

be the *Kac-Moody affinization* of $\mathfrak{a}$. It is a Lie algebra with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}\kappa(x, y)\mathbf{1}, \qquad [\mathbf{1}, \hat{\mathfrak{a}}_\kappa] = 0,$$

for all $x, y \in \mathfrak{a}$ and all $m, n \in \mathbb{Z}$, where $\delta_{i,j}$ is the Kronecker symbol.

An $\hat{\mathfrak{a}}_\kappa$-module M is called *smooth* if for any $m \in M$ there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that $xt^n m = 0$ for all $x \in \mathfrak{g}$, $n \geqslant N$ or, equivalently,

$$x(z) = \sum_{n\in\mathbb{Z}} (xt^n)z^{-n-1}$$

is a field on M for all $x \in \mathfrak{a}$.

Lemma 3.1 *For any smooth $\hat{\mathfrak{a}}_\kappa$-module M the fields $x(z)$, $y(z)$, $x, y \in \mathfrak{a}$, are mutually local, and we have*

$$x(z)y(w) \sim \frac{1}{z-w}[x,y](w) + \frac{\kappa(x,y)}{(z-w)^2}.$$

Proof The assertion is equivalent to the fact that

$$[x(z), y(w)] = [x,y](w)\delta(z-w) + \kappa(x,y)\partial_w\delta(z-w),$$

which can be checked directly. □

Let M be a smooth $\hat{\mathfrak{a}}_\kappa$-module on which the central element $\mathbf{1}$ acts as the identity. By Lemma 3.1, $\langle x(z)\colon x \in \mathfrak{a}\rangle_M$ has a structure of vertex algebras (see Section 2.11). Moreover, the correspondence

$$\hat{\mathfrak{a}}_\kappa \ni x \otimes t^n \mapsto x(z)_{(n)} \in \mathrm{End}(\langle x(z)\colon x \in \mathfrak{a}\rangle_M)$$

gives an $\hat{\mathfrak{a}}_\kappa$-module structure on the vertex algebra $\langle x(z)\colon x \in \mathfrak{a}\rangle_M$, see Proposition 2.3. By Proposition 2.3, we find that the $\hat{\mathfrak{a}}_\kappa$-module $\langle x(z)\colon x \in \mathfrak{a}\rangle_M$ is generated by the vector id_M, which satisfies the condition

$$\mathfrak{a}[t]\,\mathrm{id}_M = 0.$$

Hence, by the Frobenius reciprocity, there is an $\hat{\mathfrak{a}}_\kappa$-module homomorphism from the $\hat{\mathfrak{a}}_\kappa$-module

$$V^\kappa(\mathfrak{a}) := U(\hat{\mathfrak{a}}_\kappa) \otimes_{U(\mathfrak{a}[t]\oplus\mathbb{C}\mathbf{1})} \mathbb{C}, \tag{3.1}$$

where $\mathbb{C}$ is a one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}$ on which $\mathfrak{a}[t]$ acts trivially and $\mathbf{1}$ acts as the identity, to the $\hat{\mathfrak{a}}_\kappa$-module $\langle x(z)\colon x \in \mathfrak{a}\rangle_M$.

By the Poincaré-Birkhoff-Witt Theorem, the direct sum decomposition (as a vector space)

$$\hat{\mathfrak{a}}_\kappa = (\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus (\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1})$$

gives us the isomorphism of vector spaces

$$U(\hat{\mathfrak{a}}_\kappa) \cong U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes U(\mathfrak{a}[t] \oplus \mathbb{C}\mathbf{1}),$$

whence

$$V^\kappa(\mathfrak{a}) \cong U(\mathfrak{a} \otimes t^{-1}\mathbb{C}[t^{-1}])$$

as $\mathbb{C}$-vector spaces.

The space $V^\kappa(\mathfrak{a})$ is naturally graded,

$$V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} V^\kappa(\mathfrak{a})_\Delta, \tag{3.2}$$

where the grading is defined by

$$\deg(x^{i_1}t^{-n_1})\dots(x^{i_m}t^{-n_m})|0\rangle = \sum_{i=1}^{m} n_i,$$

where $|0\rangle = 1 \otimes 1$. We have $V^\kappa(\mathfrak{a})_0 = \mathbb{C}|0\rangle$, and we identify $\mathfrak{a}$ with $V^\kappa(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$.

Proposition 3.1 ([80, 141]) *There is a unique vertex algebra structure on $V^\kappa(\mathfrak{a})$ such that $|0\rangle = 1 \otimes 1$ is the vacuum vector and $Y(x, z) = x(z)$ for $x \in \mathfrak{g}$. Moreover, there is a surjective homomorphism $V^\kappa(\mathfrak{a}) \to \langle x(z) \colon x \in \mathfrak{a}\rangle_M$ of vertex algebras for any smooth $\hat{\mathfrak{a}}_\kappa$-module M on which* **1** *acts as the identity.*

Proof The first assertion is clear from Theorem 2.6. For the second one, first recall that there is a homomorphism of $\hat{\mathfrak{a}}_\kappa$-modules $V^\kappa(\mathfrak{a}) \to \langle x(z) \colon x \in \mathfrak{a}\rangle_M$. It is clearly a homomorphism of vertex algebras by construction. Since both $V^\kappa(\mathfrak{a})$ and $\langle x(z) \colon x \in \mathfrak{a}\rangle_M$ are generated by $|0\rangle$, the surjectivity follows. □

The vertex algebra $V^\kappa(\mathfrak{a})$ is called the *universal affine vertex algebra* associated with $\mathfrak{a}$ and κ. It is a conical vertex algebra by the grading (3.2). The unique simple graded quotient $L_\kappa(\mathfrak{a})$ of $V^\kappa(\mathfrak{a})$ is called the *simple affine vertex algebra* associated with $\mathfrak{a}$ and κ.

Proposition 3.2 *The category $V^\kappa(\mathfrak{a})$ -Mod of $V^\kappa(\mathfrak{a})$-modules is the same as that of smooth representations of $\hat{\mathfrak{a}}_\kappa$ on which* **1** *acts as the identity.*

Proof Any $V^\kappa(\mathfrak{a})$-modules is a smooth $\hat{\mathfrak{a}}_\kappa$-module by the correspondence $xt^n \mapsto \operatorname{Res}_{z=0}(z^n x(z))$. Conversely, we have a vertex algebra homomorphism $V^\kappa(\mathfrak{a}) \to \langle x(z)\rangle_M$ for any smooth $\hat{\mathfrak{a}}_\kappa$-module M on which **1** acts as the identity, and hence, M is a $V^\kappa(\mathfrak{a})$-module. It is clear that this correspondence is compatible with morphisms. □

By Proposition 3.2, the category $L_\kappa(\mathfrak{a})$ -Mod of $L_\kappa(\mathfrak{a})$-modules is a full-subcategory of the category of smooth $\hat{\mathfrak{a}}_\kappa$-module consisting of objects M on which $Y_M(v, z) = 0$ for any element v in the kernel of the natural surjection $V^\kappa(\mathfrak{a}) \to L_\kappa(\mathfrak{a})$.

Example 3.1 Let $\mathfrak{h}$ be a vector space viewed as a commutative Lie algebra, and κ be any bilinear form on $\mathfrak{h}$. Then $V^\kappa(\mathfrak{h})$ is the *Heisenberg vertex algebra associated with $\mathfrak{h}$ and κ*. In the case that $\mathfrak{h}$ is one-dimensional and κ is a nonzero bilinear form, then $V^\kappa(\mathfrak{h})$ is isomorphic to the vertex algebra π in Section 2.12.

Example 3.2 Let us consider another important example. Assume that $\mathfrak{a}$ is a simple Lie algebra $\mathfrak{g}$, and that

$$\kappa = \frac{k}{2h^\vee} \times \text{Killing form of } \mathfrak{g}, \qquad \text{for } k \in \mathbb{C},$$

where $h^\vee$ its dual Coxeter number of $\mathfrak{g}$. The reader is referred to Appendix A for main notations and standard facts about simple Lie algebras (Section A.1), and the corresponding affine Kac-Moody Lie algebras (Section A.2).

In this case, $V^\kappa(\mathfrak{a})$ is identical to the $\hat{\mathfrak{g}}$-module

$$V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ is the affine Kac-Moody algebra associated with $\mathfrak{g}$ as in Appendix A, and $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K$ on which $\mathfrak{g}[t]$ acts trivially and K acts as multiplication by k. We will preferably use the notation $V^k(\mathfrak{g})$ in this case.

The representation $V^k(\mathfrak{g})$ is a highest weight representation of $\hat{\mathfrak{g}}$ with highest weight $k\Lambda_0$, where Λ_0 is the highest weight of the *basic* representation (it corresponds to $k = 1$)[1], and highest weight vector v_k, where v_k denotes the image of $1 \otimes 1$ in $V^k(\mathfrak{g})$. According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation. As the Schur Lemma extends to a representation with countable dimension[2], the result holds for highest weight $\hat{\mathfrak{g}}$-modules.

A representation M is said to be *of level* k if K acts as $k\mathrm{Id}$ on M (see §A.5.2). Then $V^k(\mathfrak{g})$ is by construction of level k.

The vertex algebra $V^k(\mathfrak{g})$ is also called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. The simple quotient $L_\kappa(\mathfrak{g})$ is denoted also by $L_k(\mathfrak{g})$ and is called the *simple affine vertex algebra associated with* $\mathfrak{g}$ *at level* k.

Exercise 3.1 Let V be a vertex algebra, and suppose that there exists a vertex algebra homomorphism $\phi : V^\kappa(\mathfrak{g}) \longrightarrow V$, so that V is a $\hat{\mathfrak{g}}_\kappa$-module. Show that

$$\mathrm{Com}(\phi(V^\kappa(\mathfrak{g})), V) = V^{\mathfrak{g}[t]},$$

where $V^{\mathfrak{g}[t]} = \{v \in V : \mathfrak{g}[t]v = 0\}$.

3.2 The Virasoro vertex algebra

Let $Vir = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$ be the *Virasoro Lie algebra*, with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m,0}C,$$
$$[C, Vir] = 0.$$

A Vir-module M is called *smooth* if

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

is a field on M. For any smooth Vir-module M the fields $L(z)$ is local to itself, and we have

[1] that is, the dual of K in $\hat{\mathfrak{h}}^*$ with respect to a basis of $\hat{\mathfrak{h}}$ adapted to the decomposition $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$.

[2] i.e., it admits a countable set of generators.

$$L(z)L(w) \sim \frac{1}{z-w}\partial_w L(w) + \frac{2}{(z-w)^2}L(w) + \frac{C/2}{(z-w)^4}.$$

A Vir-module M is said to be of *central charge* $c \in \mathbb{C}$ if the central element C acts as multiplication by c.

Let M be a smooth Vir-module of central charge c. Then $\langle L(z)\rangle_M$ is a smooth Vir-module of central charge c by the action $L_n \mapsto L(z)_{(n+1)}$. It is generated by id_M and we have $L(z)_{(n)}\,\mathrm{id}_M = 0$ for $n \geqslant 0$. Similarly to the case of $V^\kappa(\mathfrak{a})$ we obtain that $\langle L(z)\rangle_M$ is a quotient of the induced representation

$$\mathrm{Vir}^c := U(Vir) \otimes_{U(\bigoplus_{n\geqslant -1}\mathbb{C}L_n \oplus \mathbb{C}C)} \mathbb{C}_c,$$

where C acts as multiplication by c and L_n, $n \geqslant -1$, acts by 0 on the one-dimensional module $\mathbb{C}_c$.

By the PBW Theorem, Vir^c has a basis of the form

$$L_{j_1} \dots L_{j_m}|0\rangle, \qquad j_1 \leqslant \cdots \leqslant j_m \leqslant -2,$$

where $|0\rangle$ is the image of $1 \otimes 1$ in Vir^c.

Proposition 3.3 **([80, 141])** *There is a unique vertex algebra structure on* Vir^c *such that* $|0\rangle = 1 \otimes 1$ *is the vacuum vector and* $Y(\omega, z) = L(z)$*, where* $\omega = L_{-2}|0\rangle$*. Moreover, there is a surjective homomorphism* $\mathrm{Vir}^c \to \langle L(z)\rangle_M$ *of vertex algebras for any smooth* Vir*-module* M *of central charge* c*.*

The vertex algebra Vir^c is called the *universal Virasoro vertex algebra with central charge* c.

Note that $T = L_{-1}$ on Vir^c since $L(z)_{(0)}L(z) = \partial_z L(z)$ (or equivalently, $L_{-1}L_{-2}|0\rangle = L_{-3}|0\rangle$). Also, Vir^c is conical by the Hamiltonian $H = L_0$:

$$\mathrm{Vir}^c = \bigoplus_{\Delta \in \mathbb{Z}_{\geqslant 0}} \mathrm{Vir}^c_\Delta, \quad \mathrm{Vir}^c_0 = \mathbb{C}|0\rangle,\ \mathrm{Vir}^c_1 = 0,\ \mathrm{Vir}^c_2 = \mathbb{C}\omega. \tag{3.3}$$

The unique simple quotient of Vir^c is called the *simple Virasoro vertex algebra with central charge* c and is denoted by Vir_c.

Proposition 3.4 *The category* Vir^c-Mod *of* Vir^c*-modules is the same as that of smooth representations of* Vir *of central charge* c*.*

3.3 Conformal vertex algebras

Definition 3.1 A graded vertex algebra $V = \bigoplus_\Delta V_\Delta$ is called *conformal* if there exists a vector ω, called the *stress tensor*, or the *conformal vector*, such that the corresponding field

$$Y(\omega, z) = T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfies the following conditions:

(1) $[L_n, L_m] = (n-m)L_{n+m} + \dfrac{n^3-n}{12}\delta_{n+m,0}c$, where c is a constant called the *central charge* of V,
(2) $\omega_{(0)} = L_{-1} = T$,
(3) $\omega_{(1)} = L_0 = H$, that is, $L_0|_{V_\Delta} = \Delta \mathrm{Id}_{V_\Delta}$ for all $\Delta \in \mathbb{Z}$.

A $\mathbb{Z}$-graded conformal vertex algebra such that $\dim V_\Delta < \infty$ for all $\Delta \in \mathbb{Z}$ and $V_\Delta = 0$ for sufficiently small Δ is also called a *vertex operator algebra*.

For a conformal vertex algebra of central charge c, we have a homomorphism $\mathrm{Vir}^c \to V$, $\omega \mapsto \omega$, of vertex algebras.

Let M be a module over a conformal vertex algebra V of central charge c. Then the Virasoro algebra acts on M via the vertex algebra homomorphism $\mathrm{Vir}^c \to V$. The module M is called a *positive energy representation* if L_0 acts semisimply with spectrum bounded below, that is, $M = \bigoplus_{d \geqslant h} M_d$ where

$$M_d = \{m \in M : L_0 m = dm\}.$$

A positive energy representation M is called an *ordinary representation* if each M_d is finite-dimensional. For an ordinary representation M the *normalized character*

$$\chi_M(q) = \mathrm{tr}_M(q^{L_0 - c/24}) = q^{-c/24} \sum_d (\dim M_d) q^d \tag{3.4}$$

is well-defined.

Example 3.3 The Virasoro vertex algebra Vir^c is clearly conformal with central charge c and conformal vector $\omega = L_{-2}|0\rangle$.

Example 3.4 The universal affine vertex algebra $V^k(\mathfrak{g})$, with $\mathfrak{g}$ simple, is conformal by *Sugawara construction* provided with $k \neq -h^\vee$ (here $h^\vee$ is the dual Coxeter number): Set

$$S = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{g}} x_{i,(-1)} x^i_{(-1)} |0\rangle,$$

where $\{x_i\,;\, i = 1, \dots, \dim \mathfrak{g}\}$ is the dual basis of $\{x^i\,;\, i = 1, \dots, \dim \mathfrak{g}\}$ with respect to the bilinear form $(\ |\ )$. Then for $k \neq -h^\vee$, $L = \dfrac{S}{k+h^\vee}$ is a stress tensor of $V^k(\mathfrak{g})$ with central charge

$$c(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

We refer to [77, §3.4.8] or to [76, 3.1.1] for a proof of this nontrivial statement; see also [112, Theorem 5.7] and its proof. We have

$$[L_m, x_{(n)}] = -n x_{(m+n)} \quad x \in \mathfrak{g},\ m, n \in \mathbb{Z}. \tag{3.5}$$

Exercise 3.2 Let $V^{\kappa}(\mathfrak{h})$ be the Heisenberg vertex algebra associated with the commutative Lie algebra $\mathfrak{h}$ of dimension r. Assume that κ is nondegenerate. Show that $T(z) = \frac{1}{2}\sum_{i=1}^{r} {}^{\circ}_{\circ} x_i(z)x^i(z) {}^{\circ}_{\circ}$ is a conformal field with central charge r, where $\{x_i\}_{1\leqslant i\leqslant r}$ and $\{x^i\}_{1\leqslant i\leqslant r}$ are dual basis of $\mathfrak{h}$ with respect to κ (see also Exercise 2.5).

It follows from Exercise 3.1 that $Z(V^k(\mathfrak{g})) = V^k(\mathfrak{g})^{\mathfrak{g}[t]}$, where

$$V^k(\mathfrak{g})^{\mathfrak{g}[t]} := \{a \in V^k(\mathfrak{g})\colon x_{(m)}a = 0 \text{ for all } x \in \mathfrak{g},\ m \in \mathbb{Z}_{\geqslant 0}\}.$$

The following exercise gives a description of the vertex center of $V^k(\mathfrak{g})$ which has a priori nothing to do the vertex algebra structure.

Exercise 3.3 Show that we have the following isomorphism of commutative $\mathbb{C}$-algebras (the product on the commutative vertex algebra $Z(V^k(\mathfrak{g}))$ is the normally ordered product):

$$Z(V^k(\mathfrak{g})) \cong \mathrm{End}_{\hat{\mathfrak{g}}}(V^k(\mathfrak{g})).$$

Remark 3.1 It is easily seen that $Z(V^k(\mathfrak{g})) = \mathbb{C}|0\rangle$ for $k \neq -h^\vee$ using the stress tensor L. For $k = -h^\vee$, the center

$$Z(V^{-h^\vee}(\mathfrak{g})) =: \mathfrak{z}(\hat{\mathfrak{g}})$$

is "huge", and it is usually referred as the *Feigin-Frenkel center* [73][3]: we have $\mathrm{gr}\,\mathfrak{z}(\hat{\mathfrak{g}}) \cong \mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}/\!/G))$, with $\mathfrak{g}/\!/G = \mathrm{Spec}\,\mathscr{O}(\mathfrak{g})^G$.

3.4 Chiral differential operators on a group

Let G be a affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$, κ an invariant bilinear form on $\mathfrak{g}$, and set

$$\mathcal{A}_G = U(\hat{\mathfrak{g}}_\kappa) \otimes \mathscr{O}(\mathscr{L}G),$$

where $\mathscr{L}G$ is the loop space of G (see Section 1.6), and consider $\mathcal{A}_G$ as an algebra such that the natural embeddings $U(\hat{\mathfrak{g}}_\kappa) \hookrightarrow \mathcal{A}_G$, $\mathscr{O}(\mathscr{L}G) \hookrightarrow \mathcal{A}_G$, are embeddings of algebras and

$$[xt^m, f_{(n)}] = (x_L f)_{(m+n)} \quad \text{for } x \in \mathfrak{g},\ f \in \mathscr{O}(G),\ m, n \in \mathbb{Z}. \tag{3.6}$$

We regard $\mathscr{O}(\mathscr{J}_\infty G)$ as a module over the subalgebra $\mathcal{A}_{G,+} = U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1}) \otimes \mathscr{O}(\mathscr{L}G) \subset \mathcal{A}_G$ on which $\mathscr{O}(\mathscr{L}G)$ acts via the natural surjection $\mathscr{O}(\mathscr{L}G) \to \mathscr{O}(\mathscr{J}_\infty G)$, an element of $\mathfrak{g}[t] \subset \mathfrak{g}[[t]]$ acts as a left invariant vector field on $\mathscr{O}(\mathscr{J}_\infty G)$ (see Example 1.2), and $\mathbf{1}$ acts as the identity. Define

[3] See Example 1.3 for more details about the scheme $\mathscr{J}_\infty(\mathfrak{g}/\!/G)$.

$$\mathcal{D}^{ch}_{G,\kappa} = \mathcal{A}_G \otimes_{\mathcal{A}_{G,+}} \mathscr{O}(\mathscr{J}_\infty G). \tag{3.7}$$

Note that

$$\mathcal{D}^{ch}_{G,\kappa} \cong U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} \mathscr{O}(\mathscr{J}_\infty G) \tag{3.8}$$

as $\hat{\mathfrak{g}}$-modules. We have the mutually local fields

$$x(z) = \sum_{n\in\mathbb{Z}} (xt^n) z^{-n-1} \quad (x \in \mathfrak{g}), \quad f(z) = \sum_{n\in\mathbb{Z}} f_{(n)} z^{-n-1} \quad (f \in \mathscr{O}(G))$$

on $\mathcal{D}^{ch}_{G,\kappa}$ satisfying the OPEs

$$x(z)y(w) \sim \frac{1}{z-w}[x,y](w) + \frac{\kappa(x,y)}{(z-w)^2}, \quad f(z)g(w) \sim 0, \tag{3.9}$$

$$x(z)f(w) \sim \frac{1}{z-w}(x_L f)(w) \tag{3.10}$$

for $x, y \in \mathfrak{g}$, $f, g \in \mathscr{O}(G)$.

The following assertion is clear from Theorem 2.6.

Theorem 3.1 *There is a unique vertex algebra structure on $\mathcal{D}^{ch}_{G,\kappa}$ such that the embeddings*

$$\pi_L \colon V^\kappa(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_{G,\kappa}, \quad u|0\rangle \mapsto u \otimes 1,$$
$$j \colon \mathscr{O}(\mathscr{J}_\infty G) \hookrightarrow \mathcal{D}^{ch}_{G,\kappa}, \quad f \mapsto 1 \otimes f,$$

are homomorphisms of vertex algebras, and

$$x(z)f(w) \sim \frac{1}{z-w}(x_L f)(w) \tag{3.11}$$

for $x \in \mathfrak{g}$, $f \in \mathscr{O}(G)$.

The vertex algebra $\mathcal{D}^{ch}_{G,\kappa}$ is called the algebra of (global) *chiral differential operators* (cdo) on G. It is naturally $\mathbb{Z}_{\geqslant 0}$-graded by the following conditions:

- elements of $\mathfrak{g}$, embedded in $\mathcal{D}^{ch}_{G,\kappa}$ through π_L, have weight 1,
- elements of $\mathscr{O}(G)$, embedded in $\mathcal{D}^{ch}_{G,\kappa}$ through j, have weight 0.

Let Ω be the subspace of $(\mathcal{D}^{ch}_{G,\kappa})_1$ spanned by vectors $f\partial g$, with $f, g \in \mathscr{O}(G)$, where $\partial = T$ is the translation operator on $\mathscr{O}(\mathscr{J}_\infty G)$. Recall that the embedding $\mathfrak{g} \hookrightarrow \operatorname{Der}_{\mathbb{C}}(\mathscr{O}(G))$, $x \mapsto x_L$, induces an isomorphism of left $\mathscr{O}(G)$-modules

$$\mathscr{O}(G) \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\simeq} \operatorname{Der}_{\mathbb{C}}(\mathscr{O}(G)). \tag{3.12}$$

It is easy to see that

$$(\mathcal{D}^{ch}_{G,\kappa})_1 = \Omega \oplus \operatorname{Der}_{\mathbb{C}}(\mathscr{O}(G)) \tag{3.13}$$

as vector spaces. Let $\Omega^1(G)$ be the space of global differential forms on G as in Appendix B. Recall that $\Omega^1(G)$ is generated as a $\mathscr{O}(G)$-module by the elements df, for $f \in \mathscr{O}(G)$, where d is the de Rham differential (see Appendix B), and that

$$\Omega^1(G) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$$

through the map $df \mapsto (x \mapsto x_L f)$ (see Lemma B.2).

Lemma 3.2 *The $\mathbb{C}$-linear map sending $f\partial g \in \Omega$ to the element $h \otimes x \mapsto (hx)_{(1)}(f\partial g)$ of $\mathrm{Hom}_{\mathscr{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)), \mathscr{O}(G))$, with $h \in \mathscr{O}(G)$ and $x \in \mathfrak{g}$, is an isomorphism of vector spaces. Therefore $\Omega \cong \Omega^1(G)$ as vector spaces.*

Proof First of all, note that for $x \in \mathfrak{g}$ and $f, g, h \in \mathscr{O}(G)$,

$$(hx)_{(1)}(f\partial g) = hf(x_L g) \in \mathscr{O}(G)$$

and, clearly, the map sending $h \otimes x \in \mathscr{O}(G) \otimes \mathfrak{g}$ to $(hx)_{(1)}(f\partial g) = hf(x_L g)$ is a morphism of $\mathscr{O}(G)$-modules. Hence the map of the lemma is well-defined. Let is denote it by Γ.

By the Frobenius reciprocity,

$$\mathrm{Hom}_{\mathscr{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)), \mathscr{O}(G)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G)),$$

and through this isomorphism, the map $\Gamma(f\partial g)$ sends $x \in \mathfrak{g}$ to $f(x_L g)$. Hence it suffices to show that the $\mathscr{O}(G)$-linear map sending $\partial g \in \Omega$ to the element $(x \mapsto x_L g)$ of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ is an isomorphism of $\mathscr{O}(G)$-modules. By Lemma B.2, this is equivalent to showing that the $\mathscr{O}(G)$-linear map sending $\partial g \in \Omega$ to $dg \in \Omega^1(G)$ is an isomorphism.

But $\partial g = g_{(-2)}|0\rangle$ is by construction a regular function on $\mathscr{J}_1(G) \cong TG$, where TG is the tangent bundle of G, and through this identification, ∂g is nothing but dg, so the statement is obvious. □

We denote by $\langle\, , \rangle \colon \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)) \times \Omega \to \mathscr{O}(G)$ the canonical $\mathscr{O}(G)$-bilinear pairing. The Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ acts on Ω by the Lie derivative given by (B.6).

Lemma 3.3 *Let $x \in \mathfrak{g}$ and $\omega \in \Omega$. Then $x_{(1)}\omega = \langle x, \omega\rangle$ and $x_{(0)}\omega = (\mathrm{Lie}\, x).\omega$*

Proof The identity $x_{(1)}\omega = \langle x, \omega\rangle$ is clear by Lemma 3.2. Let us prove the second one using it. The Lie derivative action can be written as:

$$y_{(1)}((\mathrm{Lie}\, x).\omega) = x_L(y_{(1)}\omega) - [x, y]_{(1)}\omega = x_{(0)}y_{(1)}\omega - [x, y]_{(1)}\omega,$$

for all $y \in \mathfrak{g}$. But

$$y_{(1)}(x_{(0)}\omega) = x_{(0)}y_{(1)}\omega - [x, y]_{(1)}\omega$$

for all $y \in \mathfrak{g}$, whence $x_{(0)}\omega = (\mathrm{Lie}\, x).\omega$. □

Let

$$\kappa^* = -\kappa - \kappa_{\mathfrak{g}}, \tag{3.14}$$

where $\kappa_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$.

Theorem 3.2 *i). There is a vertex algebra embedding*

$$\pi_R : V^{\kappa^*}(\mathfrak{g}) \hookrightarrow \mathrm{Com}(V^{\kappa}(\mathfrak{g}), \mathcal{D}^{ch}_{G,\kappa}) \subset \mathcal{D}^{ch}_{G,\kappa}$$

such that

$$[\pi_R(x)_{(m)}, f_{(n)}] = (x_R f)_{(m+n)} \quad \textit{for } x \in \mathfrak{g},\ f \in \mathscr{O}(G),\ m, n \in \mathbb{Z},$$

where x_R denotes the right invariant vector field corresponding to $x \in \mathfrak{g}$.
ii). There is a vertex algebra isomorphism

$$\mathcal{D}^{ch}_{G,\kappa} \cong \mathcal{D}^{ch}_{G,\kappa^*}$$

that sends $f \in \mathscr{O}(G)$ to $S(f) \in \mathscr{O}(G)$, where $S : \mathscr{O}(G) \to \mathscr{O}(G)$ is the antipode.

Proof i) From now, we identify $x \in \mathfrak{g}$ with its image in $\mathcal{D}^{ch}_{G,\kappa}$ through π_L.
$*$ Analysis. We assume that such a morphism π_R does exist. Then in particular

$$\pi_R(x^i)_{(1)} x^j = 0 \quad \text{for all} \quad i, j = 1, \ldots, j.$$

Fix $i, j \in \{1, \ldots, j\}$.

Let $\{x^1, \ldots, x^d\}$ be a basis of $\mathfrak{g}$, and $\{\omega^1, \ldots, \omega^d\}$ the dual $\mathscr{O}(G)$-basis of $\Omega \cong \Omega^1(G)$. The isomorphism (B.2) tells that $\{x^1, \ldots, x^d\}$ forms an $\mathscr{O}(G)$-basis of $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$. In particular,

$$x^i_R = \sum_p f^{i,p} x^p, \quad i = 1, \ldots, d,$$

for some invertible matrix $(f^{i,p})_{1 \leqslant i,p \leqslant d}$ over $\mathscr{O}(G)$. We will repeatidily make use of the identities of Lemma B.2 and Lemma B.3.

We set for all $i \in \{1, \ldots, d\}$,

$$\pi_R(x^i) = x^i_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{i,p} \omega^q. \tag{3.15}$$

We first verify that for all i, j,

$$(x^i)_{(n)} \pi_R(x^j) = 0 \tag{3.16}$$

for all i, j and $n \geqslant 0$. By (3.9), (3.10), the condition (3.16) is clearly satisfied for $n \geqslant 2$.

Fix i, j. We first verify that $(x^i)_{(1)} \pi_R(x^j) = 0$. First, by (B.3), (3.9), (3.10) and Borcherds identity (2.33), we have

$$(x_R^i)_{(1)}x^j = \sum_p (f^{i,p}_{(-1)}x^p)_{(1)}x^j = \sum_j (f^{i,p}_{(-1)}x^p_{(1)}x^j + x^p_{(0)}f^{i,p}_{(0)}x^j)$$
$$= \sum_p (f^{i,p}\kappa(x^p,x^j) - x_L^p(x_L^j f^{i,p})) \qquad (3.17)$$

Using Lemma B.2 (i) twice, we get

$$-x_L^p(x_L^j f^{i,p}) = \sum_s x_L^p(c_p^{j,s}f^{i,s}) = -\sum_{s,u} c_s^{p,u}c_p^{j,s}f^{i,u} = \sum_{s,u} c_s^{u,p}c_p^{j,s}f^{i,u}.$$

Since

$$\kappa_{\mathfrak{g}}(x^i,x^j) = \sum_{p,q} c_p^{i,q}c_q^{j,p}, \quad i,j = 1,\dots,d,$$

we deduce that

$$-\sum_p x_L^p(x_L^j f^{i,p}) = \sum_u \kappa_{\mathfrak{g}}(x^u,x^j)f^{i,u}. \qquad (3.18)$$

Combining (3.17), (3.18) and (3.14), we obtain that for $i,j = 1,\dots,d$,

$$(x_R^i)_{(1)}x^j = -\sum_p \kappa^*(x^p,x^j)f^{i,p}.$$

On the other hand, by Lemma 3.3, we have for any $p,q \in \{1,\dots,d\}$,

$$(\kappa^*(x^p,x^q)f^{i,p}\omega^q)_{(1)}x^j = \kappa^*(x^p,x^q)f^{i,p}\langle\omega^q,x^j\rangle = \kappa^*(x^p,x^j)f^{i,p},$$

whence $(x^i)_{(1)}\pi_R(x^j) = 0$.

We now wish to prove that $(x^i)_{(0)}\pi_R(x^j) = 0$. We have

$$(x^i)_{(0)}\pi_R(x^j) = (x^i)_{(0)}(x_R^j + \sum_{p,q}\kappa^*(x^p,x^q)f^{j,p}\omega^q).$$

One one hand, using Lemma B.2 (i),

$$x^i_{(0)}x^j_R = \sum_q x^i_{(0)}(f^{j,q}x^q) = \sum_q x^i_{(0)}f^{j,q}_{(-1)}x^q = \sum_q ((x_L^i f^{j,q})_{(-1)}x^q + f^{j,q}_{(-1)}[x^i,x^q]) = 0.$$

On the other hand, using Lemma B.2 (i) and Lemma 3.3, we get

$$x^i_{(0)}(\kappa^*(x^p,x^q)f^{j,p}\omega^q) = \kappa^*(x^p,x^q)\left((x_L^i f^{j,p})\omega^q + f^{j,p}(\mathrm{Lie}\,x^j).\omega^q\right)$$
$$= -\sum_{s,r}\left(\kappa^*(x^p,x^r)c_p^{i,s}f^{j,s}\omega^r + \kappa^*(x^p,x^q)c_q^{i,r}f^{j,p}\omega^r\right)$$
$$= -\sum_{s,r}\left(\kappa^*([x^i,x^s],x^r)f^{j,s}\omega^r - \kappa^*(x^p,[x^i,x^r])f^{j,p}\omega^r\right) = 0$$

due to the invariance of κ^*. This proves that $(x^i)_{(0)}\pi_R(x^j) = 0$.

In conclusion, (3.16) holds for any $i, j = 1, \dots, d$ and $n \geqslant 0$ as desired.

It remains to verify that π_R defines a vertex algebra homomorphism that is injective. Due to the decomposition (3.13), we see that the map π_R defined by (3.15) is injective since the map $\mathfrak{g} \to \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$, $x \mapsto x_L$, is.

For the vertex algebra homomorphism part, we have to show that

$$(\pi_R(x))\,(z)\,(\pi_R(y))\,(w) \sim \frac{1}{z-w}\pi_R([x,y])(w) + \frac{\kappa^*(x,y)}{(z-w)^2} \tag{3.19}$$

for all $x, y \in V^{\kappa^*}(\mathfrak{g})$, that is,

$$\pi_R(x)_{(1)}\pi_R(y) = \kappa^*(x,y), \tag{3.20}$$

$$\pi_R(x)_{(0)}\pi_R(y) = \pi_R([x,y]). \tag{3.21}$$

for all $x, y \in V^{\kappa^*}(\mathfrak{g})$.

To compute $\pi_R(x)_{(1)}\pi_R(y)$ we notice that for $i, j \in \{1, \dots, d\}$,

$$\pi_R(x^i)_{(1)}\pi_R(x^j) = \left(x^i_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{i,p}\omega^q\right)_{(1)} \left(x^j_R + \sum_{q,p} \kappa^*(x^p, x^q) f^{j,p}\omega^q\right)$$

is a sum of four terms. We have

$$\begin{aligned}(x^i_R)_{(1)}x^j_R &= \sum_{p,s} (f^{i,p}x^p)_{(1)}(f^{j,s}x^s) \\ &= \sum_{p,s} (f^{i,p}f^{j,s}\kappa(x^p,x^s) - f^{i,p}x^s_L(x^p_L f^{j,s}) - f^{j,s}x^p_L(x^s_L f^{i,p}) - (x^p_L f^{j,s})(x^s_L f^{i,p}))\end{aligned}$$

Using (3.14) and Lemma B.2 we see that

$$-f^{i,p}x^s_L(x^p_L f^{j,s}) = -f^{j,s}x^p_L(x^s_L f^{i,p}) = (x^p_L f^{j,s})(x^s_L f^{i,p}) = \kappa_{\mathfrak{g}}(x^p, x^s) f^{i,p} f^{j,s},$$

for all i, j, s, p, whence

$$(x^i_R)_{(1)}x^j_R = -\sum_{p,s} \kappa^*(x^p, x^s) f^{i,p} f^{j,s}.$$

Next, using Lemma 3.3, we get for all i, j, p, q,

$$\left(\kappa^*(x^p, x^q) f^{i,p}\omega^q\right)_{(1)} x^j_R = \kappa^*(x^p, x^q) f^{i,p} f^{j,q}.$$

Similarly,

$$(x^i_R)_{(1)} \left(\kappa^*(x^s, x^u) f^{j,s}\omega^u\right)_{(1)} x^j_R = \kappa^*(x^p, x^q) f^{i,p} f^{j,q},$$

and evidently,

$$\left(\kappa^*(x^p, x^q) f^{i,p} \omega^q\right)_{(1)} \left(\kappa^*(x^s, x^u) f^{j,s} \omega^u\right) = 0$$

Adding up, we get

$$\pi_R(x^i)_{(1)} \pi_R(x^j) = \sum_{p,q} \kappa^*(x^p, x^q) f^{i,p} f^{j,q}. \tag{3.22}$$

We differentiate the above relation. By Lemma B.2, we have for i, j, p, q, s,

$$\begin{aligned} x_L^s \left(\sum_{p,q} \kappa^*(x^p, x^q) f^{i,p} f^{j,q} \right) &= \sum_{p,q} \kappa^*(x^p, x^q) \left((x_L^s f^{i,p}) f^{j,q} + f^{i,p} (x_L^s f^{j,q}) \right) \\ &= \sum_{p,q,u,v} \kappa^*(x^p, x^q) \left(-c_p^{s,u} f^{i,u} f^{j,q} - c_q^{s,v} f^{i,p} f^{j,v} \right) \\ &= \sum_{p,q,u,v} \left(-\kappa^*([x^s, x^u], x^q) f^{i,u} f^{j,q} - \kappa^*(x^p, [x^s], x^v]) f^{i,p} f^{j,v} \right) \\ &= 0. \end{aligned}$$

Therefore, (3.22) is constant. This constant can be computed by observing that the matrix $(f^{i,j})$, considered as a function on the group G, is equal to the identity at the neutral element of G by the identity (B.3). Hence, (3.22) is equal to $\kappa^*(x^i, x^j)$, which proves (3.20).

We now compute $\pi_R(x)_{(0)} \pi_R(y)$. For $i, j \in \{1, \ldots, d\}$ we have

$$\pi_R(x^i)_{(0)} \pi_R(x^j) = \pi_R(x^i)_{(0)} \left(\sum_q f^{j,q} x^q + \sum_{s,u} \kappa^*(x^s, x^u) f^{j,s} \omega^u \right).$$

Using (3.16) with $n = 0$, we have

$$\begin{aligned} \pi_R(x^i)_{(0)} (f^{j,q} x^q) &= (\pi_R(x^i)_{(0)} f^{j,q}) x^q \\ &= \sum_p f^{i,p} (x_L^p f^{j,q}) x^q = [x_R^i, x_R^j] = [x^i, x^j]_R, \end{aligned} \tag{3.23}$$

by Lemma 3.3, Lemma B.2 (ii), Lemma B.3.

On the other hand,

$$\begin{aligned} \pi_R(x^i)_{(0)} (\kappa^*(x^s, x^u) f^{j,s} \omega^u) &= (x_R^i)_{(0)} (\kappa^*(x^s, x^u f^{j,s} \omega^u) \\ &= \kappa^*(x^s, x^u) x_R^i (f^{j,s} \omega^u) \\ &= \kappa^*(x^s, x^u) f^{j,p} x_L^p (f^{j,s}) \omega^u) = \kappa^*(x^s, x^u) c_q^{i,j} f^{q,s} \omega^u) \end{aligned} \tag{3.24}$$

by Lemma 3.3, Lemma B.2 (ii), and Lemma B.3. Adding up (3.23) and (3.24) we obtain that

$$\pi_R(x^i)_{(0)} \pi_R(x^j) = \pi_R([x^i, x^j])$$

for all $i, j = 1, \ldots, d$, which proves (3.21).

To sum up, we have proven that π_R is an injective vertex algebra homomorphism. Complete the proof about the bracket $[\pi_R(x)_{(m)}, f_{(n)}] = (x_R f)_{(m+n)} \cdots$

ii) Consider the unique vertex algebra homomorphism

$$\Phi \colon \mathcal{D}^{ch}_{G,\kappa} \longrightarrow \mathcal{D}^{ch}_{G,\kappa^*}$$

whose restriction to $\mathcal{O}(G)$ is given by the antipode S, and restriction to $V^\kappa(\mathfrak{g})$ is the map $\pi_L(x) \mapsto \pi_R(x)$. It is easy to verify that Φ is indeed a vertex algebra homomorphism by (3.19), since

$$(\Phi(x))(z)(\Phi(f))(w) \sim \frac{1}{z-w}\left(\Phi(x)_L \Phi(f)\right)(w)$$

for $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$ which holds by

$$x_R S(f) = S(x_L f)$$

for $x \in \mathfrak{g}$, $f \in \mathcal{O}(G)$.

It remains to show that Φ is an isomorphism. Consider the vertex algebra homomorphism from $\mathcal{D}^{ch}_{G,\kappa^*}$ to $\mathcal{D}^{ch}_{G,(\kappa^*)^*}$ whose restriction to $\mathcal{O}(G)$ is given by the antipode, and restriction to $V^{\kappa^*}(\mathfrak{g})$ is the map $\pi_R(x) \to \pi_L(x)$. Note that $(\kappa^*)^* = \kappa$. Similarly to Φ, we verify that Ψ is indeed a vertex algebra homomorphism. Moreover, we have $\Psi \circ \Phi = \mathrm{id}_{\mathcal{D}^{ch}_{G,\kappa}}$ and $\Phi \circ \Psi = \mathrm{id}_{\mathcal{D}^{ch}_{G,\kappa^*}}$ This concludes the proof of (ii). $\square$

Theorem 3.3 *Suppose that G is connected. The vertex algebras $V^\kappa(\mathfrak{g})$ and $V^{\kappa^*}(\mathfrak{g})$ form a dual pair in $\mathcal{D}^{ch}_{G,\kappa}$.*

Proof We have to show that

$$V^\kappa(\mathfrak{g}) = (\mathcal{D}^{ch}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])} \quad \text{and} \quad V^{\kappa^*}(\mathfrak{g}) = (\mathcal{D}^{ch}_{G,\kappa})^{\pi_L(\mathfrak{g}[[t]])}.$$

By Theorem 3.2, we have already established the inclusions $V^\kappa(\mathfrak{g}) \subset (\mathcal{D}^{ch}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])}$ and $V^{\kappa^*}(\mathfrak{g}) \subset (\mathcal{D}^{ch}_{G,\kappa})^{\pi_L(\mathfrak{g}[[t]])}$. To show the other inclusions, observe that

$$\begin{aligned}(\mathcal{D}^{ch}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])} &= \left(U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} \mathcal{O}(\mathcal{J}_\infty G)\right)^{\pi_R(\mathfrak{g}[[t]])} \\ &= U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} \mathcal{O}(\mathcal{J}_\infty G)^{\pi_R(\mathfrak{g}[[t]])}\end{aligned}$$

since the image by π_R of $V^{\kappa^*}(\mathfrak{g})$ commutes with $U(\hat{\mathfrak{g}}_\kappa)$. But since G is connected, we get that

$$\mathbb{C} \cong \mathcal{O}(\mathcal{J}_\infty G)^{\mathcal{J}_\infty(G)} = \mathcal{O}(\mathcal{J}_\infty G)^{\mathfrak{g}[[t]]} = \mathcal{O}(\mathcal{J}_\infty G)^{\pi_R(\mathfrak{g}[[t]])}.$$

As a result,

$$(\mathcal{D}^{ch}_{G,\kappa})^{\pi_R(\mathfrak{g}[[t]])} \cong V^\kappa(\mathfrak{g}).$$

Using the isomorphism $\mathcal{D}^{ch}_{G,\kappa} \cong \mathcal{D}^{ch}_{G,\kappa^*}$ of Theorem 3.2, we obtain that

$$(\mathcal{D}^{ch}_{G,\kappa})^{\pi_L(\mathfrak{g}[[t]])} \cong V^{\kappa^*}(\mathfrak{g}).$$

This concludes the proof of the theorem. □

Suppose that G is reductive. The *algebraic Peter-Weyl theorem* states that

$$\mathscr{O}(G) \cong \bigoplus_{\chi \in \hat{G}} V_\lambda \otimes V_{\lambda^*} \tag{3.25}$$

as $G \times G$-modules, where $\hat{G}$ is the set of isomorphism classes of finite-dimensional simple rational G-modules, V_λ denotes a representation of $\lambda \in \hat{G}$ and λ^* is an element of $\hat{G}$ such that V_{λ^*} is the dual G-module to V_λ (see e.g. [176, Theorem 27.3.9]). The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is reductive, so that

$$[\mathfrak{g}, \mathfrak{g}] = \bigoplus_{i=1}^{r} \mathfrak{g}_i,$$

where each $\mathfrak{g}_i$ is a simple Lie subalgebra of $\mathfrak{g}$. Then $\kappa|_{\mathfrak{g}_i}$ is a constant multiplication of the Killing form $\kappa_{\mathfrak{g}_i}$ of $\mathfrak{g}_i$. We say κ is *irrational* if and $\kappa|_{\mathfrak{g}_i}/\kappa_{\mathfrak{g}_i} \notin \mathbb{Q}$ for all i.

Proposition 3.5 ([29]) *Let G be reductive, and suppose that $\kappa|_{[\mathfrak{g},\mathfrak{g}]}$ is irrational and that $\kappa|_{\mathfrak{z}(\mathfrak{g})}$ is non-degenerate, where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Then we have*

$$\mathcal{D}^{ch}_{G,\kappa} \cong \bigoplus_{\lambda \in \hat{G}} \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda^*,\kappa^*},$$

where $\mathbb{V}_{\lambda,\kappa} = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1})} V_\lambda$ and V_λ is considered to be a $\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1}$-module on which $\mathfrak{g}[t]$ acts by the projection $\mathfrak{g}[t] \to \mathfrak{g}$ and $\mathbf{1}$ as the identity.

Proof By the assumption on κ, $\mathbb{V}_{\lambda,\kappa}$ and $\mathbb{V}_{\lambda^*,\kappa^*}$ for $\lambda \in \hat{G}$ are irreducible $\hat{\mathfrak{g}}_\kappa$-module and $\hat{\mathfrak{g}}_{\kappa^*}$-module, respectively ([123]). Moreover, $\mathcal{D}^{ch}_{G,\kappa}$ is completely reducible as $\hat{\mathfrak{g}}_\kappa \oplus \hat{\mathfrak{g}}_{\kappa^*}$-modules and a direct sum of $\mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\mu^*,\kappa^*}$ with $\lambda, \mu \in \hat{G}$. Because κ is generic (and so is κ^*), the category of integrable $\hat{\mathfrak{g}}_{\kappa^*}$-modules is equivalent to the category of integrable $\mathfrak{g}$-modules and the equivalence is given by $M \to M^{t\mathfrak{g}[[t]]}$, see [123] and [124, Section 30]. Since $\mathbb{V}^{t\mathfrak{g}[[t]]}_{\mu^*,\kappa^*} = V_{\mu^*}$, it is sufficient to show that $(\mathcal{D}^{ch}_{G,\kappa})^{\pi_R(t\mathfrak{g}[[t]])} \cong \bigoplus_{\lambda \in \hat{G}} \mathbb{V}_{\lambda,\kappa} \otimes V_{\lambda^*}$ as $\hat{\mathfrak{g}}_\kappa \times \mathfrak{g}$-modules. But we have $(\mathcal{D}^{ch}_{G,\kappa})^{\pi_R(t\mathfrak{g}[[t]])} = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})} \left(\mathscr{O}(\mathscr{J}_\infty G)^{\pi_R(t\mathfrak{g}[[t]])}\right) = U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})} \mathscr{O}(G)$. Hence the assertion follows from the algebraic Peter-Weyl theorem. □

Example 3.5 Let G be a torus T, $\{h_1, \ldots, h_r\}$ a basis of the abelian Lie algebra $\mathfrak{g} = \mathrm{Lie}(T)$. Then $\mathscr{O}(G) = \mathbb{C}[P]$, where $P = \bigoplus_{i=1}^r \mathbb{Z}\varpi_i$ is the weight lattice of $\mathfrak{g}$. The subalgebra $V^\kappa(\mathfrak{g}) \subset \mathcal{D}^{ch}_{T,\kappa}$ is the Heisenberg vertex algebra generated by the field $h_L(z)$, $h \in \mathfrak{g}$, satisfying the OPE

$$h_L(z)h'_L(w) \sim \frac{\kappa(h,h')}{(z-w)^2}.$$

The OPE (3.11) reads as

$$h_L(z)e^{\alpha}(w) \sim \frac{\alpha(h)}{z-w}e^{\alpha}(w) \quad (h \in \mathfrak{g},\ \alpha \in P).$$

Since the Killing form of $\mathfrak{g}$ is zero, we have $\kappa^* = -\kappa$. The subalgebra $\pi_R(V^{\kappa^*}(\mathfrak{g})) \subset \mathcal{D}^{ch}_{T,\kappa}$ is generated the fields $h_R(z)$, $h \in \mathfrak{g}$, defined by

$$h_R(z) := h_L(z) - \sum_{i=1}^{r} \kappa(h, h_i) e^{-\varpi_i}(z)\partial e^{\varpi_i}(z) \tag{3.26}$$

(Note that $e^{-\varpi_i}(z)\partial e^{\varpi_i}(z) = {:}e^{-\varpi_i}(z)\partial e^{\varpi_i}(z){:} = (e^{-\varpi_i}\partial e^{\varpi_i})(z)$ since $\mathscr{O}(\mathscr{J}_\infty G)$ is commutative.) We have

$$h_L(z)h'_R(w) \sim 0, \quad h_R(z)h'_R(w) \sim \frac{\kappa^*(h,h')}{(z-w)^2}.$$

The stress tensor vector of $\mathcal{D}^{ch}_{G,\kappa}$ is given by

$$T(z) = \sum_{i=1}^{r} {:}h_i(z)(e^{-\varpi_i}\partial e^{\varpi_i})(z){:} - \frac{1}{2}\sum_{i,j=1}^{r} \kappa(h_i,h_j){:}(e^{-\varpi_i}\partial e^{\varpi_i})(z)(e^{-\varpi_j}\partial e^{\varpi_j})(z){:},$$

which has central charge $2r$.

