Representations of Galois and of GL_2 in characteristic p

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Graduate course at Columbia University (fall 2007)
Monday-Wednesday 2.40 pm - 3.55 pm

WARNING: These notes are informal and are not intended to be published. I apologize for the inaccuracies, flaws and English mistakes that they surely contain.

Week 1 (September 10 and 12): Introduction

Week 2 (September 17 and 19): Classification of 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over $\overline{\mathbb{F}}_p$; classification of weights

Week 3 (September 24 and 26): Bruhat-Tits tree and Ihara-Tits Lemma; compact induction and the Hecke operator T

Week 4 (October 1 and 3): Classification of smooth admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$; semi-simple modulo p Langlands correspondence

Week 5 (October 8 and 10): Diagrams, examples coming from representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, other examples; statement of the existence theorem and beginning of proof: injective envelopes of weights

Week 6 (October 15 and 17): Proof of existence theorem: injective envelopes of diagrams

Week 7 (October 22 and 24): Results on some representations of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$: principal series and injective envelopes of weights

Week 8 (October 29 and 31): Diamond weights: background and definition; Diamond diagrams I: definition

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1 Week 1

1.1 Introduction I

Fix p a prime number and F a finite extension of $\mathbb{Q}_p$. The subject of this course is to look for a modulo p Langlands "correspondence" between on the one side (continuous) representations of the linear group $\mathrm{GL}_2(F)$ over an algebraically closed field of characteristic p (usually an algebraic closure of $\mathbb{F}_p$) and on the other side (continuous) 2-dimensional representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over the same field. Here $\overline{\mathbb{Q}}_p$ is an algebraic closure of F (which is also an algebraic closure of $\mathbb{Q}_p \subseteq F$) and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is the set of field automorphisms of $\overline{\mathbb{Q}}_p$ fixing F pointsetwise. Before stating some of the results that will be proved (or at least sketched) in this course, I first need to recall briefly some well known results about Langlands correspondences for $\mathrm{GL}_1(F)$ and $\mathrm{GL}_2(F)$.

Let me start with $\mathrm{GL}_1(F)$ and recall the main result of local class field theory.

The field $\overline{\mathbb{Q}}_p$ contains a ring of integers $\overline{\mathbb{Z}}_p$ which is a local ring with maximal ideal $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$. The field $\overline{\mathbb{Z}}_p/\mathfrak{m}_{\overline{\mathbb{Z}}_p}$ is an algebraic closure of $\mathbb{F}_p$ and we call it $\overline{\mathbb{F}}_p$. Let $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) := \{$field automorphisms of $\overline{\mathbb{F}}_p$ fixing $\mathbb{F}_p$ pointsetwise$\}$. As $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ preserves both $\overline{\mathbb{Z}}_p$ and $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$, we deduce a group morphism:

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \xrightarrow{\alpha} \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

and we denote by $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ the kernel: the subgroup $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ is called the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Recall that all the above groups are profinite

(i.e. projective limits of finite groups) and thus compact (for the projective limit topology of the discrete topology on each finite group). The group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is pro-cyclic and topologically generated by a specific element Fr called the Frobenius element which acts on $\overline{\mathbb{F}}_p$ as $\mathrm{Fr}(x) = x^p$ $(x \in \overline{\mathbb{F}}_p)$. We denote by $\mathrm{W}(\overline{\mathbb{Q}}_p/F) \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ of elements that map to a finite power of Fr in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. It is called the Weil group of F.

Theorem 1.1. *(i) If K/F is a finite Galois extension and $N_{K/F} : K \to F$ is the norm map, there is a canonical group morphism:*

$$\mathrm{Gal}(K/F) \longrightarrow F^\times / N_{K/F}(K^\times)$$

which induces an isomorphism:

$$\mathrm{Gal}(K/F)^{\mathrm{ab}} \overset{\sim}{\longrightarrow} F^\times / N_{K/F}(K^\times)$$

where $\mathrm{Gal}(K/F)^{\mathrm{ab}}$ is the maximal abelian quotient of $\mathrm{Gal}(K/F)$.

(ii) Taking the projective limit over all K and restricting to $\mathrm{W}(\overline{\mathbb{Q}}_p/F)$, the isomorphisms in (i) induce a group isomorphism:

$$r_F : \mathrm{W}(\overline{\mathbb{Q}}_p/F)^{\mathrm{ab}} \overset{\sim}{\longrightarrow} F^\times$$

where $\mathrm{W}(\overline{\mathbb{Q}}_p/F)^{\mathrm{ab}}$ is the maximal abelian quotient of $\mathrm{W}(\overline{\mathbb{Q}}_p/F)$.

Let me briefly recall how the morphism (i) is defined. Let K^{unr} be the maximal unramified extension of K inside $\overline{\mathbb{Q}}_p$. For any K as in (i), the above map α factors through $\mathrm{Gal}(K^{\mathrm{unr}}/F)$. Let $\mathrm{Fr}(K^{\mathrm{unr}}/F) \subset \mathrm{Gal}(K^{\mathrm{unr}}/F)$ be the subset of elements w such that $\alpha(w) \in \mathbb{Z}_{>0}$. The restriction map $\mathrm{Fr}(K^{\mathrm{unr}}/F) \to \mathrm{Gal}(K/F)$ turns out to be surjective. Recall that for any finite extension L of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$, the ring $\overline{\mathbb{Z}}_p \cap L$ is a local ring and that we call a uniformizer of L any element of $\overline{\mathbb{Z}}_p \cap L$ which is a generator of its maximal ideal. Then the morphism in (i) of Theorem 1.1 is determined by sending $g \in \mathrm{Gal}(K/F)$ to the class of the element $N_{K(\hat{g})/F}(\pi(\hat{g}))^{-1} \in F^\times$ where $\hat{g} \in \mathrm{Fr}(K^{\mathrm{unr}}/F)$ is a lifting of g, $K(\hat{g})$ is the subfield of K^{unr} of elements fixed by $\hat{g}$ (a finite extension of F), $\pi(\hat{g})$ is a uniformizer of $K(\hat{g})$ and $N_{K(\hat{g})/F}$ is the norm map from $K(\hat{g})$ to F.

Theorem 1.1 has the following obvious corollary, which is local Langlands correspondence for GL_1:

Corollary 1.2. *Let E be any algebraically closed field. There is a canonical bijection between isomorphism classes of locally constant 1-dimensional representations of $\mathrm{W}(\overline{\mathbb{Q}}_p/F)$ over E and isomorphism classes of locally constant irreducible admissible representations of $\mathrm{GL}_1(F)$ over E.*

By definition, a locally constant representation of a topological group on a vector space means a representation such that each vector is fixed by an open subgroup. We also say "smooth" representation. A smooth representation is called admissible if moreover the subvector space of elements fixed by an open subgroup is always finite dimensional (over E). Using that an endomorphism acting on a finite dimensional E-vector space always has non-zero eigenvectors, one easily deduces that a smooth admissible irreducible representation of $\mathrm{GL}_1(F)$ over E is necessarily 1-dimensional.

Since the end of the seventies, we have:

Theorem 1.3. *Let E be any algebraically closed field of characteristic 0. There is a canonical bijection between isomorphism classes of smooth irreducible 2-dimensional representations of $\mathrm{W}(\overline{\mathbb{Q}}_p/F)$ over E and isomorphism classes of smooth admissible irreducible supercuspidal representations of $\mathrm{GL}_2(F)$ over E.*

Theorem 1.3 involves the work of many people (Weil, Jacquet, Langlands, Kutzko). It was extended in 1998 to $\mathrm{GL}_n(F)$ by Harris-Taylor and (independently) Henniart. We write ρ for a representation of $\mathrm{W}(\overline{\mathbb{Q}}_p/F)$ and π for a representation of $\mathrm{GL}_2(F)$. So the theorem is a correspondence $\rho \leftrightarrow \pi$. It is a result of Casselman that any smooth irreducible representation of $\mathrm{GL}_n(F)$ over an algebraically closed field of characteristic zero is automatically admissible, hence we can forget "admissible" in Theorem 1.3. I have now to explain what "supercuspidal" means. Among smooth admissible representations of $\mathrm{GL}_2(F)$, there are very easy ones called parabolic inductions. They are of the form:
$$\mathrm{ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi$$
where $B(F) \subset \mathrm{GL}_2(F)$ is the subgroup of upper triangular matrices, $\chi : B(F) \to E^\times$ is a locally constant character and where:

$$\mathrm{ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi := \{f : \mathrm{GL}_2(F) \to E \text{ locally constant, } f(bg) = \chi(b)f(g)\}$$

($b \in B(F)$, $g \in \mathrm{GL}_2(F)$) with left action of $\mathrm{GL}_2(F)$ given by $(g'f)(g) := f(gg')$. Such representations are always smooth admissible (whatever $\mathrm{char}(E)$ is) and are irreducible for "most" χ.

Definition 1.4. *A smooth irreducible admissible representation of $\mathrm{GL}_2(F)$ over E is called supercuspidal if it is not a subquotient of a parabolic induction.*

Definition 1.4 works for all E. Finally, I have to explain how Theorem 1.1 is "incorporated" in Theorem 1.3. For a smooth irreducible (admissible) representation π of $\mathrm{GL}_2(F)$ over E, Schur's lemma applies, that is $F^\times \hookrightarrow \mathrm{GL}_2(F)$ acts on the underlying E-vector space by multiplication by a smooth character $\chi_\pi : F^\times \to E^\times$. Then the correspondence of Theorem 1.3, among many other deep properties, satisfies $\chi_\pi \circ r_F = \det(\rho)$ if $\rho \leftrightarrow \pi$. That is, $\det(\rho)$ and χ_π match under the correspondence of Corollary 1.5. Finally, let us mention that Theorem 1.3 can be extended to include all smooth irreducible representations of $\mathrm{GL}_2(F)$, not just those supercuspidal, but one needs to introduce the Weil-Deligne group and I don't want to go into this (as I won't use it in this course).

We now have a look at what happens when $\mathrm{char}(E) \neq 0$. Let us first assume that $\mathrm{char}(E) = \ell$ with $\ell \neq p$. Then it is a theorem of Vignéras that Theorem 1.3 goes through without change in that case (and we can as well forget the word "admissible" in the statement as admissibility again follows from irreducibility). The proof is essentially a reduction modulo ℓ proof starting from Theorem 1.3. I won't say more in that case and assume now, and till the end of that course, that $\mathrm{char}(E) = p$.

Then it is not an exaggeration to say: "we enter a new universe".

We first look at the case $F = \mathbb{Q}_p$. As usual when one enters a new universe, things look quite the same at the beginning: it turns out that Theorem 1.3 goes through in that case and this was proven by Barthel-Livné (and myself) some time ago.

Theorem 1.5. *Let E be any algebraically closed field of characteristic p. There is a canonical bijection between isomorphism classes of smooth irreducible 2-dimensional representations of $\mathrm{W}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over E and isomorphism classes of smooth admissible irreducible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E.*

We have the same compatibility as in the previous cases with class field theory (Corollary 1.2). Note however that it is not known here whether we can forget the word "admissible" as for the two other cases. Let us make explicit the correspondence of Theorem 1.5 as it is important in this course.

Fix an embedding $\overline{\mathbb{F}}_p = \overline{\mathbb{Z}}_p/\mathfrak{m}_{\overline{\mathbb{Z}}_p} \hookrightarrow E$ and consider the character ω_2 of

$I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ obtained as follows:

$$g \in I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \mapsto \frac{g\left(\sqrt[p^2-1]{p}\right)}{\sqrt[p^2-1]{p}} \in \mu_{p^2-1}(\overline{\mathbb{Z}}_p) \mapsto (\overline{\mathbb{Z}}_p/\mathfrak{m}_{\overline{\mathbb{Z}}_p})^\times \hookrightarrow E^\times$$

where $\sqrt[p^2-1]{p}$ is any (p^2-1)-root of p. The character ω_2 doesn't depend on the chosen root but depends on the chosen embedding. However, any other choice of embedding gives either ω_2 or ω_2^p and this won't matter in the sequel. The characters (ω_2, ω_2^p) were first introduced by Serre and are called Serre's fundamental characters of level 2. For each integer $r \in \{0, \cdots, p-2\}$ there exists a unique smooth irreducible 2-dimensional representation ρ_r of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over E such that:

$$\rho_r|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$$

and with determinant the modulo p cyclotomic character to the power $r+1$ (this representation actually extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$). Moreover any smooth irreducible 2-dimensional representation of $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over E is isomorphic to some $\rho_r \otimes_E \chi$ where $\chi : W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to E^\times$ is a smooth character.

Remark 1.6. In fact, up to unramified twist, we have $\rho_r \simeq \mathrm{ind}_{W(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})}^{W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \omega_2^{r+1}$ where $\mathbb{Q}_{p^2}$ is the quadratic unramified extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ (and ω_2^{r+1} is an extension of ω_2^{r+1} to $W(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^2})$).

The correspondence of Theorem 1.3 in the case $\mathrm{char}(E) = p$ and $F = \mathbb{Q}_p$ is then as follows:

$$\rho_r \otimes_E \chi \mapsto \pi_r \otimes \left(\chi \circ r_{\mathbb{Q}_p}^{-1} \circ \det\right) \tag{1}$$

with:

$$\pi_r := \left(\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2\right)/T.$$

Let me briefly explain what π_r is. First, $\mathrm{Sym}^r E^2$ is the r-th symmetric product of the standard representation E^2 of $GL_2(\mathbb{Z}_p)$ (acting via its quotient $GL_2(\mathbb{F}_p)$ on the canonical basis of E^2). Note that this is possible as E has characteristic p. This action is extended to $GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ by sending $p \in \mathbb{Q}_p^\times$ to the identity. Such a representation of $GL_2(\mathbb{Z}_p)$ is called a weight. Now if σ is a weight, the representation $\mathrm{ind}_{GL_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{GL_2(\mathbb{Q}_p)} \sigma$ means the E-vector space of functions:

$$f : GL_2(\mathbb{Q}_p) \to \text{underlying space of } \sigma$$

with compact support modulo $\mathbb{Q}_p^\times$ (that is, with compact image in the quotient $\mathrm{PGL}_2(\mathbb{Q}_p)$) and such that $f(kg) = \sigma(k)(f(g))$ ($k \in \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$, $g \in \mathrm{GL}_2(\mathbb{Q}_p)$). Such an induction is called a compact induction. As for the parabolic induction, the (left) action of $\mathrm{GL}_2(\mathbb{Q}_p)$ is given by $(g'f)(g) := f(gg')$. Finally, $T \in \mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2\big)$ is a certain entertwining operator analogous to the Hecke operator "T_p" that I won't describe here (in fact, one has $\mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2\big) = E[T]$). One important property of the representations π_r is:

Theorem 1.7. *We have* $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)} \pi_r = \mathrm{Sym}^r E^2 \oplus (\mathrm{Sym}^{p-1-r} E^2 \otimes \det^r)$.

Recall that the socle of a representation of a group G (over E) is the maximal semi-simple subrepresentation, that is the maximal subrepresentation which is a direct sum of irreducible representations.

For $F = \mathbb{Q}_p$, the representations π_r exhaust all irreducible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $E^\times$ up to twist. This turns out to completely break down when $F \neq \mathbb{Q}_p$.

1.2   Introduction II

I now give the main new results of that course that for most of them concern the case $F \neq \mathbb{Q}_p$ but F unramified over $\mathbb{Q}_p$ and were all obtained in collaboration with V. Paskunas. I will assume for simplicity that E is an algebraic closure of $\mathbb{F}_p$, but I will keep the notation E to distinguish it from $\overline{\mathbb{Z}}_p/\mathfrak{m}_{\overline{\mathbb{Z}}_p}$. This assumption has the advantage that now, smooth finite dimensional representations of $\mathrm{W}(\overline{\mathbb{Q}}_p/F)$ over E all extend to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. So I will also forget about Weil groups and only use now Galois groups.

Let me just say right away that Theorem 1.4 completely breaks down in that case: there are many more supercuspidal representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over E than 2-dimensional smooth irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over E. Although no full classification of supercuspidal is known so far when $F \neq \mathbb{Q}_p$, I will build many of them in that course, and relate some of them to 2-dimensional smooth irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over E.

I introduce now some notations that I will keep throughout. Let $\mathcal{O}_F$ be the ring of integers in F. Recall $\mathcal{O}_F$ is a local principal ring, that is every ideal can be generated by one element and there is a unique maximal ideal $\mathfrak{m}_F$. As $\mathfrak{m}_F$ is maximal, $\mathcal{O}_F/\mathfrak{m}_F$ is a field which is a finite extension of $\mathbb{F}_p$, therefore isomorphic to $\mathbb{F}_q$, $q = p^f$ for an integer $f \in \mathbb{Z}_{\geq 1}$. I assume till the

end of this introduction F unramified, that is $[F : \mathbb{Q}_p] = f$ (and will tell when this assumption is actually useless). In the unramified case, p is still a generator of $\mathfrak{m}_F$.

I let $K := \mathrm{GL}_2(\mathcal{O}_F)$, $I \subset K$ the subgroup of upper triangular matrices modulo p, $I_1 \subset I$ its maximal pro-p subgroup, that is the subgroup of upper unipotent matrices modulo p, $K_1 \subset I_1$ the subgroup of matrices that are congruent to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ modulo p and $N \subset \mathrm{GL}_2(F)$ the normalizer of I in $\mathrm{GL}_2(F)$, that is the subgroup of $\mathrm{GL}_2(F)$ generated by K, $F^\times$ and the matrix $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$. If $\chi : I \to E$ is a smooth character, I let $\chi^s := \chi(\Pi \cdot \Pi^{-1})$ and note that $\Pi g \Pi^{-1} \in I$ if $g \in I$.

The main idea to construct representations π of $\mathrm{GL}_2(F)$ over E is to first construct the triple $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ where π^{K_1} (resp. π^{I_1}) denotes the sub-vector space (of the underlying space of π) of elements fixed by K_1 (resp. I_1). More precisely π^{K_1} is seen as a representation of $K/K_1 = \mathrm{GL}_2(\mathbb{F}_q)$, π^{I_1} is seen as a representation of N (as N also normalizes I_1) and can is the canonical injection $\pi^{I_1} \subset \pi^{K_1}$.

I try now to formalize this: I call a basic diagram any triple $D := (D_0, D_1, r)$ where D_0 is a smooth representation of $KF^\times$ over E such that $p \in F^\times$ acts trivially, D_1 a smooth representation of N over E and $r : D_1 \hookrightarrow D_0$ an injection inducing an $IF^\times$-equivariant isomorphism $D_1 \xrightarrow{\sim} D_0^{I_1}$. For instance $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ is such a diagram. A basic diagram is said to be irreducible if it doesn't contain any non-zero strict basic subdiagram.

Theorem 1.8. *Assume $p > 2$. Let $D = (D_0, D_1, r)$ be a basic diagram and assume $D_0^{K_1}$ is finite dimensional.*

(i) There exists at least one smooth admissible representation π of $\mathrm{GL}_2(F)$ over E such that:

    (a) $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0$

    (b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ contains D

    (c) π is generated by D_0.

(ii) Assume D is irreducible. Then any smooth admissible π satisfying (a), (b) and (c) is irreducible.

This theorem has to be thought of as an existence theorem only, as unicity in (i) is wrong in general. Moreover, it has nothing to do with F unramified over $\mathbb{Q}_p$ and works for any finite extension of $\mathbb{Q}_p$. The idea (due

to Paskunas) is to build π inside the injective envelope $\mathrm{inj}_K D_0$ of the K-representation D_0 in the category of smooth representations of K over E. Roughly speaking, the main point is to prove one can non-canonically extend the action of I on $\mathrm{inj}_K D_0$ to an action of N such that there exists an injection $(D_0, D_1, r) \hookrightarrow (\mathrm{inj}_K D_0, \mathrm{inj}_K D_0, \mathrm{id})$, which is possible as injective envelopes are very flexible. Then the two compatible actions of K and N on the same vector space $\mathrm{inj}_K D_0$ glue to give an action of $\mathrm{GL}_2(F)$ (this is Ihara-Tits Lemma) and π is defined as the subspace generated by D_0. The whole process is highly non-canonical both because the action of K on $\mathrm{inj}_K D_0$ is only defined up to non-unique isomorphism and because the extension to an action of N involves choices. The converse to (ii) is wrong in general: reducible diagrams can lead to π as in (i) being irreducible.

Let me now fix an irreducible 2-dimensional representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over E. There is a description of ρ in terms of fundamental characters analogous to the one above for $F = \mathbb{Q}_p$. Buzzard, Diamond and Jarvis, in their quest for a generalization of Serre's conjecture to totally real field, have associated to ρ a set of weights $\mathcal{D}(\rho)$, that is a set of irreducible representations of K over E.

Example 1.9. If $F = \mathbb{Q}_p$ and $\rho = \rho_r$ with $0 \le r \le p-1$, then the set of weights $\mathcal{D}(\rho_r)$ is precisely $\{\mathrm{Sym}^r E^2, \mathrm{Sym}^{p-1-r} E^2 \otimes \det^r\}$, that is the weights in the K-socle of π_r by Theorem 1.7.

We won't however consider in that course all irreducible ρ but only those which are generic (I will give a precise definition below).

Example 1.10. If $F = \mathbb{Q}_p$ then ρ is generic provided $1 \le r \le p-2$.

Most of irreducible ρ are generic. For such ρ, one has exactly $|\mathcal{D}(\rho)| = 2^f$. Because of Theorem 1.7 (and because of the conjectures of Buzzard, Diamond and Jarvis), it is natural to get primarily interested in those irreducible supercuspidal π such that their K-socle is the direct sum of the weights of $\mathcal{D}(\rho)$ (when $f > 1$, using Theorem 1.8 one can build many irreducible supercuspidal that don't satisfy such a property for any ρ). We first build basic diagrams such that $\mathrm{soc}_K D_0$ satisfies this property:

Theorem 1.11. *Fix an irreducible generic Galois representation ρ.*

(i) There exists a unique finite dimensional representation $D_0(\rho)$ of $\mathrm{GL}_2(\mathbb{F}_q)$ over E such that:

(a) $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)} D_0(\rho) \simeq \oplus_{\sigma \in \mathcal{D}(\rho)} \sigma$

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(b) each irreducible σ in $\mathcal{D}(\rho)$ only occurs once as a Jordan-Hölder factor of $D_0(\rho)$ (hence in the socle)

(c) $D_0(\rho)$ is maximal (for inclusion) for properties (a) and (b).

(ii) Each Jordan-Hölder factor of $D_0(\rho)$ only occurs once in $D_0(\rho)$.

(iii) As an I-representation, one has:

$$D_0(\rho)^{I_1} \simeq \bigoplus_{\substack{certain \ (\chi,\chi^s) \\ \chi \neq \chi^s}} \chi \oplus \chi^s$$

(in particular $D_0(\rho)^{I_1}$ is stable under $\chi \mapsto \chi^s$).

This is a theorem on representation theory of finite groups. The assertion (i) is a general fact that works for any set of distinct weights (not just the sets $\mathcal{D}(\rho)$) but (ii) seems quite specific to the combinatorics of the weights of $\mathcal{D}(\rho)$.

Example 1.12. If $F = \mathbb{Q}_p$ and $\rho = \rho_r$ with $1 \leq r \leq p - 2$, then we have $D_0(\rho_r) = E_r \oplus (E_{p-1-r} \otimes \det^r)$ where E_s is an extension $0 \to \mathrm{Sym}^s E^2 \to E_s \to \mathrm{Sym}^{p-3-s} \otimes \det^{s+1} \to 0$ ($E_s = \mathrm{Sym}^s E^2$ if $p - 3 - s < 0$).

Let me now go back to the setting of the first theorem and assume that p acts trivially on $\det(\rho)$ (via the local reciprocity map r_F of Theorem 1.1) which is always possible up to twist. Using (ii) and (iii) of Theorem 1.1, one can uniquely extend the action of I on $D_0(\rho)^{I_1}$ to an action of N. I denote by $D_1(\rho)$ the resulting representation of N. The idea is then to use $D_0(\rho)$ and $D_1(\rho)$ to associate a basic diagram to ρ but one needs to choose an $IF^\times$-equivariant injection $r : D_1(\rho) \hookrightarrow D_0(\rho)$. Up to isomorphisms of basic diagrams, it turns out there are infinitely many such injections as soon as $f > 1$! We denote by $D(\rho, r) := (D_0(\rho), D_1(\rho), r)$ any such basic diagram: it is not irreducible in general.

Example 1.13. If $F = \mathbb{Q}_p$ and $\rho = \rho_r$ with $1 \leq r \leq p - 2$, then there is a unique possible $D_r := D(\rho_r)$ (sorry for the conflict of notations with the r's!) and we will see that $(\pi_r^{K_1}, \pi_r^{I_1}, \mathrm{can}) \cong D_r$.

I would have liked to state in general that there exists a unique (up to isomorphism) smooth admissible representation $\pi(\rho, r)$ of $\mathrm{GL}_2(F)$ over E which is generated by its K_1-invariant vectors and which is such that $(\pi(\rho, r)^{K_1}, \pi(\rho, r)^{I_1}, \mathrm{can}) \cong D(\rho, r)$. However, when $f > 1$, we could prove neither unicity nor existence. The only result we have so far is the following corollary to Theorem 1.8:

Corollary 1.14. *(i) There exists a smooth admissible representation π of $\mathrm{GL}_2(F)$ over E such that:*

(a) $\mathrm{soc}_K \pi = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$

(b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ contains $D(\rho, r)$

(c) π is generated by $D_0(\rho)$.

(ii) If $D(\rho, r)$ and $D(\rho, r')$ are two non-isomorphic basic diagrams associated to ρ, and π, π' are as in (i) respectively for $D(\rho, r)$ and $D(\rho, r')$, then π and π' are non-isomorphic.

My student Hu recently proved that a π satisfying (a), (b), (c) as in (i) of Corollary 1.14 is actually *not* unique.

We can at least prove an irreducibility result :

Theorem 1.15. *Any π as in (i) of Theorem 1.8 is irreducible and is a supercuspidal representation.*

Theorem 1.15 uses in a crucial way the fact F is unramified over $\mathbb{Q}_p$ and is the most involved result of this course. Note that one can't use (ii) of Theorem 1.8 as the basic diagrams $D(\rho, r)$ are not irreducible in general.

2 Week 2

2.1 Classification of 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ over $\overline{\mathbb{F}}_p$

We let F be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}_F$ the ring of integers in F, $\mathfrak{m}_F$ its maximal ideal and $\mathbb{F}_q = \mathbb{F}_{p^f}$ its residue field $\mathcal{O}_F/\mathfrak{m}_F$. We fix ϖ_F a uniformizer of F. We have $[F : \mathbb{Q}_p] = ef$ where $e \in \mathbb{Z}_{\geq 1}$ is called the ramification index. Recall E is an algebraic closure of $\mathbb{F}_p$ and $\overline{\mathbb{Q}}_p$ is an algebraic closure of F.

We have an exact sequence of profinite groups:

$$\mathrm{I}(\overline{\mathbb{Q}}_p/F) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$$

where $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ is the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q) \subset \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is the image of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. Let $\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$ be the maximal pro-p subgroup of $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ (called wild inertia). Recall that for any $n \in \mathbb{Z}_{\geq 1}$, the subgroup $\{x \in \overline{\mathbb{F}}_p^\times, x^{p^n-1} = 1\}$ is just $\mathbb{F}_{p^n}^\times$ where $\mathbb{F}_{p^n} \subset \overline{\mathbb{F}}_p$ is the unique subfield of cardinality p^n. If A is a commutative group and m an integer, $\mu_m(A)$ denotes the subgroup of elements a such that $a^m = 1_A$ (with multiplicative notations).

Lemma 2.1. *For $n \in \mathbb{Z}_{\geq 1}$, the maps $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \to \mathbb{F}_{p^n}^\times$ obtained by sending $g \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)$ to the image of:*

$$\frac{g\left(\sqrt[p^n-1]{\varpi_F}\right)}{\sqrt[p^n-1]{\varpi_F}} \in \mu_{p^n-1}(\overline{\mathbb{Z}}_p)$$

in $\overline{\mathbb{F}}_p^\times = (\overline{\mathbb{Z}}_p/\mathfrak{m}_{\overline{\mathbb{Z}}_p})^\times$ are well defined (and independent of any choice) and induce an isomorphism of topological groups:

$$\mathrm{I}(\overline{\mathbb{Q}}_p/F)/\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w \overset{\sim}{\longrightarrow} \varprojlim_n \mathbb{F}_{p^n}^\times$$

where the projective limit is taken with respect to the norm maps $\mathbb{F}_{p^{nm}}^\times \to \mathbb{F}_{p^n}^\times$.

Proof. Let F^{unr} be the infinite extension of F obtained by adding all m-roots of elements of $\mathcal{O}_F^\times$ for all m prime to p, then one has $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \overset{\sim}{\to} \mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{unr}})$. Let F^m be the infinite extension of F^{unr} obtained by adding all m-roots of ϖ_F for some m prime to p. Then one has:

$$\mathrm{I}(\overline{\mathbb{Q}}_p/F)/\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w \overset{\sim}{\longrightarrow} \varprojlim_m \mathrm{Gal}(F^m/F^{\mathrm{unr}})$$

where the transition maps are restriction maps (the quotient on the left is called tame inertia). But the map $g \mapsto \frac{g(\sqrt[m]{\varpi_F})}{\sqrt[m]{\varpi_F}}$ induces a group isomorphism $\mathrm{Gal}(F^m/F^{\mathrm{unr}}) \simeq \mu_m(\overline{\mathbb{Z}}_p)$ which is independent both of the choice of the uniformizer ϖ_F and of the m-root $\sqrt[m]{\varpi_F}$ (all this because the m-root of a element of $\mathcal{O}_F^\times$ is in F^{unr} and therefore $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{unr}})$ acts trivially on it). As m is prime to p, reduction modulo $\mathfrak{m}_{\overline{\mathbb{Z}}_p}$ induces a group isomorphism $\mu_m(\overline{\mathbb{Z}}_p) \overset{\sim}{\to} \mu_m(\overline{\mathbb{F}}_p)$ (this is Hensel's Lemma applied to $P(x) = x^m - 1$). Therefore, one has $\mathrm{I}(\overline{\mathbb{Q}}_p/F)/\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w \overset{\sim}{\longrightarrow} \varprojlim_m \mu_m(\overline{\mathbb{F}}_p)$ where the transition maps are $\mu \in \mu_{mm'}(\overline{\mathbb{F}}_p) \mapsto \mu^m \in \mu_{m'}(\overline{\mathbb{F}}_p)$. But any integer m prime to p divides an integer of the form $p^n - 1$ for a convenient n (take n such that $\overline{p}^n = 1$ in $(\mathbb{Z}/m\mathbb{Z})^\times$ where $\overline{p} \in (\mathbb{Z}/m\mathbb{Z})^\times$ is the image of p). It is thus enough to take the projective limits over integers of the form $p^n - 1$. The result follows then from $\mu_{p^n-1}(\overline{\mathbb{F}}_p) = \mathbb{F}_{p^n}^\times \subset \overline{\mathbb{F}}_p^\times$. $\square$

Lemma 2.2. *Any continuous character $\theta : \mathrm{I}(\overline{\mathbb{Q}}_p/F) \to E^\times$ (where the latter is endowed with the discrete topology) factors through $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \twoheadrightarrow \mathbb{F}_{p^n}^\times \overset{\theta}{\longrightarrow} E^\times$ for some $n > 0$.*

Proof. Because of the continuity assumption, the inverse image of the open subgroup $\{1\} \subset E^\times$ is an open subgroup U of $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ and therefore θ

factors through the finite quotient $\mathrm{I}(\overline{\mathbb{Q}}_p/F)/U$. The image of $\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$ in $\mathrm{I}(\overline{\mathbb{Q}}_p/F)/U$ is a finite p-group, but $E^\times$ doesn't contain any non-trivial p-torsion element, that is any x such that $x^{p^m} = 1$ with $m > 0$, therefore $\theta(\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w) = \{1\}$ and $\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w \subset U$. So we see that θ factors through a finite quotient of $\mathrm{I}(\overline{\mathbb{Q}}_p/F)/\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$, hence through some $\mathbb{F}_{p^n}^\times$ by Lemma 2.1. $\qquad\square$

Definition 2.3 (Serre)**.** *A fundamental character of level $n > 0$ is a character $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \to E^\times$ that factors through $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \twoheadrightarrow \mathbb{F}_{p^n}^\times \to E^\times$.*

Let us fix a field embedding $\iota : \mathbb{F}_{p^n} \hookrightarrow E$ and denote by ω_n the fundamental character $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \twoheadrightarrow \mathbb{F}_{p^n}^\times \overset{\iota}{\hookrightarrow} E^\times$ (which depends on ι). We see that any fundamental character of level n is of the form $\omega_n^{i_0 + p i_1 + \cdots + p^{n-1} i_{n-1}}$ with $0 \le i_j \le p - 1$. Note that, if all $i_j = p - 1$, we get $\omega_n^{p^n - 1} = 1$.

Lemma 2.4. *If m divides n, we have $\omega_n^{1 + p^m + p^{2m} + \cdots + p^{(\frac{n}{m} - 1)m}} = \omega_m$.*

Proof. This follows from Lemma 2.1 and the fact that:

$$N_{\mathbb{F}_{p^n}/\mathbb{F}_{p^m}}(x) = x^{1 + p^m + p^{2m} + \cdots + p^{(\frac{n}{m} - 1)m}}$$

for $x \in \mathbb{F}_{p^n}^\times$ (product of all conjugates). $\qquad\square$

Lemma 2.5. *The character ω_n extends (non-canonically) from $\mathrm{I}(\overline{\mathbb{Q}}_p/F) = \mathrm{Gal}(\overline{\mathbb{Q}}_p/F^{\mathrm{unr}})$ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ if and only if n divides f.*

Proof. The Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$ is topologically generated by the element $\mathrm{Fr}^f : x \mapsto x^q$ ($x \in \mathbb{F}_q$). Let $s \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ be any lifting of Fr^f that acts trivially on $F(\sqrt[p^n-1]{\varpi_F})$ (it exists because $F(\sqrt[p^n-1]{\varpi_F})$ has the same residue field as F and thus one still has a surjection $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F(\sqrt[p^n-1]{\varpi_F})) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$). As $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ is normal in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ (being the kernel of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$), one has $sgs^{-1} \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)$ for any $g \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)$. As $s(\sqrt[p^n-1]{\varpi_F}) = \sqrt[p^n-1]{\varpi_F}$ and $s(\sqrt[p^n-1]{1}) = (\sqrt[p^n-1]{1})^q$ (recall $\mu_{p^n-1}(\overline{\mathbb{Z}}_p) \overset{\sim}{\to} \mu_{p^n-1}(\overline{\mathbb{F}}_p)$), one gets:

$$\omega_n(sgs^{-1}) = \frac{sgs^{-1}(\sqrt[p^n-1]{\varpi_F})}{\sqrt[p^n-1]{\varpi_F}} = s\left(\frac{g(\sqrt[p^n-1]{\varpi_F})}{\sqrt[p^n-1]{\varpi_F}}\right) = \left(\frac{g(\sqrt[p^n-1]{\varpi_F})}{\sqrt[p^n-1]{\varpi_F}}\right)^q = \omega_n(g)^q$$

for any $g \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)$. If ω_n extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, one also has $\omega_n(sgs^{-1}) = \omega_n(s)\omega_n(g)\omega_n(s^{-1}) = \omega_n(g)$. Thus we get $\omega_n(g) = \omega_n(g)^q$ for any $g \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)$ which implies $\omega_n^{q-1} = 1$ which implies n divides f by Definition 2.3. Here is

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an obvious explicit way to extend ω_n when n divides f : use that for any $(q-1)$-root $\sqrt[q-1]{\varpi_F}$ of ϖ_F and any $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, the element:

$$\frac{g\left(\sqrt[q-1]{\varpi_F}\right)}{\sqrt[q-1]{\varpi_F}} \in \mu_{q-1}(\overline{\mathbb{Z}}_p)$$

still doesn't depend on $\sqrt[q-1]{\varpi_F}$ and induces a character $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathbb{F}_q^\times$ which is ω_f in restriction to $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$ (use that $\mu_{q-1}(\overline{\mathbb{Z}}_p) \subset F^\times$ and hence g acts trivially on it). Note that this character depends on the choice of ϖ_F. Now, take $\omega_f^{1+p^n+p^{2n}+\cdots+p^{(\frac{f}{n}-1)n}}$. $\qquad\square$

The same proof yields that a fundamental character extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ if and only if it is a power of ω_f.

Lemma 2.6. *Let G be a finite group such that $|G| = p^n$ and let V be a non-zero $\mathbb{F}_p$-vector space on which G acts. Then $V^G \neq 0$.*

Proof. This is a standard result of finite group theory. We recall the proof as it is easy: let $x \in V$, $x \neq 0$ and let $W \subseteq V$ be the non-zero sub-$\mathbb{F}_p$-vector space generated by gx, $g \in G$. As G is finite, W is a finite set of cardinality p^m with $m > 0$. As G acts on the additive group W, one has $W = \amalg_i Gw_i$ for a finite set $\{w_i\}$ of elements of W. We have $|Gw_i| = p^{n_i}$ for each i and $n_i = 0$ if and only if $w_i \in W^G$, therefore $p^m = |W| = pc + |W^G|$ where c is an integer. As $m > 0$, this implies p divides $|W^G|$, and as $W^G \neq \emptyset$ (because $0 \in W^G$), we get $|W^G| > 1$: there is $y \in V$, $y \neq 0$ such that $y \in V^G$. $\qquad\square$

We fix in the sequel a (field) embedding $\mathbb{F}_{q^2} \hookrightarrow E$ and denote by ω_{2f} the corresponding fundamental character. Recall we have $\omega_f = \omega_{2f}^{1+q}$ (Lemma 2.4).

Proposition 2.7. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous representation, then ρ is of one of the following forms:*

(i) ρ is reducible and:

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_f^{m_1} & * \\ 0 & \omega_f^{m_2} \end{pmatrix}$$

where m_i are two integers

(ii) ρ is irreducible and:

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_{2f}^{m} & 0 \\ 0 & \omega_{2f}^{qm} \end{pmatrix}$$

where m is an integer such that $q+1$ doesn't divide m.

Proof. (i) Assume first that ρ is reducible, hence in particular $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ is reducible of the form $\left(\begin{smallmatrix} \chi_1 & * \\ 0 & \chi_2 \end{smallmatrix}\right)$ where χ_1, χ_2 are continuous characters $\mathrm{I}(\overline{\mathbb{Q}}_p/F) \to E^\times$ that extend to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. By Lemma 2.2 we have $\chi_i = \omega_{n_i}^{m_i}$ for some integers n_i ($i \in \{1,2\}$) and we can choose n_i as small as possible via Lemma 2.4. As the characters extend to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, we have n_i divides f ($i \in \{1,2\}$) by Lemma 2.5 (the same proof works with $\omega_{n_i}^{m_i}$ instead of ω_{n_i}) and by Lemma 2.4 we go back to $n_1 = n_2 = f$.

(ii) The subgroup $\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$ is also normal in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Let $\rho^{\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w}$ be the subspace of (the underlying space of) ρ where $\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$ acts trivially. For $v \in \rho^{\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w}$, $g \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and $w \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$, one has $wgv = g(g^{-1}wg)v = gv$ because $g^{-1}wg \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$, hence $gv \in \rho^{\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w}$ and $\rho^{\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$-stable. But $\rho^{\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w} \neq 0$ by Lemma 2.6. As ρ is irreducible, we must thus have $\rho = \rho^{I^w}$ and $\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$ acts trivially (that is, $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ factors through $\mathrm{I}(\overline{\mathbb{Q}}_p/F)/\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$). Because $\mathrm{I}(\overline{\mathbb{Q}}_p/F)/\mathrm{I}(\overline{\mathbb{Q}}_p/F)^w$ is abelian and "prime to p" by Lemma 2.1, $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ is the direct sum of two fundamental characters $\chi_1 \oplus \chi_2$. Arguing as in the proof of Lemma 2.5, we have $\chi_i(sgs^{-1}) = \chi_i(g)^q$ for $g \in \mathrm{I}(\overline{\mathbb{Q}}_p/F)$ where $s \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ is a lift of Fr^f. As the representation $\rho^s := \rho(s \cdot s^{-1})$ is isomorphic to ρ (being a conjugate of ρ), we have $\{\chi_1, \chi_2\} = \{\chi_1^q, \chi_2^q\}$. If $\chi_1 = \chi_1^q$, then $\chi_2 = \chi_2^q$ and the characters χ_i extend to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ by Lemma 2.5. It is easy then to deduce that ρ is reducible so this case can't happen (you can do this as an exercice, you may have to distinguish the cases $\chi_1 = \chi_2$ and $\chi_1 \neq \chi_2$). Hence $\chi_1 = \chi_2^q \neq \chi_1^{p^f}$ and $\chi_2 = \chi_1^q$ which implies $\chi_i^{q^2} = \chi_i$, $i \in \{1,2\}$. We easily derive (ii) from this. $\qquad\square$

Remark 2.8. In fact, one only needs to fix an embedding $\mathbb{F}_q \hookrightarrow E$ because in the irreducible case, we see that we sum up on the fundamental characters corresponding to all field embeddings $\mathbb{F}_{q^2} \hookrightarrow E$ giving back the fixed one on restriction to $\mathbb{F}_q$ (there are 2 of them).

Corollary 2.9. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous representation, then ρ is of one of the following forms:*

(i) ρ is reducible and:

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1}(r_i+1)p^i} & * \\ 0 & 1 \end{pmatrix} \otimes \eta$$

for some character η that extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and some integers r_i with $-1 \leq r_i \leq p-2$ and $(r_0, \cdots, r_{f-1}) \neq (p-2, \cdots, p-2)$

(ii) ρ *is irreducible and:*

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i+1)p^i} \end{pmatrix} \otimes \eta$$

for some character η that extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and some integers r_i with $0 \leq r_0 \leq p-1$, $-1 \leq r_i \leq p-2$ for $i > 0$ and $(r_0, \cdots, r_{f-1}) \neq (p-1, p-2, \cdots, p-2)$.

Proof. (i) follows from (i) of Proposition 2.7 and Lemma 2.5 by twisting by $\omega_f^{-m_2}$ and using $\omega_f^{q-1} = 1$. (ii) Write first $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ as in (ii) of Proposition 2.7. We can assume that q doesn't divide m otherwise use $\omega_{2f}^{q^2} = \omega_{2f}$ to replace m by m/q and switch the two characters. Write $m = a + qb$ with $1 \leq a \leq q-1$ and $0 \leq b \leq q-1$. As $1+q$ doesn't divide m, we have either $b < a$ or $a < b$. If $b < a$, we write $m = (a-b) + (1+q)b$ and set $\eta := \omega_{2f}^{(1+q)b} = \omega_f^b$. As $1 \leq a-b \leq q-1$, there is a unique way to write $a - b = \sum_{i=0}^{f-1}(r_i + 1)p^i$ with r_i as in (ii) (easy check). If $a < b$, replace m by $b + qa$ (using again $\omega_{2f}^{q^2} = \omega_{2f}$) and argue as for the previous case. $\qquad\square$

The integers r_i and the character η in Corollary 2.9 are not quite unique. If $* = 0$ in (i), we can replace $(r_i)_{0\leq i\leq f-1}$ by $(p-3-r_i)_{0\leq i\leq f-1}$ and η by $\eta\omega_f^{\sum_{i=0}^{f-1}p^i(r_i+1)}$. In (ii), we can replace $(r_i)_{0\leq i\leq f-1}$ by $(p-1-r_0, p-3-r_1, \cdots, p-3-r_{f-1})$ and η by $\eta\omega_f^{r_0+\sum_{i=1}^{f-1}p^i(r_i+1)}$. However, there are no other possibilities (prove it!).

Exercise 2.10. Prove that $\omega = \omega_1^e$ where $\omega : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathbb{F}_p^\times$ is defined by $g(\sqrt[p]{1}) = (\sqrt[p]{1})^{\omega(g)}$ where $\sqrt[p]{1}$ is a non-trivial p-root of 1.

We will need the following definition in the sequel.