Now suppose that κ is non-degenerate. Then we have the embedding of vertex algebras

$$V^{\kappa}(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_{G,\kappa},$$

and we have

$$T(z) = T_L(z) + T_R(z),$$

where $T_L(z) = \frac{1}{2}\sum_{i=1}^{r} {:}h_{i,L}(z)h^i_L(z){:}$ and $T_R(z) = -\frac{1}{2}\sum_{i=1}^{r} {:}h_{i,R}(z)h^i_R(z){:}$ are the stress tensors of the vertex subalgebra $V^{\kappa}(\mathfrak{g})$ and $V^{\kappa^*}(\mathfrak{g})$, respectively. As a $V^{\kappa}(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$-module we have

$$\mathcal{D}^{ch}_{G,\kappa} \cong \bigoplus_{\lambda \in P_+} \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda,\kappa^*}, \tag{3.27}$$

Here $\mathbb{V}_{\lambda,\kappa}$ is the highest weight representation of the Heisenberg algebra $\hat{\mathfrak{g}}_\kappa$ with highest weight λ.

In the case that κ is non-degenerate it is possible to give the vertex algebra structure using the decomposition (3.27) ([79]). Note that the vector $|\lambda\rangle = v_\lambda \otimes v_\lambda \in$

$\mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda,\kappa^*}$, where v_λ is the highest weight vector of $\mathbb{V}_{\lambda,\kappa}$ or $\mathbb{V}_{\lambda,\kappa^*}$, corresponds to the vector $e^\lambda \in \mathscr{O}(G)$ on the left-hand-side. Observe from (3.26) that

$$\partial_z Y(|\lambda\rangle, z) = {\circ \atop \circ} (\lambda_L(z) - \lambda_R(z)) Y(|\lambda\rangle, z) {\circ \atop \circ}, \tag{3.28}$$

where we identified P_+ as a subspace of $\mathfrak{g}$ via the form κ. In view of Theorem 2.2, (3.28) together with the relation $Y(|\lambda\rangle, z)|0\rangle|_{z=0} = |\lambda\rangle$ completely determines the field $Y(|\lambda\rangle, z)$. As a result, we find that

$$Y(|\lambda\rangle, z) = e^\lambda \exp\Big(\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(\lambda_L)_{(n)} - (\lambda_R)_{(n)}}{-n} z^{-n} \Big) \quad \text{for } \lambda \in P_+,$$

where e^λ is the operator on $\bigoplus_{\lambda \in \mathbb{Z}} \mathbb{V}_{\lambda,\kappa} \otimes \mathbb{V}_{\lambda,\kappa^*}$ defined by $e^\lambda|\mu\rangle = |\lambda + \mu\rangle$, $[(h_L)_{(n)}, e^\lambda] = [(h_R)_{(n)}, e^\lambda] = 0$. Here note that $\lambda_L(z) - \lambda_R(z)$ generates a commutative vertex subalgebra and $(\lambda_L)_{(0)} - (\lambda_R)_{(0)}$ acts as zero on the whole space (compare with (3.31) below). It is straightforward to check that

$$Y(\lambda, z) Y(\mu, w) \sim 0, \quad Y(\lambda, z) Y(\mu, z) = Y(\lambda + \mu, z), \tag{3.29}$$

$$h_L(z) Y(|\lambda\rangle, w) \sim \frac{\lambda(h)}{z - w} Y(|\lambda\rangle, w), \quad h_R(z) Y(|\lambda\rangle, w) \sim \frac{\lambda(h)}{z - w} Y(|\lambda\rangle, w). \tag{3.30}$$

This construction is useful to construct $\mathcal{D}^{ch}_{G,\kappa}$-modules for a non-degenerate κ. For $\lambda \in P_+ = \hat{G}$, set

$$M_{\lambda,\kappa} = \bigoplus_{\mu \in P_+} \mathbb{V}_{\lambda+\mu,\kappa} \otimes \mathbb{V}_{\mu,\kappa^*},$$

which is naturally a $V^\kappa(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$-modules. The $V^\kappa(\mathfrak{g}) \otimes V^{\kappa^*}(\mathfrak{g})$-module structure extends to the $\mathcal{D}^{ch}_{G,\kappa}$-module structure by setting

$$Y_{M_{\lambda,\kappa}}(\alpha, z) = e^\alpha \exp\Big(\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(\alpha_L)_{(n)} - (\alpha_R)_{(n)}}{-n} z^{-n} \Big) z^{(\alpha_L)_{(0)} - (\alpha_R)_{(0)}} \tag{3.31}$$

for $\alpha \in P_+$, where e^α is defined by $e^\alpha(v_{\lambda+\mu} \otimes v_\mu) = v_{\lambda+\mu+\alpha} \otimes v_{\mu+\alpha}$, $[(h_L)_{(n)}, e^\lambda] = [(h_R)_{(n)}, e^\lambda] = 0$. (Note that $z^{(\alpha_L)_{(0)} - (\alpha_R)_{(0)}} = z^{\kappa(\lambda,\alpha)}$ on $M_{\lambda,\kappa}$.)

Exercise 3.4 Show that $M_{\lambda,\kappa}$ is a simple $\mathcal{D}^{ch}_{G,\kappa}$-module for all $\lambda \in P_+$.

Part II
Poisson vertex algebras, Li filtration and associated varieties

In this part, we introduce important objects related to vertex algebras and discuss interesting relations between them.

Chapter 4 gives a concise presentation of *Poisson vertex algebras* (a particular class of commutative vertex algebras). It will be observed that any vertex algebra is naturally filtered and that the corresponding graded space has a structure of a Poisson vertex algebra. The *Zhu C_2-functor* $V \mapsto R_V$ associates with any vertex algebra V a certain quotient that has a Poisson algebra structure and that generates the Poisson vertex algebra $\operatorname{gr} V$ as a differential algebra. The spectrum of the *Zhu C_2-algebra* R_V is called the *associated scheme*. Its maximal spectrum is called the *associated variety*. Associated schemes and associated varieties, as well as their spaces of arcs, will occupy a central place in the rest of this book. Chapter 5 is about the Zhu functor $V \mapsto \operatorname{Zhu}(V)$. It gives a correspondence between the theory of modules over a vertex algebra and the representation theory of its Zhu's algebra. This correspondence is particularly well-understood in the case of the universal affine vertex algebras, where Zhu's algebras are enveloping algebras of the corresponding finite-dimensional simple Lie algebras. In Chapter 6, we develop the theory of Poisson vertex modules and Frenkel-Zhu's bimodules. These notions generalize all above constructions to the setting of modules over a vertex algebras.

We summarize in the following diagram the main objects that we will encounter in this part:

Note that we do not claim that this diagram commutes. In general, only the upper right triangle does. We will see, however, many interesting examples where the above diagram commutes, that is, $\operatorname{gr} \operatorname{Zhu}(V) \cong R_V$. We will also discuss relations between the Poisson vertex algebra $\operatorname{gr} V$ and the coordinate ring over the arc space of the associated scheme $\operatorname{Spec} R_V$.

Chapter 4
Poisson vertex algebras

We refer the reader to Appendix C for basics on Poisson algebras and Poisson varieties. Just as the graded space of an almost-commutative filtered associative algebra with unit have naturally a structure of a Poisson algebra, we will see in this chapter that any vertex algebra is naturally filtered and that the corresponding graded space is naturally a *Poisson vertex algebra* (Definition 4.1). A nice way to construct Poisson vertex algebras is to consider the coordinate ring of the arc space of an affine Poisson variety (see Section 4.2). Actually, strong relations exists, at least conjecturally, between the arc space of the associated variety and the singular support of a vertex algebra, that is, the spectrum of the corresponding graded algebra.

In this Chapter, we also introduce the Zhu C_2-algebra R_V of a vertex algebra V. The corresponding scheme and the corresponding reduced scheme, called the *associated scheme* and the *associated variety*, respectively, are crucial invariants of the vertex algebra V which capture important information about V. The singular support of V is deeply related to the arc spaces of these schemes. Not surprisingly, the associated variety is usually easier to describe than the associated scheme as we will observe in many examples in the following parts. It already contains meaningful information. One can, for example, detect from it the *lisse condition*.

In Section 4.3, we define the Zhu C_2-algebra R_V of a vertex algebra. The notion of associated scheme and associated variety are introduced in Section 4.4. We will also discuss in this section connections between the singular support and the arc spaces of these schemes. Section 4.7 deals with the lisse condition.

4.1 Definition

Recall that a vertex algebra V is a *commutative vertex algebra* (cf. Section 2.9), that is, a unital commutative algebra equipped with a derivation, if and only if $a_{(n)} = 0$ in $\mathrm{End}(V)$ for all $n \geqslant 0$.

Definition 4.1 A commutative vertex algebra V is called a *Poisson vertex algebra* if there exists a linear map

$$V \otimes V \to V[\lambda], \quad a \otimes b \mapsto \{a_\lambda b\} = \sum_{n \geqslant 0} \frac{\lambda^n}{n!} a_{(n)} b, \tag{4.1}$$

called the λ-*bracket*, such that

$$\{(\partial a)_\lambda b\} = -\lambda\{a_\lambda b\}, \quad \{a_\lambda \partial b\} = (\lambda + \partial)\{a_\lambda b\}, \quad \text{(sesquilinearity)} \tag{4.2}$$
$$\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}, \quad \text{(skewsymmetry)} \tag{4.3}$$
$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_\mu c\}, \quad \text{(the Jacobi identity)} \tag{4.4}$$
$$\{a_\lambda (bc)\} = \{a_\lambda b\} c + b\{a_\lambda c\}, \quad \text{(left Leibniz rule).} \tag{4.5}$$

Here, in (4.1), $a_{(n)}$, for $n \geqslant 0$, are “new" operators. (The “old" ones given by the field $a(z)$ being zero for $n \geqslant 0$ since V is commutative.)

In terms of operators $a_{(n)}$, “sesquilinearity", “skewsymmetry”, the “Jacobi identity” and the “left Leibniz rule” are equivalent to the following properties, respectively:

$$(\partial a)_{(n)} = [\partial, a_{(n)}] = -n a_{(n-1)}, \tag{4.6}$$
$$a_{(n)} b = \sum_{j \geqslant 0} (-1)^{n+j+1} \frac{1}{j!} \partial^j (b_{(n+j)} a), \tag{4.7}$$
$$[a_{(m)}, b_{(n)}] = \sum_{j \geqslant 0} \binom{m}{j} (a_{(j)} b)_{(m+n-j)}, \tag{4.8}$$
$$a_{(n)} (b \cdot c) = (a_{(n)} b) \cdot c + b \cdot (a_{(n)} c) \tag{4.9}$$

for all $a, b, c \in V$ and all $n, m \geqslant 0$.

The equation (4.9) says that $a_{(n)}$, for $n \geqslant 0$, is a derivation of the ring V. (Do not confuse $a_{(n)} \in \mathrm{Der}(V)$, for $n \geqslant 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.)

It follows from the definition that we also have the “right Leibniz rule” ([113, Exercise 4.2])

$$\{(ab)_\lambda c\} = \{b_{\lambda+\partial} c\}_{\to} a + \{a_{\lambda+\partial} c\}_{\to} b, \tag{4.10}$$

where $\{b_{\lambda+\partial} c\}_{\to}$ means that ∂ is moved to the right, that is,

$$\{b_{\lambda+\partial} c\}_{\to} a = \sum_{n \geqslant 0} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \lambda^j (b_{(n)} c)(\partial^{n-j} a).$$

One finds that (4.10) is equivalent to

$$(a \cdot b)_{(n)} c = \sum_{i \geqslant 0} (a_{(-i-1)} b_{(n+i)} c + b_{(-i-1)} a_{(n+i)} c),$$

for all $a, b, c \in V$, and $n \in \mathbb{Z}_{\geqslant 0}$ (compare with (2.33), cf. Section 4.3).

4.2 Poisson vertex structure on arc spaces

Arc spaces over an affine Poisson scheme naturally give rise to a Poisson vertex algebras, as shows the following result.

Theorem 4.1 ([6, Proposition 2.3.1]) *Given an affine Poisson scheme X, that is, $X = \operatorname{Spec} R$ for some Poisson algebra R, there is a unique Poisson vertex algebra structure on $\mathscr{J}_\infty(R) = \mathscr{O}(\mathscr{J}_\infty(X))$ such that*

$$\{a_\lambda b\} = \{a, b\}$$

for all $a, b \in R$, that is,

$$a_{(n)}b = \begin{cases} \{a, b\} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases}$$

for all $a, b \in R$.

Proof The bilinear map

$$R \otimes R \to R[\lambda], \quad a \otimes b \mapsto \{a_\lambda b\} = \{a, b\}, \tag{4.11}$$

clearly satisfies $\{a_\lambda b\} = -\{b_{-\lambda-\partial} a\}$, $\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} = \{\{a_\lambda b\}_\mu c\}$. This extends uniquely to the linear map

$$\mathscr{J}_\infty R \otimes \mathscr{J}_\infty(R) \to \mathscr{J}_\infty(R)\{\lambda\}, \quad a \otimes b \mapsto \{a_\lambda b\}, \tag{4.12}$$

satisfying (4.2), (4.3) and (4.5). Here, the well-defindness of this map follows from the fact that the relations in $\mathscr{J}_\infty R$ is spanned by the relations of the form $\partial^n a$, where a is a relation in R, and that (4.11) is well-defined.

Finally we need to show that the Jacobi identify (4.4) is satisfied. By the Leibniz rule it is sufficient to show this for the generators $\partial^n a$, for $a \in R$, $n \in \mathbb{Z}$, but this is easily done. □

Remark 4.1 More generally, given a Poisson scheme X, not necessarily affine, the structure sheaf $\mathscr{O}_{\mathscr{J}_\infty(X)}$ carries a unique Poisson vertex algebra structure such that

$$f_{(n)}g = \delta_{n,0}\{f, g\}$$

for all $f, g \in \mathscr{O}_X \subset \mathscr{O}_{\mathscr{J}_\infty(X)}$, see [18, Lemma 2.1.3.1].

Example 4.1 Recall that $\mathbb{C}[\mathfrak{g}^*]$ has naturally a Poisson structure induced from the Kirillov-Kostant-Souriau Poisson structure on $\mathfrak{g}^*$ (see Example C.2). Namely, for all $f, g \in \mathscr{O}(\mathfrak{g}^*)$ and all $x \in \mathfrak{g}^*$,

$$\{f, g\}(x) = \langle x, [d_x f, d_x g]\rangle,$$

where $d_x f, d_x g$ are the differentials of f, g, respectively, at $x \in \mathfrak{g}^*$ viewed as elements of $(\mathfrak{g}^*)^* \cong \mathfrak{g}$. In particular, for $f, g \in \mathfrak{g} = (\mathfrak{g}^*)^* \subset \mathscr{O}(\mathfrak{g}^*)$,

$$\{f,g\} = [f,g].$$

Since

$$\mathscr{J}_\infty(\mathfrak{g}^*) = \operatorname{Spec}\mathbb{C}[x^i_{(-n)}\ ;\ i=1,\dots,d,\ n\geqslant 1],$$

where $\{x^1,\dots,x^d\}$ is a basis of $\mathfrak{g}$, it follows from Theorem 4.1 that $\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))$ inherits a Poisson vertex algebra from that of $\mathscr{O}(\mathfrak{g}^*)$.

We may identify $\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))$ with the symmetric algebra $S(t^{-1}\mathfrak{g}[t^{-1}])$ via

$$x_{(-n)} \longmapsto xt^{-n}, \qquad x\in\mathfrak{g},\ n\geqslant 1.$$

For $x\in\mathfrak{g}$, identify x with $x_{(-1)}|0\rangle = (xt^{-1})|0\rangle$, where $|0\rangle$ stands for the unit element in $S(t^{-1}\mathfrak{g}[t^{-1}])$. Then (4.8) gives that

$$[x_{(m)}, y_{(n)}] = (x_{(0)}y)_{m+n} = \{x,y\}_{(m+n)} = [x,y]_{(m+n)},$$

for all $x,y\in\mathfrak{g}$ and all $m,n\in\mathbb{Z}_{\geqslant 0}$. So the Lie algebra $\mathscr{J}_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$ acts on $\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))$ by:

$$\mathfrak{g}[[t]] \to \operatorname{End}(\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))), \quad xt^n \mapsto x_{(n)}, \quad n\geqslant 0,$$

where $x_{(n)}$, for $n\geqslant 0$, is the endomorphism of $\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))$ given by the Poisson vertex structure on $\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))$. This action coincides with that obtained by differentiating the action of $\mathscr{J}_\infty(G) = G[[t]]$ on $\mathscr{J}_\infty(\mathfrak{g}^*)$ induced by the coadjoint action of G (see Example 1.2). In other words, the Poisson vertex algebra structure of $\mathscr{O}(\mathscr{J}_\infty(\mathfrak{g}^*))$ comes from the $\mathscr{J}_\infty(G)$-action on $\mathscr{J}_\infty(\mathfrak{g}^*)$.

Example 4.2 Consider the cotangent bundle T^*G to an affine algebraic group G, which is a smooth affine symplectic variety. In particular, $\mathscr{O}(T^*G)$ is a Poisson algebra. Since $T^*G = G\times\mathfrak{g}^*$, we have

$$\mathscr{O}(T^*G) = \mathscr{O}(\mathfrak{g}^*)\otimes\mathscr{O}(G).$$

The Poisson algebra structure of $\mathscr{O}(T^*G)$ is described as follows. The natural embeddings

$$\mathscr{O}(\mathfrak{g}^*) \hookrightarrow \mathscr{O}(T^*G), \quad \mathscr{O}(G) \hookrightarrow \mathscr{O}(T^*G),$$

are homomorphisms of Poisson algebras, where $\mathscr{O}(\mathfrak{g}^*)$ is equipped with the Kirillov-Kostant-Souriau Poisson structure and $\mathscr{O}(G)$ is equipped with the trivial Poisson structure. Finally, the Poisson bracket between $\mathscr{O}(\mathfrak{g}^*)$ and $\mathscr{O}(G)$ is described by the following formula:

$$\{x,f\} = x_L f,$$

for $x\in\mathfrak{g}\subset\mathscr{O}(\mathfrak{g}^*)$, $f\in\mathscr{O}(G)$.

By Theorem 4.1, $\mathscr{O}(\mathscr{J}_\infty T^*G)$ is naturally a Poisson vertex algebra. Since $\mathscr{J}_\infty T^*G = \mathscr{J}_\infty G \times \mathscr{J}_\infty \mathfrak{g}^*$ by Lemma 1.4, we have

$$\mathscr{O}(\mathscr{J}_\infty T^*G) = \mathscr{O}(\mathscr{J}_\infty \mathfrak{g}^*) \otimes \mathscr{O}(\mathscr{J}_\infty G),$$

and the Poisson vertex algebra structure is given by the following formulas:

$$\begin{aligned} &\{x_\lambda y\} = [x, y], \qquad x, y \in \mathfrak{g} \subset \mathscr{O}(\mathfrak{g}^*) \subset \mathscr{O}(\mathscr{J}_\infty \mathfrak{g}^*),\\ &\{f_\lambda g\} = 0, \qquad f, g \in \mathscr{O}(G) \subset \mathscr{O}(\mathscr{J}_\infty G),\\ &\{x_\lambda f\} = x_L f \qquad x \in \mathfrak{g},\ f \in \mathscr{O}(G). \end{aligned}$$

4.3 The Li filtration and the corresponding Poisson vertex structure

Our second basic example of Poisson vertex algebras comes from the graded vertex algebra associated with the canonical filtration, that is, the *Li filtration*.

Definition 4.2 Let V be a vertex algebra. A set $\{a^i : i \in I\}$ of vectors in V is called *a set of strong generators* if V is spanned by $|0\rangle$ and the elements of the form

$$a^{i_1}_{(-n_1-1)}, \dots, a^{i_r}_{(-n_r-1)}|0\rangle$$

with $r \geqslant 0$, $i_j \in I$, $n_j \geqslant 0$. A vertex algebra V is called *finitely strongly generated* if there exist a finite set of strong generators.

Note that $\{a : a \in V\}$ is a set of strong generators.

The universal affine vertex algebra $V^\kappa(\mathfrak{a})$, the vertex algebra of cdo $\mathcal{D}^{ch}_{G,\kappa}$ on an affine algebraic group G, the Virasoro vertex algebra Vir^c and their quotient vertex algebra are strongly finitely generated.

Haisheng Li [143] has shown that every vertex algebra is canonically filtered. For a vertex algebra V, choose a set $\{a^i : i \in I\}$ of strong generators of V. Let F^pV be the subspace of V spanned by the elements

$$a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle, \tag{4.13}$$

with $i_j \in I$, $n_j \geqslant 0$, $n_1 + n_2 + \cdots + n_r \geqslant p$. Then

$$V = F^0V \supset F^1V \supset \dots.$$

It is clear from the definition that $TF^pV \subset F^{p+1}V$, where T is the translation operator of V.

Definition 4.3 The decreasing filtration $(F^pV)_p$ is called the *Li filtration*.

Set for $n \in \mathbb{Z}$,

$$(F^pV)_{(n)}F^qV := \mathrm{span}_{\mathbb{C}}\{a_{(n)}b\ ;\ a \in F^pV,\ b \in F^qV\}.$$

Lemma 4.1 *For $p \geqslant 1$ we have*

$$F^pV = \sum_{j=1}^{p}(F^0V)_{(-j-1)}F^{p-j}V.$$

In particular, the Li filtration $F^\bullet V$ is independent of the choice of the strong generators of V.

Proof Let $v \in (F^0V)_{(-j-1)}F^{p-j}V$, for $j \in \{1,\dots,p\}$. Since $\{a^i : i \in I\}$ is a set of strong generators of F^0V, one can write

$$v = (a^{i_1}_{(-n_1-1)}a^{i_2}_{(-n_2-1)}\cdots a^{i_r}_{(-n_r-1)}|0\rangle)_{(-j-1)}b,$$

with $i_j \in I$, $n_i \geqslant 0$, $b \in F^{p-j}V$. Then it follows from Borcherds identity (2.33) and induction that v is a linear combination of elements of the form

$$a^{i_1}_{(-m_1-1)}a^{i_2}_{(-m_2-1)}\cdots a^{i_r}_{(-m_r-1)}b, \tag{4.14}$$

with $m_j \geqslant 0$ such that $m_1 + \cdots + m_r = \sum\limits_{j=1}^{r} n_j + j \geqslant j$. From this, it is now easy to see that $v \in F^pV$ because b is in $F^{p-j}V$. This shows the inclusion

$$\sum_{j=1}^{p}(F^0V)_{(-j-1)}F^{p-j}V \subset F^pV.$$

To show the other inclusion, set

$$\tilde{F}_pV = \sum_{j=1}^{p}(F^0V)_{(-j-1)}F^{p-j}V.$$

It is enough to prove that any monomial of F^pV of the form (4.13) is contained in $\tilde{F}_pV$. We argue by induction on r, the length of a monomial (4.13). Let $v \in F^pV$ be a monomial as in (4.13). Then $v = a^{i_1}_{(-n_1-1)}b$, with $b \in F^{p-n_1}V$. If $n_1 \geqslant 1$, we clearly get $v \in \tilde{F}_pV$ since $F^0V = V$.

Assume that $n_1 = 0$. Then $b \in F^pV$ is a monomial of length $r-1$. By the induction hypothesis, it is a sum of elements of the form $w_{(-j-1)}c$, with $w \in F^0V$, $j \in \{1,\dots,p\}$, $c \in F^{p-j}V$. By Borcherds identity (2.32), we have

$$a^{i_1}_{(-1)}w_{(-j-1)}c = w_{(-j-1)}a^{i_1}_{(-1)}c + \sum_{i\geqslant 0}\binom{-1}{i}(a_{(i)}w)_{(-j-i-2)}c.$$

Since $a^{i_1}_{(-1)}c \in F^{p-j}V$ and $w \in F^0V$, we see that $w_{(-j-1)}a^{i_1}_{(-1)}c \in \tilde{F}^pV$. Next, $a_{(i)}w \in F^0V$ and $c \in F^{p-j}V \subset F^{p-j-i-1}V$. Therefore $(a_{(i)}w)_{(-j-i-2)}c \in \tilde{F}_pV$. This shows that $v \in \tilde{F}_pV$, whence the expected inclusion. □

Proposition 4.1 *Let $p, q \in \mathbb{Z}$. We have $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n-1}V$ for all $n \in \mathbb{Z}$. Moreover, if $n \geqslant 0$, then $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n}V$. Here we have set $F^pV = V$ for $p < 0$.*

Proof $*$ First case. $n \leqslant -1$, that is, $n = -j-1$ with $j \geqslant 0$. Any element of $(F^pV)_{(n)}(F^qV)$ is a linear combination of elements of the form

$$(a^{i_1}_{(-n_1-1)}a^{i_2}_{(-n_2-1)}\cdots a^{i_r}_{(-n_r-1)}|0\rangle)_{(-j-1)}b,$$

with $i_j \in I$, $n_i \geqslant 0$, $n_1+\cdots+n_r \geqslant p$, $b \in F^qV$. So, arguing as in the proof of Lemma 4.1 by using Borcherds identity (2.33) and induction, we easily obtain that any element of $(F^pV)_{(n)}(F^qV)$ is a linear combination of elements of the form

$$a^{i_1}_{(-m_1-1)}a^{i_2}_{(-m_2-1)}\cdots a^{i_r}_{(-m_r-1)}b,$$

with $m_j \geqslant 0$, $m_1+\cdots+m_r = \sum_{j=1}^{r} n_j + j \geqslant p+j = p-n-1$, $b \in F^qV$, whence the inclusion $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n-1}V$.

$*$ Second case. $n \geqslant 0$. Since $F^{p+q-n}V \subset F^{p+q-n-1}V$, it suffices to show that $(F^pV)_{(n)}(F^qV) \subset F^{p+q-n}V$. We prove the statement by induction on q, observing that for $q \leqslant n-p$, the inclusion is clear because $F^{p+q-n}V = V$.

Assume $q > n-p$. The space $(F^pV)_{(n)}(F^qV)$ is generated by vectors $a_{(n)}b$, with $a \in F^pV$, $b \in F^qV$. By Lemma 4.1, a vector $b \in F^qV$ is a sum of vectors $u_{(-j-1)}c$, with $u \in V$, $j \in \{1,\ldots,q\}$, $c \in F^{q-j}V$. By Borcherds identity (2.32), we have

$$a_{(n)}u_{(-j-1)}c = u_{(-j-1)}a_{(n)}c + \sum_{i\geqslant 0}\binom{n}{i}(a_{(i)}u)_{(n-i-j-1)}c.$$

By the induction hypothesis, $a_{(n)}c \in F^{p+q-j-n}V$ since $q-j < q$ and, hence, $u_{(-j-1)}a_{(n)}c \in F^{p+q-n}V$. Next, assume for awhile that $a_{(i)}u \in F^{p-i}V$. Then by the first case, $(a_{(i)}u)_{(n-i-j-1)}c \in F^{p-i+q-j-(n-i-j-1)-1}V = F^{p+q-n}V$ and, therefore, $a_{(n)}u_{(-j-1)}c \in F^{p+q-n}V$, which shows the expected conclusion.

So, it remains to show that for $p \in \mathbb{Z}$ and $n \geqslant 0$, we have $(F^pV)_{(n)}V \subset F^{p-n}V$. We prove this fact by induction on p, observing that the statement is obvious for $p \leqslant 0$ since then $p-n \leqslant 0$ and $F^{p-n}V = V$. Assume $p > 0$. By Lemma 4.1, a vector $a \in F^pV$ is a sum of vectors $u_{(-j-1)}b$, with $u \in V$, $j \in \{1,\ldots,p\}$, $b \in F^{p-j}V$. By Borcherds identity (2.33), we have for $c \in V$,

$$(u_{(-j-1)}b)_{(n)}c = \sum_{i\geqslant 0}(-1)^i\binom{-j-1}{i}(u_{(-j-i-1)}b_{(n+i)}c - (-1)^{j+1}b_{(-j-1+n-i)}u_{(i)}c).$$

By the induction hypothesis, $b_{(n+i)}c \in F^{p-j-n-i}V$ because $p-j < p$. Hence $u_{(-j-i-1)}b_{(n+i)}c \in F^{p-n}V$. On the other hand, by the first case or the induction hypothesis if $-j-1+n-i \geqslant 0$, $b_{(-j-1+n-i)}u_{(i)}c \in F^{p-j-(-j-1+n-i)-1}V = F^{p-n+i}V \subset F^{p-n}V$ because $i \geqslant 0$. In conclusion, we have shown the inclusion $(F^pV)_{(n)}V \subset F^{p-n}V$ for $n \geqslant 0$, as desired.

This concludes the proof of the lemma. □

Definition 4.4 A vertex algebra V is called *good* if the filtration $F^\bullet V$ is *separated*, that is, $\bigcap_{p\geqslant 0} F^pV = \{0\}$.

In Corollary 4.2 below we show that any positively graded vertex algebra is good.

Set

$$\mathrm{gr}^F V = \bigoplus_{p\geqslant 0} F^pV/F^{p+1}V.$$

We denote by $\sigma_p \colon F^pV \mapsto F^pV/F^{p+1}V$, for $p \geqslant 0$, the canonical quotient map. When the filtration F is obvious, we often briefly write $\mathrm{gr}\,V$ for the space $\mathrm{gr}^F V$.

We have $V \cong \mathrm{gr}\,V$ as vector space for a good vertex algebra V.

Proposition 4.2 **([143])** *The space* $\mathrm{gr}^F V$ *is a Poisson vertex algebra by*

$$\sigma_p(a)\cdot\sigma_q(b) := \sigma_{p+q}(a_{(-1)}b), \tag{4.15}$$

$$\partial\sigma_p(a) := \sigma_{p+1}(Ta), \tag{4.16}$$

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n}(a_{(n)}b), \tag{4.17}$$

for all $a \in F^pV \setminus F^{p+1}V$, $b \in F^qV$, $n \geqslant 0$.

Proof First of all, the space $\mathrm{gr}^F V$ naturally inherits a graded vertex algebra structure from the vertex algebra structure on V. The vertex operator is given by

$$Y(\sigma_p(a),z)b := \sum_{n\in\mathbb{Z}} \sigma_{p+q-n-1}(a_{(n)}b)z^{-n-1},$$

for $a \in F^pV \setminus F^{p+1}V$, $b \in F^qV$, $n \in \mathbb{Z}$, the vacuum is $|0\rangle = \sigma_0(|0\rangle)$ and the translation operator is the linear map sending $a \in F^pV \setminus F^{p+1}V$ to $\sigma_p(a)_{(-2)}|0\rangle = \sigma_{p+1}(Ta)$, since $Ta = a_{(-2)}|0\rangle$. The axioms are easy to check. The verifications are left to the reader.

Furthermore, by Proposition 4.1, $Y(\sigma_p(a),z)_{(n)}Y(\sigma_q(b),z) = 0$ for $a \in F^pV \setminus F^{p+1}V$, $b \in F^qV$, $n \geqslant 0$ and, hence, $\mathrm{gr}^F V$ is a commutative vertex algebra whose product is given by (4.15), and derivation is given by (4.16).

It remains to show that (4.17) defines a Poisson vertex algebra on $\mathrm{gr}^F V$. It is easy to check that the axioms (4.6), (4.7) and (4.8) are satisfied. We prove only (4.9). Let $a \in F^pV \setminus F^{p-1}V$, $b \in F^qV$, $c \in F^rV$, $n \geqslant 0$. By Borcherds identity (2.33), we have

$$a_{(n)}(b_{(-1)}c) = b_{(-1)}a_{(n)}c + \sum_{i\geqslant 0}\binom{n}{i}(a_{(i)}b)_{(n-1-i)}c$$
$$= b_{(-1)}a_{(n)}c + (a_{(n)}b)_{(-1)}c + \sum_{i=0}^{n-1}\binom{m}{i}(a_{(i)}b)_{(n-1-i)}c.$$

For $i \in \{0,\dots,n-1\}$, $(a_{(i)}b)_{(n-1-i)}c \in F^{p+q+r-n+1}V$ since $n-1-i \geqslant 0$, while $a_{(n)}(b_{(-1)}c)$, $b_{(-1)}a_{(n)}c$ and $(a_{(n)}b)_{(-1)}c$ are in $F^{p+q+r-n}V$. Hence,

$$\sigma_p(a)_{(n)}(\sigma_q(b).\sigma_r(c)) = \sigma_q(b).(\sigma_p(a)_{(n)}\sigma_r(c)) + (\sigma_p(a)_{(n)}\sigma_q(b)).\sigma_r(c), \tag{4.18}$$

whence the expected statement. □

Define

$$R_V := V/F^1V = F^0V/F^1V \subset \mathrm{gr}^F V. \tag{4.19}$$

Note that

$$F^1V = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}b\colon a,b\in V\} \tag{4.20}$$

by Lemma 4.1. We also write $C_2(V)$ for the space F^1V by historical reason.

Proposition 4.3 ([184]) *R_V is a Poisson algebra by*

$$\bar{a}\cdot\bar{b} = \overline{a_{(-1)}b},\quad \{\bar{a},\bar{b}\} = \overline{a_{(0)}b}$$

for $a,b\in V$, where $\bar{a} = \sigma_0(a)$.

Proof First, by (4.15) with $p=q=0$, the product $\bar{a}\cdot\bar{b} = \overline{a_{(-1)}b}$, for $a,b\in V$, gives to R_V a ommutative associative algebra structure, with unit $\overline{|0\rangle}$.

Let us prove that the bracket $\{\bar{a},\bar{b}\} = \overline{a_{(0)}b}$, for $a,b\in V$, is Poisson for the commutative algebra R_V. It verifies the skew-symmetry property by (4.7) with $n=0$ and $a,b\in F^0V$ so that $\partial^j(\bar{b}_{(j)}\bar{a}) \in F^1V$ for $j>0$, and the left Leibniz rule by (4.18) with $p=q=0$ and $n=0$. Then it also verifies the right Leibniz rule by the skew-symmetry. As for the Jacobi identity, it follows from (4.8) with $m=n=0$. □

Definition 4.5 The Poisson algebra R_V is called the *Zhu C_2-algebra* of V.

Proposition 4.4 ([143]) *As a differential algebra, $\mathrm{gr}^F V$ is generated by R_V.*

Proof Set $A = \bigoplus_{p\geqslant 0} A_p = \mathrm{gr}^F V$, $A_p = F^pV/F^{p+1}V$. We wish to show that the graded differential algebra A is generated by $A_0 = R_V$ as a differential algebra.

First, note that we have

$$A_+ := \bigoplus_{p>0} A_p = A\partial A. \tag{4.21}$$

Indeed, it is clear that $A\partial A \subset A_+$.

Conversely, let us show that $A_+ \subset A\partial A$. Let $v \in F^pV$. By Lemma 4.1, v is a sum of terms of the form $a_{(-j-1)}b$, with $j \in \{1, \dots, p\}$, $a \in F^0V$ and $b \in F^{p-j}V$. By Borcherds identity (2.32) and (2.30), we have

$$a_{(-j-1)}b = a_{(-j-1)}b_{(-1)}|0\rangle = b_{(-1)}a_{(-j-1)}|0\rangle + \sum_{l\geqslant 0}\binom{-j-1}{l}(a_{(l)}b)_{(-j-2-l)}|0\rangle$$
$$= b_{(-1)}\left(\frac{T^j a}{j!}\right) + \sum_{l\geqslant 0}\binom{-j-1}{l}\frac{T^{j+l+1}(a_{(l)}b)}{(j+l+1)!}.$$

Hence,

$$\sigma(a_{(-j-1)}b) = \sigma_{p-j}(b).\partial^j\left(\frac{\sigma_0(a)}{j!}\right) + \sum_{l\geqslant 0}\binom{-j-1}{l}\partial^{j+l+1}\left(\frac{\sigma_{p-j-l-1}(a_{(l)}b)}{(j+l+1)!}\right).$$

This shows that A_p is contained in $A\partial A$ for all $p > 0$, whence $A_+ \subset A\partial A$.

Let A' be the differential subalgebra of A generated by A_0. We will show by induction on p that $A_p \subset A'$.

Clearly, $A_0 \subset A'$. So let $p > 0$. By (4.21), $A_p = \sum_{i=0}^{p-1} A_i\partial A_{p-i-1}$, which is contained in A' by the induction hypothesis. □

Corollary 4.1 *Let $\{a^i : i \in I\}$ be a set of vectors of a good vertex algebra V. The following are equivalent.*

i). $\{a^i : i \in I\}$ are strong generators of V;
ii). the image of $\{a^i : i \in I\}$ generates R_V.

In particular, a vertex algebra V is finitely strongly generated if and only if R_V is finitely generated.

In this book we will always assume that a vertex algebra V is finitely strongly generated.

Definition 4.6 Let $\phi\colon V \to W$ be a map between two Poisson vertex algebras. We say that ϕ is a *Poisson vertex algebra homomorphism* if ϕ is a homomorphism of differential algebras such that

$$\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b),$$

for all $a, b \in V$, $n \geqslant 0$.

The following assertion is clear.

Lemma 4.2 *If $\phi\colon V \to W$ is a homomorphism of vertex algebras, then ϕ respects the canonical filtration, that is, $\phi(F^pV) \subset F^pW$. Hence it induces a homomorphism $\mathrm{gr}^F V \to \mathrm{gr}^F W$ of Poisson vertex algebra homomorphism which we denote by $\mathrm{gr}^F \phi$. The map $\mathrm{gr}^F \phi$ restricts to a Poisson algebra homomorphism $R_V \to R_W$, which we denote by $\bar{\phi}$. If in addition ϕ is surjective, then $\phi(F^pV) = F^pW$. In particular, $\mathrm{gr}\,\phi : \mathrm{gr}^F V \to \mathrm{gr}^F W$ and $\bar{\phi} : R_V \to R_W$ are surjective homomorphisms of Poisson vertex algebras and Poisson algebras, respectively.*

4.4 Associated variety and singular support

We now focus on geometrical objects associated with R_V and $\mathrm{gr}^F V$.

Definition 4.7 Define the *associated scheme* $\tilde{X}_V$ and the *associated variety* X_V of a vertex algebra V as

$$\tilde{X}_V := \mathrm{Spec}\, R_V, \qquad X_V := \mathrm{Specm}\, R_V = (\tilde{X}_V)_{\mathrm{red}}.$$

Definition 4.8 Let X be an affine Poisson variety. A vertex algebra V is called a *chiral quantization* of X if $X_V \cong X$ as Poisson varieties.

By Proposition 4.4, $\mathrm{gr}^F V$ is generated by the subring R_V as a differential algebra. Thus, we have a surjection $\mathscr{J}_\infty(R_V) \to \mathrm{gr}^F V$ of differential algebras by Lemma 1.1 since R_V generates $\mathscr{J}_\infty(R_V)$ as a differential algebra, too.

This is in fact a homomorphism of Poisson vertex algebras.

Proposition 4.5 ([6, Proposition 2.5.1]) *The identity map $R_V \to R_V$ induces a surjective Poisson vertex algebra homomorphism*

$$\mathscr{J}_\infty(R_V) = \mathscr{O}(\mathscr{J}_\infty(\tilde{X}_V)) \twoheadrightarrow \mathrm{gr}^F V.$$

Proof As noticed just above, the identity map $R_V \to R_V$ induces a surjective homomorphism of differential algebras $f\colon \mathscr{J}_\infty(R_V) \to \mathrm{gr}^F V$. Let us show that f is a Poisson vertex algebra homomorphism. It suffices to verify that $f(a_{(n)}b) = f(a)_{(n)} f(b)$, for all $a, b \in \mathscr{J}_\infty(R_V)$ and all $n \geqslant 0$.

By construction, this is true for all $a, b \in R_V$ and $n \geqslant 0$, since the restriction of f to R_V is the identity map, and $a_{(n)}b = \delta_{n,0}\{a, b\}$ for $a, b \in R_V$. The statement is then a direct consequence of Lemma 4.3 below. ◻

Remark 4.2 Suppose that the Poisson structure of R_V is trivial. Then the Poisson vertex algebra structure of $\mathscr{J}_\infty(R_V)$ is trivial, and so is that of $\mathrm{gr}^F V$ by Proposition 4.5. This happens if and only if

$$(F^pV)_{(n)}(F^qV) \subset F^{p+q-n+1}V \quad \text{for all} \quad n \geqslant 0.$$

If this is the case, one can give $\mathrm{gr}^F V$ yet another Poisson vertex algebra structure by setting

$$\sigma_p(a)_{(n)}\sigma_q(b) := \sigma_{p+q-n+1}(a_{(n)}b) \quad \text{for all} \quad n \geqslant 0.$$

(We can repeat this procedure if this Poisson vertex algebra structure is again trivial.)

Lemma 4.3 ([142, Lemma 3.3]) *Let V, W be two Poisson vertex algebras, and $\phi\colon V \to W$ an algebra homomorphism such that $\phi\partial = \partial\phi$. Suppose that*

$$\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b) \ \text{ for all } \ a, b \in R \text{ and } n \geqslant 0,$$

where R is a generating subset of V as a differential algebra. Then ϕ is a vertex Poisson algebra homomorphism.

Proof Let $a, b \in V$ be such that

$$\phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b) \ \text{ for all } \ n \geqslant 0. \tag{4.22}$$

Using (4.6) for both V and W, the assumption $\phi\partial = \partial\phi$ and (4.22), we obtain for all $n \geqslant 0$:

$$\begin{aligned}\phi(a_{(n)}\partial b) &= \phi(\partial a_{(n)}b) + n\phi(a_{(n-1)}b)\\ &= \partial\phi(a)_{(n)}\phi(b) + n\phi(a)_{(n-1)}\phi(b)\\ &= \phi(a)_{(n)}(\partial\phi(b)).\end{aligned}$$

By the left Leibniz rule (4.9) and induction we deduce that (4.22) holds for all $a \in R$ and $b \in V$.

Next, using the skew-symmetry (4.7) and $\phi\partial = \partial\phi$ we get that $\phi(\partial a_{(n)}b) = \phi(\partial a)_{(n)}\phi(b)$ for all $a \in R$, $b \in V$ and $n \geqslant 0$. Again by the left Leibniz rule (4.9) and induction, we deduce that (4.22) holds for all $a, b \in V$.

This concludes the proof of the lemma. □

Definition 4.9 Define the *singular support* $SS(V)$ of a vertex algebra V as

$$SS(V) = \operatorname{Spec}(gr^F V) \subset \mathscr{J}_\infty \tilde{X}_V.$$

Definition 4.10 We say that a vertex algebra V admits a *PBW basis* if there exists a collection $\{a^i : i = 1, \ldots, n\}$ of vectors of V such that the set of monomials

$$a^{i_1}_{(-n_1)}a^{i_2}_{(-n_2)}\ldots a^{i_r}_{(-n_r)}|0\rangle, \quad i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r,\ n_s \leqslant n_{s+1} \text{ if } i_s = i_{s+1},$$

form a basis of V.

Lemma 4.4 *The following conditions are equivalent.*

i). V admits a PBW basis.
ii). R_V is isomorphic to a polynomial ring and $SS(V) = \mathscr{J}_\infty \tilde{X}_V$.

Proof Should we provide a proof? □

4.5 Examples

The following assertion is easily seen from definition, Lemma 4.1 and Borcherds identity (2.32).

Lemma 4.5 *Let $\{a^i : i \in I\}$ be a set of strong generators of a vertex algebra V such that for all $i_1, i_2 \in I$ and all $n \geqslant 0$, $a^{i_1}_{(n)} a^{i_2}$ is a linear combination of $|0\rangle$ and the a^i's, $i \in I$. Then*

$$F^1 V = \mathrm{span}_{\mathbb{C}}\{a^i_{(-n_i-2)} v : i \in I,\ n_i \geqslant 0, v \in V\}.$$

Proof Writing v as a linear combination of elements of the form (4.13), we see that $a^i_{(-n_i-2)} v$, $i \in I$, $n_i \geqslant 0$, $v \in V$ belongs to $F^1 V$, whence one inclusion.

Conversely, show by induction on the length r of monomials v of the form (4.13) that $F^1 V$ is contained in the right-hand-side. Let $v = a^{i_1}_{(-n_1-1)} a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)} |0\rangle \in F^1 V$. At least one of the n_j's is greater than 1. If $n_1 \geqslant 1$, then the statement is clear. In particular, the statement is clear if $r = 1$. Assume $n_1 = 0$. Then $v' = a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)} |0\rangle \in F^1 V$ and by the induction hypothesis, there is $j \in I$, $m \geqslant 0$, $w \in V$ such that

$$v = a^{i_1}_{(-1)} a^j_{(-m-2)} w = a^j_{(-m-2)} a^{i_1}_{(-1)} w + \sum_{l \geqslant 0} \binom{-1}{l} (a^{i_1}_{(l)} a^j)_{(-m-3-l)} w.$$

The element $a^j_{(-m-2)} a^{i_1}_{(-1)} w$ lies in the right-hand-side set of the lemma. Moreover, by the hypothesis of the lemma, $(a^{i_1}_{(l)} a^j)_{(-m-3-l)} w$ is a linear combination of elements $a^i_{(-m-3-l)} w$, $i \in I$. Note that $|0\rangle_{(-m-3-l)} w = 0$ because $-m-3-l$ cannot be equal to -1. Since $m, l \geqslant 0$, we get that $v \in \mathrm{span}_{\mathbb{C}}\{a^i_{(-n_i-2)} v : i \in I,\ n_i \geqslant 0, v \in V\}$, as desired. □

Example 4.3 Consider the universal affine vertex algebra $V^\kappa(\mathfrak{a})$ as defined in Section 3.1. Since $V^\kappa(\mathfrak{a})$ is strongly generated by $x \in \mathfrak{a} \subset V^\kappa(\mathfrak{a})$, we have

$$F^1 V^\kappa(\mathfrak{a}) = t^{-2}\mathfrak{a}[t^{-1}] V^\kappa(\mathfrak{a})$$

by Lemma 4.5. Therefore,

$$R_{V^\kappa(\mathfrak{a})} = V^\kappa(\mathfrak{a}) / t^{-2}\mathfrak{a}[t^{-1}] V^\kappa(\mathfrak{a}).$$

By the PBW theorem we have an isomorphism linear map

$$\mathscr{O}(\mathfrak{a}^*) = S(\mathfrak{a}) \xrightarrow{\simeq} R_{V^\kappa(\mathfrak{a})} \tag{4.23}$$

that sends the monomial $x^1 x^2 \dots x^r \in S(\mathfrak{a})$, for $x^i \in \mathfrak{a}$, to the image of $x^1_{(-1)} \dots x^r_{(-1)} |0\rangle$ in $R_{V^\kappa(\mathfrak{a})}$. This is in fact an homomorphism of Poisson algebras. Therefore,

$$\tilde{X}_{V^\kappa(\mathfrak{a})} = X_{V^\kappa(\mathfrak{a})} = \mathfrak{a}^*.$$

In particular, $V^\kappa(\mathfrak{a})$ is a chiral quantization of $\mathfrak{a}^*$. Moreover, the surjection

$$\mathscr{O}(\mathscr{J}_\infty \mathfrak{a}^*) \to \mathrm{gr}^F V^\kappa(\mathfrak{a})$$

is an isomorphism since both sides have the same graded dimension with respect to

$$\deg xt^{-n} = n, \qquad n \in \mathbb{Z}_{>0}.$$

(Here we have used the fact that $V^\kappa(\mathfrak{a})$ is good, see also Example 4.4 below.) Hence

$$SS(V^\kappa(\mathfrak{a})) = \mathscr{J}_\infty \mathfrak{a}^*.$$

For the simple quotient $L_\kappa(\mathfrak{a})$ of $V^\kappa(\mathfrak{a})$, the surjection $V^\kappa(\mathfrak{a}) \twoheadrightarrow L_\kappa(\mathfrak{a})$ induces a surjection $\mathscr{O}(\mathfrak{a}^*) = R_{V^\kappa(\mathfrak{a})} \twoheadrightarrow R_{L_\kappa(\mathfrak{a})}$. Thus,

$$R_{L_\kappa(\mathfrak{a})} \cong \mathscr{O}(\mathfrak{a}^*)/I$$

for some graded Poisson ideal I of $\mathscr{O}(\mathfrak{a}^*)$, and $X_{L_\kappa(\mathfrak{a})}$ is the zero locus of I in $\mathfrak{a}^*$, which is a conic Poisson subvariety. Similarly, $SS(L_\kappa(\mathfrak{a}))$ is a $\mathbb{C}^*$-invariant closed subscheme of $\mathscr{J}_\infty \mathfrak{a}^*$.