Definition 2.11. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous irreducible representation. We say ρ is generic if in the description (ii) above one has $1 \leq r_0 \leq p-2$ and $0 \leq r_i \leq p-3$ for $i > 0$.*

Note that for $p = 2$, there are no generic ρ. In most of this course, we will thus have $p > 2$.

Exercise 2.12. Prove that the definition of genericity doesn't depend on the choice of the embedding $\mathbb{F}_q \hookrightarrow E$.

2.2 Classification of weights for GL_2

We fix once and for all an embedding $\mathbb{F}_q \hookrightarrow E$.

Definition 2.13. *A weight (for $K = \mathrm{GL}_2(\mathcal{O}_F)$) is a continuous irreducible representation of K over E.*

By the continuity assumption, a weight is trivial on restriction to an open subgroup of K and in particular factors through a finite quotient of K. The following lemma gives more:

Lemma 2.14. *A weight is trivial on restriction to the first congruence subgroup of K, i.e. the subgroup of matrices of the form $\mathrm{Id} + \varpi_F \mathrm{M}_2(\mathcal{O}_F)$. Thus, in particular, a weight factors through $\mathrm{GL}_2(\mathcal{O}_F/\mathfrak{m}_F) = \mathrm{GL}_2(\mathbb{F}_q)$.*

Proof. This follows from Lemma 2.6 (as $\mathrm{Id} + \varpi_F \mathrm{M}_2(\mathcal{O}_F)$ is a pro-p-group) and from the exact sequence:

$$1 \to \mathrm{Id} + \varpi_F \mathrm{M}_2(\mathcal{O}_F) \to \mathrm{GL}_2(\mathcal{O}_F) \to \mathrm{GL}_2(\mathcal{O}_F/(\varpi_F)) \to 1.$$

$\square$

We recall without proof the following theorem:

Theorem 2.15 (Brauer)**.** *Let G be a finite group and G_{reg} the subset of elements of order prime to p. Then the number of irreducible representations of G on a E-vector space is the number of conjugacy classes in G_{reg} (note that if $h \in G_{\mathrm{reg}}$ and $g \in G$, $ghg^{-1} \in G_{\mathrm{reg}}$ as its order is that of h).*

For any integer r in $\mathbb{Z}_{\geq 0}$, consider the $r+1$-dimensional representation of the finite group $\mathrm{GL}_2(\mathbb{F}_q)$: $\mathrm{Sym}^r E^2$ where $\mathrm{GL}_2(\mathbb{F}_q)$ acts through its natural action on the canonical basis of E^2. Here, we use the fixed embedding $\mathbb{F}_q \hookrightarrow E$. This representation can be identified with $\oplus_{i=0}^r E x^{r-i} y^i$ where the action of $\mathrm{GL}_2(\mathbb{F}_q)$ is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{r-i} y^i = (ax + cy)^{r-i}(bx + dy)^i \tag{2}$$

and where $a, b, c, d \in \mathbb{F}_q$ are seen in E via the above fixed embedding. If $0 \leq j \leq f-1$, we denote by $(\mathrm{Sym}^r E^2)^{\mathrm{Fr}^j}$ the $r+1$-dimensional representation $\oplus_{i=0}^r E x^{r-i} y^i$ with the action of $\mathrm{GL}_2(\mathbb{F}_q)$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{r-i} y^i = (a^{p^j} x + c^{p^j} y)^{r-i}(b^{p^j} x + d^{p^j} y)^i.$$

Note that $(\mathrm{Sym}^r E^2)^{\mathrm{Fr}^f} = \mathrm{Sym}^r E^2$.

Lemma 2.16. *Let $r_0, \cdots, r_{f-1}$ and m be integers such that $0 \leq r_i \leq p-1$ and $0 \leq m < q-1$. Then the representations:*

$$(\mathrm{Sym}^{r_0} E^2) \otimes_E (\mathrm{Sym}^{r_1} E^2)^{\mathrm{Fr}} \otimes_E \cdots \otimes_E (\mathrm{Sym}^{r_{f-1}} E^2)^{\mathrm{Fr}^{f-1}} \otimes \det^m \qquad (3)$$

are irreducible and non-equivalent.

Proof. The representation (3) can be identified with

$$\oplus_{i_0=0}^{r_0} \oplus_{i_1=0}^{r_1} \cdots \oplus_{i_{f-1}=0}^{r_{f-1}} E x^{\sum_{j=0}^{f-1}(r_j - i_j)p^j} y^{\sum_{j=0}^{f-1} i_j p^j}$$

where the action of $\mathrm{GL}_2(\mathbb{F}_q)$ is as in (2) above replacing r by $\sum_{j=0}^{f-1} r_j p^j$ and i by $\sum_{j=0}^{f-1} i_j p^j$ (and twisting). One can check irreducibility as follows: any non-zero stable subspace has a non-zero vector fixed by the p-group of upper unipotent matrices by Lemma 2.6. An easy calculation shows that the only such vector in the representation is $x^{\sum_{j=0}^{f-1} r_j p^j}$. The same computation using the action of the lower unipotent matrices shows this vector generates the whole representation (use that the E-vector space generated by $\sum_{i=0}^{q-1} \lambda^i v_i$ for all $\lambda \in \mathbb{F}_q$ is the E-vector space generated by v_i for all i). Therefore, any non-zero subrepresentation must be the whole representation and we have irreducibility. Assume now two representations are isomorphic, then the characters giving the action of the diagonal matrices $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ $(a \in \mathbb{F}_p^\times)$ on the vectors fixed by subgroup of upper unipotent matrices must be the same. As this character is $a^{\sum_{j=0}^{f-1} r_j p^j}$, the list of the r_i have to be the same unless may-be all r_i are 0 and all r_i are $p-1$. However, in this case, the corresponding representations obviously don't have the same dimension (1 and q). Therefore any two isomorphic representations can at most differ by a twist $\det^*$. But the characters giving the action of matrices $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ on upper unipotent fixed vectors again must be the same which immediately implies this twist is trivial. $\square$

For simplicity, we denote in the sequel by $(r_0, \cdots, r_{f-1}) \otimes \det^m$ the representation (3).

Proposition 2.17. *The representations $(r_0, \cdots, r_{f-1}) \otimes \det^m$ with $0 \leq r_i \leq p-1$ and $0 \leq m < q-1$ exhaust the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ on E.*

Proof. This is a special case of a very general theorem on modular algebraic representations of finite groups of Lie type, but I give here an elementary proof specific to GL_2. By Theorem 2.15 and Lemma 2.16, we just have to check that $\mathrm{GL}_2(\mathbb{F}_q)_{\mathrm{reg}}$ has exactly $q(q-1)$ conjugacy classes, as this is

the number of irreducible representations in Lemma 2.16. But $\mathrm{GL}_2(\mathbb{F}_q)_{\mathrm{reg}}$ is the subset of matrices that become diagonalizable over an extension of $\mathbb{F}_q$ (those which can only be made upper triangular non diagonal don't have an order prime to p). So the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)_{\mathrm{reg}}$ are of two types: those of the diagonal matrices and those of the matrices which become diagonalizable over $\mathbb{F}_{q^2}$ only. There are $q-1+((q-1)^2-(q-1))/2 = q(q-1)/2$ conjugacy classes of the first type (easy count and don't forget permutation of eigenvalues!). The matrices of the second type are all obtained up to conjugation as $\iota(x)$ where $\iota : \mathbb{F}_{q^2} \hookrightarrow \mathrm{M}_2(\mathbb{F}_q)$ is a fixed algebra embedding and $x \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times$ (any two such embeddings are conjugate in $\mathrm{GL}_2(\mathbb{F}_q)$). Moreover $\iota(x)$ and $\iota(y)$ (for $x, y \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q^\times = \mathbb{F}_{q^2} \setminus \mathbb{F}_q$) are conjugate in $\mathrm{GL}_2(\mathbb{F}_q)$ if and only if $x = sy$ where s is the unique non-trivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. An easy count gives again $q(q-1)/2$ classes. We thus finally get $q(q-1)/2 + q(q-1)/2 = q(q-1)$ conjugacy classes. $\qquad\square$

3 Week 3

3.1 Tree and amalgam

We first recall (or define) the Bruhat-Tits tree associated to GL_2. We keep the previous notations: F is a finite extension of $\mathbb{Q}_p$, ϖ_F is a fixed uniformizer in F etc. We denote by Π the matrix $\begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}$.

Fix a 2-dimensional F-vector space V and denote by X the set of equivalence classes $[L]$ of $\mathcal{O}_F$-lattices L of V for the equivalence relation: $L \sim L'$ if $L = L'x$ for some $x \in F^\times$. We endow X with the structure of a graph as follows: the vertices are the equivalence classes $[L]$ and two distinct vertices $[L]$ and $[L']$ are connected by a unique edge if there exist representatives L, L' such that $\varpi_F L \subsetneq L' \subsetneq L$. Note that this is equivalent to $L/L' \simeq \mathbb{F}_q$ and, replacing L by $\varpi_F L$ (which doesn't change $[L]$), this is also equivalent to $\varpi_F L' \subsetneq L \subsetneq L'$.

Lemma 3.1. *The graph X is a tree, that is, it is connected and has no circuit.*

Proof. Take L and L' two distinct lattices in V and assume $L' \subsetneq L$. Taking the inverse image in L of a Jordan-Hölder sequence of L/L' (seen as a $\mathcal{O}_F$-module) gives a sequence of lattices $L' = L_0 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_n \subsetneq \cdots \subsetneq L_r = L$ such that $L_n/L_{n-1} \simeq \mathbb{F}_q$. In particular, the vertices $[L_n]$ and $[L_{n-1}]$ are connected, which proves that $[L]$ and $[L']$ are always connected by a succession of edges. This proves connectedness. Let us prove that there is

no circuit. Assume we have lattices $L_0 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_n \subsetneq \cdots \subsetneq L_r$ such that $L_n/L_{n-1} \simeq \mathbb{F}_q = \mathcal{O}_F/\varpi_F$ and such that there are no "round trip", that is $[L_{n-2}] \neq [L_n]$ for all n. We prove that $L_r/L_0 \simeq \mathcal{O}_F/\varpi_F^r$, so that in particular one always has $[L_0] \neq [L_r]$ (and the graph has no circuit). There is nothing to prove if $r = 1$. Assume $r = 2$ (we know that $[L_0] \neq [L_2]$ but we prove $L_2/L_0 = \mathcal{O}_F/\varpi_F^2$). We have an exact sequence $0 \to L_1/L_0 \to L_2/L_0 \to L_2/L_1 \to 0$. If $L_2/L_0 \simeq (\mathcal{O}_F/\varpi_F) \oplus (\mathcal{O}_F/\varpi_F)$ then $L_0 = \varpi_F L_2$ which implies $[L_0] = [L_2]$ and contradicts the fact that there is no "round trip". Hence we necessarily have $L_2/L_0 \simeq \mathcal{O}_F/\varpi_F^2$. Assume $r > 2$. By induction and the case $r = 2$, we can assume $L_r/L_1 \simeq \mathcal{O}_F/\varpi_F^{r-1}$ and $L_{r-1}/L_0 \simeq \mathcal{O}_F/\varpi_F^{r-1}$. We have an exact sequence $0 \to L_1/L_0 \to L_r/L_0 \to L_r/L_1 \to 0$. Assume we have $L_r/L_0 \simeq (\mathcal{O}_F/\varpi_F)f_1 \oplus (\mathcal{O}_F/\varpi_F^{r-1})f_2$ where f_1 generates L_1/L_0. The injection $\mathcal{O}_F/\varpi_F^{r-1} \simeq L_{r-1}/L_0 \hookrightarrow L_r/L_0$ necessarily sends a generator of L_{r-1}/L_0 to $af_1 \oplus b\varpi_F f_2$ (for some $a, b \in \mathcal{O}_F$) since the composition $L_{r-1}/L_0 \hookrightarrow L_r/L_0 \twoheadrightarrow L_r/L_1 \simeq (\mathcal{O}_F/\varpi_F^{r-1})f_2$ has kernel $L_1/L_0 \simeq \mathcal{O}_F/\varpi_F$. As $\varpi_F f_1 = 0$ and $r > 2$, we have $\varpi_F^{r-2}(af_1 \oplus b\varpi_F f_2) = 0$ in L_r/L_0 which contradicts the fact $L_{r-1}/L_0 \hookrightarrow L_r/L_0$ is an injection. Therefore we must have $L_r/L_0 \simeq \mathcal{O}_F/\varpi_F^r$. $\square$

The graph X is called the Bruhat-Tits tree. The group $\mathrm{Aut}_F(V)$ naturally acts on X since it acts on the lattices. Fixing a basis (e_1, e_2) of V, we can identify this group with $\mathrm{GL}_2(F)$ and thus get a right action of $\mathrm{GL}_2(F)$ on X via:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} e_1 = ae_1 + ce_2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} e_2 = be_1 + de_2.$$

The class of the lattice $L_0 := \mathcal{O}_F e_1 \oplus \mathcal{O}_F e_2$ is called the central vertice. It is the only vertice fixed by K. The class of the lattice $L_1 := \mathcal{O}_F e_1 \oplus \mathcal{O}_F \varpi_F e_2 = \Pi L_0$ is fixed by $\Pi K \Pi^{-1}$, hence in particular fixed by I (drawing for $F = \mathbb{Q}_2$).

Let $\mathrm{GL}_2(F)^0 := \{g \in \mathrm{GL}_2(F), \ \det(g) \in \mathcal{O}_F^\times\}$.

Lemma 3.2. *Let T be the subtree of X made out of the vertices $\{[L_0], [L_1]\}$ and of the edge connecting them. Then the map $T \to \mathrm{GL}_2(F)^0 \backslash X$ (induced by $T \hookrightarrow X$) is an isomorphism.*

Proof. Any lattice in V is of the form gL_0 for some $g \in \mathrm{GL}_2(F)$. If $\mathrm{val}(\det(g)) \in 2\mathbb{Z}$, then $[gL_0] = [g^0 L_0] = g^0[L_0]$ with $g^0 \in \mathrm{GL}_2(F)^0$. If $\mathrm{val}(\det(g)) \notin 2\mathbb{Z}$, then $[gL_0] = [g\Pi\Pi L_0] = [g\Pi L_1] = [h^0 L_1] = h^0[L_1]$ with $h^0 \in \mathrm{GL}_2(F)^0$. This proves the map is surjective. But injectivity is obvious. $\square$

Let G_1, G_2, H be three groups and $H \overset{\iota_i}{\hookrightarrow} G_i$ be two injections of groups.

By definition, the amalgam $G_1 *_H G_2$ is the inductive limit of the diagram:

$$
\begin{array}{ccc}
H & \overset{\iota_1}{\hookrightarrow} & G_1 \\
\| & & \\
H & \overset{\iota_2}{\hookrightarrow} & G_2.
\end{array}
$$

Equivalently, it is the quotient of the free group generated by G_1 and G_2 (i.e. $G_1 *_{\{e\}} G_2$) by the relations $\iota_1(h)\iota_2(h^{-1})$, $h \in H$.

The following result is due to Ihara and Tits:

Theorem 3.3. *The two injections $K \hookrightarrow \mathrm{GL}_2(F)^0$ and $\Pi K \Pi^{-1} \hookrightarrow \mathrm{GL}_2(F)^0$ induce a group isomorphism:*

$$
K *_I \Pi K \Pi^{-1} \overset{\sim}{\to} \mathrm{GL}_2(F)^0
$$

where $I \subset K$ and $I \subset \Pi K \Pi^{-1}$ are the natural injections identifying I with $K \cap \Pi K \Pi^{-1}$.

Proof. The proof uses the Bruhat-Tits tree X. Following Serre, we first prove that $\mathrm{GL}_2(F)^0$ is generated by K and $\Pi K \Pi^{-1}$ and then that the map:

$$
K *_{K \cap \Pi K \Pi^{-1}} \Pi K \Pi^{-1} \to \mathrm{GL}_2(F)^0
$$

is injective. Note that K is the subgroup of $\mathrm{GL}_2(F)^0$ of elements fixing $[L_0]$ and that $\Pi K \Pi^{-1}$ is the subgroup of $\mathrm{GL}_2(F)^0$ of elements fixing $[L_1]$. Let G be the subgroup of $\mathrm{GL}_2(F)^0$ generated by K and $\Pi K \Pi^{-1}$. We have $X = GT \cup (\mathrm{GL}_2(F)^0 - G)T$ as $\mathrm{GL}_2(F)^0 T = X$ by Lemma 3.2. But GT and $(\mathrm{GL}_2(F)^0 - G)T$ are disjoint subtrees of X and GT is connected (all this easily follows by induction from the fact that gT and $g'T$ for $g, g' \in \mathrm{GL}_2(F)^0$ have a vertice in commun if and only if $g = g'h$ with $h \in K \cup \Pi K \Pi^{-1}$). We thus have $X = GT \amalg (\mathrm{GL}_2(F)^0 - G)T$ and as X is connected (Lemma 3.1), we must have $X = GT$, that is $\mathrm{GL}_2(F)^0 = G$. The map $K *_{K \cap \Pi K \Pi^{-1}} \Pi K \Pi^{-1} \to \mathrm{GL}_2(F)^0$ is not injective if and only if there exist $g_0, \cdots, g_r \in (K \cup \Pi K \Pi^{-1}) \setminus (K \cap \Pi K \Pi^{-1})$ with $r > 1$ such that $g_{i-1} \in K$ (resp. $\Pi K \Pi^{-1}$) implies $g_i \in \Pi K \Pi^{-1}$ (resp. K) and such that $g_0 g_1 \cdots g_r = 1$. If this happens, assume, say, $g_0 \in K$, r even and consider the sequence of vertices $[L_0]$, $g_0[L_1]$, $g_0 g_1[L_0]$, $g_0 g_1 g_2[L_1]$,...,$g_0 g_1 \cdots g_{r-1}[L_0]$, $g_0 g_1 \cdots g_r[L_1] = [L_1]$, $[L_0]$. It induces a circuit in X without "tround trip" (as is easily checked). But this contradicts Lemma 3.1 and thus can't happen. The other possibilities also lead to circuits in X without "round trip", and can't happen either. $\qquad\square$

We will constantly use the following corollary in the sequel:

Corollary 3.4. *Let V be an E-vector space endowed with an action of K and N which coincide on $I = K \cap N$. Then there exists a unique action of $\mathrm{GL}_2(F)$ on V extending the previous actions of K and N.*

Proof. Unicity is clear since K and N generate $\mathrm{GL}_2(F)$, the problem is existence. Let $\rho_0 : K \to \mathrm{Aut}_E(V)$ and $\rho_1 : N \to \mathrm{Aut}_E(V)$ the corresponding group homomorphisms. Define a group homomorphism $\rho_0' : \Pi K \Pi^{-1} \to \mathrm{Aut}_E(V)$ by $\rho_0'(\Pi g \Pi^{-1}) := \rho_1(\Pi) \rho_0(g) \rho_1(\Pi^{-1})$. Since $\rho_0|_I = \rho_1|_I$, one has for $g \in I$ (recall $\Pi g \Pi^{-1} \in I$ and $\Pi \in N$):

$$
\begin{aligned}
\rho_0'(\Pi g \Pi^{-1}) &= \rho_1(\Pi) \rho_0(g) \rho_1(\Pi^{-1}) \\
&= \rho_1(\Pi) \rho_1(g) \rho_1(\Pi^{-1}) \\
&= \rho_1(\Pi g \Pi^{-1}) \\
&= \rho_0(\Pi g \Pi^{-1})
\end{aligned}
$$

hence ρ_0 and ρ_0' induce a group homomorphism $K *_I \Pi K \Pi^{-1} \to \mathrm{Aut}_E(V)$. From Theorem 3.3, we deduce a group homomorphism $\rho : \mathrm{GL}_2(F)^0 \to \mathrm{Aut}_E(V)$. Now $\mathrm{GL}_2(F)$ is the semi-direct product $\mathrm{GL}_2(F)^0 \cdot F^\times \langle \Pi \rangle$ and $\mathrm{GL}_2(F)^0$ is a normal subgroup. One can extend ρ to $\mathrm{GL}_2(F)$ by $\rho(z\Pi) := \rho_1(z\Pi)$ ($z \in F^\times \subset N$). To check that ρ is a group homomorphism, it is enough to check that $\rho(\Pi g \Pi^{-1}) = \rho(\Pi)\rho(g)\rho(\Pi^{-1}) = \rho_1(\Pi)\rho(g)\rho_1(\Pi^{-1})$ for $g \in \mathrm{GL}_2(F)^0$. It is enough to check it for $g \in K$ and $g \in \Pi K \Pi^{-1}$. But in both cases it readily follows from the definition of ρ. Finally, it is clear that $\rho|_K = \rho_0$ and $\rho|_N = \rho_1$. $\qquad \square$

3.2 Compact induction and Hecke operator

I now give elementary properties of some compact induction.

Let H be a closed subgroup of $\mathrm{GL}_2(F)$, e.g. $H = B(F)$, $H = K$, $H = I$, $H = KF^\times$, etc. and σ a smooth representation of H on a finite dimensional E-vector space V_σ. As in §1.1, we consider the representation $\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma$ which is the E-vector space of functions:

$$ f : \mathrm{GL}_2(F) \to V_\sigma $$

which are locally constant, have compact support modulo H and such that $f(hg) = \sigma(h)(f(g))$ ($h \in H$, $g \in \mathrm{GL}_2(F)$). The action of $\mathrm{GL}_2(F)$ is defined as usually by $(g'f)(g) := f(gg')$.

From now on, we assume that H is compact open modulo $F^\times$, e.g. $H = K$, $H = I$, $H = KF^\times$, etc. In that case, any function $f : \mathrm{GL}_2(F) \to V_\sigma$ such

that $f(hg) = \sigma(h)(f(g))$ is locally constant as σ is smooth and H contains an open neighborhood of 1_H. For $g \in \mathrm{GL}_2(F)$ and $v \in V_\sigma$, we denote by $[g, v] : \mathrm{GL}_2(F) \to V_\sigma$ the following element of $\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma$:

$$\begin{aligned} [g,v](g') &= \sigma(g'g)v \quad \text{if} \quad g' \in Hg^{-1} \\ [g,v](g') &= \quad 0 \quad\quad \text{if} \quad g' \notin Hg^{-1}. \end{aligned}$$

That is, $[g, v]$ is the unique element of $\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma$ with support on Hg^{-1} which sends g^{-1} to v. We have $g([g', v]) = [gg', v]$ and $[gh, v] = [g, \sigma(h)v]$ if $h \in H$.