Exercise 4.1 Let Vir^c be the universal Virasoro vertex algebra of central charge $c \in \mathbb{C}$.

i). Show that $\mathrm{gr}^F \mathrm{Vir}^c \cong \mathbb{C}[L_{-2}, L_{-3}, \ldots]$.
ii). Deduce from (i) that $R_{\mathrm{Vir}^c} \cong \mathbb{C}[x]$, where x is the image of $L := L_{-2}|0\rangle$ in R_{Vir^c}, with the trivial Poisson structure.
iii). Show that one can endow $\mathrm{gr}^F \mathrm{Vir}^c$ with a non-trivial Poisson vertex algebra structure such that

$$\{L_\lambda L\} = TL + 2\lambda L.$$

4.6 The conformal weight filtration and comparison with the Li filtration

Suppose that V is positively graded:

$$V = \bigoplus_{\Delta \in \frac{1}{r_0}\mathbb{Z}_{\geqslant 0}} V_\Delta,$$

where r_0 is some positive integer. (In most cases we assume that $r_0 = 1$ or 2.) There is another natural filtration of V defined as follows [142].

Choose a set $\{a^i : i \in I\}$ of homogeneous strong generators of V. Let $G_p V$, $p \in \frac{1}{r_0}\mathbb{Z}_{\geqslant 0}$, be the subspace of V spanned by the vectors

$$a^{i_1}_{(-n_1-1)}a^{i_2}_{(-n_2-1)}\cdots a^{i_r}_{(-n_r-1)}|0\rangle \tag{4.24}$$

with $i_j \in I$, $n_j \geqslant 0$, $\Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_r}} \leqslant p$. Then $G_\bullet V$ defines an increasing filtration of V:

$$0 = G_{-1}V \subset G_0 V \subset \ldots G_1 V \subset \ldots, \qquad V = \bigcup_p G_p V.$$

Definition 4.11 The increasing filtration $G_\bullet V$ is called the *conformal weight filtration*.

Lemma 4.6 *We have*

$$TG_pV \subset G_pV, \tag{4.25}$$

$$(G_pV)_{(n)}G_qV \subset G_{p+q}V \quad \textit{for } n \in \mathbb{Z}, \tag{4.26}$$

$$(G_pV)_{(n)}G_qV \subset G_{p+q-1}V \quad \textit{for } n \in \mathbb{Z}_{\geqslant 0}. \tag{4.27}$$

Proof Since $[T, a^i_{(-n)}] = na^i_{(-n-1)}$, for any $i \in I$, $n \geqslant 0$, and $T|0\rangle = 0$, (4.25) is easily seen.

For $n < 0$, we establish (4.26) exactly as for the proof of Proposition 4.1, using Borcherds identity (2.33).

Assume $n \geqslant 0$. Since $G_{p+q-1}V \subset G_{p+q}V$ it suffices to establish (4.27). In addition, it suffices to prove that $a_{(n)}b \subset G_{p+q-1}V$ for all $a \in G_pV$, $b \in G_qV$ that are homogeneous.

Recall that by (12.7), we have for $n \geqslant 0$,

$$(V_\Delta)_{(n)}V_{\Delta'} \subset V_{\Delta+\Delta'-n-1}. \tag{4.28}$$

Therefore, (4.27) will be a consequence of the equality (4.29) in Lemma 4.7 below. Indeed, setting $F^iV_\Delta := F^iV \cap V_\Delta$, $G_iV_\Delta := G_iV \cap V_\Delta$ for $i \geqslant 0$, we obtain by (4.29) and Proposition 4.1,

$$\begin{aligned} a_{(n)}b \in (F^{\Delta_a - p}V_{\Delta_a})_{(n)}F^{\Delta_b-q}V_{\Delta_b} &\subset F^{\Delta_a-p+\Delta_b-q-n}V_{\Delta_a+\Delta_b-n-1} \\ &= G_{p+q-1}V_{\Delta_a+\Delta_b-n-1} \subset G_{p+q-1}V \end{aligned}$$

for homogenous elements $a \in G_pV$, $b \in G_qV$ and $n \geqslant 0$.

Notice that the proof of Lemma 4.7 uses (4.26) for $n < 0$, but does not use (4.27) or (4.26) for $n \geqslant 0$. □

It follows that $\mathrm{gr}_G V = \bigoplus_p G_pV/G_{p-1}V$ is naturally a Poisson vertex algebras.

Lemma 4.7 ([6, Proposition 2.6.1]) *We have*

$$F^pV_\Delta = G_{\Delta-p}V_\Delta, \tag{4.29}$$

where $F^pV_\Delta = V_\Delta \cap F^pV$, $G_pV_\Delta = V_\Delta \cap G_pV$. Therefore

$$\mathrm{gr}^F V \cong \mathrm{gr}_G V$$

as Poisson vertex algebras.

Proof The second assertion easily deduces from the first one. Let us prove the first assertion. Clearly, $V_\Delta \subset G_\Delta V_\Delta$ since for $a \in V_\Delta$, one can write $a = a_{(-1)}|0\rangle \in G_\Delta V$. The other inclusion is obvious and, hence, $V_\Delta = G_\Delta V_\Delta$, that is,

$$F^0 V_\Delta = G_\Delta V_\Delta.$$

We now show the inclusion $F^p V_\Delta \subset G_{\Delta-p} V_\Delta$ by induction on $p \geqslant 0$. Let $p > 0$. By Lemma 4.1, $F^p V_\Delta$ is generated by elements $v = a_{(-i-1)}b$, with $a \in V_{\Delta_a}$, $b \in F^{p-i} V_{\Delta_b}$, $i \geqslant 1$, $\Delta_a + \Delta_b + i = \Delta$. Hence it suffices to show that for such elements, $v \in G_{\Delta-p} V_\Delta$. By the induction hypothesis, $F^{p-i} V_{\Delta_b} \subset G_{\Delta_b - p + i} V_{\Delta_b}$. Because $a \in V_{\Delta_a} \subset G_{\Delta_a} V$, we have by (4.26) with $n = -i-1 < 0$,

$$v = a_{(-i-1)}b \in (G_{\Delta_a} V_{\Delta_a})_{(-i-1)} G_{\Delta_b - p + i} V_{\Delta_b} \subset G_{\Delta_a + \Delta_b - p + i} V_\Delta = G_{\Delta - p} V_\Delta.$$

Hence $F^p V_\Delta \subset G_{\Delta-p} V_\Delta$.

It remains to show the opposite inclusion $G_{\Delta-p} V_\Delta \subset F^p V_\Delta$. We prove that any element v of the form (4.24) belongs to $F^p V_\Delta$ by induction on $r \geqslant 0$. For $r = 0$, the statement is obvious. Assume $r > 0$. Then $v = a^{i_1}_{(-n_1-1)} w$, with $w = a^{i_2}_{(-n_2-1)} \cdots a^{i_r}_{(-n_r-1)}|0\rangle$, $n_j \geqslant 0$, $\sum_j \Delta_{a^{i_j}} \leqslant p$, $\Delta_{a^{i_1}} + \Delta_w + n_1 = \Delta$, where each a^{i_j} is homogeneous. Because $w \in G_{p - \Delta_{a^{i_1}}} V_{\Delta_w}$, the induction hypothesis gives that $w \in F^{\Delta_{a^{i_1}} + \Delta_w - p} V_{\Delta_w}$. Hence

$$v = a^{i_1}_{(-n_1-1)} w \in F^{\Delta_{a^{i_1}} + \Delta_w - p + n_1} V_{\Delta_{a^{i_1}} + \Delta_w + n_1} = F^{\Delta - p} V_\Delta$$

since $a \in F^0 V_{\Delta_{a^{i_1}}}$. □

By Lemma 4.7, it follows in particular that the conformal weight filtration is independent of the choice of the set of strong generators.

Corollary 4.2 *A vertex algebra is good if it is positively graded.*

Proof This is clear from Lemma 4.7 since $F^p V_\Delta = G_{\Delta-p} V_\Delta = 0$ if $p > \Delta$ for each Δ. □

Example 4.4 Consider the universal affine vertex algebra $V^\kappa(\mathfrak{a})$. Since $V^\kappa(\mathfrak{a})$ is strongly generated by $x \in \mathfrak{a} \subset V^\kappa(\mathfrak{a})$, which has conformal weight one, it follows that

$$G_p V^\kappa(\mathfrak{a}) = U_p(\mathfrak{a}[t^{-1}]t^{-1})|0\rangle,$$

where $U_\bullet(\mathfrak{a}[t^{-1}]t^{-1})$ is the PBW filtration of $U(\mathfrak{a}[t^{-1}]t^{-1})$ (see Example C.2). On the other hand, we have the isomorphisms (cf. Example 4.1)

$$\operatorname{gr} U(\mathfrak{a}[t^{-1}]t^{-1}) \cong S(\mathfrak{a}[t^{-1}]t^{-1}) \cong \mathbb{C}[\mathscr{J}_\infty(\mathfrak{a}^*)].$$

Hence, as a consequence of Lemma 4.7, we reconfirm the fact that

$$\operatorname{gr}^F V^\kappa(\mathfrak{a}) \cong \operatorname{gr}_G V^\kappa(\mathfrak{a}) \cong \mathbb{C}[\mathscr{J}_\infty(\mathfrak{a}^*)]$$

as Poisson vertex algebras.

Example 4.5 Consider the cdo $\mathcal{D}^{ch}_{G,\kappa}$ on G at level κ. We have

$$G_p \mathcal{D}^{ch}_{G,\kappa} = U_p(\mathfrak{g}[t^{-1}]t^{-1}) \otimes \mathscr{O}(J_\infty G).$$

Thus

$$\operatorname{gr}^F \mathcal{D}^{ch}_{G,\kappa} \cong \operatorname{gr}_G \mathcal{D}^{ch}_{G,\kappa} \cong \mathbb{C}[\mathscr{J}_\infty(\mathfrak{g}^*)] \otimes \mathscr{O}(J_\infty G) = \mathscr{O}(\mathscr{J}_\infty(T^*G)),$$

which restricts to the isomorphism

$$R_{\mathcal{D}^{ch}_{G,\kappa}} \cong \mathscr{O}(T^*G). \tag{4.30}$$

In parcticular, we have

$$\tilde{X}_{\mathcal{D}^{ch}_{G,\kappa}} \cong T^*G, \quad SS(\mathcal{D}^{ch}_{G,\kappa}) \cong \mathscr{J}_\infty T^*G. \tag{4.31}$$

4.7 The lisse condition

Geometrical properties of the associated variety X_V should reflect important information about the vertex algebra V. It is natural to first consider the simplest case where X_V has dimension 0.

Recall that we are assuming that a vertex algebra V is finitely strongly generated so that $\tilde{X}_V$ is a scheme of finite type.

Lemma 4.8 ([30, Exercise 8.3]) *Let $X = \operatorname{Spec} R$ be an affine scheme of finite type over a field K. Then the following assertion are equivalent:*

i). $\dim X = 0$*,*
ii). R is a finite dimensional K-algebra.

If so, then X is a finite discrete topological space.

Proof To prove the equivalence (i) $\iff$ (ii), recall that a Noetherian ring has dimension zero if and only if it is Artinian [152, Theorem 3.2 and Example 2 in §5]. So the converse implication (ii)$\Rightarrow$(i) is clear because a finite dimensional algebra is an Artinian ring. Indeed, if R is a finite dimensional K-vector space, then it is Artinian as K-vector space. But every ideal of R is a K-vector space and thus they satisfy the descending chain condition, which proves that R is Artinian as ring.

Conversely, assume that $\dim X = 0$, that is, R is Artinian. Then R is a finite product of Artinian local rings (cf. [30, Theorems 8.7]). So one may assume that R is an Artinian local ring, with maximal ideal $\mathfrak{m}$. Then $R/\mathfrak{m}$ if a finite extension of K by Zariski lemma. Since R is Artinian, $\mathfrak{m}$ is the radical of A ([152, proof of Theorem 3.2]) and thus $\mathfrak{m}^n = 0$ for some n. Thus we have a chain

$$R \supseteq \mathfrak{m} \supseteq \mathfrak{m}^2 \supseteq \cdots \supseteq \mathfrak{m}^n = 0.$$

Since R is Noetherian, $\mathfrak{m}$ is finitely generated and each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is a finite dimensional $R/\mathfrak{m}$-vector space, hence $\mathfrak{m}$ is a finite dimensional vector space. This completes the proof. □

Definition 4.12 A vertex algebra V is called *lisse* (or C_2*-cofinite*) if $\dim X_V = 0$ or, equivalently, if $R_V = V/F^1(V)$ is finite dimensional.

As a consequence of Proposition 4.5, we have the following result.

Theorem 4.2 *We have* $\dim SS(V) = 0$ *if and only if* $\dim X_V = 0$.

Proof The "only if" part is obvious since $\pi_\infty(SS(V)) = \tilde{X}_V$. The "if" part follows from Corollary 1.2 and Proposition 4.5. □

By Theorem 4.2 we get:

Lemma 4.9 *The vertex algebra V is lisse if and only if* $\dim SS(V) = 0$.

The C_2-cofiniteness condition, $\dim V/C_2(V) < \infty$, was introduced by Zhu [184], while the term *lisse* has been borrowed from Beilinson, Feigin and Mazur who considered the finiteness condition $\dim SS(V) = 0$ in the case of Virasoro vertex algebras. The equivalence between these two notions was established in [6]. In this book, we will be rather using the name *lisse*.

Lemma 4.10 *Suppose that V is conical, so that $V = \bigoplus_{\Delta \geqslant 0} V_\Delta$ and $V_0 = \mathbb{C}|0\rangle$. The algebras $\operatorname{gr}^F V$ and R_V are equipped with the induced grading:*

$$\operatorname{gr}^F V = \bigoplus_{\Delta \geqslant 0} (\operatorname{gr}^F V)_\Delta, \quad (\operatorname{gr}^F V)_0 = \mathbb{C},$$
$$R_V = \bigoplus_{\Delta \geqslant 0} (R_V)_\Delta, \quad (R_V)_0 = \mathbb{C}.$$

Then the following conditions are equivalent:

i). V is lisse,
ii). $X_V = \{\text{point}\}$,
iii). the image of any vector $a \in V_\Delta$ for $\Delta > 0$ in R_V is nilpotent,
iv). the image of any vector $a \in V_\Delta$ for $\Delta > 0$ in $\operatorname{gr}^F V$ is nilpotent.

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras.

Proof The equivalence (i) $\iff$ (ii) follows from Lemma 4.8. Let us prove the equivalence (i) $\iff$ (iii). One can assume that $R_V = \mathbb{C}[x^1, \dots, x^N]/I$ for some ideal I. If $X_V = \{\text{point}\}$, then $R_V/\sqrt{0} = \mathbb{C}$. So $\sqrt{I}$ is the argumentation ideal of $\mathbb{C}[x^1, \dots, x^N]$ or, equivalently, each x^i is nilpotent. Conversely, if each x^i is nilpotent, $\sqrt{I}$ is the argumentation ideal of $\mathbb{C}[x^1, \dots, x^N]$, that is, $R_V/\sqrt{0} = \mathbb{C}$ and so $\dim X_V = 0$.

To prove the equivalence (i) $\iff$ (iv), write

$$\mathrm{gr}^F V = \mathbb{C}[x^1_{(-j)}, \dots, x^N_{(-j)},\ j \geqslant 1]/J$$

for some ideal J, which is possible by Proposition 4.5. Part (i) is equivalent to that $\dim SS(V) = 0$ by Lemma 4.9. Hence one can argue as for the equivalence (i) $\iff$ (iii). Namely, if $SS(V) = \{\text{point}\}$, then $\mathrm{gr}^F V/\sqrt{0} = \mathbb{C}$. So $\sqrt{J}$ is the argumentation ideal of $\mathbb{C}[x^1_{(-j)}, \dots, x^N_{(-j)},\ j \geqslant 1]$ or, equivalently, each $x^i_{(-j)}$ is nilpotent. Conversely, if each $x^i_{(-j)}$ is nilpotent, $\sqrt{J}$ is the argumentation ideal of $\mathbb{C}[x^1_{(-j)}, \dots, x^N_{(-j)},\ j \geqslant 1]$, that is, $\mathrm{gr}^F V/\sqrt{0} = \mathbb{C}$ and so $\dim SS(V) = 0$.

Lemma 4.11 *Let V be a conical vertex algebra, $\{a^i : i \in I\}$ a set of homogenous strong generators, so that $\mathscr{O}(\mathscr{L}\tilde{X}_V)$ is a topological ring generated by the image $\bar{a}^i_{(n)}$, $i \in I$, where $\bar{a}^i$ is the the image of a^i in R_V. If V is lisse, then each $\bar{a}^i_{(n)}$ is nilpotent in $\mathscr{O}(\mathscr{L}\tilde{X}_V)$.*

Proof Recall that the ind-schemes $\mathscr{L}\tilde{X}_V$ is the direct limit of schemes $\mathscr{L}_n\tilde{X}_V$, with $\mathscr{L}_0\tilde{X}_V = \mathscr{J}_\infty\tilde{X}_V$. The canonical morphism $(\tilde{X}_V)_{\mathrm{red}} = X_V \to \tilde{X}_V$ induces morphisms $\mathscr{L}_n X_V \to \mathscr{L}_n\tilde{X}_V$ for each n and, hence, a morphism of ind-schemes $\mathscr{L}X_V \to \mathscr{L}\tilde{X}_V$. Since $\mathbb{C}(\!(z)\!)$ is a field, similarly to Lemma 1.7 we establish that

$$\mathscr{L}X_V \xrightarrow{\simeq} \mathscr{L}\tilde{X}_V,$$

whence $\mathscr{L}_n X_V \xrightarrow{\simeq} \mathscr{L}_n\tilde{X}_V$ for each n as well.

Moreover, if X_V is a point as topological space, then $\mathscr{L}X_V$ is also a point since $\mathrm{Hom}_{Alg}(\mathbb{C}, \mathbb{C}(\!(z)\!)) \cong \mathrm{Hom}_{Sch}(\mathrm{Spec}\,\mathbb{C}(\!(z)\!), X_V) \cong \mathrm{Hom}_{Sch}(\mathrm{Spec}\,\mathbb{C}, \mathscr{L}X_V)) \cong \mathrm{Hom}_{Alg}(\mathscr{O}(\mathscr{L}X_V), \mathbb{C})$ consists of only one point. It follows that if $\tilde{X}_V$ is zero-dimensional, then each $\mathscr{L}_n\tilde{X}_V$ is zero-dimensional too.

Hence, if V is lisse, then $\mathscr{O}(\mathscr{L}_n\tilde{X}_V)/\sqrt{0} = \mathbb{C}$, that is, $\mathbb{C}[\bar{a}^i_{(-j-1)} : i \in I]_{j \geqslant -n}/\sqrt{0} = \mathbb{C}$. So the augmentation ideal of $\mathbb{C}[\bar{a}^i_{(-j-1)} : i \in I]_{j \geqslant -n}$ is generated by the $\bar{a}^i_{(-j-1)}$'s. In particular each $\bar{a}^i_{(-j-1)}$, for $i \in I$ and $j \geqslant -n$, is nilpotent in $\mathscr{O}(\mathscr{L}_n\tilde{X}_V)$. Since this is true for each n we get the statement. □

Proposition 4.6 *Let V be a conformal, finitely strongly generated conical vertex algebra. If V is lisse, then any simple V-module is L_0-graded.*

Proof Let $\{a^i : i \in I\}$ be a finite set of strong generators of V, M a simple V-module, and $m_0 \in M \setminus \{0\}$. Let us first show that the L_0-module $\mathrm{span}_{\mathbb{C}}\{L_0^n m : n \in \mathbb{Z}_{\geqslant 0}\}$

generated by m_0 is finite-dimensional. Define an increasing filtration $G_\bullet M$ on M as follows. Set $G_{-1}M = \{0\}$, $G_0 M = \mathbb{C}m_{d_1}$, and

$$G_p M = \mathrm{span}_{\mathbb{C}}\{a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)} m_0 \colon i_j \in I,\ n_j \in \mathbb{Z},\ \Delta_{a^{i_1}} + \dots + \Delta_{a^{i_r}} \leqslant p\},$$

for $p > 0$. Then $M = \bigcup_p G_p M$ and for any $p, q, n \in \mathbb{Z}$, $(G_p V)_{(n)} G_q M \subset G_{p+q} M$; this follows from Borcherds identity (2.33). Set $\mathrm{gr}_G\, M := \bigoplus_{p \geqslant 0} G_p M / G_{p-1} M$. The commutative vertex algebra $\mathrm{gr}_G\, V$ acts on $\bar{M} := \mathrm{gr}_G\, M$ by setting

$$(\sigma_p a)_{(n)} \sigma_q(m) = \sigma_{p+q}(a_{(n)} m)$$

for $a \in G_p V \setminus G_{p-1} V$ and $m \in G_q M$. Here σ_p denotes the symbol map for both $\mathrm{gr}_p\, V$ and $\mathrm{gr}_p\, M$. By the correspondence between commutative $\mathscr{J}_\infty R_V$-modules and $\mathscr{O}(\mathscr{L}\tilde{X}_V)$-modules (see Theorem 2.7), this induces an action of $\mathscr{O}(\mathscr{L}\tilde{X}_V)$ on $\bar{M}$ by $\bar{a}^i_{(n)}.\sigma_q(m) = \sigma_{p+q}(a^i_{(n)} m)$ for $a^i \in G_p V \setminus G_{p-1} V$ and $m \in G_q M$, where $\bar{a}$ denote the image of $a \in V$ in R_V. Since the image of L_0 in R_V is nilpotent in $\mathscr{O}(\mathscr{L}\tilde{X}_V)$, we deduce $\bar{L}_0^n \sigma_0(m_0) = \sigma_{pn}(L_0^n m_0) = 0$ for sufficiently large n, where p such that $L_0 m_0 \in G_p V \setminus G_{p-1} V$. Hence $L_0^n m_0 = 0$ for sufficiently large n.

As a consequence, $\mathrm{span}_{\mathbb{C}}\{L_0^n m \colon n \in \mathbb{Z}_{\geqslant 0}\}$ is finite-dimensional. This proves that the action of L_0 on M is locally finite. Hence, M is a direct sum of generalized eigenspaces,

$$M = \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda\,\mathrm{Id})^{n_\lambda} \supset \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda\,\mathrm{Id}) =: M'.$$

Using (12.7) for $H = L_0$, we easily verify that M' is a vertex submodule of M which is nonzero since the action of L_0 is locally finite. Hence, $M = M'$ which concludes the proof. □

In the notation of the above proof, note that if $m \in M$ has L_0-eigenvalue λ, then by (12.7), $a_{(n)} m$ has eigenvalue $\lambda + \Delta_a - n - 1$ for homogeneous a.

Definition 4.13 Call a V-module a *positive energy representation* if there is $\lambda \in \mathbb{C}$ such that $M = \bigoplus\limits_{n \in \mathbb{Z}_{\geqslant 0}} M_{\lambda+n}$, where $M_d = \{m \in M \colon L_0^M m = dm\}$.

Note that any (nonzero) simple L_0-graded V-module is $\mathbb{Z}$-graded. Indeed, given such a module M, choose λ such $M_\lambda \neq 0$. Then

$$M' := \bigoplus_{n \in \mathbb{Z}} M_{\lambda+n}$$

is a nonzero submodule of M: the element $L_0^M.(a_{(r)} m)$ has conformal weight

$$\Delta^a + \lambda + n - r - 1 \in \lambda + \mathbb{Z}$$

for any homogenous $a \in V$, $n, r \in \mathbb{Z}$ and $m \in M_{\lambda+n}$. Hence $M' = M$, the module M being simple.

As a result, the following proposition implies that any simple module of a conformal, finitely strongly generated conical vertex algebra is an irreducible positive energy representation.

Proposition 4.7 *Let V be a conformal, finitely strongly generated conical vertex algebra. If V is lisse, then any $\mathbb{Z}$-graded simple V-module is positively graded. Add that each component is finite-dimensional!*

Proof We keep the notation of the proof of Proposition 4.6. We may assume that m_0 is an L_0-eigenvector of weight $\lambda \in \mathbb{C}$. Notice that the L_0-weight of $a^{i_1}_{(n_1)} \dots a^{i_r}_{(n_r)} m_0$ is

$$\lambda + \Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_r}} - n_1 - \cdots - n_r - r. \tag{4.32}$$

Since M is smooth, there is $N > 0$ such that for all $n \geqslant N$ and all $i \in I$, $a^i_{(n)} m_0 = 0$. Furthermore since $\bar{a}^i_{(n)}$ is nilpotent in $\mathscr{O}(\mathscr{L}\tilde{X}_V)$ for any n, we deduce that $(\bar{a}^{i_1}_{(n_1)})^{l_1} \dots (\bar{a}^{i_r}_{(n_r)})^{l_r}.m_0 = 0$ in $\bar{M}$ if $n_j \geqslant N$ and l_j large enough for $j = 1, \dots, r$, whence the statement by (5.14). $\square$

4.8 Remarks on the Poisson center of the Zhu C_2-algebra.

Let $\mathcal{Z}(R_V)$ be the Poisson center of R_V, that is,

$$\mathcal{Z}(R_V) = \{\bar{a} \in R_V \mid \{\bar{a}, \bar{b}\} = 0 \; \forall b \in V\}.$$

If V is conformal with conformal vector ω, then $\bar{\omega}$ belongs to $\mathcal{Z}(R_V)$.

Lemma 4.12 *Let $a \in V$ such that $\bar{a} \in \mathcal{Z}(R_V)$. Then*

$$D_a : R_V \to R_V, \quad \bar{b} \mapsto \overline{a_{(1)} b},$$

defines a derivation of R_V.

Proof Since $\bar{a} \in \mathcal{Z}(R_V)$, we have $a_{(n)} F^p V \subset F^{p-n+1} V$. In particular, $a_{(1)} F^1 V \subset F^1 V$. Thus, the linear map $D_a : R_V \to R_V$ is well-defined. We have

$$a_{(1)}(b_{(-1)} c) = [a_{(1)}, b_{(-1)}] c + b_{(-1)} a_{(1)} c = (a_{(0)} b)_{(0)} c + (a_{(1)} b)_{(1)} c + b_{(-1)} a_{(1)} c.$$

Since $a \in \mathcal{Z}(R_V)$, $a_{(0)} b \in F^1 V$. Hence $(a_{(0)} b)_{(0)} c \in F^1 V$, Therefore D_a is a derivation as required. $\square$

Note that $D_\omega(\bar{a}) = \Delta_a \bar{a}$ for a homogenous element a of V of conformal weight Δ_a.

Theorem 4.3 *Let V be a conical conformal vertex algebra. The following conditions are equivalent.*

i). $\bar{\omega}$ is nilpotent in R_V.

ii). the argumentation ideal of $\mathcal{Z}(R_V)$ is contained in the radical of R_V.

Proof The direction ii) $\Rightarrow$ i) is obvious. So let us show that i) $\Rightarrow$ ii). Let a be a homogenous element a of V of conformal weight $\Delta_a > 0$ such that $\bar{a} \in \mathcal{Z}(R_V)$. Since $\bar{\omega} \in \sqrt{(0)}$, we have $D_a(\bar{\omega}) \in \sqrt{(0)}$. However,

$$\begin{aligned} a_{(1)}\omega &= [a_{(1)}, \omega_{(-1)}]|0\rangle = -[\omega_{(-1)}, a_{(1)}]|0\rangle = -\sum_{j\geqslant 0}(-1)^j (\omega_{(j)}a)_{(-j)}|0\rangle \\ &= -\sum_{j\geqslant 1}(-1)^j (\omega_{(j)}a)_{(-j)}|0\rangle \equiv -(\omega_{(1)}a)_{(-1)}|0\rangle = -\Delta_a a \pmod{F^1 V}. \end{aligned}$$

Therefore, $a \in \sqrt{(0)}$. □

Corollary 4.3 *Let V be a conical conformal vertex algebra such that R_V is Poisson commutative. Then the following conditions are equivalent.*

i). V is lisse.
ii). $\bar{\omega}$ is nilpotent in R_V.

Chapter 5
Modules over vertex algebras and Zhu's functor

We introduce in this chapter the Zhu algebra $\mathrm{Zhu}(V)$ and the Zhu functor $V \mapsto \mathrm{Zhu}(V)$ assigning to a vertex operator algebra an associative algebra. Zhu established that the equivalence classes of the irreducible representations of V are in one-to-one correspondence with the equivalence classes of the irreducible representations of $\mathrm{Zhu}(V)$ (see Theorem 5.2). The Zhu algebra $\mathrm{Zhu}(V)$ has a much simpler structure than V, for example, the one-to-one correspondence theorem implies that if V is *rational* then $\mathrm{Zhu}(V)$ is semisimple. The associative algebra $\mathrm{Zhu}(V)$ also plays a crucial role in the proof of the modular invariance; see, for example, [184, 157, 15].

Section 5.1 is about the Zhu algebra of a vertex algebra, the Zhu functor and consequences for the modules over the vertex algebra. In Section 5.4, we discuss the connexion between the Zhu algebra and the Zhu C_2-algebra. Using this, we explicitly compute in Section 5.6 the Zhu algebra in some examples.

We continue to assume that a vertex algebra V is finitely strongly generated.

5.1 Zhu's algebra and Zhu's functor

We assume in this chapter that V be a $\mathbb{Z}$-graded vertex algebra (see Definition 2.7).

Definition 5.1 For homogeneous elements a, b of V, set

$$a \circ b := \mathrm{Res}_z \left(Y(a,z) b \frac{(z+1)^{\Delta_a}}{z^2} \right) = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} b,$$

and extend the products $\circ$ linearly. The expression $(z+1)^k$, for $k \in \mathbb{Z}$, means $\sum\limits_{j \geqslant 0} \binom{k}{j} z^j$. We set

$$\mathrm{Zhu}(V) := V / V \circ V,$$

where $V \circ V := \mathrm{span}\{a \circ b \ ; \ a, b \in V\}$.

Theorem 5.1 ([80, 184]) *The quotient* Zhu(V) *is an associative algebra, called the* Zhu algebra *of V, with multiplication defined as*

$$a * b := \mathrm{Res}_z\left(Y(a,z)b\frac{(z+1)^{\Delta_a}}{z}\right) = \sum_{i\geqslant 0}\binom{\Delta_a}{i}a_{(i-1)}b$$

for homogeneous elements $a, b \in V$. Its unit is the image of the vacuum $|0\rangle$ in the quotient Zhu(V).

A vertex algebra V is called a *chiralization* of an algebra A if Zhu$(V) \cong A$.
Before proving the theorem, we need some lemmas.

Lemma 5.1 *For a homogenous element a and $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$T^n a = n!\binom{-\Delta_a}{n}a \pmod{V \circ V}.$$

Proof From $a \circ |0\rangle = a_{(-2)}|0\rangle + \Delta_a a = Ta + \Delta_a a$, we deduce that $Ta = -\Delta_a a \pmod{V \circ V}$. Using this relation and $\Delta_{Ta} = \Delta_a + 1$, the identities follows from an easy induction on n. □

Lemma 5.2 *For a, b homogeneous elements,*

$$b * a = \mathrm{Res}_z\left(Y(a,z)b\frac{(z+1)^{\Delta_a-1}}{z}\right) = \sum_{i\geqslant 0}\binom{\Delta_a - 1}{i}a_{(i-1)}b \pmod{V \circ V}.$$

Proof By skew-symmetry (Proposition 2.4) and Lemma 5.1, we have

$$\begin{aligned}
Y(b,z)a = e^{zT}Y(a,-z)b &= \sum_{n\in\mathbb{Z}} e^{zT}a_{(n)}b(-z)^{-n-1}\\
&= \sum_{n\in\mathbb{Z}}\sum_{j\geqslant 0}\frac{T^j(a_{(n)}b)}{j!}z^j(-z)^{-n-1}\\
&= \sum_{n\in\mathbb{Z}}\sum_{j\geqslant 0}\binom{-\Delta_a-\Delta_b+n+1}{j}z^j a_{(n)}b(-z)^{-n-1}\\
&= \sum_{n\in\mathbb{Z}}(-z)^{-n-1}(z+1)^{-\Delta_a-\Delta_b+n+1}a_{(n)}b\\
&= (z+1)^{-\Delta_a-\Delta_b}Y(a,-\frac{z}{z+1})b.
\end{aligned}$$

Therefore, we get

$$b * a = \mathrm{Res}_z\left(Y(b,z)a\frac{(z+1)^{\Delta_b}}{z}\right) = \mathrm{Res}_z\left(Y(a,-\frac{z}{z+1})b\frac{(z+1)^{\Delta_b}}{z}(z+1)^{-\Delta_a-\Delta_b}\right).$$

Recall the formula for change of variable for residue. For $g(w) = \sum_{m\geqslant M} v_m w^m \in V((w))$ and $f(z) = \sum_{n=1}^{\infty} a_n z^n \in z\mathbb{C}[[z]]$ with $a_1 \neq 0$, the power series $g(f(z)) \in$

$V((z))$ is defined as

$$g(f(z)) = \sum_{m \geqslant M} v_m f(z)^m = \sum_{m \geqslant M} \sum_{j=1}^{\infty} v_m (a_1 z)^m \binom{m}{j} \bar{f}^j,$$

where $\bar{f} = \sum_{i=2}^{\infty} \frac{a_i}{a_1} z^{i-1}$. Then we have the following formula:

$$\mathrm{Res}_w\, g(w) = \mathrm{Res}_z (g(f(z))) \frac{d}{dz} f(z). \tag{5.1}$$

Using the formula of change of variable (5.1) with $w = -\frac{z}{z+1}$ we deduce that

$$b * a = \mathrm{Res}_w \left(Y(a, w) b \frac{(w+1)^{\Delta_a - 1}}{w} \right),$$

whence the expected result. □

Lemma 5.3 *For homogeneous elements a, b, we have*

$$a * b - b * a = \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i)} b \pmod{V \circ V}$$

In particular, the image of the conformal vector belongs to the center of $\mathrm{Zhu}(V)$.

Proof The first assertion is an easy consequence of Lemma 5.2. The last assertion follows from the fact that $\omega * a - a * \omega \equiv \sum_{i \geqslant 0} \binom{1}{i} \omega_{(i)} a \equiv Ta + Ha = a \circ |0\rangle$. □

Lemma 5.4 *For every homogeneous element $a \in V$, and $m \geqslant n \geqslant 0$,*

$$\mathrm{Res}_z \left(Y(a, z) \frac{(z+1)^{\Delta_a + n}}{z^{2+m}} b \right) \in V \circ V.$$

Proof Since

$$\frac{(z+1)^{\Delta_a + n}}{z^{2+m}} = \sum_{i=0}^{n} \binom{n}{i} \frac{(z+1)^{\Delta_a}}{z^{2+m-i}},$$

we only need to prove the lemma for the case $n = 0$ and $m \geqslant 0$. We prove the statement by induction on m, the case $m = 0$ being clear from the definition of $V \circ V$. Assume the statement true for any $m \leqslant k$, and prove it for $m = k + 1$. By induction, we have

$$\mathrm{Res}_z \left(Y(Ta, z) \frac{(z+1)^{\Delta_a + 1}}{z^{2+k}} b \right) \in V \circ V.$$

On the other hand,

$$\begin{aligned}\operatorname{Res}_z\left(Y(Ta,z)\frac{(z+1)^{\Delta_a+1}}{z^{2+k}}b\right) &= \operatorname{Res}_z\left(\frac{\partial}{\partial z}Y(a,z)\frac{(z+1)^{\Delta_a+1}}{z^{2+k}}b\right)\\ &= -\operatorname{Res}_z\left(Y(a,z)\frac{\partial}{\partial z}\frac{(z+1)^{\Delta_a+1}}{z^{2+k}}b\right)\\ &= -(\Delta_a+1)\operatorname{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z^{2+k}}b\right)\\ &\quad + (2+k)\operatorname{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z^{2+k+1}}b\right),\end{aligned}$$

whence the statement for $m = k+1$ since the first term of the right-hand-side is in $V \circ V$ by induction. □

Proof (Proof of Theorem 5.1) First, for homogenous a,

$$a * |0\rangle = a_{(-1)}|0\rangle = a \quad \text{and} \quad |0\rangle * a = |0\rangle_{(-1)}a = a. \tag{5.2}$$

To prove the theorem, we have to show that $V \circ V$ is a two-sided ideal of $\mathrm{Zhu}(V)$, so that $*$ is well-defined on $\mathrm{Zhu}(V)$, and that $\mathrm{Zhu}(V)$ is an associative algebra for $*$. It suffices to show that the following relations hold for homogeneous elements a, b, c:

$$a * (V \circ V) \subset V \circ V, \tag{5.3}$$
$$(V \circ V) * a \subset V \circ V, \tag{5.4}$$
$$(a * b) * c - a * (b * c) \in V \circ V, \tag{5.5}$$

since (5.2) will ensure that the image of $|0\rangle$ is a unit for the multiplication $*$ on $\mathrm{Zhu}(V)$.

We only detail the proof of (5.3), the other identities are proven using the same technics. The idea is to show that for homogeneous elements a, b, c of V, we have $a * (b \circ c) - b \circ (a * c) \in V \circ V$ so that $a * (b \circ c) \in V \circ V$. Using (2.35), we get

$$\begin{aligned}a * (b \circ c) - b \circ (a * c) &= \operatorname{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z}\right)\operatorname{Res}_w\left(Y(b,w)\frac{(w+1)^{\Delta_b}}{w^2}c\right)\\ &\quad - \operatorname{Res}_w\left(Y(b,w)\frac{(w+1)^{\Delta_b}}{w^2}\right)\operatorname{Res}_z\left(Y(a,z)\frac{(z+1)^{\Delta_a}}{z}c\right)\\ &= \operatorname{Res}_w\left(\operatorname{Res}_{z-w}\left(Y(Y(a,z-w)b,w)\frac{(z+1)^{\Delta_a}}{z}\frac{(w+1)^{\Delta_b}}{w^2}c\right)\right)\\ &= \sum_{i=0}^{\Delta_a}\sum_{j\geqslant 0}\binom{\Delta_a}{i}\operatorname{Res}_w\left(Y(a_{(i+j)}b,w)(-1)^j\frac{(w+1)^{\Delta_a+\Delta_b-i}}{w^{j+3}}c\right).\end{aligned}$$

Since $\Delta_{a_{(i+j)}b} = \Delta_a + \Delta_b - i - j - 1$, the right hand side of the last equality belongs to $V \circ V$ in virtue of Lemma 5.4. □

For a simple positive energy representation $M = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} M_{\lambda+n}$, $M_\lambda \neq 0$, of V, let M_{top} be the *top degree component* M_λ of M. Using (2.50), we see that for any d,

$$a_{(n)}^{M} M_d \subset M_{d+\Delta_a-n-1}. \tag{5.6}$$

For a homogeneous vector $a \in V$, let $o(a) = a_{(\Delta_a-1)} = a^M_{(\Delta_a-1)}$, so that $o(a)$ preserves the homogeneous components of any graded representation of V by (5.6).

The importance of Zhu's algebra in vertex algebra theory comes from the following fact that was established by Yonchang Zhu.

Theorem 5.2 ([184]) *For any positive energy representation M of V, $[a] \mapsto o(a)$ gives a well-defined representation of* $\mathrm{Zhu}(V)$ *on M_{top}, where $[a]$ is the image of a in* $\mathrm{Zhu}(V)$. *Moreover, the correspondence $M \mapsto M_{\mathrm{top}}$ gives a bijection between the set of isomorphism classes of irreducible positive energy representations of V and that of simple* $\mathrm{Zhu}(V)$*-modules.*

The proof of this theorem will be given after Theorem 5.3.

5.2 Current algebra and Zhu algebra

Lemma 5.5 *For a vertex algebra V, V/TV is a Lie algebra by*

$$[a+TV, b+TV] = a_{(0)}b + TV, \quad a, b \in V$$

Proof The skew symmetry property follows from the skew symmetry property of vertex algebra, which is equivalent to (4.7). The Jacobi identity follows from the Borcherds identity (2.32). □

Lemma 5.6 ([39]) *Let V be a vertex algebra, (R, ∂) a differential algebra. Then*

$$\mathrm{Lie}(V, R) := (V \otimes R)/(T \otimes 1 + 1 \otimes \partial)(V \otimes R)$$

is a Lie algebra by

$$[a \otimes r, b \otimes s] = \sum_{j \geqslant 0} a_{(j)} b \otimes \left(\frac{1}{j!}\partial^j r\right) s. \tag{5.7}$$

Proof Since R is a commutative vertex algebra, $V \otimes R$ is a vertex algebra with the translation operator $T \otimes 1 + 1 \otimes \partial$. The assertion follows by applying Lemma 5.5 to the vertex algebra $V \otimes R$. □

The *Borcherds Lie algebra* associated with a vertex algebra V is by definition the Lie algebra

$$\mathrm{Lie}(V) := \mathrm{Lie}(V, \mathbb{C}[t, t^{-1}]) = V \otimes \mathbb{C}[t, t^{-1}]/(T \otimes 1 + 1 \otimes \partial_t)(V \otimes \mathbb{C}[t, t^{-1}]),$$

where $\mathbb{C}[t, t^{-1}]$ is viewed as a differential algebra with the derivation ∂_t. We have

$$[a_{\{m\}}, b_{\{n\}}] = \sum_{j\geqslant 0} \binom{m}{j} (a_{(j)}b)_{\{m+n-j\}}, \tag{5.8}$$

where $a_{\{n\}}$ is the image of $a \otimes t^n \in V \otimes \mathbb{C}[t, t^{-1}]$ in $\mathrm{Lie}(V)$. By definition, we have $(Ta)_{\{n\}} = -na_{\{n-1\}}$.

The following is clear from (2.47).

Lemma 5.7 *Any V-module M is a* $\mathrm{Lie}(V)$*-module by $a_{\{n\}} \mapsto a_{(n)} = a^M_{(n)}$ for $a \in V$, $n \in \mathbb{Z}$.*

Note that a $\mathrm{Lie}(V)$-module needs not to be a V-module since the identities (2.47) and (2.48) may not be satisfied.

Recall that (2.48) is equivalent to the identity (2.33), which contains an infinite sum. In order to make sense of (2.33), we shall introduce a completion $U(\widetilde{\mathrm{Lie}(V)})$ of the the universal enveloping algebra $U(\mathrm{Lie}(V))$ of $\mathrm{Lie}(V)$ as follows.

Assume that V is $\mathbb{Z}$-graded by a Hamiltonian H. Then $\mathrm{Lie}(V)$ is a graded Lie algebra, by defining the action $\mathrm{ad}\, H$ of H on $\mathrm{Lie}(V)$ by

$$\mathrm{ad}\, H(a_{\{n\}}) = -(n+1)a_{\{n\}} + (Ha)_{\{n\}}.$$

We have

$$\mathrm{Lie}(V) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Lie}(V)_d, \quad \mathrm{Lie}(V)_d = \{x \in \mathrm{Lie}(V) \colon (\mathrm{ad}\, H)x = dx\}.$$

Let $U(\mathrm{Lie}(V)) = \bigoplus_{d\in\mathbb{Z}} U(\mathrm{Lie}(V))_d$ be the induced $\mathbb{Z}$-grading on $U(\mathrm{Lie}(V))$.

Define

$$\begin{aligned} U(\widetilde{\mathrm{Lie}(V)}) &= \bigoplus_{d\in\mathbb{Z}} U(\widetilde{\mathrm{Lie}(V)})_d, \\ U(\widetilde{\mathrm{Lie}(V)})_d &= \varprojlim_{r} U(\mathrm{Lie}(V))_d / \sum_{p\leqslant r} U(\mathrm{Lie}(V))_{d-p} U(\mathrm{Lie}(V))_p. \end{aligned}$$

The space $U(\widetilde{\mathrm{Lie}(V)})$ is a $\mathbb{Z}$-graded topological ring with each component $U(\widetilde{\mathrm{Lie}(V)})_d$ being complete. Now the identity

$$(a_{(m)}b)_{\{n\}} = \sum_{j\geqslant 0} (-1)^j \binom{m}{j} (a_{\{m-j\}} b_{\{n+j\}} - (-1)^m b_{\{m+n-j\}} a_{\{j\}}) \tag{5.9}$$

makes sense as an element of $U(\widetilde{\mathrm{Lie}(V)})$. Let $I = \bigoplus_{d\in\mathbb{Z}} I_d$ be the graded ideal of $U(\widetilde{\mathrm{Lie}(V)})$ generated by (5.9) and $(|0\rangle)_{\{n\}} = \delta_{n,-1}$. Let

$$\mathcal{U}(V) = \bigoplus_{d\in\mathbb{Z}} \mathcal{U}(V)_d, \quad \mathcal{U}(V)_d = U(\widetilde{\mathrm{Lie}(V)})_d / \overline{I}_d,$$

where $\overline{I}_d$ is the closure of I_d in $U(\widetilde{\mathrm{Lie}(V)})_d$. Then $\mathcal{U}(V)$ is again a $\mathbb{Z}$-graded topological ring with each component $\mathcal{U}(V)_d$ being complete, which is called the *universal enveloping algebra* [80] , or the *current algebra* [153] of V.

A $\mathcal{U}(V)$-module M is called *smooth* if the action $\mathcal{U}(V) \times M \to M$ is continuous, where M is equipped with the discrete topology.

Lemma 5.8 *A V-module is the same as a smooth $\mathcal{U}(V)$-module.*

Clearly, $\mathcal{U}(V)_0$ is a subalgebra of $\mathcal{U}(V)$. Define the algebra $A(V)$ by

$$A(V) = \mathcal{U}(V)_0 / \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}},$$

where – denotes the closure.

Theorem 5.3 *We have the isomorphism of algebras*

$$\mathrm{Zhu}(V) \cong A(V).$$

Let $\mathcal{U}(V)_{\leqslant 0} = \bigoplus_{d \leqslant 0} \mathcal{U}(V)_d \subset \mathcal{U}(V)$. For an $A(V)$-module E, define the positive energy representation $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)$ of V by

$$\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E) = \mathcal{U}(V) \otimes_{\mathcal{U}(V)_{\leqslant 0}} E, \tag{5.10}$$

where $\mathcal{U}(V)_{\leqslant 0}$ acts on E by the projection $\mathcal{U}(V)_{\leqslant 0} \to A(V)$.

Proof (Proof of Theorem 5.2) Let M be a simple positive energy representation V. Then M_{top} is a simple $\mathcal{U}(V)_0$-module on which $\mathcal{U}(V)_{-r}$ acts trivially for $r > 0$. Hence M_{top} is a simple module over $A(V) = \mathrm{Zhu}(V)$. Conversely, let E be a simple $\mathrm{Zhu}(V)$-module. Since $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)$ is a positive energy representation of such that $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)_{top} = E$, any proper graded submodule of $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)$ intersects E trivially. Indeed, if N is any such proper graded submodule such that $N_{top} = N \cap E \neq \{0\}$, then for any nonzero element v in the intersection, we have $E = A(V).v \subset N$ since E a simple $\mathrm{Zhu}(V)$-module. But E generates $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)$ as V-modules, because $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E) \cong \mathcal{U}(V)_{>0}.E$, whence $N = \mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)$. Hence there exists a unique simple graded quotient $L(E)$ of $\mathrm{Ind}_{A(V)}^{\mathcal{U}(V)}(E)$. Clearly, the maps $E \mapsto L(E)$ and $M \mapsto M_{top}$ are inverse to each other. $\square$

5.3 Proof of Theorem 5.3

This section is devoted to the proof of Theorem 5.3, following [100]. We use the following notation, which is defined for any homogeneous element $a \in V$ and extend linearly to V:

$$J_n(a) := a_{\{\Delta_a - 1 + n\}}.$$

The advantage of this notation is that $J_n(a)$ has always degree $-n$. The proof of the following combinatorial lemma essentially follows from (5.9):

Lemma 5.9 ([100, Corollary A.2]) *For all integers s, t, N satisfying $N + s \geqslant 0$, the following identity holds in the universal enveloping algebra of V:*

$$\begin{aligned} J_{-s}(a)J_t(b) &= \sum_{j=0}^{N}\sum_{i\geqslant 0}(-1)^i\binom{N+\Delta_a}{i}\binom{-N-s-1}{j}J_{t-s}(a_{(-N-s-i-j-1)}b) \\ &\quad - \sum_{k\geqslant N+1}\sum_{j=0}^{N}(-1)^j\binom{N+s+j}{j}\binom{N+s-k}{k-j}J_{-k-s}(a)J_{k+t}(b) \\ &\quad + \sum_{j=0}^{N}\sum_{i\geqslant 0}(-1)^{N+s+1}\binom{N+s+j}{j}\binom{N+s+j+i}{i}J_{t-N-s-1-i}(b)J_{N+1+i}(a). \end{aligned}$$

Lemma 5.10 *Every element $a = J_{n_1}(a_1)\dots J_{n_m}(a_m)$ can be expressed in the quotient $A(V)$ as $J_0(v)$ for some $v = v(a)$ in V depending on a.*

Proof We prove the statement by induction on the length m. If $m = 1$, there is nothing to do. Let $m \geqslant 2$, and assume the statement true for every monomial of length $< m$. Apply Lemma 5.9 to $J_{n_{m-1}}(a_{m-1})J_{n_m}(a_m)$, where

$$-s = n_{m-1}, \quad t = n_m, \quad a = a_{m-1}, \quad b = a_m.$$

In Lemma 5.9, choose N big enough so that $\min\{N + n_m, N\} > 0$. Then $J_{k+n_m}(a_m)$ and $J_{N+1+i}(a_{m-1})$ are both contained in $\bigoplus_{j<0}\mathcal{U}(V)_j$ for $k \geqslant N+1$, and so a is congruent to a linear combination of the following terms with length $< m$:

$$J_{n_1}(a_1)\dots J_{n_{m-2}}(a_{m-2})J_{n_{m-1}+n_m}((a_{m-1})_{(-N+n_{m-1}-i-j-1)}a_m).$$

By induction, these terms are congruent to monomials of the form $J_0(v')$, $v' \in V$. So a is itself congruent to some monomial $J_0(v)$. Here, notice that for any $n \in \mathbb{Z}$, $a, b \in V$, we have $J_n(a) + J_n(b) = J_n(a+b)$. $\square$

We are now in a position to prove Theorem 5.3. Let φ be the composition map of the linear map from V to $\mathcal{U}(V)_0$ sending homogeneous element a to $a_{\{\Delta_a-1\}}$ with the canonical quotient map from $\mathcal{U}(V)_0$ to $A(V)$. Lemma 5.10 ensures that this map is surjective.

Let us show now that φ factors through $\mathrm{Zhu}(V)$, that is,

$$\varphi(V \circ V) \subset \overline{\sum_{r>0}\mathcal{U}(V)_r\mathcal{U}(V)_{-r}}.$$

Let a, b be homogeneous elements $a, b \in V$. We have $\Delta_{a_{(i-2)}b} = \Delta_a + \Delta_b - i + 1$. Using the identity (5.9), we get

$$\begin{aligned}
\varphi(a \circ b) &= \sum_{i \geqslant 0} \binom{\Delta_a}{i} (a_{(i-2)} b)_{\{\Delta_a + \Delta_b - i\}} \\
&= \sum_{i \geqslant 0} (-1)^i \binom{-2}{i} (a_{\{\Delta_a - 2 - i\}} b_{\{\Delta_b + i\}} - b_{\{\Delta_b - 2 - i\}} a_{\{\Delta_a + i\}}) \\
&= \sum_{i \geqslant 0} (-1)^i \binom{-2}{i} (J_{-i-1}(a) J_{i+1}(b) - J_{-i-1}(b) J_{i+1}(a)).
\end{aligned}$$

Since $\deg J_{i+1}(b) = \deg J_{i+1}(a) = -i - 1 < 0$, we get that

$$\varphi(a \circ b) \in \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}},$$

whence the statement. As a result, we get a well-defined map, still denoted by φ, from $\mathrm{Zhu}(V)$ to $A(V)$ which is surjective.

Next, we prove that φ is an algebra homomorphism. It is enough to show that $\varphi(a * b) = \varphi(a)\varphi(b)$ for homogeneous elements $a, b \in V$. Again using the identity (5.9), we get

$$\begin{aligned}
\varphi(a * b) &= \sum_{i \geqslant 0} \binom{\Delta_a}{i} (a_{(i-1)} b)_{\{\Delta_a + \Delta_b - i - 1\}} \\
&= \sum_{i \geqslant 0} (-1)^i \binom{-1}{i} (a_{\{\Delta_a - 1 - i\}} b_{\{\Delta_b - 1 + i\}} + b_{\{\Delta_b - 2 - i\}} a_{\{\Delta_a + i\}}) \\
&= \sum_{i \geqslant 0} (-1)^i \binom{-2}{i} (J_{-i}(a) J_i(b) + J_{-i-1}(b) J_{i+1}(a)) \\
&= J_0(a) J_0(b) \quad (\mathrm{mod}\ \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}}).
\end{aligned}$$

On the other hand, by letting $s = t = N = 0$ in Lemma 5.10, we have

$$\begin{aligned}
J_0(a) J_0(b) &= \sum_{i \geqslant 0} (-1)^i \binom{\Delta_a}{i} J_0(a_{(-i-1)} b) \quad (\mathrm{mod}\ \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}}) \\
&= \sum_{i \geqslant 0} (-1)^i \binom{\Delta_a}{i} (a_{(-i-1)} b)_{\{\Delta_a + \Delta_b + i - 1\}} \quad (\mathrm{mod}\ \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}}),
\end{aligned}$$

whence $\varphi(a * b) = \varphi(a)\varphi(b)$ in $A(V)$.

It remains to construct an inverse map for φ. By Lemma 5.10 every element of $A(V)$ can be expressed as $J_0(a) + \ (\mathrm{mod}\ \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}})$. We want to define a map ψ from $A(V)$ to $\mathrm{Zhu}(V)$ sending $J_0(a) + \ (\mathrm{mod}\ \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}})$ to $a + V \circ V$. Once we can show this, it is clear that ψ and φ are inverse to each other. The well-definedness requires that whenever $J_0(a) \in \ (\mathrm{mod}\ \overline{\sum_{r>0} \mathcal{U}(V)_r \mathcal{U}(V)_{-r}})$, then $a \in V \circ V$. This can be shown using the functor L^0 constructed by Dong, Li and Mason [60]. To be completed...

5.4 Relations between the Zhu algebra and the Zhu C_2-algebra

We define an increasing filtration of the Zhu algebra. For this, we assume that V is $\mathbb{Z}_{\geqslant 0}$-graded, that is, $V = \bigoplus_{\Delta \geqslant 0} V_\Delta$. Then $V_{\leqslant p} := \bigoplus_{\Delta=0}^{p} V_\Delta$ gives an increasing filtration of V. Define

$$\mathrm{Zhu}_p(V) := \mathrm{im}(V_{\leqslant p} \to \mathrm{Zhu}(V)).$$

Obviously, we have

$$0 = \mathrm{Zhu}_{-1}(V) \subset \mathrm{Zhu}_0(V) \subset \mathrm{Zhu}_1(V) \subset \cdots, \quad \text{and} \quad \mathrm{Zhu}(V) = \bigcup_{p \geqslant -1} \mathrm{Zhu}_p(V).$$

Also, since $a_{(n)}b \in V_{\Delta_a+\Delta_b-n-1}$ for $a \in V_{\Delta_a}$, $b \in V_{\Delta_b}$, we have

$$\mathrm{Zhu}_p(V) * \mathrm{Zhu}_q(V) \subset \mathrm{Zhu}_{p+q}(V). \tag{5.11}$$

The following assertion follows from the skew symmetry.

By Lemma 5.3, we have

$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(V)] \subset \mathrm{Zhu}_{p+q-1}(V). \tag{5.12}$$

This means that the filtered associative algebra $\mathrm{Zhu}(V)$ is *almost-commutative* (see Section C.3). By (5.11) and (5.12) the associated graded space,

$$\mathrm{gr}\,\mathrm{Zhu}(V) = \bigoplus_{p \geqslant 0} \mathrm{Zhu}_p(V)/\mathrm{Zhu}_{p-1}(V),$$

is so naturally a graded Poisson algebra (see Section C.3).

Our next focus is to explore the connections between the Zhu algebra and the Zhu C_2-algebra or, equivalently, between the Poisson schemes $\tilde{X}_V$ and $\mathrm{Spec}\,\mathrm{gr}\,\mathrm{Zhu}(V)$.

First, note that $a \circ b \equiv a_{(-2)}b \pmod{\bigoplus_{\Delta \leqslant \Delta_a+\Delta_b} V_\Delta}$ for homogeneous elements a, b in V.

Lemma 5.11 (Zhu [57, Proposition 2.17(c)] and [19, Proposition 3.3]) *The following map defines a well-defined surjective Poisson algebra homomorphism:*

$$\begin{aligned} \eta_V : R_V &\longrightarrow \mathrm{gr}\,\mathrm{Zhu}(V) \\ \bar{a} &\longmapsto a \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta}. \end{aligned}$$

Proof We have $a \circ b = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} b = a_{(-2)} b + \sum_{i \geqslant 1} \binom{\Delta_a}{i} a_{(i-2)} b$. Since the degree of $a_{(i-2)} b$ is $\Delta_a + \Delta_b + 1 - i < \deg a_{(-2)} b$ if $i \geqslant 1$, we get that

$$a_{(-2)} b = a \circ b \quad (\bmod \bigoplus_{\Delta < \Delta_{a_{(-2)} b}} V_\Delta).$$

This shows that $C_2(V)$ is contained in $V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta$ and, hence, η_V is well-defined. Clearly, it is surjective. It remains to show that η_V is an algebra homomorphism. But $\eta_V(\bar{a}.\bar{b}) = \eta_V(\overline{a_{(-1)} b}) = a_{(-1)} b \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a + \Delta_b} V_\Delta}$ while the image of $a * b$ in $\operatorname{gr} \operatorname{Zhu}(V)$ is $a_{(-1)} b \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a + \Delta_b} V_\Delta}$ since the degree of $a_{(i-1)} b$ is $\Delta_a + \Delta_b - i < \Delta_a + \Delta_b$ for $i \geqslant 1$. $\square$

Remark 5.1 The map η_V is not an isomorphism in general. For example, let $\mathfrak{g}$ be the simple Lie algebra of type E_8 and $V = L_1(\mathfrak{g})$. Then $\dim R_V > \dim \operatorname{Zhu}(V) = 1$. This counter-example was discovered by Gaberdiel and Gannon[83] [1]. In other words, the diagram

$$\begin{tikzcd} V \arrow[d, "\mathrm{Zhu}"'] \arrow[dr, "\mathrm{Zhu}\ C_2"] & \\ \operatorname{Zhu}(V) \arrow[r, "\mathrm{gr}"'] & ? \end{tikzcd}$$

is not always commutative.

Remark 5.2 It was shown in [71] that $R_V \cong \operatorname{gr} \operatorname{Zhu}(V)$ for $V = L_k(\mathfrak{g})$ for any nonnegative $k \in \mathbb{Z}_{\geqslant 0}$, if $\mathfrak{g}$ is the simple Lie algebra $\mathfrak{sl}_n$.

By Lemma 5.11, we have $\operatorname{Specm}(\operatorname{gr} \operatorname{Zhu} V) \subset X_V$.

Conjecture 5.1 **([9])** *If V is a simple $\mathbb{Z}_{\geqslant 0}$-graded conformal vertex algebra, then*

$$X_V \cong \operatorname{Specm}(\operatorname{gr} \operatorname{Zhu} V).$$

Remark 5.3 One may also ask wether the following diagram is commutative.

$$\begin{tikzcd} V \arrow[r, "\mathrm{gr}^F"] \arrow[d, "\mathrm{Zhu}"'] & \mathrm{gr}^F V \arrow[d, "\mathrm{Zhu}"] \\ \operatorname{Zhu}(V) \arrow[r, "\mathrm{gr}"'] & ? \end{tikzcd}$$

[1] The equality $\dim \operatorname{Zhu}(V) = 1$ follows from the fact that $L_1(\mathfrak{g})$ is *holomorphic*, that is, the only simple module is itself, because it is the only integrable affine $\mathfrak{g}$-module of level 1. Because Zhu's algebra of any holomorphic vertex operator algebra is one-dimensional ref?, we get that $\operatorname{gr} \operatorname{Zhu}(V) \cong \mathbb{C}$. On the other hand, it is easy to check that $\dim R_V > 1$ since the unique proper maximal submodule of $V^1(\mathfrak{g})$ is generated by $(e_\theta t^{-1}|0\rangle)^2$: see Section 7.1.

In other words, one may ask wether one has $\mathrm{Zhu}(\mathrm{gr}^F V) \cong \mathrm{gr}\,\mathrm{Zhu}(V)$. Note that R_V is not isomorphic to $\mathrm{Zhu}(\mathrm{gr}^F V)$ in general since $C_2(V) \neq V \circ V$ even for commutative vertex algebras.

Although the above diagram is known to be commutative in several examples, e.g., the universal affine vertex algebra $V^k(\mathfrak{g})$ (cf. §5.6.2), the fermion Fock space (cf. §5.6.3), the *W-algebra* $\mathscr{W}^k(\mathfrak{g}, f)$, etc., it is not true in general.

Exercise 5.1 Verify that the example $V = L_1(\mathfrak{g})$, with $\mathfrak{g}$ simple of type E_8 as in Remark 5.1, furnishes an example of vertex algebra V such that $\mathrm{Zhu}(\mathrm{gr}^F V) \ncong \mathrm{gr}\,\mathrm{Zhu}(V)$.

? Open problem

Is there an example of a vertex algebra for which $R_V \ncong \mathrm{Zhu}(\mathrm{gr}^F V)$?

Corollary 5.1 *If V is lisse then* $\mathrm{Zhu}(V)$ *is finite dimensional. Hence the number of isomorphic classes of simple positive energy representations of V is finite.*

5.5 Filtration of current algebra

We continue to assume that V is $\mathbb{Z}_{\geqslant 0}$-graded.

Recall the increasing, conformal weight filtration $G_\bullet V$ (Section 4.6). This induces the increasing filtration $G_\bullet\,\mathrm{Lie}(V)$ of $\mathrm{Lie}(V)$ such that

$$[G_p\,\mathrm{Lie}(V), G_q\,\mathrm{Lie}(V)] \subset G_{p+q-1}\,\mathrm{Lie}(V),$$

where $G_p\,\mathrm{Lie}(V)$ is the image of $G_p V \otimes \mathbb{C}[t, t^{-1}]$ in $\mathrm{Lie}(V)$. Hence, $\mathrm{gr}_G\,\mathrm{Lie}(V) = \bigoplus_p G_p\,\mathrm{Lie}(V)/G_{p-1}\,\mathrm{Lie}(V)$ is naturally a commutative Lie algebra. On the other hand, $\mathrm{Lie}(\mathrm{gr}_G V)$ is also a commutative Lie algebra since $\mathrm{gr}\,V$ is commutative.

Lemma 5.12 *There is a surjective Lie algebra homomorphism*

$$\mathrm{Lie}(\mathrm{gr}_G V) \longrightarrow \mathrm{gr}_G\,\mathrm{Lie}(V)$$

that sends $\sigma_p(a)_{\{n\}}$ to $\sigma_p(a_{\{n\}})$.

Proof The map obtained by composing the quotient map $V \otimes \mathbb{C}[t, t^{-1}] \to \mathrm{Lie}(V)$ with the quotient map $\mathrm{Lie}(V) \to \mathrm{gr}_G\,\mathrm{Lie}(V)$ is clearly surjective, and it factorizes through the composition map $V \otimes \mathbb{C}[t, t^{-1}] \to \mathrm{gr}_G V \otimes \mathbb{C}[t, t^{-1}] \to \mathrm{Lie}(\mathrm{gr}_G V)$ since any element $T\sigma_p(a)_{\{n\}} + n\sigma_p(a)_{\{n-1\}}$ of $(T \otimes 1 + 1 \otimes \partial_t)\,\mathrm{gr}_G V$ is mapped to the element $\sigma_{p+1}(Ta_{\{n\}}) + n\sigma_p(a_{\{n-1\}})$ of $(T \otimes 1 + 1 \otimes \partial_t) G_p V$ for $a \in G_p V \setminus G_{p-1} V$, $n \in \mathbb{Z}$. It remains to verify that the resulting surjective map is a Lie algebra homomorphism, but this is clear from (5.8). (Note that both Lie algebras are commutative.) $\square$

The filtration $G_\bullet \operatorname{Lie}(V)$ induces a filtration $G_\bullet U(\operatorname{Lie}(V))$ of the universal enveloping algebra $U(\operatorname{Lie}(V))$. This in turn induces a filtration $G_\bullet \mathcal{U}(V)$ of the current algebra $\mathcal{U}(V)$, where $G_p\mathcal{U}(V)$ is the closure of the image of $G_pU(\operatorname{Lie}(V))$ in $\mathcal{U}(V)$. Let $\operatorname{gr}_G \mathcal{U}(V) = \bigoplus_p G_p\mathcal{U}(V)/G_{p-1}\mathcal{U}(V)$ be the associated graded topological algebra.

Lemma 5.12 immediately gives the following result:

Lemma 5.13 *There is a surjective algebra homomorphism*

$$\mathcal{U}(\operatorname{gr}_G V) \longrightarrow \operatorname{gr}_G \mathcal{U}(V).$$

The surjection in Proposition 4.5 induces a surjection $\mathcal{O}(\mathscr{L}\tilde{X}_V) \to \mathcal{U}(\operatorname{gr}_G V)$. Thus by Lemma 5.13 we have a surjection

$$\mathcal{O}(\mathscr{L}\tilde{X}_V) \twoheadrightarrow \operatorname{gr}_G \mathcal{U}(V). \tag{5.13}$$

Theorem 5.4 ([1]) *Let V be a strongly finitely generated conformal lisse vertex algebra. Then any simple V-module is an ordinary positive energy representation. Therefore the number of isomorphic classes of simple V-modules is finite.*

Proof Let $m \in M \setminus \{0\}$. Then $M = \mathcal{U}(V)m$ since M is simple. Define an increasing filtration G_pM by setting $G_pM = G_p\mathcal{U}(V)m$. Then $\operatorname{gr}_G M = \bigoplus_p G_pM/G_{p-1}M$ is naturally a module over $\operatorname{gr}_G \mathcal{U}(V)$, and hence over $\mathcal{O}(\mathscr{L}\tilde{X}_V)$. By construction, we have $\operatorname{gr}_G M = \mathcal{O}(\mathscr{L}\tilde{X}_V)\bar{m}$, where $\bar{m}$ is the image of m in $\operatorname{gr}_G M$.

Let us first now that the L_0-module $\operatorname{span}_{\mathbb{C}}\{L_0^n m : n \in \mathbb{Z}_{\geqslant 0}\}$ generated by m is finite-dimensional. Let $\{a^i : i \in I\}$ be a finite set of strong generators of V.

Since $\operatorname{gr}_G M = \mathcal{O}(\mathscr{L}\tilde{X}_V)\bar{m}$, there is $A_0 \in \mathcal{O}(\mathscr{L}\tilde{X}_V)$ such that $\overline{L_0^n m} = A_0^n \bar{m}$. By Lemma 4.11, the images of the $a^i_{(n)}$'s in R_V are nilpotent in $\mathcal{O}(\mathscr{L}\tilde{X}_V)$, whence $A_0^n\bar{m} = 0$ for sufficiently large n. As a result, $L_0^n\bar{m} = 0$ for sufficiently large n too, and so $\operatorname{span}_{\mathbb{C}}\{L_0^n m : n \in \mathbb{Z}_{\geqslant 0}\}$ is finite-dimensional. This proves that the action of L_0 on M is locally finite. Therefore, M is a direct sum of generalized eigenspaces,

$$M = \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda \operatorname{Id})^{n_\lambda} \supset \bigoplus_{\lambda \in \mathbb{C}} \ker(L_0 - \lambda \operatorname{Id}) =: M'.$$

Using (12.7) for $H = L_0$, we easily verify that M' is a vertex submodule of M which is nonzero since the action of L_0 is locally finite. Hence, $M = M'$ which proves that M is L_0-graded.

Let us now show that M is positively graded. We may assume that m is an L_0-eigenvector of weight $\lambda \in \mathbb{C}$. Notice that the L_0-weight of $a^{i_1}_{(n_1)} \ldots a^{i_r}_{(n_r)} m_0$ is

$$\lambda + \Delta_{a^{i_1}} + \cdots + \Delta_{a^{i_r}} - n_1 - \cdots - n_r - r. \tag{5.14}$$

Since M is smooth and I is finite, there is $N > 0$ such that for all $n \geqslant N$ and all $i \in I$, $a^i_{(n)} m = 0$. Furthermore using again Lemma 4.11, we deduce that $(\bar{a}^{i_1}_{(n_1)})^{l_1} \ldots (\bar{a}^{i_r}_{(n_r)})^{l_r}.m = 0$ in $\operatorname{gr}_G M$ if $n_j \geqslant N$ and l_j large enough for $j = 1, \ldots, r$, whence the statement by (5.14).

It remains to prove that each graded component $M_{\lambda+n}$ is finite-dimensional. Since $M_\lambda \neq 0$, we may assume that $m \in M_\lambda$. A L_0-weight space in $\mathrm{gr}_G M$ is generated by some $(\bar{a}^{i_1}_{(n_1)})^{t_1} \dots (\bar{a}^{i_r}_{(n_r)})^{t_r} \bar{m}$, with $(\bar{a}^{i_1}_{(n_1)})^{t_1} \dots (\bar{a}^{i_r}_{(n_r)})^{t_r} \in \mathscr{O}(\mathscr{L}\tilde{X}_V)$. Since each $\bar{a}^i_{(n)}$ is nilpotent in $\mathscr{O}(\mathscr{L}\tilde{X}_V)$ and I is finite, each L_0-weight space is finite-dimensional. □

Theorem 5.5 ([59, 153]) *Le V be lisse. Then the abelian category of V-modules is equivalent to the module category of a finite-dimensional associative algebra.*

5.6 Computation of Zhu's algebras

This section describes some technics to compute the Zhu algebra, and contains some explicit examples.

5.6.1 PBW basis

Recall that a vertex algebra V admits a PBW basis if R_V is a polynomial algebra and if the map $\mathbb{C}[\mathscr{J}_\infty(X_V)] \twoheadrightarrow \mathrm{gr}^F V$ is an isomorphism (cf. Definition 4.10).

Theorem 5.6 *If V admits a PBW basis, then $\eta_V : R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}V$ is an isomorphism.*

Proof We have $\mathrm{gr}\,\mathrm{Zhu}(V) = V/\mathrm{gr}(V \circ V)$, where $\mathrm{gr}(V \circ V)$ is the associated graded space of $V \circ V$ with respect to the filtration induced by the filtration $V_{\leqslant p}$. We wish to show that $\mathrm{gr}(V \circ V) = F^1 V$. Since $a \circ b \equiv a_{(-2)} b \pmod{V_{\leqslant \Delta_a + \Delta_b}}$ for homogeneous $a, b \in V$, it is sufficient to show that $a \circ b \neq 0$ implies that $a_{(-2)} b \neq 0$.

Suppose that $a_{(-2)} b = (Ta)_{(-1)} b = 0$ for homogeneous $a, b \in V$. Since V admits a PBW basis, $\mathrm{gr}^F V$ has no zero divisors, whence $Ta = 0$. Also, from the PBW property we find that $Ta = 0$ implies that $a = c|0\rangle$ for some constant $c \in \mathbb{C}$. Thus, a is a constant multiple of $|0\rangle$, in which case $a \circ b = 0$. □

5.6.2 Universal affine vertex algebras

The universal affine vertex algebra $V^k(\mathfrak{g})$ admits a PBW basis. Therefore

$$\eta_{V^k(\mathfrak{g})} : R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\simeq} \mathrm{gr}\,\mathrm{Zhu}V^k(\mathfrak{g}).$$

On the other hand, from Lemma 5.3 one finds that

$$\begin{aligned} U(\mathfrak{g}) &\longrightarrow \mathrm{Zhu}(V^k(\mathfrak{g})) \\ \mathfrak{g} \ni x &\longmapsto [x_{(-1)}|0\rangle] \end{aligned} \tag{5.15}$$

gives a well-defined algebra homomorphism. This map respects the filtration on both sides, where the filtration in the left side is the PBW filtration. Hence it induces a map between their associated graded algebras, which is identical to $\eta_{V^k(\mathfrak{g})}$. Therefore (5.15) is an isomorphism, that is to say, $V^k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})$.

Exercise 5.2 Extend Theorem 5.6 to the case where $\mathfrak{g}$ is a Lie superalgebra.

Theorem 5.2 gives the following in this example. The top degree component of the irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight λ is $L_{\mathfrak{g}}(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let $N_k = N_k(\mathfrak{g})$ be the maximal ideal of $V^k(\mathfrak{g})$ as in Example 3.2 so that

$$L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k,$$

where $L_k(\mathfrak{g})$ is the unique graded quotient of $V^k(\mathfrak{g})$. We have the exact sequence $J_k \to U(\mathfrak{g}) \to \mathrm{Zhu}(L_k(\mathfrak{g})) \to 0$, where J_k is the image of N_k in $\mathrm{Zhu}(V) = U(\mathfrak{g})$ through the compound map $N_k \hookrightarrow V \twoheadrightarrow \mathrm{Zhu}(V)$, and thus

$$\mathrm{Zhu}(L_k(\mathfrak{g})) = U(\mathfrak{g})/J_k.$$

Hence when the homomorphism $\eta_{L_k(\mathfrak{g})}$ of Lemma 5.11 is an isomorphism, the associated variety $X_{L_k(\mathfrak{g})}$ can be viewed as an analog of associated varieties of primitive ideals (see Section D.4). However, there are substantial differences (see Example 8.1). In general, it is a hard problem to compute N_k and I_k.

5.6.3 Free fermions

Let $\mathfrak{n}$ be a finite-dimensional vector space. We refer to Appendix E for basics on superalgebras and Clifford algebras.

Consider the *Clifford algebra* Cl associated with the vector space $\mathfrak{n} \oplus \mathfrak{n}^*$ and the non-degenerate bilinear forms $\langle\ |\ \rangle$ defined by $\langle \phi + x | \psi + y \rangle = \phi(y) + \psi(x)$ for $\phi, \psi \in \mathfrak{n}^*$, $x, y \in \mathfrak{n}$. Specifically, Cl is the unital $\mathbb{C}$-superalgebra that is isomorphic to $\bigwedge(\mathfrak{n}) \otimes \bigwedge(\mathfrak{n}^*)$ as $\mathbb{C}$-vector spaces, and

$$[x, \phi] = \phi(x), \qquad x \in \mathfrak{n} \subset \bigwedge(\mathfrak{n}),\ \phi \in \mathfrak{n}^* \subset \bigwedge(\mathfrak{n}^*).$$

(Note that $[x, \phi] = x\phi + \phi x$ since x, ϕ are odd.) Define an increasing filtration on Cl by setting $Cl_p := \bigwedge^{\leqslant p}(\mathfrak{n}) \otimes \bigwedge(\mathfrak{n}^*)$. We have

$$0 = Cl_{-1} \subset Cl_0 \subset Cl_1 \subset \cdots \subset Cl_N = Cl,$$

where $N = \dim \mathfrak{n}$, and

$$Cl_p.Cl_q \subset Cl_{p+q}, \qquad [Cl_p, Cl_q] \subset Cl_{p+q-1}.$$

As a consequence, the associated graded algebra,

$$\overline{Cl} := \operatorname{gr} Cl = \bigoplus_{p \geqslant 0} \frac{Cl_p}{Cl_{p+1}},$$

is naturally a graded Poisson superalgebra. We have $\overline{Cl} = \bigwedge(\mathfrak{n}) \otimes \bigwedge(\mathfrak{n}^*)$ as a commutative superalgebra, and its Poisson (super)bracket is given by:

$$\{x, \phi\} = \phi(x), \quad \{x, y\} = 0, \quad \{\phi, \psi\} = 0, \quad x, y \in \mathfrak{n} \subset \bigwedge(\mathfrak{n}),\ \phi, \psi \in \mathfrak{n}^* \subset \bigwedge(\mathfrak{n}^*).$$

The charged fermion Fock space

The *Clifford affinization* $\hat{C}l$ of $\mathfrak{n}$ is the Clifford algebra associated with $\mathfrak{n}[t, t^{-1}] \oplus \mathfrak{n}^*[t, t^{-1}]$ and its symmetric bilinear form defined by

$$\langle xt^m | ft^n \rangle = \delta_{m+n,0} f(x), \quad \langle xt^m | yt^n \rangle = 0 = \langle ft^m | gt^n \rangle$$

for $x, y \in \mathfrak{n}$, $f, g \in \mathfrak{n}^*$, $m, n \in \mathbb{Z}$.

Let $\{x_i\}_{1 \leqslant i \leqslant s}$ be a basis of $\mathfrak{n}$, and $\{x_i^*\}_{1 \leqslant i \leqslant s}$ its dual basis. We write $\psi_{i,m}$ for $x_i t^m \in \hat{C}l$ and $\psi^*_{i,m}$ for $x_i^* t^m \in \hat{C}l$, so that $\hat{C}l$ is the associative superalgebra with

- odd generators: $\psi_{i,m}, \psi^*_{i,m}$, $m \in \mathbb{Z}$, $i = \{1, \dots, s\}$,
- relations: $[\psi_{i,m}, \psi_{j,n}] = [\psi^*_{i,m}, \psi^*_{j,n}] = 0$, $[\psi_{i,m}, \psi^*_{j,n}] = \delta_{i,j}\delta_{m+n,0}$.

Define the *charged fermion Fock space* associated with $\mathfrak{n}$ as

$$\mathcal{F}(\mathfrak{n}) := \frac{\hat{C}l}{\sum\limits_{\substack{m \geqslant 0 \\ 1 \leqslant i \leqslant s}} \hat{C}l \psi_{i,m} + \sum\limits_{\substack{k \geqslant 1 \\ 1 \leqslant j \leqslant s}} \hat{C}l \psi^*_{j,k}} \cong \bigwedge(\psi_{i,n})_{\substack{n<0 \\ 1 \leqslant i \leqslant s}} \otimes \bigwedge(\psi^*_{j,m})_{\substack{m \leqslant 0 \\ 1 \leqslant j \leqslant s}},$$

where $\bigwedge(a_i)_{i \in I}$ denotes the exterior algebra with generators a_i, $i \in I$. It is an irreducible $\hat{C}l$-module, and as $\mathbb{C}$-vector spaces we have

$$\mathcal{F}(\mathfrak{n}) \cong \bigwedge(\mathfrak{n}^*[t^{-1}]) \otimes \bigwedge(\mathfrak{n}[t^{-1}]t^{-1}).$$

There is a unique vertex (super)algebra structure on $\mathcal{F}(\mathfrak{n})$ such that the image of 1 is the vacuum $|0\rangle$ and

$$Y(\psi_{i,-1}|0\rangle, z) = \psi_i(z) := \sum_{n \in \mathbb{Z}} \psi_{i,n} z^{-n-1}, \quad i = 1, \dots, s,$$
$$Y(\psi^*_{i,0}|0\rangle, z) = \psi^*_i(z) := \sum_{n \in \mathbb{Z}} \psi^*_{i,n} z^{-n}, \quad i = 1, \dots, s.$$

We have $F^1\mathcal{F}(\mathfrak{n}) = \mathfrak{n}^*[t^{-1}]t^{-1}\mathcal{F}(\mathfrak{n}) + \mathfrak{n}[t^{-1}]t^{-2}\mathcal{F}(\mathfrak{n})$, and it follows that there is an isomorphism

$$\begin{aligned}\overline{Cl} &\xrightarrow{\simeq} R_{\mathcal{F}(\mathfrak{n})},\\ x_i &\longmapsto \overline{\psi_{i,-1}|0\rangle},\\ x_i^* &\longmapsto \overline{\psi_{i,0}^*|0\rangle}\end{aligned}$$

as Poisson superalgebras. Thus,

$$X_{\mathcal{F}(\mathfrak{n})} = T^*(\Pi\mathfrak{n}),$$

where $\Pi\mathfrak{n}$ is the space $\mathfrak{n}$ considered as a purely odd affine space. Its arc space $\mathscr{J}_\infty(T^*(\Pi\mathfrak{n}))$ is also regarded as a purely odd affine space, such that

$$\mathbb{C}[\mathscr{J}_\infty(T^*(\Pi\mathfrak{n}))] = \wedge(\mathfrak{n}^*[t^{-1}]) \otimes \wedge(\mathfrak{n}[t^{-1}]t^{-1}).$$

The map $\mathbb{C}[\mathscr{J}_\infty(X_{\mathcal{F}(\mathfrak{n})})] \to \operatorname{gr}\mathcal{F}(\mathfrak{n})$ is an isomorphism and, hence, $\mathcal{F}(\mathfrak{n})$ admits a PBW basis. Therefore we have the isomorphism

$$\eta_{\mathcal{F}(\mathfrak{n})}\colon R_{\mathcal{F}(\mathfrak{n})} = \overline{Cl} \xrightarrow{\simeq} \operatorname{gr}\mathrm{Zhu}(\mathcal{F}(\mathfrak{n}))$$

by Exercise 5.2. On the other hand the map

$$\begin{aligned}Cl &\longrightarrow \mathrm{Zhu}(\mathcal{F}(\mathfrak{n}))\\ x_i &\longmapsto \overline{\psi_{i,-1}|0\rangle},\\ x_i^* &\longmapsto \overline{\psi_{i,0}^*|0\rangle}\end{aligned}$$

gives an algebra homomorphism that respects the filtration. Hence we have

$$\mathrm{Zhu}(\mathcal{F}(\mathfrak{n})) \cong Cl.$$

That is, $\mathcal{F}(\mathfrak{n})$ is a chiralization of Cl.

Chapter 6
Poisson vertex modules and their associated variety

In this chapter we give the definition of a Poisson vertex modules over a Poisson vertex algebra and we study some their properties. This notion will be useful to construct new Poisson vertex algebras in Chap. **??** applying the BRST reduction.

6.1 Poisson vertex modules

Definition 6.1 A *Poisson vertex module* over a Poisson vertex algebra V is a V-module M in the usual sense of vertex V-module, equipped with a linear map

$$V \mapsto (\operatorname{End} M)[[z^{-1}]]z^{-1}, \quad a \mapsto Y_{-}^{M}(a,z) = \sum_{n\geqslant 0} a_{(n)}^{M} z^{-n-1},$$

satisfying

$$a_{(n)}^{M} m = 0 \quad \text{for} \quad n \gg 0, \tag{6.1}$$

$$(Ta)_{(n)}^{M} = -na_{(n-1)}^{M}, \tag{6.2}$$

$$a_{(n)}^{M}(bv) = (a_{(n)}^{M} b)v + b(a_{(n)}^{M} v), \tag{6.3}$$

$$[a_{(m)}^{M}, b_{(n)}^{M}] = \sum_{i\geqslant 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)}^{M}, \tag{6.4}$$

$$(ab)_{(n)}^{M} = \sum_{i=0}^{\infty} (a_{(-i-1)} b_{(n+i)}^{M} + b_{(-i-1)} a_{(n+i)}^{M}) \tag{6.5}$$

for all $a, b \in V$, $m, n \geqslant 0$, $v \in M$.

A Poisson vertex algebra V is naturally a Poisson vertex module over itself.

Example 6.1 Let M be a Poisson vertex module over $\mathbb{C}[\mathscr{J}_{\infty}(\mathfrak{g}^*)]$. Then by (6.4), the assignment

$$xt^n \mapsto x^M_{(n)}, \qquad x \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*] \subset \mathbb{C}[\mathscr{J}_\infty(\mathfrak{g}^*)],\ n \geqslant 0,$$

defines a $\mathscr{J}_\infty(\mathfrak{g}) = \mathfrak{g}[[t]]$-module structure on M. In fact, a Poisson vertex module over $\mathbb{C}[\mathscr{J}_\infty(\mathfrak{g}^*)]$ is the same as a $\mathbb{C}[\mathscr{J}_\infty(\mathfrak{g}^*)]$-module M in the usual associative sense equipped with an action of the Lie algebra $\mathscr{J}_\infty(\mathfrak{g})$ such that $(xt^n)m = 0$ for $n \gg 0$, $x \in \mathfrak{g}$, $m \in M$, and

$$(xt^n) \cdot (am) = (x_{(n)}a) \cdot m + a(xt^n) \cdot m$$

for $x \in \mathfrak{g}$, $n \geqslant 0$, $a \in \mathbb{C}[\mathscr{J}_\infty(\mathfrak{g}^*)]$, $m \in M$.

Below we often write $a_{(n)}$ for $a^M_{(n)}$.

The proofs of the following assertions are straightforward. (We refer to §C.6 for the definition of *Poisson modules*.)

Lemma 6.1 *Let R be a Poisson algebra, E a Poisson module over R. There is a unique Poisson vertex $\mathscr{J}_\infty(R)$-module structure on $\mathscr{J}_\infty(R) \otimes_R E$ such that*

$$a_{(n)}(b \otimes m) = (a_{(n)}b) \otimes m + \delta_{n,0} b \otimes \{a, m\}$$

for $n \geqslant 0$, $a \in R \subset \mathscr{J}_\infty(R)$, $b \in \mathscr{J}_\infty(R)$, $m \in E$ (Recall that $\mathscr{J}_\infty(R) = \mathbb{C}[\mathscr{J}_\infty(\operatorname{Spec} R)]$.)

Proof Proof? □

Lemma 6.2 *Let R be a Poisson algebra, M a Poisson vertex module over $\mathscr{J}_\infty(R)$. Suppose that there exists a R-submodule E of M (in the usual commutative sense) such that $a_{(n)}E = 0$ for $n > 0$, $a \in R$, and that M is generated by E (in the usual commutative sense). Then there exists a surjective homomorphism*

$$\mathscr{J}_\infty(R) \otimes_R E \twoheadrightarrow M$$

of Poisson vertex modules.

Proof Proof? □

6.2 Canonical filtration of modules over vertex algebras

Let V be a vertex algebra graded by a Hamiltonian H. A *compatible filtration* of a V-module M is a decreasing filtration

$$M = \Gamma^0 M \supset \Gamma^1 M \supset \cdots$$

such that

$$\begin{aligned}
&a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n-1}M \quad \text{for } a \in F^p V,\ \forall n \in \mathbb{Z},\\
&a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n}M \quad \text{for } a \in F^p V,\ n \geqslant 0,\\
&H.\Gamma^p M \subset \Gamma^p M \quad \text{for all } p \geqslant 0,\\
&\bigcap_p \Gamma^p M = 0.
\end{aligned}$$

For a compatible filtration $\Gamma^\bullet M$, the associated graded space

$$\mathrm{gr}^\Gamma M = \bigoplus_{p \geqslant 0} \Gamma^p M / \Gamma^{p+1} M$$

is naturally a graded vertex Poison module over the graded vertex Poisson algebra $\mathrm{gr}^F V$, and hence, it is a graded vertex Poison module over $\mathscr{J}_\infty(R_V) = \mathbb{C}[\tilde{X}_V]$ by Theorem **??**.

The vertex Poisson $\mathscr{J}_\infty(R_V)$-module structure of $\mathrm{gr}^\Gamma M$ restricts to a Poisson R_V-module structure of $M/\Gamma^1 M = \Gamma^0 M/\Gamma^1 M$, and $a_{(n)}(M/\Gamma^1 M) = 0$ for $a \in R_V \subset \mathscr{J}_\infty(R_V)$, $n > 0$. It follows that there is a homomorphism

$$\mathscr{J}_\infty(R_V) \otimes_{R_V} (M/\Gamma^1 M) \to \mathrm{gr}^\Gamma M, \quad a \otimes \bar{m} \mapsto a\bar{m},$$

of vertex Poisson modules by Lemma 6.2.

Let $\{a^i \mid i \in I\}$ be a set of strong generators of V. Set

$$F^p M = \mathrm{span}_{\mathbb{C}}\{a^1_{(-n_1-1)} \dots a^r_{(-n_r-1)} m \mid a^i \in V,\ m \in M,\ n_1 + \cdots + n_r \geqslant p\}.$$

Proposition 6.1 ([143]) *$F^\bullet M$ is a compatible filtration of M. In fact, it is the finest compatible filtration of M, that is, $F^p M \subset \Gamma^p M$ for all p for any compatible filtration $\Gamma^\bullet M$ of M. In particular, $F^\bullet M$ is independent of the choice of strong generators.*

Proof Proof □

$F^\bullet M$ is called the Li filtration [143] of M.

The subspace $F^1 M$ is spanned by the vectors $a_{(-2)}m$ with $a \in V$, $m \in M$, which is often denoted by $C_2(M)$ in the literature. Set

$$\overline{M} = M/F^1 M (= M/C_2(M)), \tag{6.6}$$

which is a Poisson module over $R_V = \overline{V}$.

Proposition 6.2 ([143, Proposition 4.12]) *By [143, Proposition 4.12], the vertex Poisson module homomorphism*

$$\mathscr{J}_\infty(R_V) \otimes_{R_V} \overline{M} \to \mathrm{gr}^F M$$

is surjective.

Proof Proof? □

Let $\{a^i\ ;\ i \in I\}$ be elements of V such that their images generate R_V in the usual commutative sense, and let U be a subspace of M such that $M = U + F^1 M$. The surjectivity of the above map is equivalent to that

$$F^p M \\ = \operatorname{span}_{\mathbb{C}}\{a^{i_1}_{(-n_1-1)} \dots a^{i_r}_{(-n_r-1)} m \mid m \in U,\ n_i \geqslant 0, n_1 + \dots + n_r \geqslant p,\ i_1, \dots, i_r \in I\}. \tag{6.7}$$

Lemma 6.3 *Let V be a vertex algebra, M a V-module. The Poisson vertex algebra module structure of $\operatorname{gr}^F M$ restricts to the Poisson module structure of $\overline{M} := M/F^1 M$ over R_V, that is, $\overline{M}$ is a Poisson R_V-module by*

$$\bar{a} \cdot \bar{m} = \overline{a_{(-1)} m}, \quad \operatorname{ad}(\bar{a})(\bar{m}) = \overline{a_{(0)} m}, \qquad \bar{a} \in R_V,\ m \in M.$$

A V-module M is called *finitely strongly generated* if $\overline{M}$ is finitely generated as a R_V-module in the usual associative sense.

Definition 6.2 For a finitely strongly generated V-module M, define its *associated variety X_M* by

$$\begin{aligned} X_M &= \operatorname{supp}_{R_V}(\overline{M}) \\ &= \{\mathfrak{p} \in \operatorname{Spec} R_V\ ;\ \mathfrak{p} \supset \operatorname{Ann}_{R_V}(\overline{M})\} \subset X_V, \end{aligned}$$

equipped with the reduced scheme structure.

A finitely strongly generated V-module M is called *lisse*, or *C_2-cofinite.* if $\dim X_M = 0$.

Lemma 6.4 ([6, Lem. 3.2.2]) *Let M be a finitely strongly generated V-module. Then the following are equivalent:*

(i) *M is lisse.*
(ii) *$\bar{M}$ is finite-dimensional.*

Proof Proof? □

6.3 Example: Associated varieties of modules over affine vertex algebras

For a $V = V^\kappa(\mathfrak{a})$-module M, or equivalently (cf. §**??**), a smooth $\widehat{\mathfrak{a}}_\kappa$-module, we have

$$\overline{M} = M/\mathfrak{a}[t^{-1}]t^{-2} M,$$

and the Poisson $\mathbb{C}[\mathfrak{a}^*]$-module structure is given by

$$x \cdot \bar{m} = \overline{(xt^{-1})m}, \quad \mathrm{ad}(x)\overline{m} = \overline{xm}, \qquad x \in \mathfrak{a},\ m \in M.$$

Now suppose that G is a connected semisimple group, $\mathfrak{g} = \mathrm{Lie}(G)$.

Let $\mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$ be the full subcateogory of the category of $\hat{\mathfrak{g}}_\kappa$-modules consisting of modules on which $t\mathfrak{g}[t]$ acts locally nilpotently and $\mathfrak{g}$ acts semisimply. Clearly, $\mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$ is an abelian category, which can be regarded as full subcategories of the category of $V^\kappa(\mathfrak{g})$-modules.

For a $\mathfrak{g}$-module E, let

$$V_\kappa(E) := U(\hat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}\mathbf{1})} E,$$

where E is considered as a $\mathfrak{g}[t] \oplus \mathbb{C}\mathbf{1}$-module on which $\mathfrak{g}[t]$ acts trivially and $\mathbf{1}$ acts as the identity. Then $V_\kappa(E)$ is an object of $\mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$ for a finite dimensional representation E of $\mathfrak{g}$. Note that $V^k(\mathfrak{g}) = V_\kappa(\mathbb{C})$ and its simple quotient $L_\kappa(\mathfrak{g})$ are also objects of $\mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$.

Lemma 6.5 *For $M \in \mathbf{KL}(\hat{\mathfrak{g}}_\kappa)$ the following conditions are equivalent:*

(1) *M is finitely strongly generated as a $V^k(\mathfrak{g})$-module,*
(2) *M is finitely generated as a $\mathfrak{g}[t^{-1}]t^{-1}$-module,*
(3) *M is finitely generated as a $\hat{\mathfrak{g}}_\kappa$-module.*

Proof Proof? □

Exercise 6.1 We have $\overline{V_\kappa(E)} \cong \mathbb{C}[\mathfrak{g}^*] \otimes E$ and $X_{V_\kappa(E)} = \mathfrak{g}^*$ for a finite dimensional representation E of $\mathfrak{g}$.

6.4 Frenkel-Zhu's bimodules

Recall that for a graded vertex algebra V, its Zhu's algebra is defined by $\mathrm{Zhu}(V) = V/V \circ V$. There is a similar construction for modules due to Frenkel and Zhu [80]. For a V-module M, set

$$\mathrm{Zhu}(M) = M/V \circ M,$$

where $V \circ M$ is the subspace of M spanned by the vectors

$$a \circ m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-2)} m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, and $m \in M$.

Proposition 6.3 ([80]) $\mathrm{Zhu}(M)$ *is a bimodule over* $\mathrm{Zhu}(V)$ *by the multiplications*

$$a * m = \sum_{i \geqslant 0} \binom{\Delta_a}{i} a_{(i-1)} m, \quad m * a = \sum_{i \geqslant 0} \binom{\Delta_a - 1}{i} a_{(i-1)} m$$

for $a \in V_{\Delta_a}$, $\Delta_a \in \mathbb{Z}$, *and* $m \in M$.