Lemma 3.5. *Any element f of $\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma$ can be written $f = \sum_{i \in I} [g_i, v_i]$ where I is a finite set, $g_i \in \mathrm{GL}_2(F)$ and $v_i \in V_\sigma$.*

Proof. Indeed, as the support of such an element f must be compact modulo $F^\times$, it is a finite union of (disjoint) cosets Hg_i^{-1} and we let $v_i := f(g_i^{-1})$. $\quad\square$

Lemma 3.6. *Let π be any smooth representation of $\mathrm{GL}_2(F)$ over E, then:*

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma, \pi) = \mathrm{Hom}_H(\sigma, \pi|_H).$$

Proof. The image of $\Phi \in \mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma, \pi)$ is given by $v \mapsto \Phi([1_H, v])$ ($v \in V_\sigma$). The image of $\phi \in \mathrm{Hom}_H(\sigma, \pi|_H)$ is given by $[g, v] \mapsto g(\phi(v))$ ($g \in \mathrm{GL}_2(F)$, $v \in V_\sigma$, note that this uniquely determines Φ by Lemma 3.5). $\quad\square$

Lemma 3.6 is called Frobenus reciprocity.

By Definition, the Hecke algebra of $\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma$ is:

$$\mathcal{H}(H, \sigma) := \mathrm{End}_{\mathrm{GL}_2(F)}(\mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma).$$

It is an algebra via addition and composition of endomorphisms (it is non commutative in general).

Lemma 3.7. *The E-vector space $\mathcal{H}(H, \sigma)$ is naturally isomorphic to the E-vector space of functions $\varphi : \mathrm{GL}_2(F) \to \mathrm{End}_E V_\sigma$ with compact support modulo $F^\times$ and such that:*

$$\varphi(h_1 g h_2) = \sigma(h_1) \circ \varphi(g) \circ \sigma(h_2) \tag{4}$$

for $h_1, h_2 \in H$ and $g \in \mathrm{GL}_2(F)$.

Proof. By Lemma 3.6 we have $\mathcal{H}(H, \sigma) = \mathrm{Hom}_H(\sigma, \mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma|_H)$. To φ, one associates $T_\varphi \in \mathrm{Hom}_H(\sigma, \mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma|_H)$ by $T_\varphi(v) := \big(g \mapsto \varphi(g)(v)\big)$. To $T \in \mathrm{Hom}_H(\sigma, \mathrm{ind}_H^{\mathrm{GL}_2(F)} \sigma|_H)$, one associates φ_T by $\varphi_T(g) := \big(v \mapsto T(v)(g)\big)$. $\square$

Exercise 3.8. Making explicit the bijection of Lemma 3.7, prove the following formula:

$$T_\varphi([g, v]) \;\; = \sum_{g'H \in \mathrm{GL}_2(F)/H} [gg', \varphi(g'^{-1})(v)]. \tag{5}$$

Exercise 3.9. Prove that the multiplication on the algebra $\mathcal{H}(H, \sigma)$ corresponds to the convolution of functions: $T_{\varphi_1} \circ T_{\varphi_2} = T_{\varphi_1 * \varphi_2}$ where:

$$\varphi_1 * \varphi_2(g) \;\; := \;\; \Big(v \mapsto \sum_{y \in \mathrm{GL}_2(F)/H} (\varphi_1(y) \circ \varphi_2(y^{-1}g))(v) \Big). \tag{6}$$

We now assume that $H = KF^\times$ and that σ is an irreducible representation of K over E that we extend to $KF^\times$ by sending ϖ_F to the identity. Recall from §2.2 that we have then $\sigma = (r_0, \cdots, r_{f-1}) \otimes \det^m$ with $0 \leq r_i \leq p - 1$ and that:

$$V_\sigma = \oplus_{i_0=0}^{r_0} \oplus_{i_1=0}^{r_1} \cdots \oplus_{i_{f-1}=0}^{r_{f-1}} E x^{\sum_{j=0}^{f-1}(r_j - i_j)p^j} y^{\sum_{j=0}^{f-1} i_j p^j}.$$

Lemma 3.10. *With the previous assumptions, we have $\mathcal{H}(KF^\times, \sigma) = E[T_\varphi]$, φ being defined as follows:*

(i) $\varphi(g) = 0$ *if* $g \notin KF^\times \Pi K = KF^\times \begin{pmatrix} 1 & 0 \\ 0 & \varpi_F^{-1} \end{pmatrix} K$

(ii) $\varphi\big(\begin{pmatrix} 1 & 0 \\ 0 & \varpi_F^{-1} \end{pmatrix} \big)(x^{\sum_{j=0}^{f-1}(r_j - i_j)p^j} y^{\sum_{j=0}^{f-1} i_j p^j}) = 0$ *if* $(i_j) \neq (r_j)$

(iii) $\varphi\big(\begin{pmatrix} 1 & 0 \\ 0 & \varpi_F^{-1} \end{pmatrix} \big)(y^{\sum_{j=0}^{f-1} r_j p^j}) = y^{\sum_{j=0}^{f-1} r_j p^j}$

Proof. Set $\alpha := \begin{pmatrix} 1 & 0 \\ 0 & \varpi_F^{-1} \end{pmatrix}$, Iwasawa decomposition tells us that $\mathrm{GL}_2(F) = \amalg_{n \geq 0} KZ\alpha^{-n}K$ therefore any element of $\mathcal{H}(KF^\times, \sigma)$ is a direct sum of elements that have support in one double coset. Let us find all φ satisfying (4) that have support in $KZ\alpha^{-n}K$. If $n = 0$, then the irreducibility of σ together with Schur's lemma imply that φ is scalar. If $n > 0$, φ must satisfy $\sigma(k_1)\varphi(\alpha^{-n}) = \varphi(\alpha^{-n})\sigma(k_2)$ whenever $k_1\alpha^{-n} = \alpha^{-n}k_2$. Granting the fact that σ is trivial on K_1 (Lemma 2.14), this is equivalent to:

$$\sigma\big(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \big)\varphi(\alpha^{-n}) = \varphi(\alpha^{-n})\sigma\big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \big) \tag{7}$$

for all $a, d \in \mathcal{O}_F^\times$, $b, c \in \mathcal{O}_F$. In particular, one must have:

$$\begin{aligned}
\sigma\left(\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)\right)\varphi(\alpha^{-n}) &= \varphi(\alpha^{-n}) \\
\varphi(\alpha^{-n}) &= \varphi(\alpha^{-n})\sigma\left(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)\right)
\end{aligned}$$

for $t \in \mathcal{O}_F$. The first equality implies $\varphi(\alpha^{-n})(v) \in Ey^{\sum_{j=0}^{f-1} r_j p^j}$ for all $v \in V_\sigma$ and the second implies $\varphi(\alpha^{-n})(v) = 0$ if $v \notin Ey^{\sum_{j=0}^{f-1} r_j p^j}$. Conversely, any such φ satisfies (7). Therefore φ satisfies (ii) and (iii) up to multiplication by a non-zero scalar and with α^{-n} instead of α^{-1}. Denote by φ_n the unique such φ with $\varphi_n(\alpha^{-n})(y^{\sum_{j=0}^{f-1} r_j p^j}) = y^{\sum_{j=0}^{f-1} r_j p^j}$ and by T_n the corresponding Hecke operator (Lemma 3.7), then we have just proven that $\mathcal{H}(KF^\times, \sigma) = \oplus_{n \geq 0} ET_n$ with $T_0 = 1$. But an explicit computation using the convolution formula (6) yields $T_{n+1} = T_1 \circ T_n$ if $n > 1$ and $T_2 = T_1 \circ T_1$ if $\dim_E \sigma > 1$, $T_2 = -1 + T_1 \circ T_1$ if $\dim_E \sigma = 1$. Therefore, we have $\mathcal{H}(KF^\times, \sigma) = E[T_1]$. $\square$

In the sequel, we write T instead of T_1. When σ is trivial, there is a nice way to see T using the tree X:

Lemma 3.11. *Assume $\sigma = 1$. Then the $\mathrm{GL}_2(F)$-representation $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} 1$ can naturally be identified with the E-vector space of functions $F : X \to E$ with finite support (that is, functions on the vertices of X which send all vertices to zero except a finite number). Moreover $T(F)$ is then the function which sends $[L]$ to $\sum_{[L']} F([L'])$ where the sum runs over all vertices such that there is an edge between $[L]$ and $[L']$.*

Proof. Fixing a basis of V as in §3.1, we see that the set of equivalence classes of lattices $\{[L]\}$ can be identified with $\mathrm{GL}_2(F)/KF^\times$ by sending $gKF^\times$ to $[gL_0]$. Therefore, any $f \in \mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} 1$ defines a function on X by $[gL_0] \mapsto f(g^{-1})$. In particular, $[g, v]$ corresponds to the function on X which sends the vertice $g[L_0]$ to $v \in E$ and all other vertices to 0. The fact f has compact support modulo $F^\times$ implies this function has finite support. There is an edge between $[L_0']$ and $[L_0]$ if and only if $L_0' = k\alpha L_0$ for some $k \in KF^\times$, or equivalently $L_0' = k\alpha^{-1}L_0$ for some $k \in KF^\times$ (using $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). Hence there is an edge between $[L']$ and $[L] = g[L_0]$ if and only if $L' = gk\alpha^{-1}L_0$ for some $k \in KF^\times$ (with α as in the proof of Lemma 3.10). The formula for $T(F)$ therefore follows directly from the formula (5) and the definition of T. $\square$

When σ is not trivial, one can still interpret $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma$ as the H^0 of some "sheaf" on X, but we won't need this in the sequel.

Exercise 3.12. For σ irreducible and $\lambda \in E$, prove that $T - \lambda$ is injective on $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma$.

Exercise 3.13. For σ irreducible and $\lambda \in E$, prove that $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T-\lambda)$ has infinite dimension over E.

4 Week 4

4.1 Classification theorem for $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section and the next, we assume most of the time $F = \mathbb{Q}_p$ and $\varpi_F = p$. In that case, the classification of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E is known.

I start with two easy lemmas for which there is no need to assume $F = \mathbb{Q}_p$:

Lemma 4.1. *A smooth irreducible admissible representation of* $\mathrm{GL}_2(F)$ *over* E *always has a central character.*

Proof. Let π be such a representation and $H \subset K$ be an open subgroup such that $\pi^H \neq 0$ (e.g. any open pro-p subgroup). Because π is admissible, π^H has finite dimension over E, and there is $v \in \pi^H$ such that $F^\times$ acts on v by multiplication by a (smooth) character. As π is irreducible, we necessarily have $\pi = \langle \mathrm{GL}_2(F)v \rangle$ and we are done since $F^\times$ commutes with $\mathrm{GL}_2(F)$. $\square$

Remark 4.2. It is not known, even for $F = \mathbb{Q}_p$, whether smooth irreducible representations of $\mathrm{GL}_2(F)$ over E are admissible, or even just whether they have a central character (all this is true when E has a characteristic distinct from p). It is not known, unless $F = \mathbb{Q}_p$ (see below), whether smooth irreducible representations of $\mathrm{GL}_2(F)$ over E with a central character are admissible.

Lemma 4.3. *There is a canonical* $\mathrm{GL}_2(F)$*-equivariant surjection:*

$$\frac{\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} 1}{(T-1)} \twoheadrightarrow 1.$$

Proof. By Lemma 3.11 (or Lemma 3.6), there is a $\mathrm{GL}_2(F)$-equivariant surjection $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} 1 \twoheadrightarrow 1$ sending F to $\sum_{[L]} F([L])$ where the sum is over all vertices of X (recall F has finite support). As there are $q + 1$ vertices $[L']$ such that there is an edge between $[L']$ and a fixed $[L]$, we see by Lemma 3.11 again that TF is sent to $\sum_{[L]} TF([L]) = \sum_{[L]}(q+1)F([L]) = \sum_{[L]} F([L])$. Therefore $TF - F$ is in the kernel of the surjection. $\square$

We have introduced in §4.1 the representations $\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2$. The following theorem is due to Barthel-Livné and myself:

Theorem 4.4. *The smooth irreducible admissible representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *over E are the following:*

(i) *the one dimensional representations $\eta \circ \det$*

(ii) *the representations:*

$$\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^r E^2}{(T - \lambda)} \otimes (\eta \circ \det)$$

for $0 \leq r \leq p - 1$, $\lambda \in E^\times$ and $(r, \lambda) \notin \{(0, \pm 1), (p - 1, \pm 1)\}$

(iii) *the representations:*

$$\mathrm{Ker}\left(\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}1}{(T - 1)} \twoheadrightarrow 1\right) \otimes (\eta \circ \det)$$

(iv) *the representations:*

$$\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^r E^2}{(T)} \otimes (\eta \circ \det).$$

In the sequel, I give comments on the theorem and give some details on its proof. In the next section, I give a survey of why the representations in (iv) are irreducible. For simplicity, I will denote:

$$\pi(r, \lambda, \eta) := \frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^r E^2}{(T - \lambda)} \otimes (\eta \circ \det)$$

for any r, λ, η. If $x \in E^\times$, $\mathrm{unr}(x) : \mathbb{Q}_p^\times \to E^\times$ is the character sending p to x and $\mathbb{Z}_p^\times$ to 1.

(i) It is known that the representations in Theorem 4.4 actually exhaust all smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E with a central character (no need to assume admissibility once there is a central character).

(ii) The representations in (ii) are actually isomorphic to principal series. More precisely, if (r, λ) is as in (ii), we have:

$$\pi(r, \lambda, 1) \simeq \mathrm{ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{unr}(\lambda) \otimes \omega_1^r \mathrm{unr}(\lambda^{-1}).$$

This is not specific to the case $F = \mathbb{Q}_p$.

(iii) The representation $\mathrm{Ker}(\pi(0,1,1) \twoheadrightarrow 1)$ in (iii) is called the Steinberg representation. It is isomorphic to the quotient $(\mathrm{ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1)/1$. This representation together with its twists constitute the so called special series.

(iv) The representations in (iv) are the supercuspidal ones. They are also called supersingular.

(v) There are entertwinings between the above representations:

$$\begin{aligned}
\pi(r,\lambda,\eta) &\simeq \pi(r,-\lambda,\eta\mathrm{unr}(-1)) \\
\pi(0,\lambda,\eta) &\simeq \pi(p-1,\lambda,\eta) \quad (\lambda \neq \pm 1) \\
\pi(r,0,\eta) &\simeq \pi(p-1-r,0,\eta\omega_1^r).
\end{aligned}$$

(vi) Some of the above entertwinings are known more generally: without any assumption on F, one has:

$$\begin{aligned}
\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma/(T-\lambda) &\simeq (\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma/(T+\lambda)) \otimes (\mathrm{unr}(-1) \circ \det) \\
\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}1/(T-\lambda) &\simeq \mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}(\mathbf{p-1})/(T-\lambda)
\end{aligned}$$

where $\mathbf{p-1} := (p-1,\cdots,p-1)$, σ is a weight and $\lambda \neq \pm 1$ in the last entertwining.

Let us now start (part of) the proof of Theorem 4.4. The following two propositions don't need $F = \mathbb{Q}_p$.

Proposition 4.5. *Let π be a smooth irreducible admissible representation of $\mathrm{GL}_2(F)$ such that ϖ_F acts trivially. Then there exist (σ,λ) with $\lambda \in E$ and σ a finite dimensional irreducible representation of $KF^\times$ over E such that π is a quotient of $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma)/(T-\lambda)$.*

Proof. As π is admissible, the $K/K_1 = \mathrm{GL}_2(\mathbb{F}_q)$-representation π^{K_1} has finite dimension over E. Let us choose an irreducible subobject σ in π^{K_1}. By Lemma 4.1, $F^\times$ acts on π (hence on π^{K_1}) by a smooth character and we can extend σ to a representation of $KF^\times$ so that $F^\times$ acts on σ by the same character (with ϖ_F acting trivially). We therefore have $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma, \pi) = \mathrm{Hom}_{KF^\times}(\sigma,\pi) = \mathrm{Hom}_{KF^\times}(\sigma,\pi^{K_1}) \neq 0$ by Lemma 3.6. There is a right action of $\mathcal{H}(\mathrm{GL}_2(F),\sigma)$ on $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma,\pi)$ given by $\Phi|T := \Phi \circ T$. As $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma,\pi)$ is finite dimensional, it has a non-zero eigenvector for the action of $\mathcal{H}(\mathrm{GL}_2(F),\sigma) \simeq E[T]$, that is, there exist $\lambda \in E$ and a non-zero map $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma)/(T-\lambda) \to \pi$. As π is irreducible, it is surjective. $\qquad\square$

Note that there is always a twist $\pi \otimes (\eta \circ \det)$ such that ϖ_F acts trivially. If π is a quotient of $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T - \lambda)$ with $\lambda \neq 0$, then π is either a character, or a principal series or a twist of the Steinberg representation.

Proposition 4.6. *The supercuspidal representations of $\mathrm{GL}_2(F)$ such that ϖ_F acts trivially are the irreducible admissible quotients of $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T)$ for σ a weight.*

Proof. (sketch) Let π be such a supercuspidal. By Proposition 4.5, π is a quotient of $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T - \lambda)$. If $\lambda \neq 0$, then π is a subquotient of a principal series by what we said above. Therefore we must have $\lambda = 0$. Conversely, assume one can find a quotient π of $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T)$ that is also a subquotient of a principal series. In particular, it is a quotient of some $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma')/(T - \lambda)$ for $\lambda \neq 0$. Let us assume for simplicity that this representation is irreducible (this is the generic case) so that we identify it with π (the general case is similar). One can prove that in such a principal series, the subrepresentation of K_1-invariants is isomorphic to a principal series of K, that is, an induction of the type $\mathrm{ind}_I^K \chi$ for $\chi : I \to E^\times$ a smooth character. More precisely it is isomorphic to the unique such principal series with socle σ' (see §7.1). By Frobenius reciprocity $\mathrm{Hom}_{\mathrm{GL}_2(F)}(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma, \pi) = \mathrm{Hom}_K(\sigma, \pi|_K) = \mathrm{Hom}_K(\sigma, \pi^{K_1}|_K)$, we must therefore have $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \sigma] \xrightarrow{\sim} [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \sigma']$. Hence the image of the line $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \sigma^{I_1}]$ in π coincides with the line $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \sigma'^{I_1}]$. Using formula (5) and the decomposition:

$$K\Pi K = \Pi K \amalg \left(\amalg_{\nu \in \mathbb{F}_q} \left(\begin{smallmatrix} \varpi_F & [\nu] \\ 0 & 1 \end{smallmatrix}\right) K \right), \tag{8}$$

one has if v' generates σ'^{I_1} and $\dim_E \sigma' \neq 1$:

$$T([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v']) = \sum_{\nu \in \mathbb{F}_q} [\left(\begin{smallmatrix} \varpi_F & [\nu] \\ 0 & 1 \end{smallmatrix}\right), v'] = \lambda [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v']$$

and an analogous formula if $\dim_E \sigma' = 1$. Likewise, one has in the representation $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T)$ (if v generates σ^{I_1} and $\dim_E \sigma \neq 1$):

$$T([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v]) = \sum_{\nu \in \mathbb{F}_q} [\left(\begin{smallmatrix} \varpi_F & [\nu] \\ 0 & 1 \end{smallmatrix}\right), v] = 0.$$

But we must have $\sum_{\nu \in \mathbb{F}_q} \left(\begin{smallmatrix} \varpi_F & [\nu] \\ 0 & 1 \end{smallmatrix}\right) [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v]$ maps to $\sum_{\nu \in \mathbb{F}_q} \left(\begin{smallmatrix} \varpi_F & [\nu] \\ 0 & 1 \end{smallmatrix}\right) [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v']$ (up to scalar) as the surjection $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma)/(T) \twoheadrightarrow \pi$ is $\mathrm{GL}_2(F)$-equivariant. This is impossible as $\lambda' \neq 0$. If $\dim_E \sigma = \dim_E \sigma' = 1$, a similar argument applies. $\square$

Unfortunately, we will see that classifying the irreducible quotients of $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)}\sigma)/(T)$ is a hard task. So far, the classification of supercuspidal representations of $\mathrm{GL}_2(F)$ over E is only known for $F = \mathbb{Q}_p$ (part (iv) of Theorem 4.4).

4.2 Irreducibility for supersingular and semi-simple correspondence

The aim of this section is to sketch the proof of (iv) of Theorem 4.4, i.e. to classify supercuspidal over E for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Recall that the K-socle of a $\mathrm{GL}_2(\mathcal{O}_F)$-representation is the union of all irreducible K-subrepresentations. It is a semi-simple K-representation. Recall also that $\pi(r,0,1) := (\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2)/(T)$. By Frobenius reciprocity (or a straightforward check), we have a canonical injection $\mathrm{Sym}^r E^2 \hookrightarrow \pi(r,0,1)$ given by $v \mapsto$ image of $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v]$ (see §3.2 for notations). Recall $\Pi := \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$ and $(\mathrm{Sym}^r E^2)^{I_1} = Ex^r$ (see §2.2).