Proof Proof? □

Thus, we have a right exact functor

$$V\text{-Mod} \to \mathrm{Zhu}(V)\text{-biMod}, \quad M \mapsto \mathrm{Zhu}(M),$$

where $\mathrm{Zhu}(V)$ -biMod is the category of bimodules over $\mathrm{Zhu}(V)$.

Lemma 6.6 *Let* $M = \bigoplus_{d \in h+\mathbb{Z}_{\geqslant 0}} M_d$ *be a positive energy representation of a* $\mathbb{Z}_{\geqslant 0}$*-graded vertex algebra* V. *Define an increasing filtration* $\{\mathrm{Zhu}_p(M)\}_p$ *on* $\mathrm{Zhu}(V)$ *by*

$$\mathrm{Zhu}_p(M) = \mathrm{im}(\bigoplus_{d=h}^{h+p} M_p \to \mathrm{Zhu}(M)).$$

(i) *We have*

$$\mathrm{Zhu}_p(V) \cdot \mathrm{Zhu}_q(M) \cdot \mathrm{Zhu}_r(V) \subset \mathrm{Zhu}_{p+q+r}(M),$$
$$[\mathrm{Zhu}_p(V), \mathrm{Zhu}_q(M)] \subset \mathrm{Zhu}_{p+q-1}(M).$$

Therefore $\mathrm{gr}\,\mathrm{Zhu}(M) = \bigoplus_p \mathrm{Zhu}_p(M)/\mathrm{Zhu}_{p-1}(M)$ *is a Poisson* $\mathrm{gr}\,\mathrm{Zhu}(V)$*-module, and hence is a Poisson* R_V*-module through the homomorphism* $\eta_V : R_V \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}(V)$.

(ii) *There is a natural surjective homomorphism*

$$\eta_M : \bar{M}(= M/F^1 M) \to \mathrm{gr}\,\mathrm{Zhu}(M)$$

of Poisson R_V*-modules. This is an isomorphism if* V *admits a PBW basis and* $\mathrm{gr}\,M$ *is free over* $\mathrm{gr}\,V$.

Example 6.2 Let $M = V_E^k$. Since $\mathrm{gr}\,V_E^k$ is free over $\mathbb{C}[\mathscr{J}_\infty(\mathfrak{g}^*)]$, we have the isomorphism

$$\eta_{V_E^k} : \overline{V_E^k} = E \otimes \mathbb{C}[\mathfrak{g}^*] \xrightarrow{\simeq} \mathrm{gr}\,\mathrm{Zhu}(V_E^k).$$

On the other hand, there is a $U(\mathfrak{g})$-bimodule homomorphism

$$\begin{aligned} E \otimes U(\mathfrak{g}) &\to \mathrm{Zhu}(V_E^k), \\ v \otimes x_1 \dots x_r &\mapsto (1 \otimes v) * (x_1 t^{-1}) * \dots * (x_r t^{-1}) + V^k(\mathfrak{g}) \circ V_E^k \end{aligned} \tag{6.8}$$

which respects the filtration. Here the $U(\mathfrak{g})$-bimodule structure of $U(\mathfrak{g}) \otimes E$ is given by

$$x(v \otimes u) = (xv) \otimes u + v \otimes xu, \quad (v \otimes u)x = v \otimes (ux),$$

and the filtration of $U(\mathfrak{g}) \otimes E$ is given by $\{U_i(\mathfrak{g}) \otimes E\}$. Since the induced homomorphism between associated graded spaces (6.8) coincides with $\eta_{V_E^k}$, (6.8) is an isomorphism.

Let $\mathcal{HC}$ be the category of *Harish-Chandra bimodules*, that is, the full subcategory of the category of $U(\mathfrak{g})$-bimodules consisting of objects M on which the adjoint action of $\mathfrak{g}$ is integrable, that is, locally finite.

Lemma 6.7 *For $M \in \mathbf{KL}_k$, we have* $\mathrm{Zhu}(M) \in \mathcal{HC}$. *If M is finitely generated, then so is* $\mathrm{Zhu}(M)$.

Part III
Quasi-lisse vertex algebras

As a Poisson variety, the associated variety of a vertex algebra is a finite disjoint union of smooth analytic Poisson manifolds, and it is stratified by its symplectic leaves. The case where the associated variety has finitely many symplectic leaves is particularly interesting. They were first considered by the authors in [24], and then referred to as *quasi-lisse* vertex algebras in [15].

Definition 6.3 ([15]) A finitely strongly generated $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra V is called *quasi-lisse* if X_V has only finitely many symplectic leaves.

Lisse vertex algebras appear as special cases quasi-lisse vertex algebras: those vertex algebras whose associated variety is just a point (remember that all vertex algebras are assumed to be strongly generated and $\mathbb{Z}_{\geqslant 0}$-graded).

In this part we give various examples of quasi-lisse vertex algebras, and present remarkable properties of lisse and quasi-lisse vertex algebras. Most of our examples are particular cases of (simple) affine vertex algebras. We will see other examples Part IV in the context of W-algebras by taking the quantized Drinfeld-Sokolov reduction of quasi-lisse affine vertex algebras. There are other expected examples coming from four dimensional $\mathcal{N} = 2$ superconformal field theories.

As a first motivation, let us comment the quasi-lisse condition in the setting of affine vertex algebras.

We have seen that $V^k(\mathfrak{g})$ plays a role similar to that of the enveloping algebra of $\mathfrak{g}$ for the representation theory of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ (cf. §**??**). Because of this, it would be nice to have analogs of the associated varieties of primitive ideals in this context (see Section D.4). Unfortunately, one cannot expect exactly the same theory. One of the main reasons is that the center of $U(\hat{\mathfrak{g}})$ is trivial (unless for the *critical level* $k = -h^\vee$), and so we do not have analog of the nilpotent cone (for the critical level, the analog is played by the arc space of the nilpotent cone, see Example 1.3 and Exercice 3.3). So we need some replacements. In this context, the associated variety of the highest weight irreducible representation $L(k\Lambda_0) = L_k(\mathfrak{g})$ of $\hat{\mathfrak{g}}$, $k \in \mathbb{C}$, viewed as a vertex algebra, is a better analog. More generally, one can consider the associated variety of any irreducible highest representation $L(\lambda)$ of $\hat{\mathfrak{g}}$, by exploiting the notion of associated variety for any module over a vertex algebra; see Section 6.3. We will see next chapters some analogies between the associated variety of $L_k(\mathfrak{g})$ and the associated variety of primitive ideals. However, there are substantial differences. For example, since $L_k(\mathfrak{g}) \cong V^k(\mathfrak{g})$ for $k \notin \mathbb{Q}$ (cf. [114]), we see that $X_{L_k(\mathfrak{g})}$ is not always contained in the nilpotent cone $\mathscr{N}$ of $\mathfrak{g}$. We observe that $X_{L_k(\mathfrak{g})}$ is contained in the nilpotent cone $\mathscr{N}$ if and only if $L_k(\mathfrak{g})$ is quasi-lisse (see Proposition 8.1). Thus, in this context, the quasi-lisse condition looks very natural.

This part is structured as follows. Chap. 7 is devoted to lisse and rational vertex algebras. Chap. 8 contains various examples of quasi-lisse (simple) affine vertex algebras. Remarkable properties of quasi-lisses vertex algebras are described in Chap. 9. We also discussed in this chapter open problems (the irreducibility conjecture and the Higgs branch conjecture in physics) related to quasi-lisse vertex algebras. In fact, the vertex algebras constructed from 4d SCFTs are expected to be quasi-lisse, since their associated varieties conjecturally coincide with the Higgs branches of the corresponding four dimensional theories ([170]).

Chapter 7
Lisse and rational vertex algebras

Recall that a vertex algebra V is called lisse if $\dim X_V = 0$, or equivalently, if R_V is finite-dimensional (see Section 4.7). Examples of lisse vertex algebras are given in Section 7.1 and Section 7.2 below. Close to the lisse condition, we have the rationality condition:

Definition 7.1 A conformal vertex algebra V is called *rational* if every $\mathbb{Z}_{\geqslant 0}$-graded V-modules is completely reducible (that is, isomorphic to a direct sum of simple V-modules).

Our below examples are actually also examples of rational vertex algebras. In Section 7.3 we list a few properties of lisse vertex algebras and rational vertex algebras, and we discuss the connections between the lisse and the rationality conditions.

7.1 Integrable representations of affine Kac-Moody algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra. Recall that the irreducible $\mathfrak{g}$-representation $L_{\mathfrak{g}}(\lambda)$, with highest weight $\lambda \in \mathfrak{h}^*$, is finite-dimensional if and only if its associated variety $\mathscr{V}(\mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda)))$ is zero (Example D.6). Contrary to irreducible highest weight representations of $\mathfrak{g}$, the irreducible $\hat{\mathfrak{g}}$-representation $L(\lambda)$, where $\lambda \in \hat{\mathfrak{h}}^*$, is finite-dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation.

The notion of finite-dimensional representations has to be replaced by the notion of *integrable representations* in the category $\mathscr{O}$. (See Section A.4 and Section A.5 for the category $\mathscr{O}$ and the definition of integrable representations for affine Kac-Moody algebras.)

Let $k \in \mathbb{C}$. Recall that the simple affine vertex algebra $L_k(\mathfrak{g})$ is isomorphic to the irreducible highest weight representation $L(k\Lambda_0)$ as a $\hat{\mathfrak{g}}$-module.

Theorem 7.1 ([8, 61]) *The following are equivalent:*

(i) $L_k(\mathfrak{g})$ *is rational,*

(ii) *$L_k(\mathfrak{g})$ is lisse, that is, $X_{L_k(\mathfrak{g})} = \{0\}$,*
(iii) *$L_k(\mathfrak{g})$ is integrable as a $\hat{\mathfrak{g}}$-module (which happens if and only if $k \in \mathbb{Z}_{\geqslant 0}$).*

It should be noted a clear analogy between the equivalence (iii) $\iff$ (iii) and the equivalence mentioned in Example D.6.

The equivalence (i) $\iff$ (iii) is well-known ref ?, as well as the last equivalence in parenthesis of Part (iii). We explain below only the implication (iii) $\Rightarrow$ (ii).

Lemma 7.1 *Let (R, ∂) be a differential algebra over $\mathbb{Q}$, and let I be a differential ideal of R, i.e., I is an ideal of R such that $\partial I \subset I$. Then $\partial\sqrt{I} \subset \sqrt{I}$.*

Proof Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Since I is ∂-invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m \equiv m!(\partial a)^m \pmod{\sqrt{I}}.$$

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$. □

Recall that a *singular vector* of a $\hat{\mathfrak{g}}$-representation M is a vector $v \in M$ such that $\hat{\mathfrak{n}}.v = 0$, if $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}$ is a trianglar decomposition of $\hat{\mathfrak{g}}$ (see §A.4.3). In particular, regarding $V^k(\mathfrak{g})$ as a $\hat{\mathfrak{g}}$-representation, a vector $v \in V^k(\mathfrak{g})$ is singular if and only if $\hat{\mathfrak{n}}.v = 0$.

In the case where k is a nonnegative integer, the maximal submodule N_k of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ ([111]), where θ is the highest positive root and $e_\theta \in \mathfrak{g}_\theta \setminus \{0\}$.

Proof (Proof of the implication (iii) ⇒ (ii) in Theorem 7.1) Suppose that $L_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}_{\geqslant 0}$ and, if so, the maximal submodule $N_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$. The exact sequence $0 \to N_k(\mathfrak{g}) \to V^k(\mathfrak{g}) \to L_k(\mathfrak{g}) \to 0$ induces the exact sequence

$$0 \longrightarrow I_k \longrightarrow R_{V^k(\mathfrak{g})} \longrightarrow R_{L_k(\mathfrak{g})} \longrightarrow 0,$$

where I_k is the image of N_k in $R_{V^k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]$, and so, $R_{L_k(\mathfrak{g})} = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in I_k is given by e_θ^{k+1}. Therefore, $e_\theta \in \sqrt{I_k}$. On the other hand, by Lemma 7.1, $\sqrt{I_k}$ is preserved by the adjoint action of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g} \subset \sqrt{I_k}$. This proves that $X_{L_k(\mathfrak{g})} = \{0\}$ as required. □

The proof of the "only if" part follows from [61]. It can also be proven using W-algebras.

In view of Theorem 7.1, one may regard the lisse condition as a generalization of the integrability condition to an arbitrary vertex algebra.

7.2 Minimal series representations of the Virasoro algebra

Let $c \in \mathbb{C}$. Denote by N_c the unique maximal submodule of the Virasoro vertex algebra Vir^c, and let $\mathrm{Vir}_c := \mathrm{Vir}^c/N_c$ be the simple quotient.

Theorem 7.2 *The following are equivalent:*

(i) Vir_c *is rational,*
(ii) Vir_c *is lisse,*
(iii) $c = 1 - \dfrac{6(p-q)^2}{pq}$ *for some* $p, q \in \mathbb{Z}_{\geqslant 2}$ *such that* $(p, q) = 1$. *(These are precisely the central charge of the* minimal series representations *of the Virasoro algebra* Vir.*)*

The equivalence (ii) $\iff$ (iii) is well-known [181].

We explain below the equivalence (i) $\iff$ (iii) (see [6, Prop. 3.4.1]).

Proof (Proof of equivalence (i) $\iff$ (iii) in Theorem 7.2) It is known that the image of N_c in R_{Vir^c} is nonzero if $N_c \neq 0$ (see e.g., [181, Lem 4.2 and 4.3] or [95, Prop. 4.3.2]. Therefore $X_{\mathrm{Vir}_c} = \{0\}$ if and only if Vir^c is not irreducible. This happens if and only if the central charge is of the form in (iii) ([110, 74, 95]). □

7.2.1

Where to add the results of Heluani-Van Ekeren on arc spaces of Virasoro? Here?

7.3 On the lisse and the rational conditions

It is known ([59]) that the rationality condition implies that V has finitely many simple $\mathbb{Z}_{\geqslant 0}$-graded modules and that the graded components of each of these $\mathbb{Z}_{\geqslant 0}$-graded modules are finite dimensional. In fact lisse vertex algebras also verify this property. More precisely, we have:

Theorem 7.3 ([1, 184, 157]) *Let* V *be a* $\mathbb{Z}_{\geqslant 0}$*-graded conformal lisse vertex algebra.*

(i) *Any simple* V*-module is a positive energy representation, that is, a positively graded* V*-module. Therefore the number of isomorphic classes of simple* V*-modules is finite.*
(ii) *Let* $M_1, \dots, M_s$ *be representatives of these classes, and let for* $i = 1, \dots, s$,

$$\chi_{M_i}(\tau) = \mathrm{Tr}_{M_i}(q^{L_0 - \frac{c}{24}}) = \sum_{n \geqslant 0} \dim(M_i)_n q^{n - \frac{c}{24}}, \quad q = e^{2i\pi\tau},$$

be the normalized character of M_i. *Then* $\chi_{M_i}(\tau)$ *converges in the Poincaré half-plane* $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, *and the vector space generated by* $SL_2(\mathbb{Z}).\chi_{M_i}(\tau)$ *is finite-dimensional.*

If V is as in Theorem 7.3 and also rational, it is known [103] that under some mild assumptions, the category of V-modules forms a modular tensor category, which for instance yields an invariant of 3-manifolds, see [31]. It is actually conjectured by Zhu in [184] that rational vertex algebras must be lisse (this conjecture is still open).

The converse is not true: is there a known counter-example? Conjecturally, we have $W_k(\mathfrak{g}, f_{min})$ with $\mathfrak{g} \in$ DES and $k = -h^\vee/6 - 1 + n$ such that $k \in \mathbb{Z}_{\geqslant 0}$...

There are significant vertex algebras that do not satisfy the lisse condition. For instance, an *admissible* affine vertex algebra $L_k(\mathfrak{g})$ (see Section 8.2) has a complete reducibility property ([9]), and the modular invariance property ([117]) in the category $\mathscr{O}$ still holds, although it is not lisse unless it is integrable.

So it is natural to try to relax the lisse condition. This is the purpose of the next chapters.

Chapter 8
Examples of affine quasi-lisse vertex algebras

We now intend to give various examples of quasi-lisse vertex algebras in the context of affine vertex algebras. We start in Section 8.1 with general facts on the associated variety of affine vertex algebras. Then we focus essentially on two interesting families of quasi-lisse simple affine vertex algebras: those coming from admissible levels (Section 8.2) and those coming from the Deligne exceptional series (Section 8.3). So far, they are roughly the only known quasi-lisse simple affine vertex algebras (see Remark 8.1 for a couple of other known cases) while they are certainly much more examples.

In what follows, $\mathfrak{g}$ is a complex simple Lie algebra with adjoint group G, and $\mathcal{N}$ is the nilpotent cone of $\mathfrak{g}$, that is, the set of nilpotent elements of $\mathfrak{g}$. We identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using a non-degenerate bilinear form, for instance the bilinear form $(\,|\,) = \dfrac{1}{h^\vee} \times$ Killing form of $\mathfrak{g}$. We shall use the notations of Appendix D, particularly for the nilpotent orbits in $\mathfrak{sl}_n$ in correspondence with partition of n (Section D.2).

8.1 General facts on associated varieties of affine vertex algebras

Let $V^k(\mathfrak{g})$ be the universal affine vertex algebra associated with $\mathfrak{g}$ at the level $k \in \mathbb{C}$. Recall first that the associated variety of $V^k(\mathfrak{g})$ is $\mathfrak{g}^* \cong \mathfrak{g}$ (cf. Example 4.3). In particular, $V^k(\mathfrak{g})$ is never quasi-lisse (see Proposition 8.1).

Let us look now at the the associated variety of the simple quotient $L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k$, where N_k is the maximal proper submodule of $V^k(\mathfrak{g})$. Contrary to the associated varieties of primitive ideals of $U(\mathfrak{g})$, the associated variety of $L_k(\mathfrak{g})$ is not always contained in the nilpotent cone $\mathcal{N}$. Indeed, if $V^k(\mathfrak{g})$ is simple, for example if $k \notin \mathbb{Q}$, then $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$ and so $X_{L_k(\mathfrak{g})} = \mathfrak{g} \not\subset \mathcal{N}$. By the main result of [16] the converse is true:

Theorem 8.1 ([16]) *Let $k \in \mathbb{C}$. Then $X_{L_k(\mathfrak{g})} = \mathfrak{g}$ if and only if $L_k(\mathfrak{g}) = V^k(\mathfrak{g})$, that is, $V^k(\mathfrak{g})$ is simple.*

On the other hand, we have a simple criterion to check whether $L_k(\mathfrak{g})$ is quasi-lisse:

Proposition 8.1 *The simple affine vertex algebra $L_k(\mathfrak{g})$ is quasi-lisse if and only if $X_{L_k(\mathfrak{g})} \subset \mathscr{N}$.*

Proof Recall that the symplectic leaves of $\mathfrak{g}$ are the adjoint G-orbits of $\mathfrak{g}$, It is well-known that the nilpotent cone $\mathscr{N}$ of the simple Lie algebra is a finite union of adjoint orbits (see Section D.1). Hence, if $X_{L_k(\mathfrak{g})}$ is contained in $\mathscr{N}$ then $L_k(\mathfrak{g})$ is quasi-lisse.

Conversely, assume that $X_{L_k(\mathfrak{g})}$ contains a non-nilpotent element x, with Jordan decomposition $x = x_s + x_n$. If $x_n = 0$, then $X_{L_k(\mathfrak{g})}$ contains $\overline{G\mathbb{C}^* x} = \overline{G\mathbb{C}^* x_s}$ since $X_{L_k(\mathfrak{g})}$ is a closed G-invariant cone of $\mathfrak{g}$. But $\overline{G\mathbb{C}^* x_s}$ contains infinitely many symplectic leaves because x_s is semisimple. So $X_{L_k(\mathfrak{g})}$ is not quasi-lisse. If $x_n \neq 0$, choose an $\mathfrak{sl}_2$-triplet (x_n, h, y_n) in $\mathfrak{g}^{x_s}$ and consider the one-parameter subgroup $\rho\colon \mathbb{C}^* \to G$ generated by ad h. We have for all $t \in \mathbb{C}^*$,

$$\rho(t)x = x_s + t^2 x_n.$$

Taking the limit when t goes to 0, we deduce that $x_s \in X_{L_k(\mathfrak{g})}$ and, hence, by the first case, $X_{L_k(\mathfrak{g})}$ is not quasi-lisse. □

Proposition 8.2 *If $X_{L_k(\mathfrak{g})} \subset \mathscr{N}$, then $L_k(\mathfrak{g})$ has only finitely many simple objects in the category $\mathscr{O}$.*

Proof We know that $\mathrm{Zhu}(L_k(V))$ is a quotient $U(\mathfrak{g})/I$ of $U(\mathfrak{g})$ (see §5.6.2). Moreover, by Zhu's correspondence (Theorem 5.2), it suffices to show that there are finitely many simple highest weight $\mathfrak{g}$-modules $L_{\mathfrak{g}}(\lambda)$ annihilated by I since $L(\hat{\lambda})_{\mathrm{top}} = L_{\mathfrak{g}}(\lambda)$, where λ is the restriction to $\mathfrak{h}$ of $\hat{\lambda} \in \hat{\mathfrak{h}}^*$ since simple objects in the category $\mathscr{O}$ are precisely the simple highest weight modules $L(\lambda)$, [110, Proposition 9.3].

Let us denote by $\chi_\lambda\colon \mathscr{Z}(\mathfrak{g}) \to \mathbb{C}$ the infinitesimal character associated to $L_{\mathfrak{g}}(\lambda)$. Recall that for $\lambda, \mu \in \mathfrak{h}^*$, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same W-orbit with respect to the twisted action of W, where W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Hence, identifying the maximal spectrum of $\mathscr{Z}(\mathfrak{g})$ with the set of all homomorphisms $\mathscr{Z}(\mathfrak{g}) \to \mathbb{C}$, it is enough to show that $\mathrm{Specm}\,(\mathscr{Z}(\mathfrak{g})/\mathscr{Z}(\mathfrak{g}) \cap I)$ is finite, or equivalently, that $\mathscr{Z}(\mathfrak{g})/\mathscr{Z}(\mathfrak{g}) \cap I$ is finite-dimensional. This will show that the set of possible infinite characters of simple $U(\mathfrak{g})/I$-modules is finite and so is the set of possible simple $U(\mathfrak{g})/I$-modules.

Include recalls on infinitesimal characters this in Appendix.

Using the surjective Poisson algebra homomorphism (cf. Lemma 5.11),

$$R_{L_k(\mathfrak{g})} \twoheadrightarrow \mathrm{gr}\,\mathrm{Zhu}(L_k(\mathfrak{g})) = \mathrm{gr}\,U(\mathfrak{g})/\mathrm{gr}\,I,$$

we get that $(\mathrm{Spec}\,\mathrm{gr}\,U(\mathfrak{g})/\mathrm{gr}\,I)_{\mathrm{red}} \subset X_{L_k(\mathfrak{g}))} \subset \mathscr{N}$, whence the augmentation ideal $(\mathrm{gr}\,\mathscr{Z}(\mathfrak{g}))_+$ of $\mathrm{gr}\,\mathscr{Z}(\mathfrak{g})$ is contained in $\sqrt{\mathrm{gr}\,I}$ because $(\mathrm{gr}\,\mathscr{Z}(\mathfrak{g}))_+ \cong \mathbb{C}[\mathfrak{g}]_+$ is the defining ideal of $\mathscr{N}$. As a result, $\mathrm{gr}\,\mathscr{Z}(\mathfrak{g})/\mathrm{gr}(\mathscr{Z}(\mathfrak{g}) \cap I)$ is finite-dimensional and so is $\mathscr{Z}(\mathfrak{g})/\mathscr{Z}(\mathfrak{g}) \cap I$. □

In view of the above results, it is natural to ask whether there exist pairs $(\mathfrak{g}, k)$ such that $X_{L_k(\mathfrak{g})}$ is neither $\mathfrak{g}$ nor contained in the nilpotent cone $\mathcal{N}$. This is indeed the case, as the following examples illustrate.

Example 8.1 ([23, Theorem 1.1])

- For $n \geqslant 4$,
$$X_{L_{-1}(\mathfrak{sl}_n)} = \overline{G\mathbb{C}^* \check{\varpi}_1} \not\subset \mathcal{N},$$
where $\check{\varpi}_1$ is the fundamental co-weight associated with α_1 if $\alpha_1, \ldots, \alpha_{n-1}$ are the simple roots of $\mathfrak{sl}_n$. Note that $\overline{G\mathbb{C}^* \check{\varpi}_1} = \overline{\mathbb{S}_1}$, where $\mathbb{S}_1$ is the unique *sheet* (see the footnote 2 of Section 8.4) of $\mathfrak{sl}_n$ containing the minimal nilpotent orbit $\mathbb{O}_{\min}$ in its closure.
- For $m \geqslant 2$,
$$X_{L_{-m}(\mathfrak{sl}_{2m})} = \overline{G\mathbb{C}^* \check{\varpi}_m} \not\subset \mathcal{N},$$
where $\check{\varpi}_m$ is the fundamental co-weight associated with α_m. Note that $\overline{G\mathbb{C}^* \check{\varpi}_m} = \overline{\mathbb{S}_0}$, where $\mathbb{S}_0$ is the unique sheet of $\mathfrak{sl}_{2m}$ containing the nilpotent orbit $\mathbb{O}_{(2^m)}$ associated with the partition (2^m) in its closure.

As a next step, in the light of Theorem 8.1, one can ask whether there is a pair $(\mathfrak{g}, k)$ such that $X_{L_k(\mathfrak{g})}$ is a maximal proper G-invariant closed subcone of $\mathfrak{g}$.

- For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{N}$ is the unique maximal proper G-invariant closed subcone of $\mathfrak{g}$. In fact the only G-invariant closed subcones of $\mathfrak{sl}_2$ are: $\{0\}$, $\mathcal{N}$, $\mathfrak{g}$. All these subsets can be realized as the associated variety of some $L_k(\mathfrak{sl}_2)$ (see Section 8.2 and more specifically Exercice 8.1).
- For $\mathfrak{g} = \mathfrak{sl}_3$, one can construct a maximal proper G-invariant closed subcone of $\mathfrak{g}$ as follows. Let (e, h, f) be a *principal* $\mathfrak{sl}_2$-triple, that is, f is regular nilpotent element of $\mathfrak{g}$. Let $\mathscr{X} = \overline{G\mathbb{C}^* h}$ be the G-invariant closed cone generated by h. This set is referred to as the *principal cone* in [52]. It contains the nilpotent cone $\mathcal{N}$ and has dimension $\dim \mathcal{N} + 1$. So, for $\mathfrak{g} = \mathfrak{sl}_3$, it is maximal for dimension reasons (it has dimension 7 while $\mathfrak{g}$ has dimension 8). More generally, for any regular semisimple element $x \in \mathfrak{g}$, the set $\overline{G\mathbb{C}^* x}$ is a G-invariant closed subcone of $\mathfrak{g}$ containing $\mathcal{N}$ (see [53, Th. 2.9]). The principal cone $\mathscr{X}$ is somehow more canonical: it is precisely the closure of the set of *principal* semisimple elements (that is, the semisimple elements which are central elements of principal $\mathfrak{sl}_2$-triples).

? Open problem

Assume that $\mathfrak{g} = \mathfrak{sl}_3$. Is there a level k such that $X_{L_k(\mathfrak{g})} = \mathscr{X}$, where $\mathscr{X} = \overline{G\mathbb{C}^* h}$ is the principal cone of $\mathfrak{sl}_3$?

In the next two sections, we describe families of quasi-lisse simple affine vertex algebras. In other words, we provide pairs $(\mathfrak{g}, k)$ for which $X_k(\mathfrak{g})$ is contained in $\mathcal{N}$.

8.2 Admissible representations

Recall that the irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible in the sense of Definition A.6. For example, an irreducible integrable representation of $\hat{\mathfrak{g}}$ is admissible. More generally, the simple affine vertex algebra $L_k(\mathfrak{g})$ is called *admissible* if it is admissible as a $\hat{\mathfrak{g}}$-module, and the level k is called *admissible* if $L_k(\mathfrak{g})$ is addmisible (see Definition A.7). This happens if and only if (Proposition A.3):

$$k = -h^\vee + \frac{p}{q} \text{ with } p, q \in \mathbb{Z}_{\geqslant 1},\ (p, q) = 1,\ p \geqslant \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1, \\ h & \text{if } (q, r^\vee) = r^\vee. \end{cases}$$

Here $r^\vee$ is the *lacety* of $\mathfrak{g}$ (i.e., $r^\vee = 1$ for the types A, D, E, $r^\vee = 2$ for the types B, C, F and $r^\vee = 3$ for the type G_2), and h is the Coxeter number.

The fist statement of the following assertion was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g} = \mathfrak{sl}_2$ by Feigin and Malikov [75]. The general proof is achieved in [8].

Theorem 8.2 ([8])

i). If k is admissible, then $SS(L_k(\mathfrak{g})) \subset \mathscr{J}_\infty(\mathscr{N})$ or, equivalently, the associated variety $X_{L_k(\mathfrak{g})}$ is contained in $\mathscr{N}$.

ii). In fact, a stronger result holds: we have

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_k},$$

where $\mathbb{O}_k$ is a nilpotent orbit which only depends on the denominator q, with q as above.

Example 8.2 Let us describe explicitly the nilpotent orbit $\mathbb{O}_k$ of Theorem 8.2 in the case where $\mathfrak{g} = \mathfrak{sl}_n$. Recall that the nilpotent orbits of $\mathfrak{sl}_n$ are parameterized by the partitions of n. Let k be an admissible level for $\mathfrak{sl}_n$, that is, $k = -n + \dfrac{p}{q}$, with $p \in \mathbb{Z}$, $p \geqslant n$, and $(p, q) = 1$. Then

$$X_{L_k(\mathfrak{g})} = \{x \in \mathfrak{g} \colon (\operatorname{ad} x)^{2q} = 0\} = \overline{\mathbb{O}_k},$$

where $\mathbb{O}_k$ is the nilpotent orbit corresponding to the partition (n) is $q \geqslant n$, and to the partition $(q, q, \ldots, q, s) = (q^m, s)$, where m and s are the quotient and the rest of the Euclidean division of n by q, respectively, if $q < n$.

Next exercise gives a proof of Theorem 8.2 for $\mathfrak{g} = \mathfrak{sl}_2$. It is based on Feigin and Malikov approach (see also [8, Theorem 5.6]).

Exercise 8.1 Let N_k be the proper maximal ideal of $V^k(\mathfrak{sl}_2)$ so that $L_k(\mathfrak{sl}_2) = V^k(\mathfrak{sl}_2)/N_k$. Let I_k be the image of N_k in $R_{V^k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{sl}_2]$ so that $R_{L_k(\mathfrak{sl}_2)} = \mathbb{C}[\mathfrak{sl}_2]/I_k$. It is known that either N_k is trivial, that is, $V^k(\mathfrak{sl}_2)$ is simple, or N_k is generated by a singular vector v whose image $\overline{v}$ in I_k is nonzero ([116, 155]).

We assume in this exercise that N_k is non trivial. Thus, $N_k = U(\widehat{\mathfrak{sl}}_2)v$.

i). Using Kostant's Separation Theorem show that, up to a nonzero scalar,

$$\overline{v} = \Omega^m e^n,$$

for some $m, n \in \mathbb{Z}_{>0}$, where $\Omega = 2ef + \frac{1}{2}h^2$ is the Casimir element of the symmetric algebra of $\mathfrak{sl}_2$.

ii). Deduce from this that $X_{L_k(\mathfrak{sl}_2)}$ is contained in the nilpotent cone of $\mathfrak{sl}_2$.

It is known that N_k is nontrivial if and only k is an admissible level for $\mathfrak{sl}_2$, or $k = -2$ is critical. Thus we have shown that $X_{L_k(\mathfrak{sl}_2)}$ is contained in the nilpotent cone of $\mathfrak{sl}_2$ if and only if $k = -2$ or k is admissible, i.e., $k = -2 + \frac{p}{q}$, with $(p, q) = 1$ and $p \geqslant 2$.

On the other hand, since $X_{L_k(\mathfrak{sl}_2)} = \{0\}$ if and only if $k \in \mathbb{Z}_{\geqslant 0}$ by Theorem 7.1, we get that $X_{L_k(\mathfrak{sl}_2)} = \mathscr{N}$ if and only if $k = -2$ or k is admissible and $k \notin \mathbb{Z}_{\geqslant 0}$.

The following exercise explains how to compute the associated variety in a concrete example exploiting a singular vector (see §A.4.3). This example is covered by both Theorem 8.2 and Theorem 8.3.

Exercise 8.2 The aim of this exercice is to compute $X_{L_{-3/2}(\mathfrak{sl}_3)}$. It was shown by Perše [165] that the proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the singular vector v given by:

$$v := \frac{1}{3}\Big((h_1 t^{-1})(e_{1,3}t^{-1})|0\rangle - (h_2 t^{-1})(e_{1,3}t^{-1})|0\rangle\Big) + (e_{1,2}t^{-1})(e_{2,3}t^{-1})|0\rangle - \frac{1}{2}e_{1,3}t^{-2}|0\rangle,$$

where $h_1 := e_{1,1} - e_{2,2}$, $h_2 := e_{2,2} - e_{3,3}$ and $e_{i,j}$ is the elementary matrix of the coefficient (i, j) in $\mathfrak{sl}_3$ identified with the set of traceless 3-size square matrices.

i). Verify that v is indeed a singular vector for $\widehat{\mathfrak{sl}_3}$, that is, $e_{i,i+1}v = 0$ for $i = 1, 2$ and $(e_{3,1}t)v = 0$.

ii). Let $\mathfrak{h} := \mathbb{C}h_1 + \mathbb{C}h_2$ be the usual Cartan subalgebra of $\mathfrak{sl}_3$. Show that $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h} = \{0\}$, and deduce from this that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ is contained in the nilpotent cone of $\mathfrak{sl}_3$.

iii). Show that the nilpotent cone is not contained in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.

iv). Denoting by $\mathbb{O}_{min}$ the minimal nilpotent orbit of $\mathfrak{sl}_3$, conclude that

$$X_{L_{-3/2}(\mathfrak{sl}_3)} = \overline{\mathbb{O}_{min}}.$$

8.3 Exceptional Deligne series

There was actually a "strong Feigin-Frenkel conjecture" stating that k is admissible if and only if $X_{L_k(\mathfrak{g})} \subset \mathscr{N}$ (provided that k is not critical, that is, $k \neq -h^\vee$ in which case it is known that $X_{L_k(\mathfrak{g})} = \mathscr{N}$). Such a statement would be interesting because it would give a geometrical description of the admissible representations $L_k(\mathfrak{g})$. reference for that?

As seen in Exercise 8.1, the equivalence holds for $\mathfrak{g} = \mathfrak{sl}_2$. The stronger conjecture is wrong in general, as shown the following result.

Theorem 8.3 ([22]) *Assume that* $\mathfrak{g}$ *belongs to the* Deligne exceptional series ([56]),

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and that $k = -\frac{h^\vee}{6} - 1 + n$*, where* $n \in \mathbb{Z}_{\geqslant 0}$ *is such that* $k \notin \mathbb{Z}_{\geqslant 0}$*. Then*

$$X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{min}},$$

where $\mathbb{O}_{min}$ *is the minimal nilpotent orbit of* $\mathfrak{g}$.

Note that the level $k = -\frac{h^\vee}{6} - 1$ is not admissible for the types D_4, E_6, E_7, E_8 (it equals $-2, -3, -4, -6$, respectively). Theorem 8.3 provides the first known examples of associated varieties contained in the nilpotent cone corresponding to non-admissible levels. We will see that Theorem 8.3 also allows to produce "new" examples of lisse simple W-algebras (see Chap. **??**).

By Proposition 8.2, if $(\mathfrak{g}, k)$ is as in Theorem 8.3, then $L_k(\mathfrak{g})$ has finitely many simple objects in the category $\mathscr{O}$. One can describe them thanks to Joseph's classification [109] of irreducible highest weights representation $L_{\mathfrak{g}}(\lambda)$ whose associated variety is $\overline{\mathbb{O}_{min}}$ (see Theorem 8.4).

We give in the next section a broad sketch of a proof of Theorem 8.3.

Remark 8.1 There are a couple of other examples of simple quasi-lisse affine vertex algebras $L_k(\mathfrak{g})$, at non-admissible level k ([22, 23, 24]). Namely, for $(\mathfrak{g}, k)$ as below, the simple affine vertex algebras $L_k(\mathfrak{g})$ is quasi-lisse:

- if $\mathfrak{g}$ of type G_2 and $k = -2$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\mathrm{min}}}$,
- if $\mathfrak{g}$ of type D_r, $r \geqslant 5$ and $k = -2, -1$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\mathrm{min}}}$,
- if $\mathfrak{g}$ of type D_r, with r an even integer $\geqslant 4$, and $k = 2 - r$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{(2^{r-2},1^4)}}$,
- if $\mathfrak{g}$ of type B_r, $r \geqslant 3$ and $k = -2$, then $X_{L_k(\mathfrak{g})} = \overline{\mathbb{O}_{\mathrm{short}}}$, where $\mathbb{O}_{\mathrm{short}}$ is the nilpotent orbit associated with the $\mathfrak{sl}_2$-triple $(e_{\theta_s}, h_{\theta_s}, f_{\theta_s})$, with θ_s the highest short root ε_1 (note that $h_{\theta_s} = 2\varpi_1^\vee$). Notice that $\mathbb{O}_{\mathrm{short}} = \mathbb{O}_{(3,1^{2r-2})}$.
- Finally, for any $\mathfrak{g}$, if $k = -h^\vee$ is critical then $X_{L_k(\mathfrak{g})} = \mathscr{N}$.

Except for $\mathfrak{g} = \mathfrak{sl}_2$, the classification problem of quasi-lisse affine vertex algebras is wide open.

8.4 Joseph ideal and proof of Theorem 8.3

We refer the reader to Section D.4 for standard facts on primitive ideals and their associated varieties.

If $\mathfrak{g}$ is not of type A, it is known [107, 86] that there exists a unique *completely prime ideal*, that is, the corresponding graded ideal is prime, in $U(\mathfrak{g})$ whose associated variety is the minimal nilpotent orbit $\overline{\mathbb{O}_{min}}$, which is the unique nilpotent orbit of $\mathfrak{g}$ of minimal dimension $2h^\vee - 2$, with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$ (see Section D.1). See [168] for a more recent review on this topic.

Definition 8.1 If $\mathfrak{g}$ is not of type A, the unique completely prime ideal whose associated variety is $\overline{\mathbb{O}_{min}}$ is denoted by $\mathcal{J}_0$, and is referred to as the *Joseph ideal* of $U(\mathfrak{g})$.

For $\mathfrak{g}$ of type A, the completely prime primitive ideals I of $U(\mathfrak{g})$ with $\mathscr{V}(I) = \overline{\mathbb{O}_{min}}$ form a single family parametrized by the elements of $\mathbb{C}$ ([107, 168]).

8.4.1 Infinitesimal character

In [107], Joseph has also computed the *infinitesimal character of* $\mathcal{J}_0$, that is, the algebra homomorphism $\mathscr{Z}(\mathfrak{g}) \to \mathbb{C}$ through which the centre $\mathscr{Z}(\mathfrak{g})$ acts on the primitive quotient $U(\mathfrak{g})/\mathcal{J}_0$. In fact, Joseph has described the set of $\lambda \in \mathfrak{h}^*$ such that such that $\mathcal{J}_0 = \mathrm{Ann}_{U(\mathfrak{g})}(L_\mathfrak{g}(\lambda))$ (see [107, Tab. p.15] or [22, Tab. 1]).

Do we need of this recall?? Let us briefly recall how to deduce the *infinitesimal character* of $L_\mathfrak{g}(\lambda)$ (or of $\mathrm{Ann}_{U(\mathfrak{g})}(L_\mathfrak{g}(\lambda))$) from the knowledge of $\lambda \in \mathfrak{h}^*$. Identify $\mathrm{Spec}\,\mathscr{Z}(\mathfrak{g})$ with the set of all homomorphisms $\mathscr{Z}(\mathfrak{g}) \to \mathbb{C}$. Such morphisms are called *infinitesimal characters*. Consider the projection map from $U(\mathfrak{g})$ to $U(\mathfrak{h}) = S(\mathfrak{h})$ with respect to the decomposition

$$U(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+).$$

It is not a morphism of algebras in general, but its restriction to $U(\mathfrak{g})^\mathfrak{h} = \{u \in U(\mathfrak{g}) \colon (\mathrm{ad}\, x)u = 0 \text{ for all } x \in \mathfrak{h}\}$ is. In particular, we get a morphism

$$p \colon \mathscr{Z}(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]$$

since $S(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}^*]$, usually called the *Harish-Chandra morphism*. Its comorphism gives a map

$$\chi \colon \mathfrak{h}^* \to \mathrm{Spec}(\mathscr{Z}(\mathfrak{g})), \quad \lambda \mapsto \chi_\lambda,$$

where $\chi_\lambda(z) = p(z)(\lambda + \rho)$ for $z \in \mathscr{Z}(\mathfrak{g})$ with ρ the half-sum of positive roots. An important consequence of the Harish-Chandra Theorem is that the map χ induces a bijection

$$\mathfrak{h}^*/W \xrightarrow{\simeq} \mathrm{Spec}(\mathscr{Z}(\mathfrak{g})).$$

Here the Weyl group W acts on $\mathfrak{h}^*$ with respect to the twisted action of W:

$$w \circ \lambda = w.(\lambda + \rho) - \rho, \qquad w \in W,\ \lambda \in \mathfrak{h}^*,$$

where $\cdot$ stands for the usual action of W on $\mathfrak{h}^*$. In particular, $\chi_\lambda = \chi_\mu$ if and only if λ and μ are in the same W-orbit with respect to the twisted action of W, and the infinitesimal character associated with the irreducible representation $L_\mathfrak{g}(\lambda)$ is just χ_λ.

8.4.2 Gan and Savin's description of the Joseph ideal

Outside the type A the nilpotent orbit $\mathbb{O}_{min}$ is *rigid*[1], hence forms a single *sheet*[2] in $\mathfrak{g}^* \cong \mathfrak{g}$. So, $\mathcal{J}_0$ cannot be obtained by parabolic induction from a primitive ideal of a proper Levi subalgebra of $\mathfrak{g}$. Different realizations of $\mathcal{J}_0$ can be found in the literature for various types of $\mathfrak{g}$. Joseph's original proof of the uniqueness of $\mathcal{J}_0$ was incomplete. This led Gan and Savin [86] to give another description of the Joseph ideal $\mathcal{J}_0$. Their argument relies on some invariant theory and earlier results of Garfinkle.

Let us outline their description.

Suppose that $\mathfrak{g}$ is not of type A. According to Kostant, $\mathcal{J}_0$ is generated by the $\mathfrak{g}$-submodule $L_{\mathfrak{g}}(0) \oplus W$ in $S^2(\mathfrak{g})$, where W is such that, as $\mathfrak{g}$-modules,

$$S^2(\mathfrak{g}) = L_{\mathfrak{g}}(2\theta) \oplus L_{\mathfrak{g}}(0) \oplus W.$$

Note that the above decomposition of $S^2(\mathfrak{g})$ still holds in type A ([87, Chap. IV, Prop. 2]). Also, observe that $L_{\mathfrak{g}}(0) = \mathbb{C}\Omega$ where $\Omega = \sum_i x_i x^i$ is the Casimir element in $S(\mathfrak{g})$, with $\{x_i\}_i$ is a basis of $\mathfrak{g}$, and $\{x^i\}_i$ its dual basis with respect to $(\ |\ )$.

Lemma 8.1 *Suppose that $\mathfrak{g}$ is not of type A. The ideal J_W in $S(\mathfrak{g})$ generated by W contains Ω^2, and hence, $\sqrt{J_W} = J_0$, where J_0 is the prime ideal of $S(\mathfrak{g})$ corresponding to the minimal nilpotent orbit closure $\overline{\mathbb{O}}_{\min}$.*

Proof By the proof of [86, Th. 3.1], the ideal $\mathcal{J}_W$ of $U(\mathfrak{g})$ generated by W contains $\mathfrak{g} \cdot \Omega$, and so the assertion follows. □

The structure of W was determined by Garfinkle [87]. Consider the $\mathfrak{sl}_2$-triple $(e_\theta, h_\theta, f_\theta)$ of $\mathfrak{g}$ where $f_\theta = e_{-\theta}$ is a θ-root vector (θ denotes the highest positive root) so that it lies in $\mathbb{O}_{min}$. Set

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \colon [h_\theta, x] = 2jx\}.$$

Then (cf. Remark D.1)

$$\begin{aligned} &\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \\ &\quad \mathfrak{g}_{-1} = \mathbb{C} f_\theta, \quad \mathfrak{g}_1 = \mathbb{C} e_\theta, \quad \mathfrak{g}_0 = \mathbb{C} h_\theta \oplus \mathfrak{g}^\natural, \quad \mathfrak{g}^\natural = \{x \in \mathfrak{g}_0 \colon (h_\theta | x) = 0\}. \end{aligned}$$

The subalgebra $\mathfrak{g}^\natural$ is a reductive subalgebra of $\mathfrak{g}$ whose simple roots are the simple roots of $\mathfrak{g}$ perpendicular to θ. Write

$$[\mathfrak{g}^\natural, \mathfrak{g}^\natural] = \bigoplus_{i \geqslant 1} \mathfrak{g}_i$$

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[1] Rigid nilpotent orbits are those nilpotent orbits which cannot be properly induced from another nilpotent orbit in the sense of Lusztig-Spaltenstein [42, 40].

[2] The sheets of $\mathfrak{g}^*$ are by definition the irreducible components of the locally closed subsets $\mathfrak{g}^{(m)} = \{\xi \in \mathfrak{g}^* \colon \dim G.\xi = 2m\}$, $m \in \mathbb{Z}_{\geqslant 0}$. A nilpotent orbit is a sheet if and only if it is rigid.

as a direct sum of simple summands, and let θ_i be the highest root of $\mathfrak{g}_i$.

- If $\mathfrak{g}$ is neither of type A_r nor C_r,

$$W = \bigoplus_{i \geqslant 1} L_{\mathfrak{g}}(\theta + \theta_i). \tag{8.1}$$

- If $\mathfrak{g}$ is of type C_r, then $\mathfrak{g}^\natural$ is simple of type C_{r-1}, so that there is a unique θ_1, and we have

$$W = L_{\mathfrak{g}}(\theta + \theta_1) \oplus L_{\mathfrak{g}}\left((\theta + \theta_1)/2\right).$$

By [87, 86], $\mathcal{J}_0$ is generated by W and $\Omega - c_0$, where W is identified with a $\mathfrak{g}$-submodule of $U(\mathfrak{g})$ by the $\mathfrak{g}$-module isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$ given by the symmetrization map, and c_0 is the eigenvalue of Ω for the infinitesimal character that Joseph obtained. We have

$$\operatorname{gr} \mathcal{J}_0 = J_0 = \sqrt{J_W}$$

and this shows that $\mathcal{J}_0$ is indeed completely prime.

This should arrive before Lemma 8.1.

Let $\mathcal{J}_W$ be the two-sided ideal of $U(\mathfrak{g})$ generated by W.

Proposition 8.3 ([22]) *We have the algebra isomorphism*

$$U(\mathfrak{g})/\mathcal{J}_W \cong \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

Proof By the proof of [86, Th. 3.1], $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$. Hence it contains $(\Omega - c_0)\Omega$. Since $c_0 \neq 0$, we have the isomorphism of algebras

$$U(\mathfrak{g})/\mathcal{J}_W \xrightarrow{\simeq} U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega \rangle \times U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega - c_0 \rangle.$$

As we have explained above, $\langle \mathcal{J}_W, \Omega - c_0 \rangle = \mathcal{J}_0$. Also, since $\mathcal{J}_W$ contains $(\Omega - c_0)\mathfrak{g}$, $\langle \mathcal{J}_W, \Omega \rangle$ contains $\mathfrak{g}$. Therefore $U(\mathfrak{g})/\langle \mathcal{J}_W, \Omega \rangle = \mathbb{C}$ as required. □

8.4.3 Proof sketch of Theorem 8.3

We now outline the proof of Theorem 8.3 following [22]. The proof is closely related to the Joseph ideal and its description by Gan and Savin. The key point is that this description was successful in constructing singular vectors of $V^k(\mathfrak{g})$ with $\mathfrak{g}, k$ as in Theorem 8.3.

Recall that the vertex algebra $V^k(\mathfrak{g})$ is naturally graded:

$$V^k(\mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_d, \quad V^k(\mathfrak{g})_d = \{v \in V^k(\mathfrak{g}) \colon Dv = -dv\}.$$

The following lemma, for $d = 2$, will be useful.

Lemma 8.2 *Let $d \in \mathbb{Z}_{>0}$. We have a $\mathfrak{g}$-module embedding*

$$\sigma_d \colon S^d(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g})_d, \quad x_1 \dots x_d \mapsto \frac{1}{d!} \sum_{\tau \in \mathfrak{S}_d} (x_{\tau(1)} t^{-1}) \dots (x_{\tau(d)} t^{-1}) |0\rangle,$$

where $S(\mathfrak{g}) = \bigoplus_d S^d(\mathfrak{g})$ is the usual grading of $S^d(\mathfrak{g})$.

Let v be a singular vector for $\mathfrak{g}$ in $S^d(\mathfrak{g})$. Then $\sigma_d(v)$ is a singular vector of $V^k(\mathfrak{g})$ if and only if $(f_\theta t)\sigma_d(v) = 0$.

For $\mathfrak{g}$ of type A_1, A_2, G_2, F_4, the number $n - h^\vee/6 - 1$ is admissible for $n \in \mathbb{Z}_{\geqslant 0}$, and Theorem 8.3 is a special case of Theorem 8.2. So there is no loss of generality in assuming that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8.

Recall from §8.4.2 that the Joseph ideal $\mathcal{J}_0$ is generated by the $\mathfrak{g}$-submodule $L_\mathfrak{g}(0) \oplus W$ in $S^2(\mathfrak{g})$, where W is such that, as $\mathfrak{g}$-modules,

$$S^2(\mathfrak{g}) = L_\mathfrak{g}(2\theta) \oplus L_\mathfrak{g}(0) \oplus W.$$

Let $W = \bigoplus_i W_i$ be the decomposition of W into irreducible submodules, and let w_i be a highest weight vector of W_i. Recall also that by (8.1), we have $W_i = L_\mathfrak{g}(\theta + \theta_i)$. Note that for $\mathfrak{g}$ of type E_6, E_7, E_8, $W = W_1$ is simple. Moreover, according to [87, Chap. IV, Prop. 11] if $\mathfrak{g}$ is not of type E_8, we have[3]:

$$w_i = e_\theta e_{\theta_i} - \sum_{j=1}^{\frac{h^\vee}{6}+1} e_{\beta_j+\theta_i} e_{\delta_j+\theta_i},$$

where (β_j, δ_j) runs through the pairs of positive roots such that

$$\beta_j + \delta_j = \theta - \theta_i.$$

The number of such pairs turns out to be equal to $h^\vee/6+1$. Choose a Chevalley basis $\{h_i\}_i \cup \{e_\alpha,\ f_\alpha\}_\alpha$ of $\mathfrak{g}$ so that the conditions of [87, Chap. IV, Def. 6] are fulfilled, that is

$$\forall\, j, \quad [e_{\delta_j}, [e_{\beta_j}, e_{\theta_1}]] = e_\theta, \quad [e_{\beta_j}, e_{\theta_1}] = e_{\beta_j+\theta_1}, \quad [e_{\delta_j}, e_{\theta_1}] = e_{\delta_j+\theta_1}.$$

Exercise 8.3 Assume that $\mathfrak{g}$ is of type D_4, E_6 or E_7, and let $n \in \mathbb{Z}_{\geqslant 0}$. Show that for each i, $\sigma_2(w_i)^{n+1}$ is a singular vector of $V^k(\mathfrak{g})$ if and only if

$$k = n - \frac{h^\vee}{6} - 1.$$

(The statement holds for $\mathfrak{g}$ of type E_8 but one needs to consider a slightly different description of w_1.)

We are now in a position to prove Theorem 8.3.

---

[3] The construction is slightly different if $\mathfrak{g}$ is of type E_8 due to the fact that E_8 is not of *depth one* (cf. [87, Chap. IV, Def. 1]), and that $(\theta - \theta_1)/2$ is not a root: see [22, proof of Th. .4.2].

Proof (Sketch of proof of Theorem 8.3) Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8 and that

$$k = n - \frac{h^\vee}{6} - 1 \quad \text{with} \quad n \in \mathbb{Z}_{\geqslant 0}.$$

Let N_k be the submodule of $V^k(\mathfrak{g})$ generated by $\sigma_2(w_i)^{n+1}$ for all i, and set

$$\tilde{L}_k(\mathfrak{g}) := V^k(\mathfrak{g})/N_k.$$

The exact sequence $0 \to N_k \to V^k(\mathfrak{g}) \to \tilde{L}_k(\mathfrak{g}) \to 0$ induces an exact sequence

$$N_k/\mathfrak{g}[t^{-1}]t^{-2}N_k \to V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \to \tilde{L}_k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}\tilde{L}_k(\mathfrak{g}) \to 0.$$

Under the isomorphism $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g}) \cong S(\mathfrak{g})$, the image of $N_k/\mathfrak{g}[t^{-1}]t^{-2}N_k$ in $V^k(\mathfrak{g})/\mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$ is identified with the ideal J_k of $S(\mathfrak{g})$ generated by w_i for all i. Hence $J_k \subset J_W \subset \sqrt{J_k}$. Therefore by Lemma 8.1,

$$\sqrt{J_k} = \sqrt{J_W} = J_0,$$

where J_0 is the defining ideal of $\overline{\mathbb{O}_{min}}$. Hence $X_{\tilde{L}_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}}$ by Lemma 8.1.

Next, since $L_k(\mathfrak{g})$ is a quotient of $\tilde{L}_k(\mathfrak{g})$, we get that

$$X_{L_k(\mathfrak{g})} \subset X_{\tilde{L}_k(\mathfrak{g})} = \overline{\mathbb{O}_{\min}} = \mathbb{O}_{\min} \cup \{0\}.$$

Therefore $X_{L_k(\mathfrak{g})}$ is either $\{0\}$ and $\mathbb{O}_{\min}$. The theorem follows since $X_{L_k(\mathfrak{g})} = \{0\}$ if and only if $k \in \mathbb{Z}_{\geqslant 0}$ (cf. Theorem 7.1). □

8.4.4 Consequences of Theorem 8.3 and its proof

First, in the previous notation, we formulate a conjecture.

Conjecture 8.1 *Assume that $\mathfrak{g}$ is of type D_4, E_6, E_7, or E_8 and that $k = n - h^\vee/6 - 1$. Then $\tilde{L}_k(\mathfrak{g}) = L_k(\mathfrak{g})$, that is, $\tilde{L}_k(\mathfrak{g})$ is simple, if $k < 0$.*

Conjecture 8.1 was proven in [22, Proof of Theorem 3.1] for $n = 0$. Note that if $k \geqslant 0$, $\tilde{L}_k(\mathfrak{g})$ is obviously not simple as, if so, the maximal submodule of $V^k(\mathfrak{g})$ is generated by $(e_\theta t^{-1})^{k+1}|0\rangle$.

As a consequence of Lemma 8.1, Lemma 8.2 and the proof of Conjecture 8.1 for $n = 0$, we obtain the following result. Recall that $\mathcal{J}_W$ is the two-sided ideal of $U(\mathfrak{g})$ generated by W.

Theorem 8.4 *Assume that $\mathfrak{g}$ belongs to the Deligne exceptional series outside the type A and that $k = -\dfrac{h^\vee}{6} - 1$. Then $L_k(\mathfrak{g})$ is a chiralization of $U(\mathfrak{g})/\mathcal{J}_W$, that is,*

$$\mathrm{Zhu}(L_k(\mathfrak{g})) \cong U(\mathfrak{g})/\mathcal{J}_W = \mathbb{C} \times U(\mathfrak{g})/\mathcal{J}_0.$$

In particular since $\mathcal{J}_0$ is maximal, the irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ is a $L_k(\mathfrak{g})$-module if and only if

$$\bar{\lambda} = 0 \quad \textit{or} \quad \operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\bar{\lambda}) = \mathcal{J}_0,$$

and such λ are described by Joseph (see [107, Tab. p.15] or [22, Tab. 1]).

Chapter 9
Properties of quasi-lisse vertex algebras and irreducibility conjecture

In this chapter, it is assumed that V is a strongly generated $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra such that $V_0 \cong \mathbb{C}|0\rangle$. Recall that X_V is called quasi-lisse is X_V has only finitely many symplectic leaves. The quasi-lisse obviously condition generalizes the lisse condition. We have already noticed that lisse vertex algebras are very nice (see e.g., Lemma 4.9 and Theorem 7.3). This chapter explores interesting properties of the larger class of quasi-lisse vertex algebras.

It is known that Poisson varieties with only finitely many symplectic leaves have special properties. For example, Brown and Gordon have proved [45] that the finiteness of the symplectic leaves in a Poisson variety X implies that the symplectic leaf $\mathscr{L}_x$ at $x \in X$ coincides with the regular locus of the zero variety of the maximal Poisson ideal contained in the maximal ideal $\mathfrak{m}_x$ corresponding to x (see Section 9.2 for more details about this). Thus, each symplectic leaf $\mathscr{L}_x$ is a smooth connected locally-closed algebraic subvariety in X. In particular, every irreducible component of X is the closure of a symplectic leaf [90, Cor. 3.3]. On the other hand, it has been established by Etingof and Schedler [68] that if R is a finitely generated Poisson algebra such that $X = \mathrm{Specm}(R)$ has finitely many symplectic leaves, then

$$\dim R/\{R, R\} < \infty.$$

As one can expect, these important facts play an important role in the study of quasi-lisse vertex algebras. That is what we shall see in this chapter.

Section 9.1 is about the modular invariance properties of quasi-lisse vertex algebras (Theorem 9.1). In Section 9.2, we introduce the notion of *chiral symplectic leaf* and exploit them to show that any quasi-lisse vertex algebra V is a quantization of the reduced arc space of its associated variety, in the sense that its reduced singular support $\mathrm{Specm}(\mathrm{gr}\, V)$ coincides with $\mathscr{J}_\infty(X_V)$ as topological spaces (Theorem 9.2). Finally, Section 9.3 concerns the irreducibility conjecture for the associated variety of quasi-lisse vertex algebras (Conjecture 9.1) and its connection with the *Higgs branch conjecture* in four-dimensional $\mathcal{N} = 2$ super-conformal theories (Conjecture**??**).

9.1 Modular invariance property of quasi-lisse vertex algebras

A V-module is called *ordinary* if it is a positive energy representation and each homogeneous space is finite-dimensional, so that the normalized character

$$\chi_M(\tau) = \mathrm{Tr}_M(q^{L_0 - \frac{c}{24}})$$

is well-defined.

Theorem 9.1 ([15]) *A quasi-lisse conformal $\mathbb{Z}_{\geqslant 0}$-graded vertex algebra has only finitely many simple* ordinary representations. *Moreover, the normalized character of any ordinary module has a modular invariance property, in the sense that it satisfies a modular linear differential equation.*

Using a modular linear differential equation, the explicit character formulas of the simple quasi-lisse affine vertex algebras associated with the Deligne exceptional series at level $k = -h^\vee/6 - 1$ (cf. Theorem 8.3) are obtained in [15].

It is known that the admissible representations $L_k(\mathfrak{g})$ have only finitely many simple objects in the category $\mathscr{O}$ and that their normalized characters satisfy a modular invariance property. Note that the above result means different things, and was new even for an admissible affine vertex algebras.

9.2 Chiral symplectic leaves and applications

Let $X = \operatorname{Spec} R$ be a reduced Poisson scheme. For I an ideal of R, we denote by $\mathscr{P}_R(I)$ the biggest Poisson ideal of R contained in I. The *symplectic core* $\mathscr{C}_R(x)$ of a point $x \in X$ is the equivalence class of x for $\sim$, with

$$x \sim y \iff \mathscr{P}_R(\mathfrak{m}_x) = \mathscr{P}_R(\mathfrak{m}_y).$$

Here, $\mathfrak{m}_x$ stands for the maximal ideal of R corresponding to x. The notion of symplectic cores, introduced in [45], are expected to be the finest possible *algebraic* stratification in which the Hamiltonian vector fields are tangent. Brown and Gordon showed that the symplectic cores coincide with the symplectic leaves, if there is only finitely many numbers of symplectic leaves.

9.2.1 Chiral symplectic leaves

It is natural to try to adapt this notion to the context of vertex Poisson algebras. Assume for awhile that V is a vertex Poisson algebra. Let I be an ideal of V in the associative sense.

Definition 9.1 We say that I is a *chiral Poisson ideal* of V if $a_{(n)}I \subset I$ for all $a \in V$, $n \in \mathbb{Z}_{\geqslant 0}$.

Thus a vertex Poisson ideal of V is a chiral Poisson ideal that is stable under the action of T. The quotient space V/I inherits a vertex Poisson algebra structure from V if I is a vertex Poisson ideal. Note that if I is a vertex (resp. chiral) Poisson ideal of V, then so is its radical $\sqrt{I}$ ([58, §3.3.2]).

Denote by $\mathscr{P}_V(I)$ the biggest chiral Poisson ideal of V contained in I. It exists since the sum of two chiral Poisson ideals is chiral Poisson. Set

$$\mathcal{L} := \mathrm{Specm}(V),$$

and define a relation $\sim$ on $\mathcal{L}$ by

$$x \sim y \iff \mathscr{P}_V(\mathfrak{m}_x) = \mathscr{P}_V(\mathfrak{m}_y),$$

where $\mathfrak{m}_x$ denotes the maximal ideal corresponding to $x \in \mathcal{L}$. Clearly $\sim$ is an equivalence relation. We will write $\mathscr{C}_{\mathcal{L}}(x)$ for the equivalence class in $\mathcal{L}$ of x, so that

$$\mathcal{L} = \bigsqcup_x \mathscr{C}_{\mathcal{L}}(x).$$

We call the set $\mathscr{C}_{\mathcal{L}}(x)$ the *chiral symplectic leaf*[1] of x in $\mathcal{L}$. Chiral symplectic leaves are expected to be the finest possible algebraic stratification in which the *chiral* Hamiltonian vector fields are tangent.

Let us return to the case where V is arbitrary (not necessarily a vertex Poison algebra). Recall that $SS(V)$ stands for the singular support of V, that is, the spectrum of $\mathrm{gr}^F V$ (cf. Definition **??**).

Theorem 9.2 ([25]) *Assume that V is a quasi-lisse vertex algebra. Then $SS(V) \cong J_\infty X_V$ as topological spaces, that is,*

$$SS(V)_{\mathrm{red}} \cong (\mathscr{J}_\infty X_V)_{\mathrm{red}}.$$

Moreover, the reduced singular support $SS(V)_{\mathrm{red}}$ have finitely many irreducible components, and each of them is the closure of some chiral symplectic leaf.

Let R_V be the C_2-algebra of V (Section **??**), and denote by $\tilde{X}_V = \mathrm{Spec}\, R_V$ the associated scheme of V (cf. Definition 4.7).

Corollary 9.1 ([25, Cor. 9.3]) *Suppose that $\tilde{X}_V$ is smooth, reduced and symplectic. Then $\mathrm{gr}^F V$ is simple as a vertex Poisson algebra, and hence, V is simple.*

Proof If X_V is a smooth symplectic variety then $\mathscr{J}_\infty X_V$ consists of a single chiral symplectic leaf. So $\mathscr{J}_\infty X_V = \mathscr{C}_{\mathscr{J}_\infty X_V}(x)$ for any $x \in \mathscr{J}_\infty X_V$. It follows that there is no nonzero proper chiral Poisson subscheme in $\mathscr{J}_\infty X_V$. From Theorem 9.2, we

[1] In [25] we call this set the *chiral symplectic core* of x but in fact we think that that term *leaf* is more appealing than the term *core*.

conclude that there is no nonzero proper chiral Poisson subscheme in Spec $\mathrm{gr}^F V$, too. Therefore $\mathrm{gr}^F V$ is simple as a vertex Poisson algebra. This shows that V is simple, because any vertex ideal $I \subset V$ yields a vertex Poisson (and so chiral Poisson) ideal $\mathrm{gr}^F I$ in $\mathrm{gr}^F V$. □

We now derive some applications of the above results.

9.2.2 Chiral differential operators

So far, all our examples of quasi-lisse, non lisse, vertex algebras are all affine vertex algebras. Here is a different kind of example.

Given a smooth affine variety X, the global section of the *chiral differential operators* $\mathcal{D}_X^{ch}$ ([151, 93, 36]) is quasi-lisse because its associated scheme is canonically isomorphic to the cotangent bundle T^*X. As a consequence of Corollary 9.1 the vertex algebra $\mathcal{D}_X^{ch}$ is simple, since the associated scheme is smooth, reduced ans symplectic. In particular, the global section of the chiral differential operators $\mathcal{D}_{G,k}^{ch}$ on the group G ([92, 29]) is simple at any level k. This example is important since $\mathcal{D}_{G,-h^\vee}^{ch}$ appears in the 4d/2d duality for the class $\mathcal{S}$ theory (cf. [14] or, here, Part. **??**).

9.2.3 Vertex Poisson center of the arc spaces of Slodowy slices

Assume that $\mathfrak{g}$ is a complex simple Lie algebra with adjoint group G. Identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the bilinear form $(\ |\ )$ as before. Denote by $\mathscr{S}_f$ the *Slodowy slice* $f + \mathfrak{g}^e$ associated with an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. The affine variety $\mathscr{S}_f$ has a Poisson structure obtained from that of $\mathfrak{g}^*$ by Hamiltonian reduction (see Section 10.2). Consider the adjoint quotient morphism

$$\psi_f : \mathscr{S}_f \to \mathfrak{g}^*//G.$$

It is known [166] that any fiber $\psi_f^{-1}(\xi)$ of this morphism is the closure of a symplectic leave, which is irreducible and reduced.

Theorem 9.3 ([25, Th. 11.1])

(i) *Any fiber of the induced vertex Poisson algebra morphism*

$$\mathscr{J}_\infty\psi_f : \mathscr{J}_\infty\mathscr{S}_f \to \mathscr{J}_\infty(\mathfrak{g}^*//G)$$

is an irreducible and reduced chiral Poisson subscheme of $\mathscr{J}_\infty\mathscr{S}_f$.

(ii) *The comorphism* $(\mathscr{J}_\infty\psi_f)^*$ *induces an isomorphism of vertex Poisson algebras between* $\mathbb{C}[\mathscr{J}_\infty\mathfrak{g}^*]^{\mathscr{J}_\infty G}$ *and the vertex Poisson center of* $\mathbb{C}[\mathscr{J}_\infty\mathscr{S}_f]$. *Moreover,* $\mathbb{C}[\mathscr{J}_\infty\mathscr{S}_f]$ *is free over its vertex Poisson center.*

Theorem 9.3 is proved similarly to Theorem 10.6, thanks to our results on chiral symplectic leaves.

Shall we give a proof?

As a consequence of Theorem 9.3, we obtain[2] that the center of the *affine W-algebra* $\mathscr{W}^{-h^\vee}(\mathfrak{g}, f)$ (see Part IV) associated with $(\mathfrak{g}, f)$ at the critical level is identified with the *Feigin-Frenkel center* $\mathfrak{z}(\hat{\mathfrak{g}})$ (see Exercice 3.3), that is, the center of the affine vertex algebra $V^{-h^\vee}(\mathfrak{g})$ at the critical level.

9.3 Irreducibility conjecture

Taking all these examples of simple quasi-lisse vertex algebras into consideration, and other ones, particularly, the (generalized) Drinfeld-Sokolov reduction of these examples of simple quasi-lisse affine vertex algebra provided that it is nonzero (cf. Part IV), we formulate a conjecture.

Conjecture 9.1 ([23, Conj. 1]) *Let $V = \oplus_{d \geqslant 0} V_d$ be a simple, finitely strongly generated, positively graded conformal vertex operator algebra such that $V_0 \cong \mathbb{C}|0\rangle$. Assume that X_V has finitely many symplectic leaves, that is, V is quasi-lisse. Then X_V is irreducible. In particular, if $X_{L_k(\mathfrak{g})} \subset \mathcal{N}$, then $X_{L_k(\mathfrak{g})}$ is the closure of some nilpotent orbit in $\mathfrak{g}$.*

The conjecture is a natural affine analog of the irreducibility theorem (cf. Theorem D.3) for the associated variety of primitive ideals of $U(\mathfrak{g})$, which has been generalized to a larger class of Noetherian algebras by Ginzburg [89]:

Theorem 9.4 ([89]) *Let A be a filtered unital $\mathbb{C}$-algebra. Assume furthermore that* $\operatorname{gr} A \cong \mathbb{C}[X]$ *is the coordinate ring of a reduced irreducible affine algebraic variety X, and assume that the Poisson variety* $\operatorname{Spec}(\operatorname{gr} A)$ *has only finitely many symplectic leaves. Then for any primitive ideal $I \subset A$, the zero locus $\mathcal{V}(I)$ of* $\operatorname{gr} I$ *in X is the closure of a single symplectic leaf. In particular, it is irreducible.*

Ginzburg's proof of Theorem 9.4 is an adaptation of a more direct proof of Theorem D.3 discovered subsequently by Vogan [180], combined with the results by Brown and Gordon [45] on symplectic cores. We hope that Theorem 9.2 can serve as a first step in proving Conjecture 9.1. The main difficulty is that the algebras considered in Conjecture 9.1 are not Noetherian.

The reader is referred to Remark 13.1 for more about this conjecture in the context of W-algebras.

[2] This fact was claimed in [7] but the proof was incomplete.

Appendices

Appendix A
Simple Lie algebras and affine Kac-Moody algebras

The ground field is the field $\mathbb{C}$ of complex numbers. Recall that a Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear form $[\ ,\ ]\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$ satisfying the following conditions:

- (skew-symmetry) $[x, y] = [y, x]$, for all $x, y \in \mathfrak{g}$,
- (Jacobi identity) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for all $x, y, z \in \mathfrak{g}$.

It is assumed that the reader is familiar with the basics on Lie algebras. We review in this appendix some of the standard facts on the structure of semisimple Lie algebras, and corresponding affine Kac-Moody algebras. This appendix is also used to fix the main notations relative to these structures.

Recall that the *enveloping algebra* of a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is the quotient

$$U(\mathfrak{g}) := T(\mathfrak{g})/\mathscr{J}(\mathfrak{g}),$$

where

$$T(\mathfrak{g}) := \bigoplus_{i=0}^{\infty} T^i(\mathfrak{g}), \qquad T^i(\mathfrak{g}) = \underbrace{\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}}_{i \text{ times}},$$

is the tensor algebra of $\mathfrak{g}$ and $\mathscr{J}(\mathfrak{g})$ is the two-sided ideal of $T\mathfrak{g}$ generated by elements $x \otimes y - y \otimes -[x, y]$, for $x, y \in \mathfrak{g}$. It is a unital associative $\mathbb{C}$-algebra. The enveloping algebra $U(\mathfrak{g})$ is naturally filtered by the *PBW filtration* $U_\bullet(\mathfrak{g})$, where $U_i(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ spanned by the products of at most i elements of $\mathfrak{g}$, for $i \geqslant 0$, and $U_0(\mathfrak{g}) = \mathbb{C}1$. By the PBW theorem, we have

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}), \tag{A.1}$$

as graded commutative algebras, where $S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$ is the *symmetric algebra* of $\mathfrak{g}$, that is, the quotient $T(\mathfrak{g})/J(\mathfrak{g})$, where $J(\mathfrak{g})$ is the two-sided ideal of $T(\mathfrak{g})$ generated by elements $x \otimes y - y \otimes x$, for $x, y \in \mathfrak{g}$.

Our main references for this chapter are [50, 136, 137, 176, 158, 110, 114]. See also [99] for a survey.

A.1 Quick review on semisimple Lie algebras

Let $\mathfrak{g}$ be a complex finite dimensional *semisimple* Lie algebra, i.e., $\{0\}$ is the only abelian ideal of $\mathfrak{g}$. Its *adjoint group* G is the smallest algebraic subgroup of $GL(\mathfrak{g})$ whose Lie algebra contains $\operatorname{ad}\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $G = \operatorname{Aut}_e(\mathfrak{g})$, where $\operatorname{Aut}_e(\mathfrak{g})$ is the subgroup of $GL(\mathfrak{g})$ generated by the elements $\exp(\operatorname{ad} x)$ with x a *nilpotent element* of $\mathfrak{g}$ (i.e., $(\operatorname{ad} x)^n = 0$ for n large enough). Hence

$$\operatorname{Lie}(G) = \operatorname{ad}\mathfrak{g} \cong \mathfrak{g}$$

since the adjoint representation $\operatorname{ad}\colon \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$, $x \mapsto (\operatorname{ad} x)(y) = [x, y]$ is faithful, $\mathfrak{g}$ being semisimple.

A.1.1 Main notations

For $\mathfrak{a}$ a subalgebra of $\mathfrak{g}$ and $x \in \mathfrak{g}$, we shall denote by $\mathfrak{a}^x$ the *centralizer* of x in $\mathfrak{a}$, that is,

$$\mathfrak{a}^x = \{y \in \mathfrak{a}\colon\ [x, y] = 0\},$$

which is also the intersection of $\mathfrak{a}$ with the kernel of the map

$$\operatorname{ad} x\colon \mathfrak{g} \longrightarrow \mathfrak{g}, \quad y \longmapsto [x, y].$$

Let $\kappa_{\mathfrak{g}}$ be the *Killing form* of $\mathfrak{g}$,

$$\kappa_{\mathfrak{g}}\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto \operatorname{tr}(\operatorname{ad} x \ \operatorname{ad} y).$$

It is a nondegenerate symmetric bilinear form of $\mathfrak{g}$ which is G-invariant, that is,

$$\kappa_{\mathfrak{g}}(g.x, g.y) = \kappa_{\mathfrak{g}}(x, y) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } g \in G,$$

or else,

$$\kappa_{\mathfrak{g}}([x, y], z) = \kappa_{\mathfrak{g}}(x, [y, z]) \quad \text{for all } x, y, z \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is semisimple, any other such bilinear form is a nonzero multiple of the Killing form.

Example A.1 Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n$, for $n \geqslant 2$, which is the set of traceless complex n-size square matrices, with bracket $[A, B] = AB - BA$. The Lie algebra $\mathfrak{sl}_n$ is known to be *simple* , that is, $\{0\}$ and $\mathfrak{g}$ are the only ideals of $\mathfrak{g}$ and $\dim\mathfrak{g} \geqslant 3$. Its Killing form is given by

$$(A, B) \mapsto 2n \operatorname{tr}(AB).$$

The bilinear form $(A, B) \mapsto \operatorname{tr}(AB)$ is more naturally used.

In fact, mathematicians usually consider a certain normalization $(\ |\ )$ of the Killing form which will coincide with this bilinear form for $\mathfrak{sl}_n$ (see Section A.2).

A.1.2 Cartan matrix and Chevalley generators

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha := \{y \in \mathfrak{g} \colon [x, y] = \alpha(x) y \text{ for all } x \in \mathfrak{h}\},$$

be the corresponding root decomposition of $(\mathfrak{g}, \mathfrak{h})$, where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a basis of Δ, with r the *rank* of $\mathfrak{g}$, and let $\alpha_1^\vee, \dots, \alpha_r^\vee$ be the coroots of $\alpha_1, \dots, \alpha_r$, respectively. The element $\alpha_i^\vee$, for $i = 1, \dots, r$, viewed as an element of $(\mathfrak{h}^*)^* \cong \mathfrak{h}$, will be often denoted by h_i.

Recall that the *Cartan matrix* of Δ is the matrix $C = (C_{i,j})_{1 \leqslant i,j \leqslant r}$, where $C_{i,j} := \alpha_j(h_i)$. The Cartan matrix C does not depend on the choice of the basis Π. It verifies the following properties:

$$C_{i,j} \in \mathbb{Z} \text{ for all } i, j, \tag{A.2}$$
$$C_{i,i} = 2 \text{ for all } i, \tag{A.3}$$
$$C_{i,j} \leqslant 0 \text{ if } i \neq j, \tag{A.4}$$
$$C_{i,j} = 0 \text{ if and only if } C_{j,i} = 0. \tag{A.5}$$

Moreover, all principal minors of C are strictly positive,

$$\det\big((C_{i,j})_{0 \leqslant i,j \leqslant s}\big) > 0 \quad \text{for} \quad 1 \leqslant s \leqslant r.$$

The semisimple Lie algebra $\mathfrak{g}$ has a presentation in term of Chevalley generators. Namely, consider the generators $(e_i)_{1 \leqslant i \leqslant r}$, $(f_i)_{1 \leqslant i \leqslant r}$, $(h_i)_{1 \leqslant i \leqslant r}$ with relations

$$[h_i, h_j] = 0, \tag{A.6}$$
$$[e_i, f_j] = \delta_{i,j} h_i, \tag{A.7}$$
$$[h_i, e_j] = C_{i,j} e_j, \tag{A.8}$$
$$[h_i, f_j] = -C_{i,j} f_j, \tag{A.9}$$
$$(\operatorname{ad} e_i)^{1 - C_{i,j}} e_j = 0 \text{ for } i \neq j, \tag{A.10}$$
$$(\operatorname{ad} f_i)^{1 - C_{i,j}} f_j = 0 \text{ for } i \neq j, \tag{A.11}$$

where $\delta_{i,j}$ is the Kronecker symbol. The last two relations are called the *Serre relations*. By (A.8) and (A.9), $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ for all i.

It is well-known that $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in \Delta$. One can choose nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$, for all α, such that $\{h_i \colon i = 1, \dots, r\} \cup \{e_\alpha \colon \alpha \in \Delta\}$ forms a *Chevalley basis* of $\mathfrak{g}$. This means, apart from the above relations, that:

$$[e_\beta, e_\gamma] = \pm(p+1)e_{\beta+\gamma}, \tag{A.12}$$

for all $\beta, \gamma \in \Delta$, where p is the greatest positive integer such that $\gamma - p\beta$ is a root. Here we consider that $e_{\beta+\gamma} = 0$ if $\beta + \gamma$ is not a root, and that $e_{\alpha_i} = e_i$, $e_{-\alpha_i} = f_i$ for $i = 1, \ldots, r$.

Let Δ_+ be the positive root system corresponding to Π, and let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \tag{A.13}$$

be the corresponding triangular decomposition. Thus $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$ and $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha$ are both nilpotent Lie subalgebras of $\mathfrak{g}$.

A.1.3 Verma modules

Let $\lambda \in \mathfrak{h}^*$ and set

$$K_{\mathfrak{g}}(\lambda) := U(\mathfrak{g})\mathfrak{n}_+ + \sum_{x\in\mathfrak{h}} U(\mathfrak{g})(x - \lambda(x)).$$

Since $K_{\mathfrak{g}}(\lambda)$ is a left $U(\mathfrak{g})$-module,

$$M_{\mathfrak{g}}(\lambda) := U(\mathfrak{g})/K_{\mathfrak{g}}(\lambda)$$

is naturally a left $U(\mathfrak{g})$-module, called a *Verma module* .

Theorem A.1 ([50, Theorem 10.6])

i). Each element of $M_{\mathfrak{g}}(\lambda)$ is uniquely written in the form um_λ for some $u \in U(\mathfrak{g})$ where $m_\lambda := 1 + K_{\mathfrak{g}}(\lambda)$.
ii). The elements $f_{\beta_1}^{n_1} \ldots f_{\beta_s}^{n_s} m_\lambda$, for all $n_i \geqslant 0$, form a basis of $M_{\mathfrak{g}}(\lambda)$.

Note that $M_{\mathfrak{g}}(\lambda)$ can also be described as follows (up to isomorphism of $U(\mathfrak{g})$-modules):

$$M_{\mathfrak{g}}(\lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda =: \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda),$$

where $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathbb{C}_\lambda$ is a one-dimensional $\mathfrak{b}$-module whose $\mathfrak{b}$-action is given by: $(x+n).z = \lambda(x)z$ for $x \in \mathfrak{h}$, $n \in \mathfrak{n}_+$ and $z \in \mathbb{C}_\lambda$. Then, up to scalars, $m_\lambda = 1 \otimes 1$.

For each $\mu \in \mathfrak{h}^*$, set

$$M_{\mathfrak{g}}(\lambda)_\mu := \{m \in M_{\mathfrak{g}}(\lambda) : xm = \mu(x)m \text{ for all } x \in \mathfrak{h}\}.$$

For $\lambda, \mu \in \mathfrak{h}^*$ we write $\mu \leqslant \lambda$ if $\lambda - \mu$ belongs to the *root lattice*

$$Q := \sum_{i=1}^{r} \mathbb{Z}\alpha_i.$$

This defines a partial order on $\mathfrak{h}^*$.

Theorem A.2 ([50, Theorem 10.7])

i). $M_{\mathfrak{g}}(\lambda) = \bigoplus_{\mu\in\mathfrak{h}^*} M_{\mathfrak{g}}(\lambda)_\mu$.
ii). $M_{\mathfrak{g}}(\lambda)_\mu \neq 0$ if and only if $\mu \preccurlyeq \lambda$, and $\dim M_{\mathfrak{g}}(\lambda)_\mu$ is the number of ways of expressing $\lambda - \mu$ as a sum of positive roots. In particular, $\dim M_{\mathfrak{g}}(\lambda)_\lambda = 1$.

If $M_{\mathfrak{g}}(\lambda)_\mu \neq 0$, then μ called a *weight* of $M_{\mathfrak{g}}(\lambda)$, and $M_{\mathfrak{g}}(\lambda)_\mu$ is called the *weight space* of $M_{\mathfrak{g}}(\lambda)$ with weight μ.

Theorem A.2 says that the weights of $M_{\mathfrak{g}}(\lambda)$ are precisely the elements $\mu \in \mathfrak{h}^*$ such that $\mu \preccurlyeq \lambda$. Thus λ is the *highest weight* of $M_{\mathfrak{g}}(\lambda)$ with respect to the partial order $\preccurlyeq$. We say that $M_{\mathfrak{g}}(\lambda)$ is the *Verma module with highest weight* λ.

One of the important fact about $M_{\mathfrak{g}}(\lambda)$ is that it has a unique maximal submodule $N_{\mathfrak{g}}(\lambda)$. It is constructed as follows: since $M_{\mathfrak{g}}(\lambda)_\lambda = \mathbb{C}m_\lambda$ and that $M_{\mathfrak{g}}(\lambda)$ is generated by m_λ, any proper submodule N of $M_{\mathfrak{g}}(\lambda)$ satisfy $N_\lambda = 0$. In particular the sum $N_{\max}$ of all proper submodules of M satisfies $(N_{\max})_\lambda = 0$. This proves the existence and the unicity of the maximal proper submodule of $M_{\mathfrak{g}}(\lambda)$: just set $N_{\mathfrak{g}}(\lambda) := N_{\max}$.

Since $N_{\mathfrak{g}}(\lambda)$ is a maximal submodule of $M_{\mathfrak{g}}(\lambda)$,

$$L_{\mathfrak{g}}(\lambda) := M_{\mathfrak{g}}(\lambda)/N_{\mathfrak{g}}(\lambda)$$

is a simple $U(\mathfrak{g})$-module, that is, an irreducible representation of $\mathfrak{g}$. There is $v_\lambda \in L_{\mathfrak{g}}(\lambda) \setminus \{0\}$ such that

- $h_i v = \lambda(h_i)v$ for all $i = 1, \dots, r$,
- $e_i v = 0$ for all $i = 1, \dots, r$, that is, $\mathfrak{n}_+ v = 0$,
- $L_{\mathfrak{g}}(\lambda) = U(\mathfrak{n}_-)v_\lambda$,
- λ is the highest weight of $L_{\mathfrak{g}}(\lambda)$.

Let

$$P := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z} \text{ for all } i = 1, \dots, r\},$$
$$P^+ := \{\lambda \in \mathfrak{h}^* : \lambda(h_i) \in \mathbb{Z}_{\geqslant 0} \text{ for all } i = 1, \dots, r\},$$

be the *weight lattice* of $\mathfrak{h}^*$ and the set of *dominant integral weights* , respectively. The elements $\varpi_i \in \mathfrak{h}^*$, $i = 1, \dots, r$, satisfying $\varpi_i(h_j) = \delta_{i,j}$ for all j are called the *fundamental weights*. We denote by $\varpi_1^\vee, \dots, \varpi_r^\vee$ the fundamental coweights. They are the elements of $\mathfrak{h}$ such that $\{\varpi_1^\vee, \dots, \varpi_r^\vee\}$ is the dual basis of $\{\alpha_1, \dots, \alpha_r\}$.

We conclude this section by the following crucial result.

Theorem A.3 ([50, Theorem 10.21]) *The simple $U(\mathfrak{g})$-module $L_{\mathfrak{g}}(\lambda)$ is finite dimensional if and only if $\lambda \in P^+$. Moreover, all simple finite dimensional $U(\mathfrak{g})$-modules are of the form $L_{\mathfrak{g}}(\lambda)$ for some $\lambda \in P^+$. These modules are pairwise non-isomorphic.*

The highest weight modules $M_{\mathfrak{g}}(\lambda)$ and $L_{\mathfrak{g}}(\lambda)$ are both elements of the *category* $\mathscr{O}$ of $\mathfrak{g}$. To avoid repetitions, we will define the category $\mathscr{O}$ only for affine Kac-Moody algebras (see Section A.4); the definition and properties are very similar.

For more about semisimple Lie algebras and their representations, possible references are [50, 136, 176]; see [137] about the category $\mathscr{O}$.

For the category $\mathscr{O}$ in the affine Kac-Moody algebras setting, we refer to Moody-Pianzola's book [158].

A.2 Affine Kac-Moody algebras

Our basic reference about affine Kac-Moody algebras is [110]. We assume from now on that $\mathfrak{g}$ is simple, that is, the only ideals of $\mathfrak{g}$ are $\{0\}$ or $\mathfrak{g}$ and $\dim \mathfrak{g} \geqslant 3$.

A.2.1 The loop algebra

Consider the *loop algebra of* $\mathfrak{g}$ which is the Lie algebra

$$L\mathfrak{g} := \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

with commutation relations

$$[xt^m, yt^n] = [x, y]t^{m+n}, \qquad x, y \in \mathfrak{g},\ m, n \in \mathbb{Z},$$

where xt^m stands for $x \otimes t^m$.

Remark A.1 The Lie algebra $L\mathfrak{g}$ is the Lie algebra of polynomial functions from the unit circle to $\mathfrak{g}$. This is the reason why it is called the loop algebra.

A.2.2 Definition of affine Kac-Moody algebras

Define the bilinear form $(\ |\ )$ on $\mathfrak{g}$ by:

$$(\ |\ ) = \frac{1}{2h^\vee} \kappa_{\mathfrak{g}},$$

where $h^\vee$ is the *dual Coxeter number* (see §A.3.3 for the definition). For example, if $\mathfrak{g} = \mathfrak{sl}_n$ then $h^\vee = n$. Thus, with respect to the induced bilinear form on $\mathfrak{h}^*$, $(\theta|\theta) = 2$, where θ is the *highest positive root* of $\mathfrak{g}$, that is, the unique (positive) root $\theta \in \Delta$ such that $\theta + \alpha_i \notin \Delta \cup \{0\}$ for $i = 1, \dots, r$.

Definition A.1 We define a bilinear map ν on $L\mathfrak{g}$ by setting:

$$\nu(x \otimes f, y \otimes g) := (x|y)\mathrm{Res}_{t=0}(\frac{df}{dt} g),$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[t, t^{-1}]$, where the linear map $\mathrm{Res}_{t=0} \colon \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ is defined by $\mathrm{Res}_{t=0}(t^m) = \delta_{m,-1}$ for $m \in \mathbb{Z}$.

The bilinear ν is a *2-cocycle* on $L\mathfrak{g}$, that is, for any $a, b, c \in L\mathfrak{g}$,

$$\nu(a, b) = -\nu(b, a), \tag{A.14}$$

$$\nu([a, b], c) + \nu([b, c], a) + \nu([c, a], b) = 0. \tag{A.15}$$

Definition A.2 We define the *affine Kac-Moody algebra* $\hat{\mathfrak{g}}$ as the vector space $\hat{\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{C}K$, with the commutation relations $[K, \hat{\mathfrak{g}}] = 0$ (so K is a central element), and

$$[x \otimes f, y \otimes g] = [x, y]_{L\mathfrak{g}} + \nu(x \otimes f, y \otimes g)K, \quad x, y \in \mathfrak{g},\ f, g \in \mathbb{C}[t, t^{-1}], \tag{A.16}$$

where $[\ ,\ ]_{L\mathfrak{g}}$ is the Lie bracket on $L\mathfrak{g}$. In other words the commutation relations are given by:

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K,$$
$$[K, \hat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.

A.2.3 Chevalley generators

The following result shows that affine Kac-Moody algebras are natural generalizations of finite dimensional semisimple Lie algebras.

Theorem A.4 *The Lie algebra $\hat{\mathfrak{g}}$ can be presented by generators $(E_i)_{0\leqslant i\leqslant r}$, $(F_i)_{0\leqslant i\leqslant r}$, $(H_i)_{0\leqslant i\leqslant r}$, and relations*

$$[H_i, H_j] = 0, \tag{A.17}$$

$$[E_i, F_j] = \delta_{i,j} H_i, \tag{A.18}$$

$$[H_i, E_j] = C_{i,j} E_j, \tag{A.19}$$

$$[H_i, F_j] = -C_{i,j} F_j, \tag{A.20}$$

$$(\operatorname{ad} E_i)^{1-C_{i,j}} E_j = 0 \text{ for } i \neq j, \tag{A.21}$$

$$(\operatorname{ad} F_i)^{1-C_{i,j}} F_j = 0 \text{ for } i \neq j, \tag{A.22}$$

where $\hat{C} = (C_{i,j})_{0\leqslant i\leqslant r}$ is an affine Cartan matrix*, that is, $\hat{C}$ satisfies the relations (A.2)–(A.5) of a Cartan matrix, all proper principal minors are strictly positive,*

$$\det\left((C_{i,j})_{1\leqslant i,j\leqslant s}\right) > 0 \quad \textit{for} \quad 0 \leqslant s \leqslant r-1,$$

and $\det(\hat{C}) = 0$.

Moreover, we can choose the labeling $\{0, \ldots, r\}$ so that the subalgebra generated by $(E_i)_{1\leqslant i\leqslant r}$, $(F_i)_{1\leqslant i\leqslant r}$, $(H_i)_{1\leqslant i\leqslant r}$ is isomorphic to $\mathfrak{g}$, that is, $(C_{i,j})_{1\leqslant i\leqslant r}$ is the Cartan matrix C of $\mathfrak{g}$.

Let us give the general idea of the construction of the Chevalley generators of $\hat{\mathfrak{g}}$ (see [99][1]). Set for $i = 1, \dots, r$,

$$E_i := e_i = e_i \otimes 1, \quad F_i := f_i = f_i \otimes 1, \quad H_i := h_i = h_i \otimes 1.$$

The point is to define E_0, F_0, H_0. Recall that θ is the highest root of Δ. Consider the *Chevalley involution* ω which is the linear involution map of $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h_i) = -h_i$ for $i = 1, \dots, r$. Then pick $f_0 \in \mathfrak{g}_\theta$ such that

$$(f_0|\omega(f_0)) = -\frac{h^\vee}{(\theta|\theta)} = -\frac{h^\vee}{2}.$$

Then we set $e_0 := -\omega(f_0) \in \mathfrak{g}_{-\theta}$ and,

$$E_0 := e_0 t = e_0 \otimes t, \quad F_0 := f_0 t^{-1} = f_0 \otimes t^{-1}, \quad H_0 := [E_0, F_0].$$

Example A.2 Assume that $\mathfrak{g} = \mathfrak{sl}_2$. Then the Cartan matrix C is $C = (2)$. Let us check that the affine Cartan matrix of $\hat{\mathfrak{sl}}_2$ is $\hat{C} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We have

$$\hat{\mathfrak{sl}}_2 = e \otimes \mathbb{C}[t, t^{-1}] \oplus f \otimes \mathbb{C}[t, t^{-1}] \oplus h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K,$$

where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We follow the above construction. We set $E_1 := e$, $F_1 := f$ and $H_1 := h$. We have $h^\vee = 2$ and $\Delta = \{\alpha, -\alpha\}$ with $\alpha(h) = 2$. The highest root is $\theta = \alpha$ and $(\mathfrak{sl}_2)_\theta = \mathbb{C}e$. So f_0 is of the form $f_0 = \lambda e$, $\lambda \in \mathbb{C}^*$ and verifies:

$$-1 = (f_0, \omega(f_0)) = -\lambda^2,$$

whence $\lambda^2 = \pm 1$. Let us fix $\lambda = 1$. So we have

$$E_0 = ft \quad \text{and} \quad F_0 = et^{-1}.$$

Then

$$H_0 = [E_0, F_0] = [f, e] + (f|e)K = K - H_1.$$

We can verify the relations of Chevalley generators. In particular, $[H_1, E_0] = -2E_0$ and $[H_0, E_1] = -2E_1$, whence the expected affine Cartan matrix $\hat{C}$.

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[1] Since our normalization of (|) is slightly different, we give the details here.

A.3 Root systems and triangular decomposition

In order to construct analogs of highest weight representations, we need a triangular decomposition for $\hat{\mathfrak{g}}$ and the corresponding combinatoric, that is, a system of roots.

A.3.1 Triangular decomposition

Recall the triangular decomposition (A.13) of $\mathfrak{g}$, and consider the following subspaces of $\hat{\mathfrak{g}}$:

$$\begin{aligned}
\hat{\mathfrak{n}}_+ &:= (\mathfrak{n}_- \oplus \mathfrak{h}) \otimes t\mathbb{C}[t] \oplus \mathfrak{n}_+ \otimes \mathbb{C}[t] = \mathfrak{n}_+ + t\mathfrak{g}[t], \\
\hat{\mathfrak{n}}_- &:= (\mathfrak{n}_+ \oplus \mathfrak{h}) \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{n}_- \otimes \mathbb{C}[t^{-1}] = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}], \\
\hat{\mathfrak{h}} &:= (\mathfrak{h} \otimes 1) \oplus \mathbb{C}K = \mathfrak{h} + \mathbb{C}K.
\end{aligned}$$

They are Lie subalgebras of $\hat{\mathfrak{g}}$ and we have

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+. \tag{A.23}$$

In fact, $\hat{\mathfrak{n}}_+$ (resp., $\hat{\mathfrak{n}}_-$, $\hat{\mathfrak{h}}$) is generated by the E_i (resp., F_i, H_i), for $i = 0, \ldots, r$. The verifications are left to the reader.

A.3.2 Extended affine Kac-Moody algebras

We now intend to define a corresponding root system, and simple roots. The simple roots $\alpha_i \in \hat{\mathfrak{h}}^*$ are defined by $\alpha_j(H_i) = C_{i,j}$ for $0 \leqslant i, j \leqslant r$. As $\det(\hat{C}) = 0$, the simple roots $\alpha_0, \ldots, \alpha_r$ are not linearly independent. For example, for $\hat{\mathfrak{sl}}_2$, we have $\alpha_0 + \alpha_1 = 0$.

For the following constructions, we need linearly independent simple roots. This is the reason why we consider the *extended affine Lie algebra* :

$$\tilde{\mathfrak{g}} := \hat{\mathfrak{g}} \oplus \mathbb{C}D,$$

with commutation relations (apart from those of $\hat{\mathfrak{g}}$),

$$[D, x \otimes f] = x \otimes t\frac{df}{dt}, \qquad [D, K] = 0, \quad x \in \mathfrak{g},\ f \in \mathbb{C}[t, t^{-1}],$$

that is,

$$[D, xt^m] = mxt^m, \qquad [D, K] = 0, \qquad x \in \mathfrak{g},\ m \in \mathbb{Z}.$$

We have the new Cartan subalgebra

$$\tilde{\mathfrak{h}} := \hat{\mathfrak{h}} \oplus \mathbb{C}D.$$

It is a commutative Lie subalgebra of $\tilde{\mathfrak{g}}$ of dimension $r+2$, and we have the corresponding triangular decomposition :

$$\tilde{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+.$$

Let us define the new simple roots $\alpha_i \in \tilde{\mathfrak{h}}^*$, for $i = 0, \dots, r$. The action of α_i on $\hat{\mathfrak{h}}$ has already been defined, and so we only have to specify $\alpha_i(D)$, for $i = 0, \dots, r$. From the relations

$$\alpha_i(D)E_i = [D, E_i] = [D, e_i] = 0, \qquad i = 1, \dots, r,$$

we deduce that $\alpha_i(D) = 0$ for $i = 1, \dots, r$. From the relation

$$\alpha_0(D)E_0 = [D, E_0] = [D, e_0 t] = E_0,$$

we deduce that $\alpha_0(D) = 1$.