Proposition 4.7. *Assume $F = \mathbb{Q}_p$. The K-subrepresentation of $\pi(r,0,1)$ generated by the image of $[\Pi, (\mathrm{Sym}^r E^2)^{I_1}]$ is isomorphic to $\mathrm{Sym}^{p-1-r}E^2 \otimes \det^r$.*

Proof. As Π normalizes I_1, $[\Pi, (\mathrm{Sym}^r E^2)^{I_1}] = [\Pi, Ex^r]$ is fixed by I_1 and I acts on it by the character $\left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \mapsto \overline{d}^r = \overline{a}^{p-1-r}(\overline{ad}^r)$. Consider the induction $\mathrm{ind}_I^K(1 \otimes \overline{d}^r) = \mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(1 \otimes d^r)$. As for compact inductions (c.f. §3.2) any element of $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(1 \otimes d^r)$ can be written $[g, v]$ where $g \in \mathrm{GL}_2(\mathbb{F}_p)$ and $v \in E$ (for instance, $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v]$ is the unique function with support in $B(\mathbb{F}_p)$ sending $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to v) and there is an analogous Frobenius reciprocity. It is a standard result of the representation theory of $\mathrm{GL}_2(\mathbb{F}_p)$ that $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} 1 = 1 \oplus \mathrm{Sym}^{p-1}E^2$ and, if $r \neq 0$, $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(1 \otimes d^r)$ is the unique non-split extension:

$$0 \to \mathrm{Sym}^r E^2 \to \mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(1 \otimes d^r) \to \mathrm{Sym}^{p-1-r}E^2 \otimes \det^r \to 0. \qquad (9)$$

If $r > 0$, the subspace $(\mathrm{Sym}^r E^2)^{I_1} \subset \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)}(1 \otimes d^r)$ is $E\left(\sum_{\lambda \in \mathbb{F}_p}[\left(\begin{smallmatrix} \lambda & 1 \\ 1 & 0 \end{smallmatrix}\right), v]\right)$ (for any $v \in E^\times$) and if $r = 0$, it is the subspace of constant functions $E\left([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), v] + \sum_{\lambda \in \mathbb{F}_p}[\left(\begin{smallmatrix} \lambda & 1 \\ 1 & 0 \end{smallmatrix}\right), v]\right)$. Now, let σ be the K-subrepresentation of $\pi(r,0,1)$ generated by the image of $[\Pi, Ex^r]$. Note that $\sigma \neq 0$ as $\mathrm{Sym}^r E^2$ and hence $[\Pi, Ex^r]$ generate $\pi(r,0,1)$. Since I acts on $[\Pi, Ex^r]$ by $\left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \mapsto \overline{d}^r$, we have by Frobenius reciprocity an equivariant surjection $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}(1 \otimes d^r) \twoheadrightarrow$

σ, $[g,v] \mapsto \hat{g}[\Pi, vx^r] = [\hat{g}\Pi, vx^r]$ where $\hat{g}$ is any lifting of g in K and $v \in E$. This surjection sends $\sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} \lambda & 1 \\ 1 & 0 \end{smallmatrix} \right), v]$ to the image of:

$$\sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} [\lambda] & 1 \\ 1 & 0 \end{smallmatrix} \right) \Pi, vx^r] = \sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), vx^r]$$

where $[\lambda] \in \mathbb{Z}_p^{\times}$ is the multiplicative representative of λ (Teichmüller representative). But using formula (5) and the decomposition (8), as in the proof of Proposition 4.6 one has in $\pi(r, 0, 1)$ for $r > 0$:

$$\sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), vx^r] = T([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), vx^r]) = 0$$

and for $r = 0$:

$$[\Pi, v] + \sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), v] = T([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right), v]) = 0.$$

Therefore, the image of $\operatorname{Sym}^r E^2$ in σ is 0. As $\operatorname{Sym}^{p-1-r} E^2 \otimes \det^r$ is irreducible and $\sigma \neq 0$, we must have $\operatorname{Sym}^{p-1-r} E^2 \otimes \det^r \xrightarrow{\sim} \sigma$ by (9). $\square$

Note that the two weights $\operatorname{Sym}^r E^2$ and $\operatorname{Sym}^{p-1-r} E^2 \otimes \det^r$ are always distinct. We therefore have a canonical injection $\operatorname{Sym}^r E^2 \oplus (\operatorname{Sym}^{p-1-r} E^2 \otimes \det^r) \hookrightarrow \pi(r, 0, 1)$.

Theorem 4.8. *Assume $F = \mathbb{Q}_p$. For $0 \leq r \leq p-1$, $\operatorname{soc}_K \pi(r, 0, 1) = \operatorname{Sym}^r E^2 \oplus (\operatorname{Sym}^{p-1-r} E^2 \otimes \det^r)$.*

Remark 4.9. As we have already mentionned in the proof of Proposition 4.6, when π is an irreducible principal series or special series and π is not the twist of an unramified principal series, one can prove that $\operatorname{soc}_K \pi$ is irreducible (i.e. contains just one weight). When π is the twist of an unramified principal series, one can prove that $\operatorname{soc}_K \pi = 1 \oplus \operatorname{Sym}^{p-1} E^2$ up to twist.

I am not going to really prove Theorem 4.8 here, just indicate the steps of the proof. Before this, here is an important corollary:

Corollary 4.10. *Assume $F = \mathbb{Q}_p$. For $0 \leq r \leq p-1$, the representations $\pi(r, 0, 1)$ are irreducible and admissible.*

Proof. Denote $\pi := \pi(r, 0, 1)$. Let $0 \subsetneq \pi' \subseteq \pi$ be a sub-representation. As $0 \subsetneq \operatorname{soc}_K \pi' \subseteq \operatorname{soc}_K \pi$, we have either $\operatorname{Sym}^r E^2 \subseteq \operatorname{soc}_K \pi'$ or $\operatorname{Sym}^{p-1-r} E^2 \otimes \det^r \subseteq \operatorname{soc}_K \pi'$. But Proposition 4.7 tells us that both generate $\pi(r, 0, 1)$. Therefore, we must have $\pi' = \pi$ and π is irreducible. To check admissibility,

it is enough to check that π^{K_n} is finite dimensional for $K_n := \mathrm{Ker}\big(K \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})\big)$. We always have $\pi^{K_n} \hookrightarrow \mathrm{inj}_{K/K_n}(\mathrm{soc}_{K/K_n}(\pi^{K_n}))$ where the latter is the injective envelope in the category of $E[K/K_n]$-modules (I'll come back to that later). But $\mathrm{soc}_{K/K_n}(\pi^{K_n}) = \mathrm{soc}_K \pi$ is finite dimensional by Theorem 4.8 and therefore so is its injective envelope (as for instance it is contained in $E[K/K_n]$). Hence π^{K_n} is finite dimensional and π is admissible. $\qquad\square$

One can also prove there is a $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant isomorphism $\pi(r, 0, 1) \simeq \pi(p - 1 - r, 0, \omega_1^r)$ (see (v) in §4.1).

Now I would like to give the main steps of the proof of Theorem 4.8. For $n \geq 0$, let $K_0(p^n) \subseteq K$ be the subgroup of matrices of the form $\big(\begin{smallmatrix} a & b \\ p^n c & d \end{smallmatrix} \big)$, $a, b, c, d \in \mathbb{Z}_p$ (so $K_0(1) = K$ and $K_0(p) = I$). Set $\sigma := \mathrm{Sym}^r E^2$ and for $n \geq 0$ denote by σ_n the following representation of $K_0(p^n)$ over E:

$$\sigma_n\big(\big(\begin{smallmatrix} a & b \\ p^n c & d \end{smallmatrix} \big) \big) := \sigma\big(\big(\begin{smallmatrix} d & c \\ p^n b & a \end{smallmatrix} \big) \big)$$

(note that for $n = 0$, σ_0 is a conjugate of σ and thus isomorphic to σ). Let $R_n := \mathrm{ind}_{K_0(p^n)}^K \sigma_n$. Note also that we have a K-equivariant isomorphism:

$$\mathrm{ind}_K^{K\big(\begin{smallmatrix} 0 & 1 \\ p^n & 0 \end{smallmatrix} \big)K} \sigma \xrightarrow{\sim} \mathrm{ind}_{K_0(p^n)}^K \sigma_n$$

sending f on the left hand side to the function $k \mapsto f(\big(\begin{smallmatrix} 0 & 1 \\ p^n & 0 \end{smallmatrix} \big) k)$ for $k \in K$. Let us assume $r > 0$ (the case $r = 0$ is analogous but slightly different). We prove:

(i) $\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \sigma = \oplus_{n \geq 0} R_n$ (this has nothing to do with $F = \mathbb{Q}_p$: we just decompose functions over the "concentric circles" in the Bruhat-Tits tree of §3.1)

(ii) the Hecke operator $T|_{R_n} : R_n \to R_{n+1} \oplus R_{n-1}$ is the sum of a K-equivariant injection $T^+ : R_n \hookrightarrow R_{n+1}$ and (for $n > 0$) a K-equivariant surjection $T^- : R_n \twoheadrightarrow R_{n-1}$ (ibid.)

(iii) we have an isomorphism of K-representations:

$$\pi(r, 0, 1) \simeq \big(\mathop{\mathrm{inj\,lim}}_{n \text{ even}} R_0 \oplus_{R_1} R_2 \oplus_{R_3} \cdots \oplus_{R_{n-1}} R_n \big) \oplus$$
$$\big(\mathop{\mathrm{inj\,lim}}_{n \text{ odd}} R_1/R_0 \oplus_{R_2} R_3 \oplus_{R_4} \cdots \oplus_{R_{n-1}} R_n \big)$$

where the maps $R_i \to R_{i \pm 1}$ are $T^\pm$ (ibid.)

(iv) the K-socle of the first inductive limit is that of R_0, that is $\sigma_0 = \sigma = \mathrm{Sym}^r E^2$, and the K-socle of the second is that of R_1/R_0, that is $\mathrm{Sym}^{p-1-r} E^2 \otimes \det^r$ (this is totally specific to $F = \mathbb{Q}_p$).

The more important step is (iv). It is based on an explicit computation (which is analogous to the one of Lemma 11.8 below) and is definitely wrong when F is not $\mathbb{Q}_p$ (that is, the K-socles are different and contain more weights).

We can now at least state the "semi-simple modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$". Denote by $\mathrm{ind}(\omega_2^{r+1})$ the unique irreducible representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over E such that its restriction to $\mathrm{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is $\omega_2^{r+1} \oplus \omega_2^{p(r+1)}$ and its determinant is ω_1^{r+1} (see §2.1). Also let $\mathrm{unr}(x)$ be the character $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to E^\times$ sending $\mathrm{Fr}^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ to $x \in E^\times$.

Definition 4.11. *Let $r \in \{0,\ldots,p-1\}$, $\lambda \in E$, $\eta : \mathbb{Q}_p^\times \to E^\times$ and $[p-3-r]$ the unique integer in $\{0,\ldots,p-2\}$ congruent to $p-3-r$ modulo $p-1$. With the previous notations, we define the following "semi-simple modulo p correspondence":*

(i) if $\lambda = 0$:

$$
(\mathrm{ind}(\omega_2^{r+1})) \otimes \eta \longleftrightarrow \frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2}{(T)} \otimes \eta
$$

(ii) if $\lambda \neq 0$:

$$
\begin{pmatrix} \omega_1^{r+1}\mathrm{unr}(\lambda) & 0 \\ 0 & \mathrm{unr}(\lambda^{-1}) \end{pmatrix} \otimes \eta \longleftrightarrow \left(\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2}{(T-\lambda)} \right)^{\mathrm{ss}} \otimes \eta \oplus
$$
$$
\left(\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^{[p-3-r]} E^2}{(T-\lambda^{-1})} \otimes \omega_1^{r+1} \right)^{\mathrm{ss}} \otimes \eta
$$

where "ss" means "semi-simplification".

Here, η is a smooth character of $\mathbb{Q}_p^\times$ over E and I should write $\eta \circ \det$ on the right hand side and $\eta \circ r_{\mathbb{Q}_p}$ on the left hand side. This correspondence can be refined into a non-semi-simple correspondence. More precisely, there is generically a unique non-split extension $\begin{pmatrix} \omega_1^{r+1}\mathrm{unr}(\lambda) & * \\ 0 & \mathrm{unr}(\lambda^{-1}) \end{pmatrix}$. Accordingly, one can prove there is generically a unique non-split extension:

$$
0 \longrightarrow \frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^r E^2}{(T-\lambda)} \longrightarrow * \longrightarrow \frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{Sym}^{[p-3-r]} E^2}{(T-\lambda^{-1})} \otimes \omega_1^{r+1} \longrightarrow 0
$$

"corresponding" to the non-split Galois extension. We won't deal with that in this course. Moreover, Colmez has found an important "functorial" way to reinterpret this correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ using (φ, Γ)-modules that makes it (surprisingly) deep.

Next time, we switch to $\mathrm{GL}_2(F)$.

5 Week 5

5.1 Basic diagrams: definition and examples

In this section, we define group theoretic structures called (basic) diagrams and give their first properties. We will use them in the next sections to build smooth admissible representations of $\mathrm{GL}_2(F)$ over E with a given K-socle.

Diagrams were first introduced by Schneider and Stuhler (although in a somewhat different form) years ago for characteristic 0 coefficient fields E and first used in the characteristic p context by Paskunas. To motivate their definition, let us go back to $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let us consider a supercuspidal representation $\pi(r, 0, 1)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ as in §4.2. Let $\chi : I \to E$ be the character giving the action of I on $(\mathrm{Sym}^r E^2)^{I_1}$ and recall $\chi^s := \chi(\Pi \cdot \Pi^{-1})$.

Lemma 5.1. *(i) We have $(\mathrm{soc}_K \pi(r, 0, 1))^{I_1} = \chi \oplus \chi^s$.*

(ii) The action of Π preserves $(\mathrm{soc}_K \pi(r, 0, 1))^{I_1}$ and interchanges χ and χ^s.

Proof. (i) follows readily from Theorem 4.8. (ii) follows from Proposition 4.7 where we have already noticed that $\Pi(\mathrm{Sym}^r E^2)^{I_1} = [\Pi, (\mathrm{Sym}^r E^2)^{I_1}]$ is fixed by I_1 and generates $\mathrm{Sym}^{p-1-r} E^2 \otimes \det^r$ (recall that Π^2 acts trivially). $\square$

Thanks to Lemma 5.1, we can see $(\mathrm{soc}_K \pi(r, 0, 1))^{I_1}$ as a representation of N (the action of $\mathbb{Q}_p^\times$ is that on $\pi(r, 0, 1)$). We can then consider the following triple $(\mathrm{soc}_K \pi(r, 0, 1), (\mathrm{soc}_K \pi(r, 0, 1))^{I_1}, \mathrm{can})$ where $\mathrm{soc}_K \pi(r, 0, 1)$ is seen as a representation of $K\mathbb{Q}_p^\times$, $(\mathrm{soc}_K \pi(r, 0, 1))^{I_1}$ is seen as a representation of N and can is the canonical injection $(\mathrm{soc}_K \pi(r, 0, 1))^{I_1} \hookrightarrow \mathrm{soc}_K \pi(r, 0, 1)$. It turns out this triple completely characterizes the representation $\pi(r, 0, 1)$:

Lemma 5.2. *Let π be an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E. Assume that π "contains" $(\mathrm{soc}_K \pi(r, 0, 1), (\mathrm{soc}_K \pi(r, 0, 1))^{I_1}, \mathrm{can})$, that is, π^{K_1} contains $\mathrm{soc}_K \pi(r, 0, 1)$ and the action of N on π^{I_1} is compatible*

with that on the subspace $(\mathrm{soc}_K \pi(r,0,1))^{I_1}$ given by Lemma 5.1. Then $\pi \simeq \pi(r,0,1)$.

Proof. Assume first $r > 0$. By Theorem 4.4, Theorem 4.8 and Remark 4.9, we necessarily have $\pi \simeq \pi(r,0,1)$ and we are done. Assume $r = 0$, then from the same statements we get either $\pi \simeq \pi(0,0,1)$ or π is an unramified principal series, that is $\pi \simeq (\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} 1)/(T - \lambda)$ with $\lambda \in E^\times \setminus \{\pm 1\}$. But we can rule out the latter case because the action of Π is not as in Lemma 5.1 (more precisely, Proposition 4.7 is wrong in that case: the image of $[\Pi, 1]$ is isomorphic to $1_K \oplus \mathrm{Sym}^{p-1} E^2$). $\qquad\square$

Remark 5.3. One can actually prove that $(\mathrm{soc}_K \pi(r,0,1))^{I_1} \simeq \pi(r,0,1)^{I_1}$.

Let us now come to the general definition of diagrams (for any F).

Definition 5.4. (i) *A diagram is a triple (D_0, D_1, r) where D_0 is a smooth representation of $KF^\times$ over E, D_1 is a smooth representation of N over E and $r : D_1 \to D_0$ is an $IF^\times$-equivariant map.*

(ii) *A basic diagram is a diagram (D_0, D_1, r) such that ϖ_F acts trivially and r induces an isomorphism $D_1 \xrightarrow{\sim} D_0^{I_1} \hookrightarrow D_0$.*

One defines morphisms of diagrams in the obvious way and gets an abelian category. In this course, we will mainly consider basic diagrams. The full subcategory of basic diagrams is of course not abelian any-more. A basic diagram (D_0, D_1, r) is said to be irreducible if it doesn't contain any non-zero strict basic subdiagram, that is if there is no non-zero (D_0', D_1', r') with $D_0' \subsetneq D_0$, $D_1' \subseteq D_1$, $D_1' \xrightarrow{\sim} D_0'^{I_1}$ and with:

$$
\begin{array}{ccc}
D_1' & \xrightarrow{r'} & D_0' \\
\downarrow & & \downarrow \\
D_1 & \xrightarrow{r} & D_0
\end{array}
$$

commutative. Equivalently, there is no K-subrepresentation $0 \subsetneq D_0' \subsetneq D_0$ such that $D_0'^{I_1}$ is preserved by N inside $D_0^{I_1}$. If (D_0, D_1, r) is irreducible, note that $D_0 = \langle K D_0^{I_1} \rangle = D_0^{K_1}$ and thus K_1 acts trivially on D_0.

Example 5.5. Let π be any smooth representation of $\mathrm{GL}_2(F)$ over E such that ϖ_F acts trivially:

(i) (π, π, id) is a diagram (not basic)

(ii) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ is a basic diagram where can is the canonical injection $\pi^{I_1} \hookrightarrow \pi^{K_1}$ (here of course π^{I_1} is seen as a representation of N)

(iii) $(\langle K\pi^{I_1}\rangle, \pi^{I_1}, \mathrm{can})$ is a basic diagram which is a subdiagram of $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$.

Example 5.6. We see any weight as a representation of $KF^\times$ by sending ϖ_F to 1. Let σ be a weight, χ the character giving the action of I on σ^{I_1} and $\sigma^s \neq \sigma$ the only weight such that I acts on $(\sigma^s)^{I_1}$ by χ^s. Let $D_0 := \sigma \oplus \sigma^s$ and let Π act on $D_0^{I_1} = Ev_\chi \oplus Ev_{\chi^s}$ by interchanging v_χ and v_{χ^s} (Π^2 must act trivially and v_χ (resp. v_{χ^s}) is a basis of σ^{I_1} (resp. $(\sigma^s)^{I_1}$)). This defines an irreducible basic diagram $D_\sigma = D_{\sigma^s}$. When $F = \mathbb{Q}_p$, Lemma 5.2 together with Remark 5.3 imply that the map $\pi \mapsto (\langle K\pi^{I_1}\rangle, \pi^{I_1}, \mathrm{can})$ induces a bijection between the set of equivalence classes of supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that p acts trivially and the set of equivalence classes of diagrams D_σ.

Example 5.7. Let $\chi : I \to E^\times$ such that $\chi \neq \chi^s$. Consider the induction $D_0 := \mathrm{ind}_I^K \chi$ as in the proof of Proposition 4.6 (where we make ϖ_F act trivially as usual). We will prove later (Lemma 7.3) that $D_0^{I_1} = (\mathrm{ind}_I^K \chi)^{I_1} = Ef_\chi \oplus Ef_{\chi^s}$ where I acts on f_χ (resp. f_{χ^s}) by χ (resp. χ^s). We then set $\Pi f_\chi := \lambda f_{\chi^s}$ and $\Pi f_{\chi^s} := \lambda^{-1} f_\chi$ where $\lambda \in E^\times$. This defines a family of non-isomorphic basic irreducible diagrams $D(\lambda)$ parametrized by $E^\times$. One can actually prove that each basic diagram $D(\lambda)$ is of type (iii) (or equivalently (ii)) of Example 5.5 for π an irreducible principal series.

For $q \neq p$, there are many more irreducible basic diagrams than the ones of Examples 5.6 and 5.7.

Example 5.8. Assume $F/\mathbb{Q}_p$ is quadratic unramified (so $q = p^2$) and fix an embedding $\mathbb{F}_{p^2} \hookrightarrow E$. We give an example of a family of irreducible basic diagrams that can't show up in the case $F = \mathbb{Q}_p$. Let r_0, r_1 be two integers such that $1 \leq r_0 \leq p-2$, $0 \leq r_1 \leq p-3$ and consider the following weights (with the notations of §2.2):

$$\begin{aligned}
\sigma_1 &:= (r_0, r_1) \\
\sigma_2 &:= (r_0 - 1, p - 2 - r_1) \otimes \det^{p(r_1+1)} \\
\sigma_3 &:= (p - 1 - r_0, p - 3 - r_1) \otimes \det^{r_0 + p(r_1+1)} \\
\sigma_4 &:= (p - 2 - r_0, r_1 + 1) \otimes \det^{r_0 + p(p-1)}.
\end{aligned}$$

For $1 \leq i \leq 4$, the representation theory of $\mathrm{GL}_2(\mathbb{F}_{p^2})$ over E (see later, in particular §10.1 and Proposition 10.4) tells us there are unique non-split K/K_1-extensions:
$$0 \to \sigma_i \to X_i \to \sigma_{i-1}^s \to 0$$
with $\sigma_{-1} := \sigma_4$. Moreover, the K/K_1-representations X_i (that we see as $KF^\times$-representations by making $\varpi_F = p$ act trivially) are such that $X_i^{I_1} =$

$\chi_i \oplus \chi_{i-1}^s$ (with obvious notations). For each i, let us fix an E-basis e_i of χ_i and e_{i-1}^s of χ_{i-1}^s so that $X_i^{I_1} = Ee_i \oplus Ee_{i-1}^s$. Let $\lambda \in E^\times$ and define an action of Π (hence of N) on $\oplus X_i^{I_1}$ as follows: for $1 \leq i \leq 3$, $\Pi e_i := e_i^s$ and $\Pi e_4 := \lambda e_4^s$ (since $\Pi^2 = 1$, this determines Πe_i^s). This defines a family of basic diagrams by setting $D_0 := \oplus X_i$, $D_1 := \oplus X_i^{I_1}$ with the above action of N and $r : D_1 \to D_0$ is the canonical injection. Any change of λ gives another isomorphism class of basic dagrams as is easily checked. Any other choice of E-basis on $X_i^{I_1}$ gives rise to a basic diagram isomorphic to one in the previous family.