A.3.3 Root system

The bilinear form $(\ |\ )$ extends from $\mathfrak{g}$ to a symmetric bilinear form on $\tilde{\mathfrak{g}}$ by setting for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$:

$$\begin{aligned}
&(xt^m | yt^n) = \delta_{m+n,0}(x|y), \qquad (L\mathfrak{g} | \mathbb{C}K \oplus \mathbb{C}D) = 0,\\
&(K|K) = (D|D) = 0, \qquad (K|D) = 1.
\end{aligned}$$

Since the restriction of the bilinear form $(\ |\ )$ to $\tilde{\mathfrak{h}}$ is nondegenerate, we can identify $\tilde{\mathfrak{h}}^*$ with $\tilde{\mathfrak{h}}$ using this form. Through this identification, $\alpha_0 = K - \theta$. For $\alpha \in \tilde{\mathfrak{h}}^*$ such that $(\alpha|\alpha) \neq 0$, we set $\alpha^\vee = \dfrac{2\alpha}{(\alpha|\alpha)}$. Note that $\alpha^\vee$ obviously corresponds to $\alpha_i^\vee = h_i$ for $\alpha = \alpha_i$, $i = 1, \dots, r$.

The set of roots $\hat{\Delta}$ of $\tilde{\mathfrak{g}}$ with basis $\hat{\Pi} := \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ is

$$\hat{\Delta} = \hat{\Delta}^{\mathrm{re}} \cup \hat{\Delta}^{\mathrm{im}},$$

where the set of *real roots* is

$$\hat{\Delta}^{\mathrm{re}} := \{\alpha + nK \colon \alpha \in \Delta,\ n \in \mathbb{Z}\},$$

and the set of *imaginary roots* is

$$\hat{\Delta}^{\mathrm{im}} := \{nK \colon n \in \mathbb{Z},\ n \neq 0\}.$$

Then we set $\hat{\Delta}^\vee := \hat{\Delta}^{\vee,\mathrm{re}} \cup \hat{\Delta}^{\vee,\mathrm{im}}$, with

$$\hat{\Delta}^{\vee,\mathrm{re}} := \{\alpha^\vee : \alpha \in \hat{\Delta}^{\mathrm{re}}\}, \qquad \hat{\Delta}^{\vee,\mathrm{im}} := \{\alpha^\vee : \alpha \in \hat{\Delta}^{\mathrm{im}}\}.$$

The positive integers

$$h := (\rho^\vee|\theta) + 1 \quad \text{and} \quad h^\vee = (\rho|\theta^\vee) + 1$$

are called the *Coxeter number* and the *dual Coxeter number* of $\mathfrak{g}$, respectively, where ρ (resp., $\rho^\vee$) is the half sum of positive roots (resp., coroots), that is defined by $(\rho|\alpha_i^\vee) = 1$ (resp., $(\rho^\vee|\alpha_i) = 1$), for $i = 1, \ldots, r$. Defining $\hat{\rho} := h^\vee D + \rho \in \tilde{\mathfrak{h}}$ and $\hat{\rho}^\vee := hD + \rho^\vee \in \tilde{\mathfrak{h}}$ we have the following formulas: $(\hat{\rho}|\alpha_i^\vee) = 1$ and $(\hat{\rho}^\vee|\alpha_i) = 1$, for $i = 0, \ldots, r$.

A.4 Representations of affine Kac-Moody algebras, category $\mathscr{O}$

We extend some notations and definitions of Section A.1 to $\tilde{\mathfrak{g}}$. For example, for M a $\tilde{\mathfrak{g}}$-module and $\lambda \in \tilde{\mathfrak{h}}^*$, we set

$$M_\lambda := \{m \in M : xm = \lambda(x)m \text{ for all } x \in \tilde{\mathfrak{h}}\}.$$

The space M_λ is called the *weight space* of weight λ of M. The set of weights of M is

$$\mathrm{wt}(M) := \{\lambda \in \tilde{\mathfrak{h}}^* : M_\lambda \neq 0\}.$$

The partial order $\preccurlyeq$ is extended to $\tilde{\mathfrak{h}}^*$ as follows: we write $\mu \preccurlyeq \lambda$ if $\lambda - \mu = \sum_{i=0}^r m_i \alpha_i$ with $m_i \in \mathbb{Z}$, $m_i \geqslant 0$. For $\lambda \in \tilde{\mathfrak{h}}^*$, we set $D(\lambda) := \{\mu \in \tilde{\mathfrak{h}}^* : \mu \preccurlyeq \lambda\}$.

A.4.1 The category $\mathscr{O}$

Let $U(\tilde{\mathfrak{g}})$ -Mod be the category of left $U(\tilde{\mathfrak{g}})$-modules.

Definition A.3 The *catgeory* $\mathscr{O}$ is defined to be the full subcategory of $U(\tilde{\mathfrak{g}})$ -Mod whose objects are the modules M satisfying the following conditions:

($\mathscr{O}1$) M is $\tilde{\mathfrak{h}}$-diagonalizable, that is, $M = \oplus_{\lambda \in \tilde{\mathfrak{h}}^*} M_\lambda$,
($\mathscr{O}2$) all weight spaces of M are finite dimensional,
($\mathscr{O}3$) there exists a finite number of $\lambda_1, \ldots, \lambda_s \in \tilde{\mathfrak{h}}^*$ such that

$$\mathrm{wt}(M) \subset \bigcup_{1 \leqslant i \leqslant s} D(\lambda_i).$$

The category $\mathscr{O}$ is stable by submodules and quotients. For M_1, M_2 two representations of $\tilde{\mathfrak{g}}$ we can define a structure of $\tilde{\mathfrak{g}}$-module on $M_1 \otimes M_2$ by using the coproduct $\tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$, $x \mapsto x \otimes 1 + 1 \otimes x$ for $x \in \tilde{\mathfrak{g}}$. Then if M_1 and M_2 are objects of $\mathscr{O}$, then so are $M_1 \oplus M_2$ and $M_1 \otimes M_2$.

A.4.2 Verma modules

We now give important examples of modules in the category $\mathscr{O}$. For $\lambda \in \tilde{\mathfrak{h}}^*$, set:

$$K(\lambda) := U(\tilde{\mathfrak{g}})\hat{\mathfrak{n}}_+ + \sum_{x \in \tilde{\mathfrak{h}}^*} U(\tilde{\mathfrak{g}})(x - \lambda(x)).$$

As it is a left ideal of $U(\tilde{\mathfrak{g}})$,

$$M(\lambda) := U(\tilde{\mathfrak{g}})/K(\lambda)$$

has a natural structure of a left $U(\tilde{\mathfrak{g}})$-module. It is called a *Verma module* .

Proposition A.1 *The $U(\tilde{\mathfrak{g}})$-module $M(\lambda)$ is in the category $\mathscr{O}$ and has a unique proper submodule $N(\lambda)$.*

We construct $N(\lambda)$ in the same way as $N_{\mathfrak{g}}(\lambda)$ for $\mathfrak{g}$ (see §A.1.3).

As a consequence of the proposition, $M(\lambda)$ has a unique simple quotient

$$L(\lambda) := M(\lambda)/N(\lambda).$$

Proposition A.2 *The simple module $L(\lambda)$ is in the category $\mathscr{O}$ and all simple modules of the category $\mathscr{O}$ are of the form $L(\lambda)$ for some $\lambda \in \tilde{\mathfrak{h}}^*$.*

The *character* of a module M in the category $\mathscr{O}$ is by definition

$$\mathrm{ch}(M) = \sum_{\lambda \in \tilde{\mathfrak{h}}^*} (\dim M_\lambda) e(\lambda),$$

where the $e(\lambda)$ are formal elements.

In general a representation M in $\mathscr{O}$ does not have a finite composition series. However, the multiplicity $[M : L(\lambda)]$ of $L(\lambda)$ in M makes sense ([114]). As a consequence, we have

$$\mathrm{ch}\, M = \sum_{\lambda} [M : L(\lambda)]\, \mathrm{ch}\, L(\lambda), \quad [M : L(\lambda)] \in \mathbb{Z}_{\geqslant 0},$$

A.4.3 Singular vectors

A *singular vector* of a $\mathfrak{g}$-representation M is a vector $v \in M$ such that $\mathfrak{n}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$. A *singular vector* of a $\hat{\mathfrak{g}}$-representation M is a vector $v \in M$ such that $\hat{\mathfrak{n}}_+.v = 0$, that is, $e_i.v = 0$ for $i = 1, \dots, r$, and $(f_\theta t).v = 0$, with $f_\theta \in \mathfrak{g}_{-\theta} \setminus \{0\}$.

A.5 Integrable and admissible representations

A.5.1 Integrable representations

The representation $L(\lambda)$, for $\lambda \in \tilde{\mathfrak{h}}^*$, is finite dimensional if and only if $\lambda = 0$, that is, $L(\lambda)$ is the trivial representation. The notion of finite dimensional representations has to be replaced by the notion of in the category $\mathscr{O}$.

Definition A.4 A representation M of $\tilde{\mathfrak{g}}$ is said to be *integrable* if

(1) M is $\tilde{\mathfrak{h}}$-diagonalizable,
(2) for $\lambda \in \tilde{\mathfrak{h}}^*$, M_λ is finite dimensional,
(3) for all $\lambda \in \mathrm{wt}(M)$, for all $i = 0, \ldots, r$, there is $N \geqslant 0$ such that for $m \geqslant N$, $\lambda + m\alpha_i \notin \mathrm{wt}(M)$ and $\lambda - m\alpha_i \notin \mathrm{wt}(M)$.

Remark A.2 As an $\mathfrak{a}_i$-module, $i = 0, \ldots, r$, an integrable representation M decomposes into a direct sum of finite dimensional irreducible $\hat{\mathfrak{h}}$-invariant modules, where $\mathfrak{a}_i \cong \mathfrak{sl}_2$ is the Lie algebra generated by the Chevalley generators E_i, F_i, H_i. Hence the action of $\mathfrak{a}_i$ on M can be "integrated" to the action of the group $SL_2(\mathbb{C})$.

The character of the simple integrable representations in the category $\mathscr{O}$ satisfy remarkable combinatorial identities (related to Macdonald identities).

A.5.2 Level of a representation

According to the well-known Schur Lemma, any central element of a Lie algebra acts as a scalar on a simple finite dimensional representation L. As the Schur Lemma extends to a representation with countable dimension, the result holds for highest weight $\tilde{\mathfrak{g}}$-modules. In particular, $K \in \tilde{\mathfrak{g}}$ acts as a scalar $k \in \mathbb{C}$ on the simple representations of the category $\mathscr{O}$.

Definition A.5 A representation M is said to be *level* k if K acts as $k\mathrm{Id}$ on M.

All simple representations of the category $\mathscr{O}$ have a level. Namely, $L(\lambda)$ has level $k = \lambda(K) \in \mathbb{C}$, and so $k = \mu(K)$ for all $\mu \in \mathrm{wt}(L(\lambda))$. Note that

$$k = \lambda(K) = \sum_{i=0}^{r} a_i \lambda(\alpha_i^\vee)$$

where the a_i are defined by $K = \sum_{i=0}^{r} a_i \alpha_i^\vee$.

Lemma A.1 *The simple representation $L(\lambda)$ is integrable if and only if λ is dominant and integrable, that is, $\lambda(H_i) \in \mathbb{Z}_{\geqslant 0}$ for all $i = 0, \ldots, r$. It has level 0 if and only if* $\dim L(\lambda) = 1$.

Recall that $\tilde{\mathfrak{h}}^*$ is identified with $\tilde{\mathfrak{h}}$ through $(\,|\,)$, and that through this identification the dual of K is D. Then, as a particular case of Lemma A.1, $L(kD)$ is integrable if and only if $k \in \mathbb{Z}_{\geqslant 0}$.

The category of modules of the category $\mathscr{O}$ of level k will be denoted by $\mathscr{O}_k$ ([110]).

The level $k = -h^\vee$ is particular since the center of $\tilde{U}(\hat{\mathfrak{g}})/\tilde{U}(\hat{\mathfrak{g}})(K-k)$ is large and the representation theory changes drastically at this level. Here, $\tilde{U}(\hat{\mathfrak{g}})$ is the completion of the enveloping algebra $U(\hat{\mathfrak{g}})$. This level is called the *critical level*.

Although the category $\mathscr{O}$ is stable by tensor product, the category $\mathscr{O}_k$ is not stable by tensor product (except for $k = 0$). Indeed from the coproduct, we get that for M_1, M_2 representations in $\mathscr{O}_{k_1}, \mathscr{O}_{k_2}$ respectively, the module $M_1 \otimes M_2$ is in $\mathscr{O}_{k_1+k_2}$. This is one motivation to study the *fusion product*; see [31], [99, Section 5] for more details on this topic.

A.5.3 Admissible representations

We now introduce a class of representations, called *admissible representations*, which includes the class of integrable representations. The definition goes back to Kac and Wakimoto [117]. While the notion of integrable representations has a geometrical meaning, the notion of admissible representations is purely combinatorial. However, conjecturally, admissible representations are precisely the representations which satisfy a certain modular invariant property (see below).

Retain the notations of §A.3.3, and recall the definition of the affine and extended affine Weyl groups (see e.g., [119]). Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and extend it to $\hat{\mathfrak{h}}$ by setting $w(K) = K$, $w(D) = D$ for all $w \in W$. Let $Q^\vee = \sum_{i=1}^r \mathbb{Z}\alpha_i^\vee$ be the coroot lattice of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}$, define the translation ([111]),

$$t_\alpha(v) = v + (v|K)\alpha - \left(\frac{1}{2}|\alpha|^2(v|K) + (v|\alpha)\right)K, \qquad v \in \hat{\mathfrak{h}},$$

and for a subset $L \subset \mathfrak{h}$, let

$$t_L := \{t_\alpha \colon \alpha \in L\}.$$

The *affine Weyl groups* $\hat{W}$ and the *extended affine Weyl group* $\tilde{W}$ are then defined by:

$$\hat{W} := W \ltimes t_{Q^\vee}, \qquad \tilde{W} := W \ltimes t_{P^\vee},$$

so that $\hat{W} \subset \tilde{W}$. Here $P^\vee = \{\lambda \in \mathfrak{h} \colon \langle \lambda, \alpha\rangle \in \mathbb{Z} \text{ for all } \alpha \in Q\}$, with $Q = \sum_{i=1}^r \mathbb{Z}\alpha_i$ the root lattice.

The group $\tilde{W}_+ := \{w \in \tilde{W} \colon w(\hat{\Pi}^\vee) = \hat{\Pi}^\vee\}$ acts transitively on orbits of $\operatorname{Aut}\hat{\Pi}^\vee$ and simply transitively acts on the orbit of $\alpha_0^\vee$. Moreover $\tilde{W} = \tilde{W}_+ \ltimes \hat{W}$. Here, $\hat{\Pi}^\vee := \{\alpha^\vee \colon \alpha \in \hat{\Pi}\}$.

Definition A.6 ([117, 119]) A weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if

(1) λ is *regular dominant*, that is,

$$\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \notin -\mathbb{Z}_{\geqslant 0} \quad \text{for all} \quad \alpha \in \hat{\Delta}^{\mathrm{re}}_+,$$

(2) the $\mathbb{Q}$-span of $\hat{\Delta}_\lambda$ contains $\hat{\Delta}^{\mathrm{re}}$, where $\hat{\Delta}_\lambda := \{\alpha \in \hat{\Delta}^{\mathrm{re}} : (\lambda|\alpha^\vee) \in \mathbb{Z}\}$.

The irreducible highest weight representation $L(\lambda)$ of $\hat{\mathfrak{g}}$ with highest weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* if λ is admissible. Note that an irreducible integrable representation of $\hat{\mathfrak{g}}$ is admissible.

Proposition A.3 ([119, Prop. 1.2]) *For $k \in \mathbb{C}$, the weight $\lambda = kD$ is admissible if and only if k satisfies one of the following conditions:*

i). $k = -h^\vee + \dfrac{p}{q}$ where $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, and $p \geqslant h^\vee$,

ii). $k = -h^\vee + \dfrac{p}{r^\vee q}$ where $p, q \in \mathbb{Z}_{>0}$, $(p, q) = 1$, $(p, r^\vee) = 1$ and $p \geqslant h$.

Here $r^\vee$ is the lacety *of $\mathfrak{g}$ (i.e., $r^\vee = 1$ for the types A, D, E, $r^\vee = 2$ for the types B, C, F and $r^\vee = 3$ for the type G_2), h and $h^\vee$ are the Coxeter and dual Coxeter numbers.*

Definition A.7 If k satisfies one of the conditions of Proposition A.3, we say that k is an *admissible level*.

For an admissible representation $L(\lambda)$ we have [116]

$$\mathrm{ch}(L(\lambda)) = \sum_{w \in \hat{W}(\lambda)} (-1)^{\ell_\lambda(w)} \mathrm{ch}(M(w \circ \lambda)) \tag{A.24}$$

since λ is regular dominant, where $\hat{W}(\lambda)$ is the *integral Weyl group* ([129, 158]) of λ, that is, the subgroup of $\hat{W}$ generated by the reflections s_α associated with $\alpha \in \hat{\Delta}_\lambda$, $w \circ \lambda = w(\lambda + \rho) - \rho$, and ℓ_λ is the length function of the Coxeter group $\hat{W}(\lambda)$. Further, Condition (ii) of Proposition A.3 implies that $\mathrm{ch}(L(\lambda))$ is written in terms of certain *theta functions* [111, Chap. 13]. Kac and Wakimoto [117] showed that admissible representations are *modular invariant*, that is, the characters of admissible representations form an $SL_2(\mathbb{Z})$ invariant subspace.

Let λ, μ be distinct admissible weights. Then Condition (1) of Proposition A.3 implies that

$$\mathrm{Ext}^1_{\hat{\mathfrak{g}}}(L(\lambda), L(\mu)) = 0.$$

Further, the following fact is known by Gorelik and Kac [94].

Theorem A.5 ([94]) *Let λ be admissible. Then*

$$\mathrm{Ext}^1_{\hat{\mathfrak{g}}}(L(\lambda), L(\lambda)) = 0.$$

Therefore admissible representations form a semisimple full subcategory of the category of $\hat{\mathfrak{g}}$-modules.

Appendix B
Differential operators

In this chapter, $X = \operatorname{Spec} A$ is an affine algebraic variety over the complex number field of dimension n. We are particularly interested in the case where X is an affine algebraic group G.

Our main references are [102, 152].

B.1 Tangent sheaf and cotangent sheaf

Let $\mathscr{O}_X$ be the sheaf of rings of regular functions, that is, the structure sheaf on X. We denote briefly the algebra $\mathscr{O}_X(X)$ of global sections by $\mathscr{O}(X)$.

We say that a section $\theta \in (\operatorname{End}_{\mathbb{C}} \mathscr{O}_X)(X)$ is a *vector field* on X if for each open subset $U \subset X$, $\theta(U) := \theta|_U \in (\operatorname{End}_{\mathbb{C}} \mathscr{O}_X)(U)$ satisfies the condition

$$\theta(U)(fg) = \theta(U)(f)g + f\theta(U)(g), \qquad f, g \in \mathscr{O}_X(U).$$

For an open subset U of X, denote the set of vector fields θ on U by $\Theta(U)$. Then $\Theta(U)$ is an $\mathscr{O}_X(U)$-module, and the presheaf $U \mapsto \Theta(U)$ turns out to be a sheaf of $\mathscr{O}_X$-modules. We denote this sheaf by Θ_X and call it the *tangent sheaf* of X. Thus

$$\Theta_X = \operatorname{Der}_{\mathbb{C}}(\mathscr{O}_X).$$

It is a coherent sheaf of $\mathscr{O}_X$-modules. Indeed, if $X = \operatorname{Spec} A$, with $A = \mathbb{C}[x_1, \ldots, x_n]/I$, with I an ideal of $\mathbb{C}[x_1, \ldots, x_n]$, then

$$\operatorname{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\partial_i, \quad \text{where} \quad \partial_i := \frac{\partial}{\partial x_i},$$

is a free $\mathbb{C}[x_1, \ldots, x_n]$-module of rank n, and

$$\operatorname{Der}_{\mathbb{C}}(A) \cong \{\theta \in \operatorname{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n]) \colon \theta(I) \subset I\}.$$

Hence $\mathrm{Der}_{\mathbb{C}}(A)$ is finitely generated over A.

We define the *cotangent sheaf* of X by $\Omega_X^1 = \delta^{-1}(\mathcal{I}/\mathcal{I}^2)$, where $\delta\colon X \to X \times X$ is the diagonal embeddings, $\mathcal{I}$ is the ideal sheaf of $\delta(X)$ in $X \times X$ defined by

$$\mathcal{I}(V) = \{f \in \mathscr{O}_{X\times X}(V)\colon f(V \cap \delta(X)) = 0\}$$

for any open subset V of $X \times X$, and δ^{-1} stands for the sheaf-theoretical inverse image functor. (We usually keep the notation Ω_X for the sheaf $\bigwedge^n \Omega_X^1$ of differential forms of top degree.)

Sections of the sheaf Ω_X^1 are called *differential forms*. By the canonical morphism $\mathscr{O}_X \to \delta^{-1}\mathscr{O}_{X\times X}$ of sheaf of $\mathbb{C}$-algebras, Ω_X^1 is naturally an $\mathscr{O}_X$-module.

We have a morphism $d\colon \mathscr{O}_X \to \Omega_X^1$ of $\mathscr{O}_X$-modules defined by

$$df = f \otimes 1 - 1 \otimes f \mod \delta^{-1}\mathcal{I}^2.$$

It satisfies $d(fg) = d(f)g + f(dg)$ for any $f, g \in \mathscr{O}_X$.

We denote briefly the $\mathscr{O}(X)$-modules $\Theta_X(X) = \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(X))$ and $\Omega_X^1(X)$ by $\Theta(X)$ and $\Omega^1(X)$, respectively.

Thus $\Omega^1(X) = \mathcal{I}(X)/\mathcal{I}(X)^2$, and $\mathcal{I}(X)$ is the kernel of the morphism

$$\varepsilon\colon \mathscr{O}(X) \otimes_{\mathbb{C}} \mathscr{O}(X) \longrightarrow \mathscr{O}(X), \quad f \otimes g \longmapsto fg.$$

The $\mathscr{O}(X) \otimes_{\mathbb{C}} \mathscr{O}(X)$-modules $\mathcal{I}(X)$, $\mathcal{I}(X)^2$ and $\Omega^1(X)$ are viewed as $\mathscr{O}(X)$-modules via the homomorphism $\mathscr{O}(X) \to \mathscr{O}(X) \otimes_{\mathbb{C}} \mathscr{O}(X)$, $f \mapsto f \otimes 1$.

In $\mathscr{O}(X) \otimes_{\mathbb{C}} \mathscr{O}(X)$ we have

$$f \otimes g = fg \otimes 1 + f(1 \otimes g - g \otimes 1) = \varepsilon(f \otimes g) + f(dg \mod \mathcal{I}(X)^2).$$

Therefore, if $\sum_i f_i \otimes g_i \in \mathcal{I}(X) = \ker \varepsilon$, then

$$\sum_i f_i \otimes g_i = \sum_i f_i dg_i \mod \mathcal{I}(X)^2,$$

and so any element of $\Omega^1(X) = \mathcal{I}(X)/\mathcal{I}(X)^2$ has the form $\sum_i f_i dg_i$, for $f_i, g_i \in \mathscr{O}(X)$.

In conclusion, we obtain the following fact.

Lemma B.1 *As $\mathscr{O}(X)$-module, Ω_X^1 is generated by df, for $f \in \mathscr{O}(X)$.*

For $\alpha \in \mathrm{Hom}_{\mathscr{O}(X)}(\Omega_X^1, \mathscr{O}_X)$ we have $\alpha \circ d \in \Theta_X$, which gives an isomorphism

$$\mathrm{Hom}_{\mathscr{O}(X)}(\Omega_X^1, \mathscr{O}_X) \cong \Theta_X$$

as $\mathscr{O}_X$-modules.

Theorem B.1 *Assume that X is smooth. For each point $x \in X$, there exist an affine open neighbourhood V of x, regular functions $x_i \in \mathscr{O}_X(V)$, and vector fields $\partial_i \in \Theta_X(V)$, for $i \in \{1, \ldots, n\}$, satisfying the conditions:*

$$[\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{i,j}, \quad 1 \leqslant i, j \leqslant n,$$
$$\Theta_V = \bigoplus_{i=1}^{n} \mathscr{O}_V \partial_i.$$

Moreover, one can choose the functions $x_1, x_2, \ldots, x_n$ *so that they generate the maximal ideal* $\mathfrak{m}_x$ *of the local ring* $\mathscr{O}_{X,x}$ *at* x*.*

Proof Since the local ring $\mathscr{O}_{X,x}$ is regular, there exist $n = \dim X$ functions $x_1, \ldots, x_n \in \mathfrak{m}_x$ generating the ideal $\mathfrak{m}_x$. Then $dx_1, \ldots, dx_n$ is a basis of the free $\mathscr{O}_{X,x}$-module $\Omega^1_{X,x}$. Hence we can take an affine open neighbourhood V of x such that $\Omega^1_X(V)$ is a free module with basis $dx_1, \ldots, dx_n$ over $\mathscr{O}_X(V)$. Taking the dual basis $\partial_1, \ldots, \partial_n \in \Theta_X(V) \cong \mathrm{Hom}_{\mathscr{O}_X(V)}(\Omega^1_X(V), \mathscr{O}_X(V))$ we get $\partial_i(x_j) = \delta_{i,j}$. Write $[\partial_i, \partial_j]$ as $[\partial_i, \partial_j] = \sum_{l=1}^{n} g^l_{i,j} \partial_l \in \mathscr{O}_X(V)$. Then we have $g^l_{i,j} = [\partial_i, \partial_j] x_l = \partial_i \partial_j x_l - \partial_j \partial_i x_l = 0$. Hence $[\partial_i, \partial_j] = 0$. □

The set $\{x_i, \partial_i : 1 \leqslant i \leqslant n\}$ defined over an affine open neighborhood of x satisfying the conditions of Theorem B.1 is called a *local coordinate system* at x.

B.2 Sheaf of differential operators

We define the sheaf $\mathcal{D}_X$ as the sheaf of $\mathbb{C}$-subalgebras of $\mathrm{End}_{\mathbb{C}}(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X. Here we identify $\mathscr{O}_X$ with a subsheaf of $\mathrm{End}_{\mathbb{C}}(\mathscr{O}_X)$ by identifying $f \in \mathscr{O}_X$ with the element $g \mapsto fg$ of $\mathrm{End}_{\mathbb{C}}(\mathscr{O}_X)$.

Because $\mathcal{D}_X$ is a quantization of $\pi_* \mathcal{O}_{T^*G}$, I think this section should come after Poisson algebra section.

It is explained in Example C.3: is it OK?

We call the sheaf $\mathcal{D}_X$ the *sheaf of differential operators* on X. For any point of X we can take an affine open neighborhood U and a local coordinate system $\{x_i, \partial_i : 1 \leqslant i \leqslant n\}$. Hence we have

$$\mathcal{D}_U := \mathcal{D}_X(U) = \bigoplus_{\alpha \in \mathbb{Z}^n_{\geqslant 0}} \mathscr{O}_U \partial_x^\alpha, \quad \partial_x^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We define the *order filtration* $F_\bullet \mathcal{D}_U$ of $\mathcal{D}_U$ by

$$F_l \mathcal{D}_U = \sum_{|\alpha| \leqslant l} \mathscr{O}_U \partial_x^\alpha, \qquad l \in \mathbb{Z}_{\geqslant 0}, \ |\alpha| := \sum_i \alpha_i.$$

More generally, for an arbitrary open subset V of X we define the order filtration $F_\bullet \mathcal{D}_X$ over V by

$$(F_l \mathcal{D}_X)(V) = \{P \in \mathcal{D}_X(V) : \mathrm{res}^V_U P \in (F_l \mathcal{D}_X)(U) \text{ for any affine open subset } U \text{ of } V\},$$

where $\mathrm{res}^V_U : \mathcal{D}_X(V) \to \mathcal{D}_X(U)$ is the restriction map.

For convenience we set $F_p \mathcal{D}_X = 0$ for $p < 0$.

Proposition B.1

i). $F_\bullet \mathcal{D}_X$ *is an increasing filtration of* $\mathcal{D}_X$ *such that* $\mathcal{D}_X = \bigcup_{l \geqslant 0} F_l \mathcal{D}_X$ *and each* $F_l \mathcal{D}_X$ *is a locally free module over* $\mathscr{O}_X$.
ii). $F_0 \mathcal{D}_X := \mathscr{O}_X$ *and* $(F_l \mathcal{D}_X)(F_m \mathcal{D}_X) = F_{l+m} \mathcal{D}_X$.
iii). $[F_l \mathcal{D}_X, F_m \mathcal{D}_X] \subset F_{l+m-1} \mathcal{D}_X$.

Remark B.1 One can alternatively define $F_\bullet \mathcal{D}_X$ by the recursive formula:

$$F_l \mathcal{D}_X = \{P \in \mathrm{End}_{\mathbb{C}}(\mathscr{O}_X) \colon [P, f] \in F_{l-1} \mathcal{D}_X \text{ for all } f \in \mathscr{O}_X\}, \quad l \in \mathbb{Z}_{\geqslant 0}.$$

Let us consider the sheaf of graded rings

$$\mathrm{gr}\, \mathcal{D}_X = \mathrm{gr}^F \mathcal{D}_X = \bigoplus_{l \geqslant 0} \mathrm{gr}_l\, \mathcal{D}_X,$$

where $\mathrm{gr}_l\, \mathcal{D}_X := F_l \mathcal{D}_X / F_{l-1} \mathcal{D}_X$, $F_{-1} \mathcal{D}_X = 0$. By Proposition B.1, $\mathrm{gr}\, \mathcal{D}_X$ is a sheaf of commutative algebras finitely generated over $\mathscr{O}_X$. Take an affine chart U with a coordinate system $\{x_i, \partial_i\}$ and set

$$\xi_i := (\partial_i \mod F_0 \mathcal{D}_U = \mathscr{O}_U) \in \mathrm{gr}_1\, \mathcal{D}_U.$$

Then we have

$$\mathrm{gr}_l\, \mathcal{D}_U = \bigoplus_{|\alpha| = l} \mathscr{O}_U \xi^\alpha,$$
$$\mathrm{gr}\, \mathcal{D}_U = \mathscr{O}_U[\xi_1, \ldots, \xi_n].$$

We can globalize this notion as follows. Let T^*X be the cotangent bundle of X and let $\pi \colon T^*X \to X$ be the projection. We may regard $\xi_1, \ldots, \xi_n$ as the coordinate system of the cotangent space $\bigoplus_{i=1}^n \mathbb{C} dx_i$ and hence $\mathscr{O}_U[\xi_1, \ldots, \xi_n]$ is canonically identified with the sheaf $\pi_* \mathscr{O}_{T^*X}$ of algebras. Thus we obtain a canonical identification

$$\mathrm{gr}\, \mathcal{D}_X \cong \pi_* \mathscr{O}_{T^*X}. \tag{B.1}$$

The algebra $\mathcal{D}(X) := \mathcal{D}_X(X)$ is called the *algebra of differential operators* on X.

B.3 Derivations and differential forms on a group

Let G be an affine algebraic group. By definition, the Lie algebra of G is the Lie algebra of *left invariant vector fields* on G, that is,

$$\mathrm{Lie}(G) = \{\theta \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)) \colon \Delta \circ \theta = (1 \otimes \theta) \circ \Delta\},$$

(see e.g. [173, Proposition 10.29]), where $\Delta\colon \mathscr{O}(G) \to \mathscr{O}(G) \otimes \mathscr{O}(G)$ is the coproduct induced by the multiplication G. Thus, $\theta \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ is in $\mathrm{Lie}(G)$ if and only if for all $g \in G$, $\lambda_g \theta = \theta \lambda_g$, where $(\lambda_g f)(y) = f(g^{-1}y)$ for $f \in \mathscr{O}(G)$ and $y \in G$.

The Lie algebra of G is canonically isomorphic, as a Lie algebra, to the Lie algebra $\mathrm{Lie}_r(G)$ of right invariant vector fields θ, that is, the Lie algebra consisted of $\theta \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ such that $\rho_g \theta = \theta \rho_g$, where $(\rho_g f)(y) = f(yg)$ for $f \in \mathscr{O}(G)$ and $y \in G$. It is also canonically isomorphic to $T_e(G)$, the tangent space at the identity to G, via the isomorphism,

$$\mathrm{Lie}(G) \longrightarrow T_e(G),$$

sending $\theta \in \mathrm{Lie}(G)$ to $\mathrm{ev}_e \circ \theta$, where ev_e is the evaluation map at the neutral element e, in which we have identified $\Theta(G) = \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ with the tangent bundle TG. We denote by $\mathfrak{g}$ this Lie algebra.

Thus, we have

$$TG \cong G \times \mathfrak{g} \quad \text{and} \quad T^*G \cong G \times \mathfrak{g}^*.$$

For $x \in \mathfrak{g}$, we write x_L (resp., x_R) the corresponding left (resp., right) invariant vector field on G. Note that $(x_L f)(a) = x(\lambda_{a^{-1}} f)$ for $f \in \mathscr{O}(G)$ and $a \in G$.

Remark B.2 Concretely, viewing G as a complex analytic space, we have for $x \in \mathfrak{g}$ and $f \in \mathscr{O}(G)$,

$$(x_L f)(a) = \left.\frac{d}{dt}\right|_{t=0} f(a\exp(tx)), \qquad a \in G,$$
$$(x_R f)(a) = \left.\frac{d}{dt}\right|_{t=0} f(\exp(tx)a), \qquad a \in G,$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map.

The embedding $\mathfrak{g} \hookrightarrow \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$, $x \mapsto x_L$, induces an isomorphism of left $\mathscr{O}(G)$-modules

$$\mathscr{O}(G) \otimes_{\mathbb{C}} \mathfrak{g} \xrightarrow{\simeq} \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)). \tag{B.2}$$

Indeed, both sides are free $\mathscr{O}(G)$-modules of rank the dimension of $\mathfrak{g}$ since G is smooth.

We denote by $\langle\ ,\ \rangle\colon \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)) \times \Omega^1(G) \to \mathscr{O}(G)$ the canonical $\mathscr{O}(G)$-bilinear pairing.

Let us collect some useful identities. Let $\{x^1, \ldots, x^d\}$ be a basis of $\mathfrak{g}$, and $\{\omega^1, \ldots, \omega^d\}$ the dual $\mathscr{O}(G)$-basis of $\Omega^1(G)$. Write

$$[x^i, x^j] = \sum_p c_p^{i,j} x^p, \quad \text{for} \quad i, j = 1, \ldots, d,$$

with $(c_p^{i,j}) \in \mathbb{C}$. The isomorphism (B.2) tells that $\{x^1, \ldots, x^d\}$ forms an $\mathscr{O}(G)$-basis of $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$. In particular,

$$x_R^i = \sum_p f^{i,p} x^p, \quad i = 1, \ldots, d, \tag{B.3}$$

for some invertible matrix $(f^{i,p})_{1\leqslant i,p\leqslant d}$ over $\mathscr{O}(G)$.

Lemma B.2 *The following identities hold:*

i). for all $i, j, s \in \{1, \ldots, d\}$,

$$x_L^i f^{j,s} + \sum_p c_s^{i,p} f^{j,p} = 0,$$

ii). for all $i, j, s \in \{1, \ldots, d\}$,

$$\sum_p f^{i,p} (x_L^p f^{j,s}) = \sum_q c_q^{i,j} f^{q,s}.$$

Proof The identities of (i) hold because they are equivalent to the commutation relations

$$[x_L^i, x_R^j] = 0 \tag{B.4}$$

for all i, j, s.

To prove (ii), we write down the relations

$$[x_R^i, x_R^j] = [x^i, x^j]_R, \tag{B.5}$$

for $i, j = 1, \ldots, d$, in coordinates. We have

$$[x_R^i, x_R^j] = \sum_s [x_R^i, f^{j,s} x^s] = \sum_s (x_R^i f^{j,s}) x^s = \sum_{s,p} f^{i,p} (x_L^p f^{j,s}) x^s$$

by (B.4) and (B.3). Plugging this into (B.5), we get the identities of (ii). □

The Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ acts on $\Omega^1(G)$ by the Lie derivative as follows:

$$((\mathrm{Lie}\,\theta).\omega)(\theta_1) = \theta(\langle \theta_1, \omega \rangle) - \langle [\theta, \theta_1], \omega \rangle, \tag{B.6}$$

for $\omega \in \Omega_G^1$ and $\theta, \theta_1 \in \mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$. (In fact, the Lie algebra $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G))$ acts on $\Omega(G) = \bigwedge^d \Omega^1(G)$ by the Lie derivative; see [102, §1.2].)

So for $i, j = 1, \ldots, d$, we have

$$(\mathrm{Lie}\, x^i).\omega^j = \sum_s \alpha_s^{i,j} \omega^p,$$

for some $\alpha_s^{i,j} \in \mathbb{C}$.

Lemma B.3 *The following identities hold:*

i). for all $i, j \in \{1, \ldots, d\}$,

$$(\mathrm{Lie}\, x^i).\omega^j = \sum_s c_j^{s,i} \omega^s,$$

ii). for all $i, j \in \{1, \dots, d\}$,

$$(\mathrm{Lie}\, x_R^i).\omega^j = 0.$$

Proof For $i, j, s \in \{1, \dots, d\}$, we have

$$\alpha_s^{i,j} = \langle x^s, (\mathrm{Lie}\, x^i).\omega^j \rangle = (\mathrm{Lie}\, x^i).\langle x^s, \omega^j \rangle + \langle [x^s, x^i], \omega^j \rangle = c_j^{s,i},$$

whence (i).

Similarly, for $i, j, s \in \{1, \dots, d\}$, we have

$$\begin{aligned}\langle x_R^s, (\mathrm{Lie}\, x_R^i).\omega^j \rangle &= (\mathrm{Lie}\, x_R^i).\langle x_R^s, \omega^j \rangle + \langle [x^s, x^i]^R, \omega^j \rangle \\ &= \sum_p f^{s,p}(x_L^p f^{i,j}) + \sum_p c_p^{i,s} f^{p,s} = 0\end{aligned}$$

by (B.3) and Lemma B.2, whence (ii). whence (ii). ◻

By the Frobenius reciprocity, we have

$$\mathrm{Hom}_{\mathscr{O}(G)}(\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)), \mathscr{O}(G)) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$$

since $\mathrm{Der}_{\mathbb{C}}(\mathscr{O}(G)) \cong \mathscr{O}(G) \otimes_{\mathbb{C}} \mathfrak{g}$. Hence, as a $\mathbb{C}$-vector spaces,

$$\Omega^1(G) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G)).$$

The above isomorphism has to be understand as follows. Write $\omega \in \Omega^1(G)$ as $\omega = \sum_i f_i dg_i$ by Lemma B.1. To such an ω we attach the element of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ which maps an element $x \in \mathfrak{g}$ to $\sum_i f_i(x_L g_i) \in \mathscr{O}(G)$.

As a consequence we obtain the following proposition.

Proposition B.2 *The linear map from* Ω_G^1 *to* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ *sending* dg *to the element* $x \mapsto x_L g$ *of* $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathscr{O}(G))$ *is an isomorphism of* $\mathscr{O}(G)$*-modules.*

B.4 The algebra of differential operators on a group

We keep the notation of the previous section.

Let $\mathcal{D}(G)$ be the algebra of differential operators on G. We have a natural embedding

$$\mathscr{O}(G) \hookrightarrow \mathcal{D}(G).$$

Moreover, from the embedding $\mathfrak{g} \hookrightarrow \Theta(G)$, $x \mapsto x_L$, given by the left invariant vector fields, we get an embedding

$$U(\mathfrak{g}) \hookrightarrow \mathcal{D}(G),$$

where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$ (see Appendix A). This induces a map

$$\iota \colon \mathcal{O}(G) \otimes U(\mathfrak{g}) \hookrightarrow \mathcal{D}(G) \tag{B.7}$$

of $\mathcal{O}(G)$-modules. We have a structure of G-equivariant sheaf on both sides, with respect to the left translation action of G on itself. The G-equivariant structure on the left-hand-side comes from the G-action on $\mathcal{O}(G)$ induces by the left translation action of G on itself, that is, the G-action on $U(\mathfrak{g})$ is trivial; the G-action on the right-hand-side is described as follows: for $g \in G$, $f \in \mathcal{O}(G)$ and $\theta \in \mathfrak{g} \subset \Theta(G)$ then $g.(f\theta) = (g.f)\theta$.

Let $\mathcal{D}_l(G)$ be the algebra of left invariant differential operators on G, that is, the algebra of elements $\alpha \in \mathcal{D}(G)$ such that for all $g \in G$ and all $f \in \mathcal{O}(G)$, $\lambda_g(\alpha f) = \alpha(\lambda_g f)$.

Proposition B.3 *The map ι induces an isomorphism of $\mathcal{O}(G)$-modules,*

$$\mathcal{O}(G) \otimes U(\mathfrak{g}) \cong \mathcal{D}(G).$$

Moreover,

$$U(\mathfrak{g}) \cong \mathcal{D}_l(G) \cong \mathcal{D}(G)^G.$$

Proof Let us first show that ι is an isomorphism. The algebra $\mathcal{D}(G)$ is filtered by the order filtration $F_\bullet\mathcal{D}(G)$. On the other hand, the PBW filtration $F_\bullet U(\mathfrak{g})$ on $U(\mathfrak{g})$ induces a filtration $F_\bullet(\mathcal{O}(G) \otimes U(\mathfrak{g}))$ on $\mathcal{O}(G) \otimes U(\mathfrak{g})$ by setting

$$F_l(\mathcal{O}(G) \otimes U(\mathfrak{g})) := \mathcal{O}(G) \otimes F_l U(\mathfrak{g}), \quad l \in \mathbb{Z}_{\geqslant 0}.$$

The map ι sends $F_l(\mathcal{O}(G) \otimes U(\mathfrak{g}))$ to $F_l\mathcal{D}(G)$, and both filtrations are exhaustive. So it suffices to check that the map on associated graded space is an isomorphism. The associated graded of the right-hand-side is

$$\mathcal{O}_{T^*G} \cong \mathcal{O}_{G\times\mathfrak{g}^*} \cong \mathcal{O}(G) \otimes \mathcal{O}(\mathfrak{g}^*),$$

by (B.1), while by the PBW theorem the associated graded of the left-hand-side is

$$\mathcal{O}(G) \otimes S(\mathfrak{g}),$$

whence the statement since $\mathcal{O}(\mathfrak{g}^*) \cong S(\mathfrak{g})$.

Next, since the map ι is G-equivariant,

$$(\mathcal{D}(G))^G \cong (\mathcal{O}(G) \otimes U(\mathfrak{g}))^G \cong \mathcal{O}(G)^G \otimes U(\mathfrak{g}) \cong U(\mathfrak{g}) \hookrightarrow \mathcal{D}_l(G).$$

To show the other inclusion, observe that $\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathcal{O}(\mathfrak{g}^*)$ while

$$\operatorname{gr} \mathcal{D}_l(G) \cong (\mathcal{O}(G) \otimes \mathcal{O}(\mathfrak{g}^*))^G \cong \mathcal{O}(G)^G \otimes \mathcal{O}(\mathfrak{g}^*) \cong \mathcal{O}(\mathfrak{g}^*),$$

where G acts on on $\mathcal{O}(G)$ by λ_g, $g \in G$, and trivially on $\mathcal{O}(\mathfrak{g}^*)$. Hence, $\mathcal{D}_l(G) \cong (\mathcal{D}(G))^G \cong U(\mathfrak{g})$ as desired.

Appendix C
Poisson algebras, Poisson varieties and Hamiltonian reduction

We have compiled in this appendix some basic facts on Poisson algebras and Poisson varieties.

C.1 Poisson algebras and Poisson varieties

Let A be a commutative associative $\mathbb{C}$-algebra with unit.

Definition C.1 Suppose that A is endowed with an additional $\mathbb{C}$-bilinear bracket $\{\ ,\ \} : A \times A \to A$. Then A is called a *Poisson algebra* if the following conditions holds:

i). A is a Lie algebra with respect to $\{\ ,\ \}$,
ii). (*Leibniz rule*) $\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}$, for all $a, b, c \in A$.

The Lie bracket $\{\ ,\ \}$ is called a *Poisson bracket* on A.

Similarly, one can define the notion of *Poisson superalgebra*: see Appendix E.

Example C.1 Let (X, ω) be a symplectic variety. Then the algebra $(\mathscr{O}(X), \{\ ,\ \})$ of regular functions, with pointwise multiplication, is a Poisson algebra.

As an example, let $\mathfrak{g} = \mathrm{Lie}(G)$ be a complex algebraic finite-dimensional Lie algebra. and pick a coadjoint orbit $\mathbb{O} = G.\xi$ of $\mathfrak{g}^*$. Then $\mathbb{O}$ has a natural structure of symplectic structure, see e.g. [54, Proposition 1.1.5]; for $\xi \in \mathfrak{g}^*$, we have

$$T_\xi(\mathbb{O}) = T_\xi(G/G^\xi) \simeq \mathfrak{g}/\mathfrak{g}^\xi$$

and the bilinear form $\omega_\xi : (x, y) \mapsto \xi([x, y])$ descends to $\mathfrak{g}/\mathfrak{g}^\xi$. This gives the symplectic structure. Hence, together with a coadjoint orbit in $\mathfrak{g}^*$, we have a natural Poisson algebra.

C.2 Tensor products of Poisson algebras

C.3 Almost commutative algebras

In another direction, we have examples of Poisson algebras coming from some noncommutative algebras. Let B be an associative filtered (noncommutative) algebra with unit,

$$0 = B_{-1} \subset B_0 \subset B_1 \subset \cdots, \quad \bigcup_{i \geqslant 0} B_i = B,$$

such that $B_i.B_j \subset B_{i+j}$ for any $i, j \geqslant 0$. Let

$$A := \operatorname{gr} B = \bigoplus_i B_i/B_{i-1}$$

be its graded algebra (the multiplication in B gives rise a well-defined product $B_i/B_{i-1} \times B_j/B_{j-1} \to B_{i+j}/B_{i+j-1}$, making A an associative algebra). We said that B is *almost commutative* if A is commutative: this means that $a_i b_j - b_j a_i \in B_{i+j-1}$ for $a_i \in B_i, b_j \in B_j$.

Assume that B is almost commutative. Then $\operatorname{gr} B$ has a natural structure of Poisson algebra. We define the Poisson bracket

$$\{\ ,\ \} : B_i/B_{i-1} \times B_j/B_{j-1} \longrightarrow B_{i+j-1}/B_{i+j-2}$$

as follows: for $a_1 \in B_i/B_{i-1}$ and $a_2 \in B_j/B_{j-1}$, let b_1 (resp. b_2) be a representative of a_1 in B_i (resp. B_j) and set

$$\{a_1, a_2\} := b_1 b_2 - b_2 b_1 \mod B_{i+j-2}\,.$$

Then we can check the required properties.

Example C.2 Let $\mathfrak{g}$ be any complex finite-dimensional Lie algebra. Let $U_\bullet\mathfrak{g}$ be the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, that is, $U_i(\mathfrak{g})$ is the subspace of $U(\mathfrak{g})$ spanned by the products of at most i elements of $\mathfrak{g}$, and $U(\mathfrak{g})_0 = \mathbb{C}1$ (see Appendix A). Then

$$0 = U_{-1}(\mathfrak{g}) \subset U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \dots, \quad U(\mathfrak{g}) = \bigcup_i U_i(\mathfrak{g}),$$
$$U_i(\mathfrak{g}) \cdot U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}), \quad [U_i(\mathfrak{g}), U_j(\mathfrak{g})] \subset U_{i+j-1}(\mathfrak{g}).$$

The associated graded space $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i \geqslant 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})$ is naturally a Poisson algebra, and the PBW Theorem states that

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]$$

as Poisson algebras, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$.

Let us describe explicitly the Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$ (see [54, Proposition 1.3.18]). Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$, with structure constants $c_{i,j}^k$, that is, $[x_i, x_j] = \sum_k c_{i,j}^k x_k$. Through the canonical isomorphism $(\mathfrak{g}^*)^* \cong \mathfrak{g}$, any element of $\mathfrak{g}$ is regarded as a linear functions on $\mathfrak{g}^*$, and thus as an element of $\mathbb{C}[\mathfrak{g}^*]$. We get for $f, g \in \mathbb{C}[\mathfrak{g}^*]$,

$$\{f, g\} = \sum_{i,j,k} c_{i,j}^k x_k \frac{\partial f}{\partial x_i^*} \frac{\partial g}{\partial x_j^*}.$$

In a more concise way, we have:

$$\{f, g\} \colon \mathfrak{g}^* \longrightarrow \mathbb{C}, \quad \xi \longmapsto \langle \xi, [d_\xi f, d_\xi g] \rangle$$

where $d_\xi f, d_\xi g \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$ denote the differentials of f and g at ξ. In particular, if $x, y \in \mathfrak{g} \cong (\mathfrak{g}^*)^* \subset \mathbb{C}[\mathfrak{g}^*]$, then

$$\{x, y\} = [x, y].$$

Moreover, if $\mathbb{O}$ is a coadjoint orbit of $\mathfrak{g}^*$,

$$\{f, g\}|_{\mathbb{O}} = \{f|_{\mathbb{O}}, g|_{\mathbb{O}}\}_{\text{symplectic}}.$$

The above Poisson structure on $\mathbb{C}[\mathfrak{g}^*]$ is referred to as the *Kirillov-Kostant-Souriau Poisson structure*.

Example C.3 Let G be an affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, and $\mathcal{D}(G)$ the algebra of (global) differential operators on G. It is filtered by the order filtration $F_\bullet \mathcal{D}(G)$, see Section B.2.

According to Proposition B.1, the filtered algebra $\mathcal{D}(G)$ is almost commutative. It fact, by (B.1), one knows that

$$\operatorname{gr} \mathcal{D}(G) \cong \mathscr{O}_{T^*G},$$

where T^*G is the cotangent bundle of G. Thus $\mathscr{O}_{T^*G}$ inherits a Poisson algebra structure from that of $\operatorname{gr} \mathcal{D}(G)$.

One the other hand, T^*G is a symplectic variety and therefore $\mathscr{O}_{T^*G}$ has a Poisson algebra structure arising from its symplectic structure. It turns out that these two Poisson structures coincide (see, for example, [54, Theorem 1.3.10]).

A *affine Poisson scheme* (resp., *affine Poisson variety)* is an affine scheme $X = \operatorname{Spec} A$ (resp. $X = \operatorname{Specm} A$) such that A is a Poisson algebra. A *Poisson scheme* (resp. *Poisson variety)* is a scheme (resp. reduced scheme) such that the structure sheaf $\mathscr{O}_X$ is a sheaf of Poisson algebras.

For example, let B be as above and continue to assume that B is almost commutative, that is, $A = \operatorname{gr} B$ is commutative. Assume furthermore that A is a finitely generated commutative algebra without zero-divisors. In other words, $A = \mathbb{C}[X]$ is the coordinate ring of a (reduced) irreducible affine algebraic variety X. So the Poisson structure on A makes X a Poisson variety.

C.4 Symplectic leaves

If X is smooth, then one may view X as a complex-analytic manifold equipped with a holomorphic Poisson structure. For each point $x \in X$ one defines the *symplectic leaf* $\mathscr{L}_x$ through x to be the set of points that could be reached from x by going along Hamiltonian flows[1].

If X is not necessarily smooth, let $\mathrm{Sing}(X)$ be the singular locus of X, and for any $k \geqslant 1$ define inductively $\mathrm{Sing}^k(X) := \mathrm{Sing}(\mathrm{Sing}^{k-1}(X))$. We get a finite partition $X = \bigsqcup_k X^k$, where the strata $X^k := \mathrm{Sing}^{k-1}(X) \setminus \mathrm{Sing}^k(X)$ are smooth analytic varieties (by definition we put $X^0 = X \setminus \mathrm{Sing}(X)$). It is known (cf. e.g., [45]) that each X^k inherits a Poisson structure. So for any point $x \in X^k$ there is a well defined symplectic leaf $\mathscr{L}_x \subset X^k$. In this way one defines symplectic leaves on an arbitrary Poisson variety. In general, each symplectic leaf is a connected smooth analytic (but not necessarily algebraic) subset in X. However, if the algebraic variety X consists of finitely many symplectic leaves only, then it was shown in [45] that each leaf is a smooth irreducible locally-closed algebraic subvariety in X, and the partition into symplectic leaves gives an algebraic stratification of X.

Example C.4 If $\mathfrak{g} = \mathrm{Lie}(G)$ is an algebraic Lie algebra, the space $\mathfrak{g}^*$ is a (smooth) Poisson variety and the symplectic leaves of $\mathfrak{g}^*$ are the coadjoint orbits of $\mathfrak{g}^*$, cf. [179, Prop. 3.1]. The Poisson structure on the coadjoint orbits of $\mathfrak{g}^*$ is known as the *Kirillov-Kostant Poisson structure*.

If $\mathfrak{g}$ is simple, the nilpotent cone $\mathscr{N}$ of $\mathfrak{g}$, which is the (reduced) subscheme of $\mathfrak{g}^*$ associated with the augmentation ideal $\mathbb{C}[\mathfrak{g}^*]^G_+$ of the ring of invariants $\mathbb{C}[\mathfrak{g}^*]$, is an example of Poisson variety with finitely many symplectic leaves. These are precisely the nilpotent orbits of $\mathfrak{g}^* \cong \mathfrak{g}$.

C.5 Induced Poisson structures and Hamiltonian reduction

There are roughly two ways to construct a new Poisson variety from a Poisson manifold X: the *induction* and the *Hamiltonian reduction*.

Recall first a result of Weinstein about the induction; see [179, Prop. 3.10]:

Theorem C.1 (Weinstein) *Let Y be a submanifold of a Poisson manifold X such that:*

i). Y is transversal to the symplectic leaves, i.e., for any symplectic leaf S and any $x \in Y \cap S$, $T_xY + T_xS = T_xX$,
ii). for any $x \in Y$, $T_xY \cap T_xS$ is a symplectic subspace of T_xS, where S is the leaf of X containing x.

[1] A *Hamiltonian flow* in X from x to x' is a curve γ defined on an open neighborhood of $[0, 1]$ in $\mathbb{C}$, with $\gamma(0) = x$ and $\gamma(1) = x'$, which is an integral curve of a Hamiltonian vector field ξ_f, for some $f \in \mathscr{O}(X)$, defined on an open neighborhood of $\gamma([0, 1])$. See for example [138, Chap. 1] for more details.

Then, there is a natural induced Poisson structure on Y, and the symplectic leaf of Y through $x \in Y$ is $Y \cap S$ if S is the symplectic leaf through x in X.

Let us now turn to the classical Hamiltonian reduction. Let G be a Lie group, with Lie algebra $\mathfrak{g}$, acting on an affine Poisson variety $(X, \{\ ,\ \})$.

Definition C.2 The action of G in X is said to be *Hamiltonian* if there is a Lie algebra homomorphism

$$H\colon \mathfrak{g} \longrightarrow \mathscr{O}_X(X), \quad x \longmapsto H_x$$

such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr, dashed, "H"'] & \mathscr{X}(X) \\ & \mathscr{O}_X(X) \arrow[u] \end{tikzcd}$$

where $\mathscr{X}(X)$ is the Lie algebra of vector fields on X and the vertical map is the natural map from $\mathscr{O}_X(X)$ to $\mathscr{X}(X)$ given by $f \mapsto \{f,\ \}$. As for the horizontal map, it comes from the G-action on X. Namely, it is the map

$$\mathfrak{g} \longrightarrow \mathscr{X}(X), \quad a \longmapsto \big(x \mapsto \frac{d}{dt}(\exp(t \operatorname{ad} a).x)|_{t=0} \in T_x X\big).$$

The map $H\colon \mathfrak{g} \to \mathscr{O}_X(X)$ is called the *Hamiltonian*. Define the *moment map*

$$\mu\colon X \longrightarrow \mathfrak{g}^*$$

by assigning to $x \in X$ the linear function $\mu(x)\colon \mathfrak{g} \to \mathbb{C}$, $a \mapsto H_a(x)$. The moment map induces a Poisson algebra homomorphism

$$\mathbb{C}[\mathfrak{g}^*] \longrightarrow \mathscr{O}_X(X).$$

Moroever, if the group G is connected, then μ is G-equivariant with respect to the coadjoint action on $\mathfrak{g}^*$.

We refer to [179, Theorem 7.31] or [138, Proposition 5.39 and Definition 5.9] for the following result.

Theorem C.2 (Marsden-Weinstein) *Assume that G is connected and that the action of G in X is Hamiltonian. Let $\gamma \in \mathfrak{g}^*$. Assume that γ is a regular value*[2] *of μ, that $\mu^{-1}(\gamma)$ is G-stable and that $\mu^{-1}(\gamma)/G$ is a variety. Let $\iota\colon \mu^{-1}(\gamma) \hookrightarrow X$ and $\pi\colon \mu^{-1}(\gamma) \twoheadrightarrow \mu^{-1}(\gamma)/G$ be the natural maps: ι is the inclusion and π is the quotient map. Then the triple*

$$(X, \mu^{-1}(\gamma), \mu^{-1}(\gamma)/G)$$

[2] If $f : X \to Y$ is a smooth map between varieties, we say that a point y is a *regular value* of f if for all $x \in f^{-1}(y)$, the map $d_x f : T_x(X) \to T_y(Y)$ is surjective. If so, then $f^{-1}(y)$ is a subvariety of X and the codimension of this variety in X is equal to the dimension of Y.

is Poisson-reducible*, i.e., there exists a Poisson structure* $\{\,,\}'$ *on* $\mu^{-1}(\gamma)/G$ *such that for all open subset* $U \subset X$ *and for all* $f, g \in \mathscr{O}_X(\pi(U \cap \mu^{-1}(\gamma))$*, on has*

$$\{f, g\}' \circ \pi(u) = \{\tilde{f}, \tilde{g}\} \circ \iota(u)$$

at any point $u \in U \cap \mu^{-1}(\gamma)$*, where* $\tilde{f}, \tilde{g} \in \mathscr{O}_X(U)$ *are arbitrary extensions of* $f \circ \pi|_{U\cap\mu^{-1}(\gamma)}, g \circ \pi|_{U\cap\mu^{-1}(\gamma)}$ *to* U*.*

C.6 Poisson modules

Let R be a Poisson algebra. A *Poisson R-module* is a R-module M in the usual associative sense equipped with a bilinear map

$$R \times M \to M, \quad (r, m) \mapsto \operatorname{ad} r(m) = \{r, m\},$$

which makes M a Lie algebra module over R satisfying

$$\{r_1, r_2 m\} = \{r_1, r_2\}m + r_2\{r_1, m\}, \quad \{r_1 r_2, m\} = r_1\{r_2, n\} + r_2\{r_1, m\}$$

for $r_1, r_2 \in R$, $m \in M$.

Lemma C.1 *For any Lie algebra* $\mathfrak{g}$*, a Poisson module over* $\mathbb{C}[\mathfrak{g}^*]$ *is the same as a* $\mathbb{C}[\mathfrak{g}^*]$*-module* N *in the usual associative sense equipped with a Lie algebra module structure* $\mathfrak{g} \to \operatorname{End} M$*,* $x \mapsto \operatorname{ad}(x)$*, such that*

$$\operatorname{ad}(x)(fm) = \{x, f\}.m + f.\operatorname{ad}(x)(m)$$

for $x \in \mathfrak{g}$*,* $f \in \mathbb{C}[\mathfrak{g}^*]$*,* $m \in M$*.*

Example C.5 If $\mathfrak{g} = \operatorname{Lie}(G)$ is a simple Lie algebra, let $\overline{\mathcal{HC}}(\mathfrak{g})$ be the full subcategory of the category of Poisson $\mathbb{C}[\mathfrak{g}^*]$-modules on which the Lie algebra $\mathfrak{g}$-action is *integrable*, that is, locally finite. If X is an affine Poisson scheme equipped with a Hamiltonian G-action, then $\mathbb{C}[X]$ is an object of $\overline{\mathcal{HC}}(\mathfrak{g})$. Note that the action of $\mathbb{C}[\mathfrak{g}^*]$ on $\mathbb{C}[X]$ is given by $\{f, g\} = \{\mu^*(f), \mathfrak{g}\}$, for $f \in \mathbb{C}[\mathfrak{g}^*]$ and $g \in \mathbb{C}[X]$, where $\mu^* \colon \mathbb{C}[\mathfrak{g}^*] \to \mathbb{C}[X]$ is the comorphism of the moment map $\mu \colon X \to \mathfrak{g}^*$.

Appendix D
Nilpotent orbits and associated varieties of primitive ideals

In this appendix, $\mathfrak{g}$ is a complex simple Lie algebra with adjoint group G. We keep all the related notations used in Appendix A. Our main references for the results of Section D.1 are [106, 55, 176].

D.1 Nilpotent cone

Let $\mathscr{N} = \mathscr{N}(\mathfrak{g})$ be the *nilpotent cone of* $\mathfrak{g}$, that is, the set of all nilpotent elements of $\mathfrak{g}$. If $\mathfrak{g}$ is a simple Lie algebra of matrices, note that $\mathscr{N}$ coincides with the set of nilpotent matrices of $\mathfrak{g}$. For $e \in \mathfrak{g}$, we denote by $G.e$ its adjoint G-orbit. The nilpotent cone is a finite union of nilpotent G-orbits and it is itself the closure of the *regular nilpotent orbit* , denoted by $\mathbb{O}_{\text{reg}}$. It is the unique nilpotent orbit of codimension the rank r of $\mathfrak{g}$. An element $x \in \mathfrak{g}$ is *regular* if its centralizer $\mathfrak{g}^x$ has the minimal dimension, that is, the rank r of $\mathfrak{g}$. Thus, $\mathbb{O}_{\text{reg}}$ is the set of all regular nilpotent elements of $\mathfrak{g}$. Regular nilpotent elements are sometimes called *principal*.

Example D.1 If $\mathfrak{g} = \mathfrak{sl}_n$, then the rank of $\mathfrak{g}$ is $n-1$ and $\mathbb{O}_{\text{reg}}$ is the conjugacy class of the n-size Jordan block J_n, i.e., $\mathbb{O}_{\text{reg}} = SL_n.J_n$ with

$$J_n := \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} = \sum_{i=1}^{n-1} e_{i,i+1},$$

where $e_{i,j}$ is the elementary matrix whose entries are all zero, except the one in position (i, j) which equals 1.

Next, there is a unique dense open orbit in $\mathscr{N} \setminus \mathbb{O}_{\text{reg}}$ which is called the *subregular nilpotent orbit* of $\mathfrak{g}$, and denoted by $\mathbb{O}_{\text{subreg}}$. Its codimension in $\mathfrak{g}$ is the rank of $\mathfrak{g}$ plus two. At the extreme opposite, there is a unique nilpotent orbit of smallest positive dimension called the *minimal nilpotent orbit* of $\mathfrak{g}$, and denoted by $\mathbb{O}_{\min}$. Its dimension is $2h^\vee - 2$ ([182]).

D.2 Chevalley order

The set of nilpotent orbits in $\mathfrak{g}$ is naturally a poset $\mathscr{P}$ with partial order $\leqslant$, called the *Chevalley order*, or *closure order*, defined as follows: $\mathbb{O}' \leqslant \mathbb{O}$ if and only if $\mathbb{O}' \subseteq \overline{\mathbb{O}}$. The regular nilpotent orbit $\mathbb{O}_{\mathrm{reg}}$ is maximal and the zero orbit is minimal with respect to this order. Moreover, $\mathbb{O}_{\mathrm{subreg}}$ is maximal in the poset $\mathscr{P} \setminus \mathbb{O}_{\mathrm{reg}}$ and $\mathbb{O}_{\min}$ is minimal in the poset $\mathscr{P} \setminus \{0\}$.

The Chevalley order on $\mathscr{P}$ corresponds to a partial order on the set $\mathscr{P}(n)$ of partitions of n, for $n > 1$, for $\mathfrak{g} = \mathfrak{sl}_n$, first described by Gerstenhaber. More generally, the Chevalley order corresponds to a partial order on some subset of $\mathscr{P}(n)$ when $\mathfrak{g}$ is of classical type as we explain below.

Let $n \in \mathbb{Z}_{>0}$. As a rule, unless otherwise specified, we write an element $\boldsymbol{\lambda}$ of $\mathscr{P}(n)$ as a decreasing sequence $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ omitting the zeroes. Thus,

$$\lambda_1 \geqslant \cdots \geqslant \lambda_s \geqslant 1 \quad \text{and} \quad \lambda_1 + \cdots + \lambda_s = n.$$

We shall denote the dual partition of a partition $\boldsymbol{\lambda} \in \mathscr{P}(n)$ by ${}^t\boldsymbol{\lambda}$.

Let us denote by $\geqslant$ the partial order on $\mathscr{P}(n)$ relative to the dominance. More precisely, given $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_s), \boldsymbol{\eta} = (\mu_1, \dots, \mu_t) \in \mathscr{P}(n)$, we have $\boldsymbol{\lambda} \geqslant \boldsymbol{\eta}$ if

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i \quad \text{for} \quad 1 \leqslant k \leqslant \min(s,t).$$

D.2.1 Case $\mathfrak{sl}_n$

Every nilpotent matrix in $\mathfrak{sl}_n$ is conjugate to a Jordan block diagonal matrix. Therefore, the nilpotent orbits in $\mathfrak{g}$ are parameterized by $\mathscr{P}(n)$. We shall denote by $\mathbb{O}_{\boldsymbol{\lambda}}$ the corresponding nilpotent orbit of $\mathfrak{sl}_n$. Then $\mathbb{O}_{\boldsymbol{\lambda}}$ is represented by the standard Jordan form $\mathrm{diag}(J_{\lambda_1}, \dots, J_{\lambda_s})$, where J_k is the k-size Jordan block. If we write ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_t)$, then

$$\dim \mathbb{O}_{\boldsymbol{\lambda}} = n^2 - \sum_{i=1}^{t} d_i^2.$$

If $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}(n)$, then $\mathbb{O}_{\boldsymbol{\eta}} \subset \overline{\mathbb{O}_{\boldsymbol{\lambda}}}$ if and only if $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$.

The regular, subregular, minimal and zero nilpotent orbits of $\mathfrak{sl}_n$ correspond to the partitions (n), $(n-1, 1)$, $(2, 1^{n-2})$ and (1^n) of n, respectively.

We give in Figure D.1 the description of the poset $\mathscr{P}(n)$ for $n = 6$. The column on the right indicates the dimension of the orbits appearing in the same row. Such diagram is called a *Hasse diagram*.

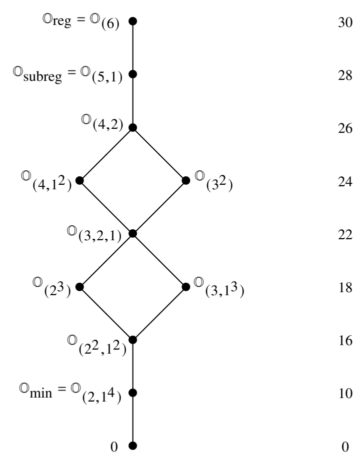

Fig. D.1 Hasse diagram for $\mathfrak{sl}_6$

D.2.2 Cases $\mathfrak{o}_n$ and $\mathfrak{so}_n$

For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_1(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n)\ ;\ \text{number of parts of each even number is even}\}.$$

The nilpotent orbits of $\mathfrak{so}_n$ are parametrized by $\mathscr{P}_1(n)$, with the exception that each *very even* partition $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$ (i.e., $\boldsymbol{\lambda}$ has only even parts) corresponds to two nilpotent orbits. For $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, not very even, we shall denote by $\mathbb{O}_{\mathbf{1},\boldsymbol{\lambda}}$, or simply by $\mathbb{O}_{\boldsymbol{\lambda}}$ when there is no possible confusion, the corresponding nilpotent orbit of $\mathfrak{so}_n$. For very even $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, we shall denote by $\mathbb{O}^{I}_{1,\boldsymbol{\lambda}}$ and $\mathbb{O}^{II}_{1,\boldsymbol{\lambda}}$ the two corresponding nilpotent orbits of $\mathfrak{so}_n$. In fact, their union forms a single $O(n)$-orbit. Thus nilpotent orbits of $\mathfrak{o}_n$ are parametrized by $\mathscr{P}_1(n)$.

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \mathscr{P}_1(n)$ and ${}^t\boldsymbol{\lambda} = (d_1, \ldots, d_t)$, then

$$\dim \mathbb{O}^{\bullet}_{1,\boldsymbol{\lambda}} = \frac{n(n-1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{t} d_i^2 - \#\{i; \lambda_i \text{ odd}\}\right),$$

where $\mathbb{O}^{\bullet}_{1,\lambda}$ is either $\mathbb{O}_{1,\lambda}$, $\mathbb{O}^{I}_{1,\lambda}$ or $\mathbb{O}^{II}_{1,\lambda}$ according to whether $\boldsymbol{\lambda}$ is very even or not. Using the same notations, If $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}_1(n)$, then $\overline{\mathbb{O}^{\bullet}_{1,\eta}} \subsetneq \overline{\mathbb{O}^{\bullet}_{1,\lambda}}$ if and only if $\boldsymbol{\eta} < \boldsymbol{\lambda}$, where $\mathbb{O}^{\bullet}_{1,\lambda}$ is either $\mathbb{O}_{1,\lambda}$, $\mathbb{O}^{I}_{1,\lambda}$ or $\mathbb{O}^{II}_{1,\lambda}$ according to whether $\boldsymbol{\lambda}$ is very even or not.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^{+} \in \mathscr{P}_1(n)$ such that $\boldsymbol{\lambda}^{+} \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\eta} \in \mathscr{P}_1(n)$ verifies $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}^{+}$. More precisely, let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ (adding zeroes if necessary). If $\boldsymbol{\lambda} \in \mathscr{P}_1(n)$, then $\boldsymbol{\lambda}^{+} = \boldsymbol{\lambda}$. Otherwise if $\boldsymbol{\lambda} \notin \mathscr{P}_1(n)$, set

$$\boldsymbol{\lambda}' = (\lambda_1, \dots, \lambda_s, \lambda_{s+1} - 1, \lambda_{s+2}, \dots, \lambda_{t-1}, \lambda_t + 1, \lambda_{t+1}, \dots, \lambda_n),$$

where s is maximum such that $(\lambda_1, \dots, \lambda_s) \in \mathscr{P}_1(\lambda_1 + \cdots + \lambda_s)$, and t is the index of the first even part in $(\lambda_{s+2}, \dots, \lambda_n)$. Note that $s = 0$ if such a maximum does not exist, while t is always defined. If $\boldsymbol{\lambda}'$ is not in $\mathscr{P}_1(n)$, then we repeat the process until we obtain an element of $\mathscr{P}_1(n)$ which will be our $\boldsymbol{\lambda}^{+}$.

D.2.3 Case $\mathfrak{sp}_n$

For $n \in \mathbb{N}^*$, set

$$\mathscr{P}_{-1}(n) := \{\boldsymbol{\lambda} \in \mathscr{P}(n) \; ; \text{ number of parts of each odd number is even}\}.$$

The nilpotent orbits of $\mathfrak{sp}_n$ are parametrized by $\mathscr{P}_{-1}(n)$. For $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}_{-1}(n)$, we shall denote by $\mathbb{O}_{-1,\lambda}$, or simply by $\mathbb{O}_{\lambda}$ when there is no possible confusion, the corresponding nilpotent orbit of $\mathfrak{sp}_n$, and if we write ${}^t\boldsymbol{\lambda} = (d_1, \dots, d_t)$, then

$$\dim \mathbb{O}_{-1,\lambda} = \frac{n(n+1)}{2} - \frac{1}{2}\left(\sum_{i=1}^{s} d_i^2 + \#\{i; \lambda_i \text{ odd}\}\right).$$

As in the case of $\mathfrak{sl}_n$, if $\boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathscr{P}_{-1}(n)$, then $\mathbb{O}_{-1,\eta} \subset \overline{\mathbb{O}_{-1,\lambda}}$ if and only if $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$.

Given $\boldsymbol{\lambda} \in \mathscr{P}(n)$, there exists a unique $\boldsymbol{\lambda}^{-} \in \mathscr{P}_{-1}(n)$ such that $\boldsymbol{\lambda}^{-} \leqslant \boldsymbol{\lambda}$, and if $\boldsymbol{\eta} \in \mathscr{P}_{-1}(n)$ verifies $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}$, then $\boldsymbol{\eta} \leqslant \boldsymbol{\lambda}^{-}$. The construction of $\boldsymbol{\lambda}^{-}$ is the same as in the orthogonal case except that t is the index of the first odd part in $(\lambda_{s+2}, \dots, \lambda_n)$.

D.3 Jacobson-Morosov Theorem and Dynkin grading

A $\frac{1}{2}\mathbb{Z}$-*grading* of the Lie algebra $\mathfrak{g}$ is a decomposition $\Gamma\colon \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ which verifies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i, j.

Lemma D.1 (cite[Proposition 20.1.5]Tauvel-Yu) *If Γ is a $\frac{1}{2}\mathbb{Z}$-grading of $\mathfrak{g}$, then for some semisimple element h_Γ of $\mathfrak{g}$,*

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \; ; \; [h_\Gamma, x] = 2jx\}.$$

Let $(\ |\ ) = \dfrac{1}{2h^\vee}\kappa_{\mathfrak{g}}$ be the non-degenerate symmetric bilinear form on $\mathfrak{g}$ as in Appendix A. Since the bilinear form $(\ |\ )$ of $\mathfrak{g}$ is ad h_Γ-invariant and nondegenerate, we get

$$(\mathfrak{g}_i, \mathfrak{g}_j) = 0 \iff i + j \neq 0.$$

Hence $\mathfrak{g}_j$ and $\mathfrak{g}_{-j}$ are in pairing. In particular, they have the same dimension.

Fix a nonzero nilpotent element $e \in \mathfrak{g}$. By the Jacobson-Morosov Theorem (cf. e.g., [55, §3.3]), there exist $h, f \in \mathfrak{g}$ such that the triple (e, h, f) verifies the $\mathfrak{sl}_2$-*triple* relations:

$$[h, e] = 2e, \qquad [e, f] = h, \qquad [h, f] = -2f.$$

In particular, h is semisimple and the eigenvalues of ad h are integers. Moreover, e and f belong to the same nilpotent G-orbit.

Example D.2 Let $\mathfrak{g} = \mathfrak{sl}_n$, and set,

$$e := J_n, \qquad h := \sum_{i=1}^{n}(n + 1 - 2i)e_{i,i}, \qquad f := \sum_{i=1}^{n-1} i(n - i)e_{i+1,i}.$$

Then (e, h, f) is an $\mathfrak{sl}_2$-triple. From this observation, we readily construct $\mathfrak{sl}_2$-triples for any standard Jordan form $\mathrm{diag}(J_{\lambda_1}, \ldots, J_{\lambda_n})$ with $(\lambda_1, \ldots, \lambda_n) \in \mathscr{P}(n)$.

The group G acts on the collection of $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ by simultaneous conjugation. This defines a natural map:

$$\Omega : \{\mathfrak{sl}_2\text{-triples}\}/G \longrightarrow \{\text{nonzero nilpotent orbits}\}, \quad (e, h, f) \mapsto G.e.$$

Theorem D.1 ([55, Theorem 3.2.10]) *The map Ω is bijective.*

The map Ω is surjective according to Jacobson-Morosov Theorem. The injectivity is a result of Kostant ([55, Theorem 3.4.10]); see [183, §2.6] for a sketch of proof.

Since h is semisimple and since the eigenvalues of ad h are integers, we get a $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$ defined by h, called the *Dynkin grading* associated with h:

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j, \qquad \mathfrak{g}_j := \{x \in \mathfrak{g} \colon [h, x] = 2jx\}. \tag{D.1}$$

We have $e \in \mathfrak{g}_1$. Moreover, it follows from the representation theory of $\mathfrak{sl}_2$ that $\mathfrak{g}^e \subset \bigoplus_{j \geqslant 0} \mathfrak{g}_j$ and that $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{\frac{1}{2}}$.

One can draw a picture to visualize the above properties. Decompose $\mathfrak{g}$ into simple $\mathfrak{sl}_2$-modules $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ and denote by d_k the dimension of V_k for $k = 1, \ldots, s$. We can assume that $d_1 \geqslant \cdots \geqslant d_s \geqslant 1$. We have $\dim V_k \cap \mathfrak{g}_j \leqslant 1$ for any $j \in \frac{1}{2}\mathbb{Z}$. We represent the module V_k on the k-th row with d_k boxes, each box corresponding to a nonzero element of $V_k \cap \mathfrak{g}_j$ for j such that $V_k \cap \mathfrak{g}_j \neq \{0\}$. We organize the rows so that the $2j$-th column corresponds to a generator of $V_k \cap \mathfrak{g}_j$. Then the boxes appearing on the right position of each row lie in $\mathfrak{g}^e$.

Example D.3 Consider the element $e = \mathrm{diag}(J_3, J_1)$ of $\mathfrak{sl}_4$. Here, we get $\dim \mathfrak{g}_0 = 5$, $\dim \mathfrak{g}_{\frac{1}{2}} = 0$, $\dim \mathfrak{g}_1 = 4$ and $\dim \mathfrak{g}_2 = 1$. The corresponding picture is given in Fig. D.2. In the Fig. D.2, the empty boxes □ correspond to nonzero elements lying

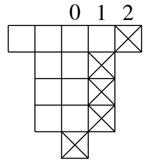

Fig. D.2 Decomposition into $\mathfrak{sl}_2$-modules for $(3, 1)$

in $[f, \mathfrak{g}]$. The boxes ⊠ correspond to nonzero elements lying in $\mathfrak{g}^e$.

This is an example of *even nilpotent element*, which means that $\mathfrak{g}_i = \{0\}$ for all half-integers i. The nilpotent orbit of an even nilpotent element is called an *even nilpotent orbit*. Note that the regular nilpotent orbit is always even.

Example D.4 Consider the element $e = \mathrm{diag}(J_2, J_1, J_1)$ of $\mathfrak{sl}_4$ which lies in the minimal nilpotent orbit of $\mathfrak{sl}_4$. Here, we get $\dim \mathfrak{g}_0 = 5$, $\dim \mathfrak{g}_{\frac{1}{2}} = 4$, $\dim \mathfrak{g}_1 = 1$. The corresponding picture is given in Fig. D.3:

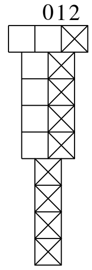

Fig. D.3 Decomposition into $\mathfrak{sl}_2$-modules for $(2, 1^2)$

We observe that $\bigoplus_{i \geqslant 1} \mathfrak{g}_i$ equals $\mathfrak{g}_1$ and has dimension 1.

Remark D.1 This is actually a general fact: if e lies in the minimal nilpotent orbit of a simple $\mathfrak{g}$, then $\oplus_{i \geqslant 1} \mathfrak{g}_i = \mathfrak{g}_1 = \mathbb{C}e$ and thus $\bigoplus_{i \geqslant 1} \mathfrak{g}_i$ has dimension 1.

One can assume that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is also a Cartan subalgebra of the reductive Lie algebra $\mathfrak{g}_0$.

Lemma D.2 *i). For any $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_j$ for some $j \in \frac{1}{2}\mathbb{Z}$.*
ii). Fix a root system Δ_0 of $(\mathfrak{g}_0, \mathfrak{h})$, and set $\Delta_{0,+} = \Delta_+ \cap \Delta_0$. Then

$$\Delta_+ = \Delta_{0,+} \cup \{\alpha \; ; \; \mathfrak{g}_\alpha \subset \mathfrak{g}_{>0}\}.$$

Denoting by Π the set of simple roots of Δ_+, we get

$$\Pi = \bigcup_{j \in \frac{1}{2}\mathbb{Z}} \Pi_j \quad \text{with} \quad \Pi_j := \{\alpha \in \Pi \; ; \; \mathfrak{g}_\alpha \subset \mathfrak{g}_j\}.$$

Lemma D.3 *We have* $\Pi = \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$.

Proof Assume that there exists $\beta \in \Pi_s$ for $s > 1$. A contradiction is expected. Since $e \in \mathfrak{g}_1$ and since $\mathfrak{g}_1$ is contained in the subalgebra generated by the root spaces $\mathfrak{g}_\alpha$ with $\alpha \in \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$, we get $[e, \mathfrak{g}_{-\beta}] = \{0\}$. In other words, $\mathfrak{g}_{-\beta} \subset \mathfrak{g}^e$. This contradicts the fact that $\mathfrak{g}^e \subset \mathfrak{g}_{\geqslant 0}$. □

From Lemma D.3 we define the *weighted Dynkin diagram*, or *characteristic*, of the nilpotent orbit $G.e$ when $\mathfrak{g}$ is simple as follows. Consider the Dynkin diagram of the simple Lie algebra $\mathfrak{g}$. Each node of this diagram corresponds to a simple root $\alpha \in \Pi$. Then the weighted Dynkin diagram is obtained by labeling the node corresponding to α with the value $\alpha(h) \in \{0, 1, 2\}$.

By convention, the zero orbit has a weighted Dynkin diagram with every node labeled with 0.

Example D.5 In type E_6, the characteristics of the regular, subregular and minimal nilpotent orbits are respectively:

An important consequence of Lemma D.3 is that there are only finitely many nilpotent orbits, namely at most $3^{\operatorname{rank}\mathfrak{g}}$. Also, the weighted Dynkin diagram is a complete invariant, i.e., two such diagrams are equal if and only if the corresponding nilpotent orbits are equal, [55, Theorem 3.5.4].

The regular nilpotent orbit always corresponds to the weighted Dynkin diagram with only 2's (this result is not obvious, cf. e.g., [55, Theorem 4.1.6]). More generally, a nilpotent orbit is even if and only if the weighted Dynkin diagram have only 2's or 0's (see Example D.3 for the definition of *even*).

D.4 Digression on primitive ideals

Let I be a two-sided ideal of the enveloping algebra $U(\mathfrak{g})$. The PBW filtration on $U(\mathfrak{g})$ induces a filtration on I, so that $\operatorname{gr} I$ becomes a graded Poisson ideal in $\mathbb{C}[\mathfrak{g}^*]$ (see Example C.2). Denote by $\mathscr{V}(I)$ the zero locus of $\operatorname{gr} I$ in $\mathfrak{g}^*$,

$$\mathscr{V}(I) := \operatorname{Specm}(\mathbb{C}[\mathfrak{g}^*]/\operatorname{gr} I) \subset \mathfrak{g}^*.$$

The set $\mathscr{V}(I)$ is usually referred to as the *associated variety* of I. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through a non-degenerate bilinear symmetric form on $\mathfrak{g}$, we shall often view associated varieties of two-sided ideals of $U(\mathfrak{g})$ as subsets of $\mathfrak{g}$.

A proper two-sided ideal I of $U(\mathfrak{g})$ is called *primitive* if it is the annihilator of a simple left $U(\mathfrak{g})$-module. Let us mention two important results on primitive ideals of $U(\mathfrak{g})$.

Theorem D.2 (Duflo Theorem [62]) *Any primitive ideal in* $U(\mathfrak{g})$ *is the annihilator* $\operatorname{Ann}_{U(\mathfrak{g})} L_{\mathfrak{g}}(\lambda)$ *of some irreducible highest weight representation* $L_{\mathfrak{g}}(\lambda)$ *of* $\mathfrak{g}$*, where* $\lambda \in \mathfrak{h}^*$.

Theorem D.3 (Irreducibility Theorem [41, 128, 108]) *The associated variety $\mathscr{V}(I)$ of a primitive ideal I in $U(\mathfrak{g})$ is irreducible, specifically, it is the closure $\overline{\mathbb{O}}$ of some nilpotent orbit $\mathbb{O}$ in $\mathfrak{g}$.*

In particular, the associated variety of a primitive ideal in contained in the nilpotent cone $\mathscr{N}$, which is a crucial property. Theorem D.3 was first partially proved (by a case-by-case argument) in [41], and in a more conceptual way in [128] and [108] (independently), using many earlier deep results due to Joseph, Gabber, Lusztig, Vogan and others.

It is possible that different primitive ideals share the same associated variety. At the same time, not all nilpotent orbit closures appear as associated variety of some primitive ideal of $U(\mathfrak{g})$.

Let $\lambda \in \mathfrak{h}^*$. The *associated variety* of the irreducible highest weight representation $L_{\mathfrak{g}}(\lambda)$ of $\mathfrak{g}$ is

$$\mathscr{V}(L_{\mathfrak{g}}(\lambda)) := \mathscr{V}(\mathrm{Ann}_{U(\mathfrak{g})}(L_{\mathfrak{g}}(\lambda))).$$

Naturally, the geometry of $\mathscr{V}(L_{\mathfrak{g}}(\lambda))$ is expected to reflect some properties of the representation $L_{\mathfrak{g}}(\lambda)$.

Example D.6 It is known that the irreducible highest weight representation $L_{\mathfrak{g}}(\lambda)$ is finite-dimensional if and only if its associated variety $\mathscr{V}(L_{\mathfrak{g}}(\lambda))$ is reduced to $\{0\}$.

Appendix E
Superalgebras and Clifford algebras

A *superspace* is a $\mathbb{C}$-vector space E equipped with a $\mathbb{Z}_2$-grading, $E = E^{\bar{0}} \oplus E^{\bar{1}}$. Elements in $E^{\bar{0}}$ are called *even*, elements of $E^{\bar{1}}$ are called *odd*. We denote by $|v| \in \{\bar{0}, \bar{1}\}$ the parity of homogeneous elements $v \in E$. A *morphism of superspaces* is a linear map preserving $\mathbb{Z}_2$-gradings. It is itself a superspace by:

$$\mathrm{Hom}(E, F)^{\bar{0}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{0}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{1}}),$$
$$\mathrm{Hom}(E, F)^{\bar{1}} = \mathrm{Hom}(E^{\bar{0}}, F^{\bar{1}}) \oplus \mathrm{Hom}(E^{\bar{1}}, F^{\bar{0}}).$$

The category of superspaces is a tensor category. Then one may define superalgebras, Lie superalgebras, Poisson superalgebras, etc. as the algebra objects, Lie algebra objects, Poisson algebra objects etc. in this tensor category.

For example, a *Lie superalgebra* is a superspace A together with a bracket $[\ ,\ ] \colon A \times A \to A$ such that for all homogeneous elements $a, b \in A$,

$$[a, b] = -(-1)^{|a||b|}[b, a],$$
$$[[a, b], c] = [a, [b, c]] - (-1)^{|a||b|}[b, [a, c]].$$

Note that any superalgebra A is naturally a Lie superalgebra by setting for all homogeneous elements $a, b \in A$,

$$[a, b] = ab - (-1)^{|a||b|}ba.$$

It is *supercommutative* if $[A, A] = 0$.

A superspace A is a *Poisson superalgebra* if it is equipped with a bracket $\{\ ,\ \} \colon A \times A \to A$ such that $(A, \{\ ,\ \})$ is a Lie superalgebra and for any $a \in A$, the operator $\{a,\ .\ \} \colon A \to A$ is a *superderivation*: for all homogeneous elements $a, b \in A$,

$$\{a, bc\} = \{a, b\}c + (-1)^{|a||b|}b\{a, c\}.$$

Let E be a $\mathbb{C}$-vector space. The *exterior algebra* $\bigwedge E$ is the quotient of the tensor algebra $T(E) = \bigoplus_{k \in \mathbb{Z}} T^k(E)$, with $T^k(E) = E \otimes \cdots \otimes E$ the k-fold tensor product, by the two-sided ideal $I(E)$ generated by elements of the form $v \otimes w + w \otimes v$ with $v, w \in E$. The product in $\bigwedge E$ is usually denoted by $v \wedge w$. Since $I(E)$ is graded, the exterior algebra inherits a grading

$$\bigwedge E = \bigoplus_{k \in \mathbb{Z}} \bigwedge^k E.$$

Clearly, $\bigwedge^0 E = \mathbb{C}$ and $\bigwedge^1 E = E$. We may thus think of $\bigwedge E$ as the associative algebra linearly generated by E, subject to the relations $v \wedge w + w \wedge v = 0$. We will regard $\bigwedge E$ as a graded superalgebra, where the $\mathbb{Z}_2$-grading is the mod 2 reduction of the $\mathbb{Z}$-grading. Since

$$[u_1, u_2] = u_1 \wedge u_2 - (-1)^{k_1 k_2} u_2 \wedge u_1 = 0$$

for $u_1 \in \bigwedge^{k_1} E$ and $u_2 \in \bigwedge^{k_2} E$, we see that $\bigwedge E$ is supercommutative.

Assume that E is endowed with a symmetric bilinear form $B \colon E \times E \to E$ (possibly degenerate).

Definition E.1 The *Clifford algebra*[1] $Cl(E, B)$ is the quotient of $T(E)$ by the two-sided ideal $\mathscr{I}(E, B)$ generated by all elements of the form

$$v \otimes w + w \otimes v - B(v, w)1, \qquad v, w \in E.$$

Clearly, $Cl(E, 0) = \bigwedge V$.

The inclusions $\mathbb{C} \to T(E)$ and $E \to T(E)$ descend to inclusions $\mathbb{C} \to Cl(E, B)$ and $E \to Cl(E, B)$ respectively. We will always view E as a subspace of $Cl(E, B)$.

Let us view $T(E) = \bigoplus_{k \in \mathbb{Z}} T^k(E)$ as a filtered superalgebra, with the $\mathbb{Z}_2$-grading and filtration inherited from the $\mathbb{Z}$-grading. Since the elements $v \otimes w + w \otimes v - B(v, w)1$ are even, of filtration degree 2, the ideal $\mathscr{I}(E, B)$ is a filtered super subspace of $T(E)$, and hence $Cl(E, B)$ inherits the structure of a filtered superalgebra. The $\mathbb{Z}_2$-grading and filtration on $Cl(E, B)$ are defined by the condition that the generators $v \in E$ are odd, of filtration degree 1. In the decomposition

$$Cl(E, B) = Cl(E, B)^{\bar{0}} \oplus Cl(E, B)^{\bar{1}}$$

the two summands are spanned by products $v_1 \dots v_k$ with k even, respectively odd. We will always regard $Cl(E, B)$ as a filtered superalgebra. Then the defining relations for the Clifford algebra become

$$[v, w] = vw + wv = B(v, w), \qquad v, w \in E.$$

The *quantization map*, given by the anti-symmetrization:

$$q \colon \bigwedge(E) \to Cl(E, B), \quad v_1 \wedge \dots \wedge v_k \mapsto \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) v_{\sigma 1} \dots v_{\sigma k},$$

---

[1] In [154], there is a factor 2. For some reasons, we prefer here a different normalization.

with $\mathfrak{S}_k$ the permutation group of order k, is an isomorphism of superspaces. Its inverse is called the *symbol map*.

Proposition E.1 *The symbol map $\sigma\colon Cl(E,B) \to \bigwedge E$ induces an isomorphism of graded superalgebras,*

$$\operatorname{gr} Cl(E,B) \xrightarrow{\simeq} \bigwedge E.$$

Since $\bigwedge(E)$ is supercommutative, $\operatorname{gr} Cl(E,B)$ inherits a Poisson superalgebra structure[2], and the graded symbol map is an isomorphism of graded Poisson superalgebras. The Poisson bracket on $\bigwedge E$ can be described by:

$$\{v,w\} = B(v,w), \qquad v,w \in E = \bigwedge^1 E.$$

For more about Clifford algebras, we refer to the recent book of Eckhard Meinrenken (it also adsresses *Weil algebras* and *quantized Weil algebras*) [154].

[2] The arguments are similar to the case of almost commutative algebras; see §C.1.

Hints for the exercises

Hints for the exercises of Chapter 2

2.3 Notice that the locality axiom is automatically satisfied by the OPE (cf. Proposition 2.1, (ii)$\Rightarrow$(i)).

Hints for the exercises of Chapter 3

3.3 Apply the "Frobenius reciprocity", which asserts that

$$\mathrm{Hom}_{\hat{\mathfrak{g}}}(U(\hat{\mathfrak{g}}) \otimes_{\mathfrak{g}[t]\oplus\mathbb{C}K} \mathbb{C}_k, V^k(\mathfrak{g})) \cong \mathrm{Hom}_{\mathfrak{g}[t]\oplus\mathbb{C}K}(\mathbb{C}_k, V^k(\mathfrak{g})).$$

Hints for the exercises of Chapter 4

4.1

i). Describe $F^p \mathrm{Vir}^c_\Delta$, where $\Delta \in \mathbb{Z}_{\geqslant 0}$, using the PBW Theorem.
ii). Just use (1).
iii). Remember that by Remark 4.2, one can go one step further, and then compute $\sigma_1(L_{(0)}L)$, $\sigma_0(L_{(1)}L)$ using the commuting relations.

Hints for the exercises of Chapter 4

Hints for the exercises of Chapter 5

5.1 Note that the maximal submodule of $L_1(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ to show that $R_V \cong \mathrm{Zhu}(\mathrm{gr}^F V)$ and use Remark 5.1.

Hints for the exercises of Chapter 8

8.1

i). Kostant's Separation Theorem [133, Th. 0.2 and 0.11] says that $S = ZH$, where $Z \cong \mathbb{C}[\Omega]$ is the center of the symmetric algebra S of $\mathfrak{sl}_2$, and H is the space of invariant harmonic polynomials which decomposes, as an $\mathfrak{sl}_2$-module, as $H = \bigoplus_{\lambda \in \mathbb{Z}_{\geqslant 0}} V_\lambda^{m_\lambda}$, with $m_\lambda = 1$ for all λ. Therefore, $S^{\mathrm{ad}\, e} = \bigoplus_{\lambda \in \mathbb{Z}_{\geqslant 0}} Z V_\lambda^{\mathrm{ad}\, e}$. To conclude, observe that, v being a singular vector, it has a fixed weight and, hence, a fixed degree.

ii). Note that from (i), $\Omega e \in \sqrt{I_k}$ and, so, $\Omega\, \mathfrak{sl}_2 \in \sqrt{I_k}$, whence $\Omega \in \sqrt{I_k}$. But in $\mathfrak{sl}_2$, the nilpotent cone is precisely the zero locus of Ω.

8.2

i). Just use the commuting relations in $V^{-3/2}(\mathfrak{sl}_3)$.

ii). Observe that the image I_k of the maximal proper maximal ideal of $V^{-3/2}(\mathfrak{sl}_3)$ is generated by the vector $\bar{v}$ as an $(\mathrm{ad}\, \mathfrak{sl}_3)$-module, where

$$\bar{v} = \frac{1}{3}\,(h_1 - h_2)\, e_{1,3} + e_{1,2} e_{2,3}$$

is the image of v in $R_{V^{-3/2}(\mathfrak{sl}_3)} \cong \mathbb{C}[h_i, e_{k,l}\,;\, i = 1, 2, k \neq l]$. Verify that

$$(\mathrm{ad}\, e_{3,2})(\mathrm{ad}\, e_{2,1})\bar{v} = -e_{1,2}e_{2,1} + e_{1,3}e_{3,1} + \frac{1}{3}\,(2h_1 + h_2)\, h_2,$$
$$(\mathrm{ad}\, e_{2,1})(\mathrm{ad}\, e_{3,2})\bar{v} = -e_{2,3}e_{3,2} + e_{1,3}e_{3,1} + \frac{1}{3}\,(h_1 + 2h_2)\, h_1,$$

and deduce from this that the intersection $X_{L_{-3/2}(\mathfrak{sl}_3)} \cap \mathfrak{h}$ is zero. For the last part, resume the arguments of the proof of Proposition 8.1.

iii). Verify that $e_{1,2} + e_{2,3}$ is not in $X_{L_{-3/2}(\mathfrak{sl}_3)}$.

iv). Observe that $X_{L_{-3/2}(\mathfrak{sl}_3)}$ cannot be reduced to zero.

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