Claim: All of the above basic diagrams are irreducible.

Proof. Let $0 \neq D_0' \subseteq D_0$ such that $D_0'^{I_1}$ is preserved by Π. As $D_0' \neq 0$ and as the σ_i are distinct, there is i such that $e_i \in D_0'$. Hence D_0' contains $\langle K\Pi e_i \rangle = \langle K e_i^s \rangle = X_{i+1}$. This implies $e_{i+1} \in D_0'$. Starting again with e_{i+1}, we see in the end that $\oplus X_i \subseteq D_0'$, hence $D_0' = D_0$. $\square$

I now finish with an example of reducible basic diagrams.

Example 5.9. Keep the notations of Examples 5.6 and 5.7. Let $D_0 := \sigma \oplus \mathrm{ind}_I^K \chi \oplus \sigma^s$. We have $D_0^{I_1} = Ev_\chi \oplus Ef_\chi \oplus Ef_{\chi^s} \oplus Ev_{\chi^s}$. Set $\Pi v_\chi := \lambda f_{\chi^s}$ and $\Pi v_{\chi^s} := f_\chi$ with $\Pi^2 = 1$ (which determines the rest). We will see in §7.1 that f_{χ^s} generates $\sigma^s \subset \mathrm{ind}_I^K \chi$ (which is the K-socle). Thus the corresponding basic diagram is reducible as it strictly contains the basic diagram of Example 5.6.

5.2 Basic diagrams: the existence theorem I

We use basic diagrams to build supercuspidal representations of $\mathrm{GL}_2(F)$ over E with a given K-socle. This was suggested by a recent conjecture of Buzzard, Diamond and Jarvis (we will come to it later) predicting some non-trivial K-socles in the case F is unramified. For the moment, there is no restriction on F.

The main theorem that we will prove in this section and the next (already mentionned in §1.2) is the following theorem (contained in joint work with Paskunas but actually due to Paskunas):

Theorem 5.10. *Assume $p > 2$. Let $D = (D_0, D_1, r)$ be a basic diagram and assume $D_0^{K_1}$ is finite dimensional. Then there exists at least one smooth admissible representation π of $\mathrm{GL}_2(F)$ over E such that:*

(i) $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0$

(ii) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *contains* D

(iii) π *is generated by* D_0.

We can already prove an irreducibility result.

Proposition 5.11. *Let D be as in Theorem 5.10 (no restriction on p) and assume D is irreducible. Let π be any smooth admissible representation of $\mathrm{GL}_2(F)$ over E satisfying (i), (ii) and (iii) of Theorem 5.10. Then π is irreducible.*

Proof. Let $0 \subsetneq \pi' \subseteq \pi$ be a subrepresentation. Because $0 \neq \mathrm{soc}_K \pi' \subseteq \mathrm{soc}_K \pi = \mathrm{soc}_K D_0$, we have $\pi' \cap D_0 \neq 0$. As $(\pi' \cap D_0)^{I_1} = \pi' \cap D_1$ is preserved by Π, the diagram $(\pi' \cap D_0, \pi' \cap D_1, \mathrm{can})$ is basic and non-zero, hence equals D as D is irreducible. Thus we have $D_0 \subset \pi'$ which implies $\pi' = \pi$ as π is generated by D_0. $\qquad\square$

Unfortunately $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ or even $(\langle K\pi^{I_1}\rangle, \pi^{I_1}, \mathrm{can})$ is usually not irreducible if π is irreducible (the second is if $F = \mathbb{Q}_p$).

We now start the proof of Theorem 5.10 which will keep us busy for quite a while. For this, we need to introduce (or recall) injective envelopes. Recall that an injective object in an abelian category $\mathcal{C}$ is an object I such that, given objects A, B and morphisms $f : A \hookrightarrow B$, $j : A \to I$ in $\mathcal{C}$ where the first morphism is injective, there is a morphism $h : B \to I$ in $\mathcal{C}$ such that $j = h \circ f$.

Definition 5.12. *Let $\mathcal{C}$ be an abelian category and X an object in $\mathcal{C}$. An object I of $\mathcal{C}$ is called an injective envelope (or injective hull) of X if it satisfies the following conditions:*

(i) I is an injective object

(ii) there is an injection $i : X \hookrightarrow I$ (in $\mathcal{C}$)

(iii) for any non-zero injection $Y \hookrightarrow I$ in $\mathcal{C}$ the composed map $Y \hookrightarrow I \twoheadrightarrow \mathrm{coKer}(i)$ has a non-zero kernel (that is "$Y \cap X \neq 0$").

Injections $i : X \hookrightarrow I$ satisfying (iii) in Definition 5.12 are called essential injections (for any objects X, I of $\mathcal{C}$).

Proposition 5.13. *Let $\mathcal{C}$ and X be as in Definition 5.12. An injective envelope of X is unique up to (non-unique) isomorphism.*

Proof. Assume one has $i : X \hookrightarrow I$ and $i' : X \hookrightarrow I'$ with I, I' as in Definition 5.12. Because I' is an injective object and i is an injective map, there exists $j : I \to I'$ such that $i = j \circ i'$. Because $i : X \hookrightarrow I$ is an essential injection, if $\mathrm{Ker}(j) \neq 0$ then we have $\mathrm{Ker}(j) \cap X \neq 0$ which is impossible as i is injective. Therefore $\mathrm{Ker}(j) = 0$ and j is injective. Because I is an injective object and j is an injective map, there exists $h : I' \to I$ such that $\mathrm{id} = h \circ j$. Because $i' : X \hookrightarrow I'$ is an essential injection, if $\mathrm{Ker}(h) \neq 0$ then we have $\mathrm{Ker}(h) \cap X \neq 0$ which is impossible as i is injective. Therefore h is injective and hence j and h are inverse isomorphisms. $\square$

The main result of this section is the following well-known theorem:

Theorem 5.14. *Let G be a finite group and σ a finite dimensional representation of G over E. Then σ admits an injective envelope in the category of finite dimensional representations of G over E.*

Proof. Step 1: reduce to the case σ irreducible.
Indeed, assume every irreducible representation of G has an injective envelope. Then $\mathrm{soc}_G\, \sigma$ has an injective envelope, denoted $\mathrm{inj}_G(\mathrm{soc}_G\, \sigma)$, which is the direct sum of the injective envelopes of the summands. From the injectivity property of $\mathrm{inj}_G(\mathrm{soc}_G\, \sigma)$, the embedding $\mathrm{soc}_G\, \sigma \hookrightarrow \mathrm{inj}_G(\mathrm{soc}_G\, \sigma)$ extends to a morphism $\sigma \to \mathrm{inj}_G(\mathrm{soc}_G\, \sigma)$. This morphism is necessarily injective because the injection $\mathrm{soc}_G\, \sigma \hookrightarrow \mathrm{inj}_G(\mathrm{soc}_G\, \sigma)$ is essential. It is then *a fortiori* also an essential injection and $\mathrm{inj}_G(\mathrm{soc}_G\, \sigma)$ is thus an injective envelope of σ.

Step 2: the regular representation $E[G]$ (with G acting by $g'[g] := [g'g]$) is injective.
Indeed, dualizing everything and because $E[G]$ is self-dual (map $f \in E[G]^*$ to $\sum_{g \in G} f([g])[g] \in E[G]$), it is equivalent to prove that $E[G]$ is a projective object. Given a G-equivariant map $j : E[G] \to \sigma$, necessarily of the form $[g] \mapsto gv$ where v is the image of $[1_G]$, and a surjection $\sigma' \twoheadrightarrow \sigma$, it is straightforward to lift j to $\widehat{j} : E[G] \to \sigma'$ by sending $[g]$ to $g\widehat{v}$ where $\widehat{v} \in \sigma'$ maps to $v \in \sigma$.

Step 3: there exists an embedding $\sigma \hookrightarrow E[G]$.
Indeed, let $0 \neq f \in \sigma^*$, the map $E[G] \to \sigma^*$, $[g] \mapsto gf$ is surjective as σ^* is irreducible and one takes its dual (recall $E[G]$ is self-dual).

Step 4: construction of a candidate for an injective envelope.
Let $I \subseteq E[G]$ be a maximal subrepresentation containing σ and such that the injection $\sigma \hookrightarrow I$ is essential. Such a representation always exists as $E[G]$ is finite dimensional although it might not be unique: in the non-empty set

of subrepresentations R containing σ and such that the injection $\sigma \hookrightarrow R$ is essential, pick up one which has maximal dimension.

Step 5: proof that the candidate is injective.
I claim that I in Step 4 is an injective object, and is thus an injective envelope of σ. Let $\bar{I}$ be a maximal quotient of $E[G]$ (that is, which has minimal dimension) such that the injection $\sigma \hookrightarrow E[G]$ still induces an injection $\sigma \hookrightarrow \bar{I}$. Again, arguing as in Step 4, such a maximal quotient always exists as $E[G]$ has finite dimension, eventhough it might not be unique. The maximality property of $\bar{I}$ implies that the injection $\sigma \hookrightarrow \bar{I}$ is then essential. Because $\sigma \hookrightarrow I$ is an essential injection, the induced map $I \to \bar{I}$ remains injective. Because $E[G]$ is an injective object, there exists a map $\bar{I} \to E[G]$ such that the composition $I \hookrightarrow \bar{I} \to E[G]$ is the inclusion $I \subseteq E[G]$. Because $\sigma \hookrightarrow \bar{I}$ is an essential injection and $\sigma \to E[G]$ is an injection, the map $\bar{I} \to E[G]$ is again an injection. By maximality of I, the injection $I \hookrightarrow \bar{I}$ is therefore an isomorphism, that is we have a direct summand $E[G] = I \oplus J$ for some representation J. It is straightforward from this and the injectivity of $E[G]$ to deduce that I is injective, and therefore is an injective envelope of σ. $\quad\square$

Theorem 5.14 of course holds whatever the characteristic of E is. I denote by $\mathrm{inj}_G \sigma$ the injective envelope of σ.

Example 5.15. If p doesn't divide the order of G, more generally if $\mathrm{char}(E)$ doesn't divide the order of G (e.g. $\mathrm{char}(E) = 0$), then the injective envelope of σ is σ itself as the category of finite dimensional representations of G over E is semi-simple.

Example 5.16. If on the other hand the order of G is a power of p (i.e. G is a p-group), then $\mathrm{inj}_G \sigma$ is isomorphic to a direct sum of $E[G]$.

Now let σ be a weight and $K_n \subset K$ as in Corollary 4.10 for integers $n > 0$. We denote by $\mathrm{inj}_{K/K_n} \sigma$ an injective envelope of σ in the category of finite dimensional representations of $K/K_n = \mathrm{GL}(\mathcal{O}_F/\varpi_F^n)$ over E. Because a representation of K/K_n can be seen as a representation of K/K_{n+1}, the injectivity property of $\mathrm{inj}_{K/K_{n+1}} \sigma$ applied to $\sigma \hookrightarrow \mathrm{inj}_{K/K_n} \sigma$ and $\sigma \to \mathrm{inj}_{K/K_{n+1}} \sigma$ yields (non-canonical) injections $\mathrm{inj}_{K/K_n} \sigma \hookrightarrow \mathrm{inj}_{K/K_{n+1}} \sigma$ in the category of K/K_{n+1}-representations. Let $\mathrm{inj}_K \sigma := \mathrm{inj}\lim_n \mathrm{inj}_{K/K_n} \sigma$.

Proposition 5.17. *The K-representation $\mathrm{inj}_K \sigma$ is an injective envelope of σ in the abelian category of smooth representations of K over E (and therefore doesn't depend up to isomorphism on the transition maps).*

Proof. The K-injection $\sigma \hookrightarrow \mathrm{inj}_K \sigma$ is essential as $\mathrm{soc}_K \mathrm{inj}_K \sigma = \sigma$ (which follows from $\mathrm{soc}_{K/K_n} \mathrm{inj}_{K/K_n} \sigma = \sigma$ for every n). Therefore it is enough to prove that $\mathrm{inj}_K \sigma$ is an injective object in the above category. For any $n, m > 0$, it is straightforward to check that $(\mathrm{inj}_{K/K_{n+m}} \sigma)^{K_n}$ is an injective object in the category of K/K_n-representations (just use the fact $\mathrm{inj}_{K/K_{n+m}} \sigma$ is an injective object for K/K_{n+m}-representations). As it contains $\mathrm{inj}_{K/K_n} \sigma$, we thus have $\mathrm{inj}_{K/K_n} \sigma \xrightarrow{\sim} (\mathrm{inj}_{K/K_{n+m}} \sigma)^{K_n}$. Taking inductive limit, this implies $\mathrm{inj}_{K/K_n} \sigma \xrightarrow{\sim} (\mathrm{inj}_K \sigma)^{K_n}$. Let $\pi \hookrightarrow \pi'$ and $\pi \to \mathrm{inj}_K \sigma$ be K-equivariant morphisms with π, π' smooth K-representations over E and the first map injective. By the injectivity property of $(\mathrm{inj}_K \sigma)^{K_1}$, the map $\pi^{K_1} \to \mathrm{inj}_K \sigma$ extends to $\pi'^{K_1} \to \mathrm{inj}_K \sigma$. Applying the injectivity property of $(\mathrm{inj}_K \sigma)^{K_n}$ with $\pi^{K_n} \oplus_{\pi^{K_{n-1}}} \pi'^{K_{n-1}} \hookrightarrow \pi'^{K_n}$ for $n \geq 2$ together with an induction on n, we get a compatible system of maps $\pi'^{K_n} \to \mathrm{inj}_K \sigma$ extending $\pi^{K_n} \to \mathrm{inj}_K \sigma$. As π' is smooth, we have $\pi' = \mathrm{inj} \lim_n \pi'^{K_n}$ and taking the inductive limit yields a map $\pi' \to \mathrm{inj}_K \sigma$ which extends $\pi \to \mathrm{inj}_K \sigma$. $\qquad\square$

Be aware that $\mathrm{inj}_K \sigma$ is an infinite dimensional representation. As usual, we see all the above injective K-representations as $KF^\times$-representations by making ϖ_F act trivially. We are now going to make use of them.

6 Week 6

6.1 Basic diagrams: the existence theorem II

In this lecture and the next, we prove Theorem 5.10. Let us first explain the rough strategy of the proof:

(i) We extend (non-canonically) the action of I on $\mathrm{inj}_K D_0$ to an action of N such that there exists an injection of diagrams $(D_0, D_1, r) \hookrightarrow (\mathrm{inj}_K D_0, \mathrm{inj}_K D_0, \mathrm{id})$.

(ii) We use Corollary 3.4 to glue the two compatible actions of K and N on $\mathrm{inj}_K D_0$ and get an action of $\mathrm{GL}_2(F)$. We then take $\pi \subseteq \mathrm{inj}_K D_0$ to be the subrepresentation generated by D_0.

Let us start with (i). Note first that the construction of the injective envelopes $\mathrm{inj}_K \sigma$ given in §5.2 also works for other compact groups. In particular, if $\chi : I \to E^\times$ is a smooth character, one can define in an analogous way $\mathrm{inj}_I \chi = \mathrm{inj} \lim_n \mathrm{inj}_{I/K_n} \chi$.

In this lecture, we prove several technical lemmas of group theory (that we will use in the next lecture).

Lemma 6.1. *Let G be a finite group, σ a finite dimensional representation of G over E and $\mathrm{inj}_G \sigma$ an injective envelope of σ. Let $D \subseteq G$ be a normal subgroup, then $(\mathrm{inj}_G \sigma)^D = \mathrm{inj}_{G/D} \sigma^D$.*

Proof. It is straightforward to check that $(\mathrm{inj}_G \sigma)^D$ is an injective object in the category of G/D-representations over E. Therefore we have to prove that the injection $\sigma^D \hookrightarrow (\mathrm{inj}_G \sigma)^D$ is essential. Let $\tau \hookrightarrow (\mathrm{inj}_G \sigma)^D$ be a non-zero injection in the category of G/D-representations over E. As $\sigma \hookrightarrow \mathrm{inj}_G \sigma$ is essential and as any G/D-representation can be seen as a G-representation, we have $\tau \cap \sigma \neq \{0\}$, hence $\tau \cap \sigma^D \neq \{0\}$ since D acts trivially on τ. $\qquad\square$

Lemma 6.2. *Let G be a finite group, σ a finite dimensional representation of G over E and $\mathrm{inj}_G \sigma$ an injective envelope of σ. Let $H \subseteq G$ be a subgroup, then $(\mathrm{inj}_G \sigma)|_H$ is an injective object in the category of finite dimensional representations of H over E.*

Proof. Let A, B be objects in this category together with H-equivariant maps $A \hookrightarrow B$ and $A \to (\mathrm{inj}_G \sigma)|_H$. By Frobenius reciprocity, we get a G-equivariant map $\mathrm{ind}_H^G A \to \mathrm{inj}_G \sigma$. As $\mathrm{inj}_G \sigma$ is an injective object, this map extends to a G-equivariant map $\mathrm{ind}_H^G B \to \mathrm{inj}_G \sigma$. As $B \hookrightarrow (\mathrm{ind}_H^G B)|_H$ (functions with support on H), we get an H-equivariant map $B \to (\mathrm{inj}_G \sigma)|_H$ extending $A \to (\mathrm{inj}_G \sigma)|_H$. $\qquad\square$

In general, it is not true that $(\mathrm{inj}_G \sigma)|_H$ is an injective envelope of $\sigma|_H$ in the category of finite dimensional representations of H over E.

Lemma 6.3. *Let G be a finite group and I an injective object in the category of finite dimensional representations of G over E. Then $I \simeq \oplus_\sigma n_\sigma \mathrm{inj}_G \sigma$ where σ runs over all irreducible representations of K over E and $n_\sigma \geq 0$ are integers such that $\mathrm{soc}_G I = \oplus_\sigma n_\sigma \sigma$.*

Proof. As the injection $\mathrm{soc}_G I \hookrightarrow I$ is essential, we have $I \simeq \mathrm{inj}_G(\mathrm{soc}_G I) = \mathrm{inj}_G(\oplus_\sigma n_\sigma \sigma) \simeq \oplus_\sigma n_\sigma \mathrm{inj}_G \sigma$. $\qquad\square$

In particular applying Lemmas 6.2 and 6.3, we get that $(\mathrm{inj}_G \sigma)|_H$ is a direct summand of representations $\mathrm{inj}_H \tau$ for some irreducible representations τ of H over E.

Corollary 6.4. *Let σ be a weight, then $(\mathrm{inj}_K \sigma)|_I = \oplus_\chi n_\chi \mathrm{inj}_I \chi$ where χ runs over all smooth characters of I over E and n_χ are integers ≥ 0 such that $(\mathrm{inj}_{K/K_1} \sigma)^{I_1}|_I = \oplus_\chi n_\chi \chi$.*

Proof. For $n \geq 0$ and as I/I_1 has order prime to p, we have by Lemma 6.1:

$$\mathrm{soc}_I((\mathrm{inj}_{K/K_n} \sigma)|_I) \subseteq ((\mathrm{inj}_{K/K_n} \sigma)|_I)^{I_1} = ((\mathrm{inj}_{K/K_n} \sigma)^{K_1})^{I_1}$$
$$= (\mathrm{inj}_{K/K_1} \sigma)^{I_1} = \oplus_\chi n_\chi \chi$$

which implies thus $\mathrm{soc}_I((\mathrm{inj}_{K/K_n} \sigma)|_I) = \oplus_\chi n_\chi \chi$. The result then follows from Lemma 6.3, Lemma 6.2 and the equalities $\mathrm{inj}_K \sigma = \mathrm{inj}\lim_n \mathrm{inj}_{K/K_n} \sigma$, $\mathrm{inj}_I \chi = \mathrm{inj}\lim_n \mathrm{inj}_{I/K_n} \chi$. $\qquad\square$

Lemma 6.5. *Let G be a finite group, σ a finite dimensional representation of G over E and $\mathrm{inj}_G \sigma$ an injective envelope of σ. Let $D \subseteq G$ be a normal p-group and assume G/D has order prime to p. Then there exists a unique (up to isomorphism) action of G on $\mathrm{inj}_D(\sigma|_D)$ extending the given action of D and such that the injection $\sigma|_D \hookrightarrow \mathrm{inj}_D(\sigma|_D)$ is G-equivariant. This action makes $\mathrm{inj}_D(\sigma|_D)$ isomorphic to $\mathrm{inj}_G \sigma$.*

Proof. In particular, in that special situation, $(\mathrm{inj}_G \sigma)|_D$ is really an injective envelope of $\mathrm{inj}_D(\sigma|_D)$. By Lemma 6.1 and Example 5.15, $(\mathrm{inj}_G \sigma)^D = \mathrm{inj}_{G/D}(\sigma^D) = \sigma^D$. I claim that the injection $\sigma|_D \hookrightarrow (\mathrm{inj}_G \sigma)|_D$ is essential. Indeed, let $\tau \subseteq (\mathrm{inj}_G \sigma)|_D$ be a D-subrepresentation such that $\sigma|_D \oplus \tau \hookrightarrow (\mathrm{inj}_G \sigma)|_D$, therefore $\sigma^D \oplus \tau^D \hookrightarrow (\mathrm{inj}_G \sigma)^D = \sigma^D$ which implies $\tau^D = 0$ which implies $\tau = 0$ as D is a p-group (Lemma 2.6). As $(\mathrm{inj}_G \sigma)|_D$ is injective (as a D-representation) by Lemma 6.2, we thus have $\mathrm{inj}_D(\sigma|_D) \simeq (\mathrm{inj}_G \sigma)|_D$. So there is an action of G on $\mathrm{inj}_D(\sigma|_D)$ as in the statement. Let us now prove it is unique. Assume we have another action and denote by $(\mathrm{inj}_G \sigma)'$ this second representation of G. The G-injection $\sigma \hookrightarrow (\mathrm{inj}_G \sigma)'$ is again essential because the D-injection $\sigma|_D \hookrightarrow (\mathrm{inj}_G \sigma)'|_D = \mathrm{inj}_D(\sigma|_D)$ is. By the injectivity property of $\mathrm{inj}_G \sigma$, we thus have a G-equivariant injection $i : (\mathrm{inj}_G \sigma)' \hookrightarrow \mathrm{inj}_G \sigma$. Using that $(\mathrm{inj}_G \sigma)'|_D = \mathrm{inj}_D(\sigma|_D)$ is an injective D-representation, we have another D-equivariant injection $j : \mathrm{inj}_G \sigma \hookrightarrow (\mathrm{inj}_G \sigma)'$ such that $j \circ i = \mathrm{id}$. This implies j is also surjective, hence an isomorphism. Thus i is also an isomorphism. $\qquad\square$

Example 6.6. Let $\chi : I \to E^\times$ be a smooth character, then $(\mathrm{inj}_I \chi)|_{I_1} = \mathrm{inj}_{I_1} 1 \simeq \mathcal{F}(I_1, E)$ where $\mathcal{F}(I_1, E)$ denotes the E-vector space of smooth functions $f : I_1 \to E$ with usual action of I_1 by right translation. The first equality follows from Lemma 6.5 applied to $G = I/K_n$, $D = I_1/K_n$ and using $\mathrm{inj}_I \chi = \mathrm{inj}\lim_n \mathrm{inj}_{I/K_n} \chi$. The second equality follows from Example 5.16, the fact I_1 is pro-p and the same inductive limit argument.

Corollary 6.7. *Let τ be a smooth admissible representation of N over E such that ϖ_F acts trivially and assume $p > 2$. Then there exists a unique*

(up to isomorphism) action of N on $\mathrm{inj}_{I_1}(\tau|_{I_1})$ (resp. $\mathrm{inj}_I(\tau|_I)$) extending the given action of I_1 (resp. I) and such that the injection $\tau|_{I_1} \hookrightarrow \mathrm{inj}_{I_1}(\tau|_{I_1})$ (resp. $\tau|_I \hookrightarrow \mathrm{inj}_I(\tau|_I)$) is N-equivariant.

Proof. For $n \geq 1$ let $I_n \subseteq I$ be the subgroup of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $a, d \equiv 1 \ (\varpi_F^n)$, $c \equiv 0 \ (\varpi_F^n)$ and $b \equiv 0 \ (\varpi_F^{n-1})$. Then I_n is an open normal compact pro-p subgroup in N. Since ϖ_F acts trivially, we can replace N by $N/\varpi_F^{\mathbb{Z}}$. The result for I_1 in the corollary then follows from Lemma 6.5 applied to $G = N/I_n\varpi_F^{\mathbb{Z}}$ and $D = I_1/I_n$ (note that $N/I_1\varpi_F^{\mathbb{Z}}$ is of order prime to p as $p > 2$) using the equalities $\mathrm{inj}_{I_1}(\tau|_{I_1}) = \mathrm{inj\,lim}_n (\mathrm{inj}_{I_1}(\tau|_{I_1}))^{I_n} = \mathrm{inj\,lim}_n \mathrm{inj}_{I_1/I_n}((\tau|_{I_1})^{I_n})$ (Lemma 6.1). The result for I is analogous. $\qquad\square$

6.2 Basic diagrams: the existence theorem III

We finish the proof of Theorem 5.10.

We need yet two other lemmas.

Lemma 6.8. *Let σ be a weight and χ a character of I on E. Then we have:*
$$\dim_E \mathrm{Hom}_I(\chi, (\mathrm{inj}_K \sigma)^{I_1}|_I) = \dim_E \mathrm{Hom}_I(\chi^s, (\mathrm{inj}_K \sigma)^{I_1}|_I) \leq 1.$$

Proof. We have:
$$\mathrm{Hom}_I(\chi, (\mathrm{inj}_K \sigma)^{I_1}|_I) = \mathrm{Hom}_I(\chi, (\mathrm{inj}_K \sigma)|_I) = \mathrm{Hom}_K(\mathrm{ind}_I^K \chi, \mathrm{inj}_K \sigma)$$

where the last equality follows from Frobenius reciprocity as in the proof of Proposition 4.7. Assume σ doesn't occur as a subquotient of $\mathrm{ind}_I^K \chi$. Then we have $\mathrm{Hom}_K(\mathrm{ind}_I^K \chi, \mathrm{inj}_K \sigma) = 0$ as $\sigma = \mathrm{soc}_K(\mathrm{inj}_K \sigma)$. But it is a result on the representation theory of $\mathrm{GL}_2(\mathbb{F}_q)$ that the irreducible constituants of $\mathrm{ind}_I^K \chi$ are the same as those of $\mathrm{ind}_I^K \chi^s$ (although in a different order) and that they occur in both representations with multiplicity 1: see the coming Theorem 7.6. Hence we also have $\mathrm{Hom}_I(\chi^s, (\mathrm{inj}_K \sigma)^{I_1}|_I) = \mathrm{Hom}_K(\mathrm{ind}_I^K \chi^s, \mathrm{inj}_K \sigma) = 0$. Assume σ occurs in $\mathrm{ind}_I^K \chi$, or equivalently in $\mathrm{ind}_I^K \chi^s$, then because $\mathrm{ind}_I^K \chi$ is multiplicity free, $\mathrm{inj}_K \sigma$ is an injective object and $\sigma = \mathrm{soc}_K(\mathrm{inj}_K \sigma)$, we have $\dim_E \mathrm{Hom}_K(\mathrm{ind}_I^K \chi, \mathrm{inj}_K \sigma) = 1$, and likewise $\dim_E \mathrm{Hom}_K(\mathrm{ind}_I^K \chi^s, \mathrm{inj}_K \sigma) = 1$ (assume there are two non-colinear homomorphisms, take a linear combination sending σ to 0 and find a contradiction). This proves the lemma. $\quad\square$

Lemma 6.9. *Assume $p > 2$. Let $\sigma = \oplus_{i=1}^m \sigma_i$ where $(\sigma_i)_{1 \leq i \leq m}$ are irreducible representations of K over E and let $e \in \mathrm{End}_I(\mathrm{inj}_K \sigma)$ be an I-equivariant idempotent (that is $e^2 = e$). Suppose that there exists an action of N on $e((\mathrm{inj}_K \sigma)|_I)$ extending the given action of I with ϖ_F acting trivially. Then there exists an action of N on $(1 - e)((\mathrm{inj}_K \sigma)|_I)$ extending the given action of I with ϖ_F acting trivially.*

Proof. Set $V := e((\mathrm{inj}_K \sigma)^{I_1})$ and $W := (1-e)((\mathrm{inj}_K \sigma)^{I_1})$. Denote by V_χ and W_χ the respective χ-isotypic subspaces for the action of I where χ runs over the smooth characters of I over E. We thus have $V = \bigoplus_\chi V_\chi$ and $W = \bigoplus_\chi W_\chi$. The action of Π on V induces an isomorphism $V_\chi \cong V_{\chi^s}$ as $\chi^s = \chi(\Pi \cdot \Pi^{-1})$ and hence $\dim_E V_\chi = \dim_E V_{\chi^s}$ for all χ. It follows from Lemma 6.8 and $\mathrm{inj}_K \sigma \cong \oplus_{i=1}^m \mathrm{inj}_K \sigma_i$ that $\dim_E (\mathrm{inj}_K \sigma)^{I_1}_\chi = \dim_E (\mathrm{inj}_K \sigma)^{I_1}_{\chi^s}$. As $(\mathrm{inj}_K \sigma)^{I_1}_\chi = V_\chi \oplus W_\chi$ for all χ, we have $\dim_E W_\chi = \dim_E W_{\chi^s}$ for all χ. For every ordered pair (χ, χ^s) such that $\chi \neq \chi^s$, choose an isomorphism of vector spaces $\phi_{\chi,\chi^s} : W_\chi \to W_{\chi^s}$ so that $\phi_{\chi,\chi^s} = \phi_{\chi^s,\chi}^{-1}$. If $\chi = \chi^s$ then $W_\chi = W_{\chi^s}$ and we set $\phi_{\chi,\chi^s} := \mathrm{id}_{W_\chi}$. Define $\phi \in \mathrm{End}_E(W)$ by:

$$\phi(w_\chi) := \phi_{\chi,\chi^s}(w_\chi), \quad \forall w_\chi \in W_\chi, \quad \forall \chi.$$

Then $\phi^2 = \mathrm{id}_W$ and:

$$(\phi u \phi^{-1}) w = (\phi u \phi^{-1})(\oplus_\chi w_\chi) = \oplus_\chi \phi(\chi^s(u)\phi^{-1}(w_\chi)) = \oplus_\chi \chi^s(u) w_\chi =$$
$$\oplus_\chi \chi(\Pi u \Pi^{-1}) w_\chi = \oplus_\chi (\Pi u \Pi^{-1}) w_\chi = (\Pi u \Pi^{-1}) w$$

where $u \in I$, $w \in W$. Hence by sending Π to ϕ we obtain an action of N on W extending the action of I such that ϖ_F acts trivially. Since $(\mathrm{inj}_K \sigma)|_I$ is an injective I-representation (Lemma 6.2) so is $(1-e)((\mathrm{inj}_K \sigma)|_I)$ as it is a direct I-summand. Since:

$$W = (1-e)((\mathrm{inj}_K \sigma)^{I_1}) = ((1-e)((\mathrm{inj}_K \sigma)|_I))^{I_1} = \mathrm{soc}_I((1-e)((\mathrm{inj}_K \sigma)|_I)),$$

we have that $(1-e)((\mathrm{inj}_K \sigma)|_I)$ is an injective envelope of W. Corollary 6.7 applied to $\tau = W$ implies there exists an action of N on $(1-e)((\mathrm{inj}_K \sigma)|_I) = \mathrm{inj}_I W$ extending the given action of I and such that the injection $W \hookrightarrow (1-e)((\mathrm{inj}_K \sigma)|_I)$ is N-equivariant. $\square$

Now we can prove the following key proposition:

Proposition 6.10. *Assume $p > 2$. Let (D_0, D_1, r) be a basic diagram such that $D_0^{K_1}$ is finite dimensional. Then there exists a smooth action of N on $\mathrm{inj}_K(\mathrm{soc}_K D_0)$ satisfying the following two conditions:*

(i) the induced action of I is the one already defined on $\mathrm{inj}_K(\mathrm{soc}_K D_0)$

(ii) the map $D_1 \xrightarrow{r} D_0 \hookrightarrow \mathrm{inj}_K(\mathrm{soc}_K D_0)$ is N-equivariant.

Proof. Let $\sigma := \mathrm{soc}_K D_0 \subseteq D_0^{K_1}$. By Proposition 5.17, we have a K-equivariant injection $i_0 : D_0 \hookrightarrow \mathrm{inj}_K \sigma$ which induces an I-equivariant injection $i_1 = i_0 \circ r : D_1|_I \hookrightarrow (\mathrm{inj}_K \sigma)|_I$. This injection factors through

$(\operatorname{inj}_K \sigma)|_I^{I_1} = (\operatorname{inj}_{K/K_1} \sigma)|_I^{I_1} = \oplus_\chi n_\chi \chi$ with the notations of Corollary 6.4. As $D_1|_I$ is also a direct sum of characters of I, we can write:

$$(\operatorname{inj}_{K/K_1} \sigma)|_I^{I_1} = D_1|_I \oplus X$$

where X is again a direct sum of characters of I (I/I_1 has order prime to p). As $(\operatorname{inj}_K \sigma)|_I = \operatorname{inj}_I((\operatorname{inj}_{K/K_1} \sigma)|_I^{I_1})$ (see proof of Corollary 6.4), we have:

$$(\operatorname{inj}_K \sigma)|_I \simeq \operatorname{inj}_I(D_1|_I) \oplus Y$$

where $Y = \operatorname{inj}_I X$ is an I-direct factor. Since $\operatorname{inj}_I(D_1|_I)|_{I_1} = \operatorname{inj}_{I_1}(D_1|_{I_1})$ (Example 6.6), Corollary 6.7 applied to $\tau = D_1$ tells us that there is a unique action of N on $\operatorname{inj}_I(D_1|_I)$ compatible with that of I and compatible with the action of N on D_1. Let $e \in \operatorname{End}_I(\operatorname{inj}_K \sigma)$ be the projector onto $\operatorname{inj}_I(D_1|_I)$ parallel to Y. Then Lemma 6.9 tells us that there exists an action of N on $(1-e)((\operatorname{inj}_K \sigma)|_I) = Y$ compatible with that of I and such that ϖ_F acts trivially. Summing up the actions of N on both summands, we get an action on $\operatorname{inj}_K \sigma = \operatorname{inj}_K(\operatorname{soc}_K D_0)$ as in the statement. $\qquad\square$

Remark 6.11. In Proposition 6.10, we use that $D_1 \hookrightarrow D_0^{I_1}$ but we don't use that this is an isomorphism.

Usually, this action of N is not unique. This finishes the proof of step (i) (cf. beginning of §6.1). We now prove step (ii) and thus finish the proof of Theorem 5.10. Consider the E-vector space $\operatorname{inj}_K(\operatorname{soc}_K D_0)$ as in Proposition 4.8 together with two actions of K and N that coincide on $K \cap N = I$. By Corollary 3.4, there is a unique smooth action of $\operatorname{GL}_2(F)$ on $\operatorname{inj}_K(\operatorname{soc}_K D_0)$ extending these actions. Define:

$$\pi := \langle \operatorname{GL}_2(F) D_0 \rangle \subseteq \operatorname{inj}_K(\operatorname{soc}_K D_0).$$

We have $\operatorname{soc}_K D_0 \subseteq \operatorname{soc}_K \pi \subseteq \operatorname{soc}_K(\operatorname{inj}_K(\operatorname{soc}_K D_0)) = \operatorname{soc}_K D_0$ hence $\operatorname{soc}_K \pi = \operatorname{soc}_K D_0$. We have $\pi^{K_n} \subseteq (\operatorname{inj}_K(\operatorname{soc}_K D_0))^{K_n} = \operatorname{inj}_{K/K_n}(\operatorname{soc}_K D_0)$ by Proposition 5.17 (more precisely its proof). As the latter space is finite dimensional by construction (Theorem 5.14), we get that π is admissible. Finally, the two other properties "$(\pi^{K_1}, \pi^{I_1}, \operatorname{can})$ contains D" and "π is generated by D_0" hold by construction.

We will soon apply this theorem to some explicit diagrams coming from the "weight part" of the recent conjectures of Buzzard, Diamond and Jarvis generalizing Serre's conjecture. Before that, we need some more representation theory of $\operatorname{GL}_2(\mathbb{F}_q)$.

7 Week 7

7.1 Principal series of $\mathrm{GL}_2(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_p$

We now fix an embedding $\mathbb{F}_q \hookrightarrow E$ till the end of the course.

We give the structure of principal series of $\mathrm{GL}_2(\mathbb{F}_q)$ over E, mostly without proof. Let $\chi : B(\mathbb{F}_q) \to E^\times$ be a character. Principal series are the parabolic inductions $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ with the usual action of $\mathrm{GL}_2(\mathbb{F}_q)$ by right translation on functions. We already met them several times, e.g. in the proof of Proposition 4.7. We start with a few easy lemmas on them.

Lemma 7.1. *The E-representations* $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ *have dimension $q + 1$.*

Proof. This comes from the fact that $B(\mathbb{F}_q)$ is of index $q + 1$ in $\mathrm{GL}_2(\mathbb{F}_q)$. $\square$

Let $U(\mathbb{F}_q) \subset \mathrm{GL}_2(\mathbb{F}_q)$ be the subgroup of upper unipotent matrices. We let $\phi \in \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ be the unique function with support on $B(\mathbb{F}_q)$ such that $\phi(u) = 1$ for all $u \in U(\mathbb{F}_q)$. For $0 \le j \le q - 1$, set:

$$f_j := \sum_{\lambda \in \mathbb{F}_q} \lambda^j \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \phi$$

with the convention $0^0 = 1$ and $0^{q-1} = 0$ (and where we have used the fixed embedding $\mathbb{F}_q \hookrightarrow E$).

Lemma 7.2. *The set $\{f_j, 0 \le j \le q - 1, \phi\}$ is a basis of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ of eigenvectors for the subgroup $\begin{pmatrix} \mathbb{F}_q^\times & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix}$ of diagonal matrices.*

Proof. The fact that the f_j and ϕ are eigenvectors for diagonal matrices is a straightforward computation. The non-zero function $\begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \phi$ has support in $B(\mathbb{F}_q) \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}^{-1}$. As these supports are disjoint in $\mathrm{GL}_2(\mathbb{F}_q)$ when $\lambda \in \mathbb{F}_q$ varies and are disjoint from $B(\mathbb{F}_q)$ (see (10) below), the functions $(\begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \phi, \lambda \in \mathbb{F}_q, \phi)$ are linearly independent. Since there are $q + 1$ of them, they form a basis of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ by Lemma 7.1. Since the $q \times q$-matrix $(\lambda^j)_{\lambda,j}$ is invertible in E, the functions $\{f_j, 0 \le j \le q - 1, \phi\}$ also form a basis. $\square$

Lemma 7.3. *The vector space $(\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi)^{U(\mathbb{F}_q)}$ has dimension 2 over E and a basis is (ϕ, f_0).*

Proof. Using the Bruhat-type decomposition:

$$\mathrm{GL}_2(\mathbb{F}_q) = B(\mathbb{F}_q) \amalg B(\mathbb{F}_q) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) U(\mathbb{F}_q)$$

we see that any $U(\mathbb{F}_q)$-invariant function is a linear combination of ϕ and the unique $U(\mathbb{F}_q)$-invariant function with support on $B(\mathbb{F}_q) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) U(\mathbb{F}_q)$ sending $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to 1. Using:

$$B(\mathbb{F}_q) \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) U(\mathbb{F}_q) = \amalg_{\lambda \in \mathbb{F}_q} B(\mathbb{F}_q) \left(\begin{smallmatrix} \lambda & 1 \\ 1 & 0 \end{smallmatrix}\right)^{-1} \tag{10}$$

and $\left(\begin{smallmatrix} 1 & \lambda' \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \lambda & 1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} \lambda+\lambda' & 1 \\ 1 & 0 \end{smallmatrix}\right)$ $(\lambda, \lambda' \in \mathbb{F}_q)$, one checks that this function is precisely $\sum_{\lambda \in \mathbb{F}_q} \left(\begin{smallmatrix} \lambda & 1 \\ 1 & 0 \end{smallmatrix}\right) \phi = f_0$. $\qquad\square$

As irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ over E have dimension at most q, we see that principal series must be reducible as $\mathrm{GL}_2(\mathbb{F}_q)$-representations. We now study their decomposition. The character χ can be uniquely written:

$$\chi : \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto a^r (ad)^m$$

where $r = \sum_{i=0}^{f-1} p^i r_i$ and $m = \sum_{i=0}^{f-1} p^i m_i$ with $0 \leq r_i \leq p-1$, $0 \leq m_i \leq p-1$, not all $m_i = p-1$. We denote by χ^s the character:

$$\chi^s : \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto d^r (ad)^m = a^{q-1-r} (ad)^{r+m}.$$

One case is easy and we can get rid of it right away:

Lemma 7.4. *Assume* $\chi = \chi^s$ *(that is* $r = 0$*), then:*

$$\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi = \det{}^m \oplus (p-1, \cdots, p-1) \otimes \det{}^m.$$

Proof. Recall $(p-1, \cdots, p-1) \otimes \det^m$ means the weight $(\mathrm{Sym}^{p-1} E^2) \otimes_E (\mathrm{Sym}^{p-1} E^2)^{\mathrm{Fr}} \otimes_E \cdots \otimes_E (\mathrm{Sym}^{p-1} E^2)^{\mathrm{Fr}^{f-1}} \otimes \det^m$ (see §2.2). Twisting everything by $\det^{-m}$, we can assume $m = 0$. It is obvious that $1 \subset \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1$ (just take the subspace of constant functions on $\mathrm{GL}_2(\mathbb{F}_q)$). For $f \in \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1$ define $S(f) \in E$ by:

$$S(f) := \sum_{\bar{g} \in B(\mathbb{F}_q) \backslash \mathrm{GL}_2(\mathbb{F}_q)} f(\bar{g}).$$

Then the map $f \mapsto S(f)$ induces an E-linear $\mathrm{GL}_2(\mathbb{F}_q)$-equivariant surjection $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1 \twoheadrightarrow 1$. Because $1 \notin \mathrm{Ker}(S)$, one has $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1 = \mathrm{Ker}(S) \oplus 1$. By Lemma 7.1, this implies $\mathrm{Ker}(S)$ has dimension q. It is thus sufficient to prove

that $\mathrm{Ker}(S)$ is irreducible, as it must be then isomorphic to $(p-1, \cdots, p-1)$ by Proposition 2.17. By Lemma 7.3, we have:

$$\mathrm{Ker}(S)^{U(\mathbb{F}_q)} = (\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1)^{U(\mathbb{F}_q)} \cap \mathrm{Ker}(S) = E f_0 = E(\phi - 1) \qquad (11)$$

(recall 1 is the constant function). Since ϕ generates $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$, any function $f \in \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1$ can be written $\sum_i \lambda_i g_i \phi$. When $f \in \mathrm{Ker}(S)$, we have $S(f) = \sum_i \lambda_i = 0$ and f can be written $f = \sum_i \lambda_i g_i (\phi - 1)$. Thus $\phi - 1$ generates $\mathrm{Ker}(S)$. By (11) this implies $\mathrm{Ker}(S)$ is irreducible as any non-zero subrepresentation has a non-zero invariant vector under $U(\mathbb{F}_q)$ by Lemma 2.6. $\qquad \square$

Before stating the result for $\chi \neq \chi^s$, I need to recall a few definition of finite groups representation theory. Let G be a finite group and R be a representation of G on a finite dimensional E-vector space (actually, we just need that E is a field). Then we define by induction a G-invariant increasing filtration $(\mathrm{soc}\, R)_i$ on R called the socle filtration as follows. We set $(\mathrm{soc}\, R)_0 := 0$ and $(\mathrm{soc}\, R)_{i+1}$ to be the inverse image in R of $\mathrm{soc}(R/(\mathrm{soc}\, R)_i)$. We also define by induction a decreasing filtration $(\mathrm{cosoc}\, R)_i$ on R called the co-socle (or radical) filtration as follows. We set $(\mathrm{cosoc}\, R)_0 := R$ and $(\mathrm{cosoc}\, R)_{i+1}$ to be $\mathrm{Ker}\big((\mathrm{cosoc}\, R)_i \twoheadrightarrow \mathrm{cosoc}((\mathrm{cosoc}\, R)_i)\big)$. Recall that the co-socle $\mathrm{cosoc}\, R$ of a representation R of G is the maximal semi-simple quotient of R. Usually, these two filtrations are distinct even up to renumbering (drawing). I denote by $R_i := (\mathrm{soc}\, R)_{i+1}/(\mathrm{soc}\, R)_i$.

I assume now $\chi \neq \chi^s$. I will assume $m = 0$ to make things simpler. Let σ be the unique weight such that χ is the character giving the action of $B(\mathbb{F}_q)$ on $\sigma^{U(\mathbb{F}_q)}$, that is $\sigma = (r_0, \cdots, r_{f-1})$. By Frobenius reciprocity $\mathrm{Hom}_{B(\mathbb{F}_q)}(\chi, \sigma|_{B(\mathbb{F}_q)}) = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi, \sigma)$, we aready know that σ will appear as a quotient of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$. For conveniency, I will actually rather study $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ (this is just a "change of variable").

To describe the constituents of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$, I introduce combinatorial notations that I will use at many places in the rest of this course. Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{P}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p - 1 - x_0\}$. If $f > 1$, then:

(i) $\lambda_i(x_i) \in \{x_i, x_i - 1, p - 2 - x_i, p - 1 - x_i\}$ for $i \in \{0, \cdots, f-1\}$

(ii) if $\lambda_i(x_i) \in \{x_i, x_i - 1\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p - 2 - x_{i+1}\}$

(iii) if $\lambda_i(x_i) \in \{p-2-x_i, p-1-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-1-x_{i+1}, x_{i+1}-1\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. Concretely, we see that $(\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ is a succession of sequences like $p-2-x_j, p-1-x_{j+1}, \cdots, p-1-x_{j+l}, x_{j+l+1}-1$ among the x_i.

For $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$, define:

$$e(\lambda) \;\; := \;\; \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1}-1\}$$

$$e(\lambda) \;\; := \;\; \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\Big) \text{ otherwise.}$$

Exercise 7.5. Prove that $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$.

For $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$, define also:

$$\mathcal{S}(\lambda) := \{i \in \{0, \cdots, f-1\}, \lambda_i(x_i) \in \{p-1-x_i, x_i-1\}\}$$

and set $\ell(\lambda) := |\mathcal{S}(\lambda)|$. If $\lambda, \lambda' \in \mathcal{P}(x_0, \cdots, x_{f-1})$, we write $\lambda' \leq \lambda$ if $\mathcal{S}(\lambda') \subseteq \mathcal{S}(\lambda)$.

Theorem 7.6. *Let* $\chi : \big(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\big) \mapsto a^r$ *and assume* $\chi \neq \chi^s$ *(that is* $r \notin \{0, q-1\}$*).*

(i) *The irreducible subquotients of* $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ *and of* $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ *are the same, and are exactly the all distinct weights:*

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det{}^{e(\lambda)(r_0, \cdots, r_{f-1})}$$

for $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$ *forgetting the weights such that* $\lambda_i(r_i) < 0$ *for some* i.

(ii) *If* τ *is an irreducible subquotient of* $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ *and* $\lambda \in \mathcal{P}(x_0, \cdots, x_{f-1})$ *its associated* f-*tuple by (i), we set* $\ell(\tau) := \ell(\lambda)$. *The socle and co-socle filtrations on* $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ *are the same (up to renumbering), with graded pieces:*

$$(\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s)_i = \bigoplus_{\ell(\tau)=i} \tau$$

for $0 \leq i \leq f$.

(iii) *We have* $(\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi)_i = (\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s)_{f-i}$.

(iv) *If τ, τ' are irreducible subquotients of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$, we write $\tau' \leq \tau$ if the corresponding f-tuples λ', λ by (i) satisfy $\lambda' \leq \lambda$. Let τ be an irreducible subquotient of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ and $U(\tau)$ the unique subrepresentation with co-socle τ. Then the socle and co-socle filtrations on $U(\tau)$ are the same (up to renumbering), with graded pieces:*

$$(U(\tau))_i = \bigoplus_{\substack{\ell(\tau')=i \\ \tau' \leq \tau}} \tau'$$

for $0 \leq i \leq \ell(\tau)$.

(v) *Let τ be an irreducible subquotient of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ and $Q(\tau)$ the unique quotient with socle τ. Then the socle and co-socle filtrations on $Q(\tau)$ are the same (up to renumbering), with graded pieces:*

$$(Q(\tau))_i = \bigoplus_{\substack{\ell(\tau')=i+\ell(\tau) \\ \tau \leq \tau'}} \tau'$$

for $0 \leq i \leq f - \ell(\tau)$.

Note that (i) of Theorem 5 is still true if $\chi = \chi^s$. Note also that (i) implies that all irreducible constituents of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ appear at most once and thus the representations $U(\tau)$ and $Q(\tau)$ in (iv) and (v) are well defined. We say that $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ is multiplicity free.

7.2 Injective envelopes of weights

Let σ be a weight of $\mathrm{GL}_2(\mathbb{F}_q)$, we describe the constituents of the injective envelope $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ (see Theorem 5.14) without their multiplicities.

Twisting if necessary, we can assume $\sigma = (r_0, \cdots, r_{f-1})$. Again, there is one case which is simple:

Lemma 7.7. *If $\sigma = (p-1, \cdots, p-1)$ then $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma = \sigma$.*

Proof. We have to prove that $(p-1, \cdots, p-1)$ is an injective object. Note first that we actually have $(p-1, \cdots, p-1) \simeq \mathrm{Sym}^{q-1} E^2$ in the notations of §2.2. Let us first prove that $\sigma|_{B(\mathbb{F}_q)}$ is injective as a $B(\mathbb{F}_q)$-representation. Let $v \in \sigma$ be a non-zero vector fixed by the lower unipotent matrices (for instance $v = y^{q-1}$ in the notations of *loc.cit.*). An easy computation shows that the map $E[U(\mathbb{F}_q)] \to \sigma$, $[g] \mapsto gv$ is surjective (recall $U(\mathbb{F}_q)$ is the subgroup of *upper*

unipotent matrices), hence bijective as source and target have dimension q. But since $U(\mathbb{F}_q)$ is a p-group, we know that $E[U(\mathbb{F}_q)]$ is isomorphic to $\mathrm{inj}_{U(\mathbb{F}_q)} 1$ (see Example 5.16). Hence $\sigma|_{U(\mathbb{F}_q)} \simeq \mathrm{inj}_{U(\mathbb{F}_q)} 1$. By Lemma 6.5 applied to $G = B(\mathbb{F}_q)$ and $D = U(\mathbb{F}_q)$, we thus have $\sigma|_{B(\mathbb{F}_q)} \simeq \mathrm{inj}_{B(\mathbb{F}_q)} 1$. As σ is self-dual, we can as well prove it is projective. Let $B \twoheadrightarrow A$ be a surjection of $E[\mathrm{GL}_2(\mathbb{F}_q)]$-modules and $\sigma \to A$ a non-zero $\mathrm{GL}_2(\mathbb{F}_q)$-equivariant map. As $\sigma|_{B(\mathbb{F}_q)}$ is projective, there exists a $B(\mathbb{F}_q)$-equivariant lift $\sigma \to B$. As $\sigma^{B(\mathbb{F}_q)} = E x^{q-1}$ is one dimensional and generates σ, its image in B must be non-zero and we have $\mathrm{Hom}_{B(\mathbb{F}_q)}(1, B|_{B(\mathbb{F}_q)}) \neq 0$. By Frobenius reciprocity, this implies there is a non-zero $\Phi \in \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1, B)$ which composed with $B \twoheadrightarrow A$ gives back the non-zero map $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1 \twoheadrightarrow (\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1)/1 = \sigma \to A$. By Lemma 7.4, the induced map $\sigma \hookrightarrow \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} 1 \xrightarrow{\Phi} B$ is $\mathrm{GL}_2(\mathbb{F}_q)$-equivariant and lifts $\sigma \to A$. $\qquad\square$

For other weights σ, it is much less trivial to work out $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ and its full structure doesn't seem to be known in general. However, we can list its irreducible constituents. As in §7.1, let $(x_0, \cdots, x_{f-1})$ be f variables and define a set $\mathcal{I}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p - 1 - x_0, p - 3 - x_0\}$. If $f > 1$, then:

(i) $\lambda_i(x_i) \in \{x_i, x_i - 1, x_i + 1, p - 2 - x_i, p - 3 - x_i, p - 1 - x_i\}$ for $i \in \{0, \cdots, f - 1\}$

(ii) if $\lambda_i(x_i) \in \{x_i, x_i - 1, x_i + 1\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p - 2 - x_{i+1}\}$

(iii) if $\lambda_i(x_i) \in \{p - 2 - x_i, p - 3 - x_i, p - 1 - x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1} - 1, x_{i+1} + 1, p - 3 - x_{i+1}, p - 1 - x_{i+1}\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. Concretely, we see that $(\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ is a succession of sequences like $p - 2 - x_j, p - 2 - \pm 1 - x_{j+1}, \cdots, p - 2 - \pm 1 - x_{j+l}, x_{j+l+1} \pm 1$ among the x_i.

As previously, we define for $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$:

$$e(\lambda) \; := \; \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i (x_i - \lambda_i(x_i))\Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1} - 1, x_{f-1} + 1\}$$

$$e(\lambda) \; := \; \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i (x_i - \lambda_i(x_i))\Big) \text{ otherwise.}$$

Exercise 7.8. Prove that $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z} x_i$.

The following lemma makes explicit the weights which appear as subquotients of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$.

Theorem 7.9. *Assume* $\sigma = (r_0, \cdots, r_{f-1}) \neq (p-1, \cdots, p-1)$.

(i) *Assume* $\sigma \neq (0, \cdots, 0)$. *The irreducible subquotients of* $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ *(without multiplicities) are the all distinct weights:*

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0, \cdots, r_{f-1})}$$

for $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$ *forgetting the weights such that* $\lambda_i(r_i) < 0$ *or* $\lambda_i(r_i) > p-1$ *for some* i.

(ii) *Assume* $\sigma = (0, \cdots, 0) = 1$. *The irreducible subquotients of* $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ *(without multiplicities) are the all distinct weights:*

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0, \cdots, r_{f-1})}$$

for $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$ *forgetting the weights such that* $\lambda_i(r_i) < 0$ *for some* i *and forgetting the weight* $(p-1, \cdots, p-1)$.

Example 7.10. The case $\mathbb{F}_q = \mathbb{F}_p$ is easy. Let $\sigma = \mathrm{Sym}^r E^2$ with $0 \leq r < p-1$. If $r \neq 0$, we have:

$$\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_p)}\,\sigma = \mathrm{Sym}^r E^2 \ \rule[0.5ex]{1.5em}{0.4pt}\ \begin{matrix} \mathrm{Sym}^{p-1-r} E^2 \otimes \det^r \\ \oplus \\ \mathrm{Sym}^{p-3-r} E^2 \otimes \det^{r+1} \end{matrix} \ \rule[0.5ex]{1.5em}{0.4pt}\ \mathrm{Sym}^r E^2$$

(forgetting Sym^{p-3-r} if $r = p-2$) and if $r = 0$, we have:

$$\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_p)}\,\sigma = 1 \ \rule[0.5ex]{1.5em}{0.4pt}\ \mathrm{Sym}^{p-3} E^2 \otimes \det \ \rule[0.5ex]{1.5em}{0.4pt}\ 1$$

(forgetting Sym^{p-3} if $p = 2$) where we write a finite dimensional indecomposable representation R of $\mathrm{GL}_2(\mathbb{F}_p)$ over E as follows:

$$R = R_0 \ \rule[0.5ex]{1em}{0.4pt}\ R_1 \ \rule[0.5ex]{1em}{0.4pt}\ R_2 \ \rule[0.5ex]{1em}{0.4pt}\cdots\rule[0.5ex]{1em}{0.4pt}\ R_n$$

where $(R_i)_i$ are the graded pieces of the socle filtration (see §7.1).

As we can already see with $\mathbb{F}_p$, it is not true that the constituents of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ appear there with multiplicity 1 in general. Although this doesn't happen for $q = p$, it is not even true in general that the socle σ appears only twice. So the situation is different and more complicated than what happens with principal series which are multiplicity free. However, there is a smaller representation V_σ inside $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ which behaves just as well as principal series if one adds a small assumption on σ:

Theorem 7.11. *(i) There is a unique maximal subrepresentation* $V_\sigma \subset \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ *such that* σ *occurs in* V_σ *with multiplicity* 1 *(hence as its socle).*

(ii) A weight occurs as a subquotient of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}\,\sigma$ *if and only if it occurs as a subquotient of* V_σ.

(iii) If moreover $\sigma = (r_0, \cdots, r_{f-1})$ *is such that* $0 \le r_i \le p - 2$ *for all* i, *then* V_σ *is multiplicity free.*

Example 7.12. Assume $q = p$ and let $\sigma = \mathrm{Sym}^r E^2$ with $0 \le r < p - 1$. If $r \ne 0$, we have:

$$V_\sigma = \ \mathrm{Sym}^r E^2 \ \underline{\quad\quad} \ \begin{matrix} \mathrm{Sym}^{p-1-r} E^2 \otimes \det^r \\ \oplus \\ \mathrm{Sym}^{p-3-r} E^2 \otimes \det^{r+1} \end{matrix}$$

(forgetting Sym^{p-3-r} if $r = p - 2$) and if $r = 0$, we have:

$$V_\sigma = \ 1 \ \underline{\quad\quad} \ \mathrm{Sym}^{p-3} E^2 \otimes \det$$

(forgetting Sym^{p-3} if $p = 2$).

We will at least prove (i) of Theorem 7.11 in §9.1 (see Proposition 9.1). When $0 \le r_i \le p - 2$ for all i, one can then describe the graded pieces of the socle and co-socle filtrations on the representation V_σ in a similar way to what is done in Theorem 7.6 for principal series. For $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$, define:

$$\mathcal{S}(\lambda) := \{i \in \{0, \cdots, f-1\}, \lambda_i(x_i) \in \{p - 2 - x_i - \pm 1, x_i \pm 1\}\}$$

and set $\ell(\lambda) := |\mathcal{S}(\lambda)|$. If $\tau = (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0, \cdots, r_{f-1})}$, set $\ell(\tau) := \ell(\lambda)$. We need yet another definition:

Definition 7.13. *Let* $\lambda, \lambda' \in \mathcal{I}(x_0, \cdots, x_{f-1})$. *We say* λ *and* λ' *are compatible if, whenever* $\lambda_i(x_i) \in \{p - 2 - x_i - \pm 1, x_i \pm 1\}$ *and* $\lambda'_i(x_i) \in \{p - 2 - x_i - \pm 1, x_i \pm 1\}$ *for the same* i, *then the signs of the* ± 1 *are the same in* $\lambda_i(x_i)$ *and* $\lambda'_i(x_i)$.

We write $\tau \le \tau'$ if the corresponding λ, λ' satisfy $\mathcal{S}(\lambda) \subseteq \mathcal{S}(\lambda')$ and λ and λ' are compatible.

Theorem 7.14. *Let* $\sigma = (r_0, \cdots, r_{f-1})$ *with* $0 \le r_i \le p - 2$ *for all* i. *If* $\sigma = 1$, *we forget below the weight* $(p - 1, \cdots, p - 1)$.

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(i) *The socle and co-socle filtrations on V_σ are the same, with graded pieces:*

$$(V_\sigma)_i = \bigoplus_{\ell(\tau)=i} \tau$$

for $0 \le i \le f$.

(ii) *Let τ be an irreducible subquotient of V_σ and $U(\tau)$ the unique subrepresentation with co-socle τ. Then the socle and co-socle filtrations on $U(\tau)$ are the same, with graded pieces:*

$$(U(\tau))_i = \bigoplus_{\substack{\ell(\tau')=i \\ \tau' \le \tau}} \tau'$$

for $0 \le i \le \ell(\tau)$.

(iii) *Let τ be an irreducible subquotient of V_σ and $Q(\tau)$ the unique quotient with socle τ. Then the socle and co-socle filtrations on $Q(\tau)$ are the same, with graded pieces:*

$$(Q(\tau))_i = \bigoplus_{\substack{\ell(\tau')=i+\ell(\tau) \\ \tau \le \tau'}} \tau'$$

for $0 \le i \le f - \ell(\tau)$.

If σ is not one dimensional, then the representation V_σ in particular contains the unique principal series $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ with K-socle σ and Theorem 7.14 is immediately checked to be consistent with Theorem 7.6.

8   Week 8

8.1   Diamond weights: definition

We now assume till the end of that course that F is unramified. We define Diamond weights associated to a continuous generic irreducible representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ (which were actually defined by Buzzard, Diamond and Jarvis). The combinatorics of these weights is similar to that of the irreducible constituents of principal series (§7.1) or injective envelopes (§7.2). They will be described by a set $\mathcal{D}(x_0, \cdots, x_{f-1})$ analogous to the sets $\mathcal{P}(x_0, \cdots, x_{f-1})$ and $\mathcal{I}(x_0, \cdots, x_{f-1})$. Recall we have fixed an embedding $\mathbb{F}_q \hookrightarrow E$.

Before defining Diamond weights, let us go back to $F = \mathbb{Q}_p$. Remember that if ρ is such that $\rho|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \omega_2^{r+1} \oplus \omega_2^{p(r+1)}$ with $0 \leq r \leq p-1$, then the modulo p local Langlands correspondence associates to it the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\pi(r, 0, 1)$ up to unramified twist. Remember also from Theorem 4.8 that the socle of $\pi(r, 0, 1)$ is $\mathrm{Sym}^r E^2 \oplus (\mathrm{Sym}^{p-1-r} E^2) \otimes \det^r$. So we have:

$$\mathrm{Sym}^r E^2 \leftrightarrow \omega_2^{r+1} \oplus \omega_2^{p(r+1)}. \tag{12}$$

Let us look at the other weight $(\mathrm{Sym}^{p-1-r} E^2) \otimes \det^r$. We have:

$$(\omega_2^{p((p-1-r)+1)} \oplus \omega_2^{(p-1-r)+1}) \otimes \omega_1^r = \omega_2^{p(p-r)+(1+p)r} \oplus \omega_2^{p-r+(1+p)r} =$$
$$\omega_2^{p^2+r} \oplus \omega_2^{p+pr} = \omega_2^{r+1} \oplus \omega_2^{p(r+1)} = \rho|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$$

where we have used $\omega_1 = \omega_2^{1+p}$ and $\omega_2^{p^2} = \omega_2$ (Lemma 2.4). What this shows is that we can somehow "see" the weight $(\mathrm{Sym}^{p-1-r} E^2) \otimes \det^r$ in the same way we saw $\mathrm{Sym}^r E^2$ in (12) but replacing ω_2 by ω_2^p.

We try to generalize this to $f > 1$. Start with ρ irreducible as in (ii) of Corollary 2.9 and assume moreover that ρ is generic (Definition 2.11). Let us try to find other ways to write down $\rho|_{I(\overline{\mathbb{Q}}_p/F)}$ by replacing ω_{2f} by its conjugate $\omega_{2f}^{p^f}$ at arbitrary places. For each $i \in \{0, \cdots, f-1\}$, choose $q_i \in \{p^i, qp^i\}$. Then can one find integers r_i' such that there is, say, a twist ρ' of ρ satisfying $\rho'|_{I(\overline{\mathbb{Q}}_p/F)} = \omega_{2f}^{\sum_{i=0}^{f-1}(r_i'+1)q_i} \oplus \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i'+1)q_i}$? The following lemma (which extends (ii) of Corollary 2.9) gives the answer:

Lemma 8.1. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous irreducible generic representation and for each $i \in \{0, \cdots, f-1\}$ choose an element $q_i \in \{p^i, qp^i\}$, then $\rho|_{I(\overline{\mathbb{Q}}_p/F)}$ can be written:*

$$\rho|_{I(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i'+1)q_i} & 0 \\ 0 & \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i'+1)q_i} \end{pmatrix} \otimes \eta'$$

for some character η' that extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and some integers r_i' which are such that:

> *if $i = 0$ and $(q_{f-1}, q_0) \in \{(p^{f-1}, 1), (qp^{f-1}, q)\}$ then $1 \leq r_0' \leq p-2$ otherwise $0 \leq r_0' \leq p-3$*

> *if $i > 0$ and $(q_{i-1}, q_i) \in \{(p^{i-1}, p^i), (qp^{i-1}, qp^i)\}$ then $0 \leq r_i' \leq p-3$ otherwise $1 \leq r_i' \leq p-2$.*

Proof. Write $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ as in (ii) of Corollary 2.9. Let us explain how one can replace $(r_i + 1)p^i$ by $(r_i' + 1)qp^i$ (at the expense of twisting). One has the congruences:

$$\begin{aligned}
(r_i + 1)p^i &\equiv (p - 1 - r_i)qp^i + p^{i+1} \quad (q+1) \quad \text{if } 0 \le i \le f - 2 \\
(r_{f-1} + 1)p^{f-1} &\equiv (p - 1 - r_{f-1})qp^{f-1} - 1 \quad (q+1).
\end{aligned}$$

This implies if $0 \le i \le f - 2$:

$$\begin{aligned}
\sum_{j=0}^{f-1}(r_j + 1)p^j &= \sum_{j \ne i, i+1}(r_j + 1)p^j + (r_i + 1)p^i + (r_{i+1} + 1)p^{i+1} \\
&= \sum_{j \ne i, i+1}(r_j + 1)p^j + (p - 1 - r_i)qp^i + (r_{i+1} + 2)p^{i+1} + a(q+1)
\end{aligned}$$

and if $i = f - 1$:

$$\begin{aligned}
\sum_{j=0}^{f-1}(r_j + 1)p^j &= \sum_{j=1}^{f-2}(r_j + 1)p^j + (r_{f-1} + 1)p^{f-1} + (r_0 + 1) \\
&= \sum_{j=1}^{f-2}(r_j + 1)p^j + (p - 1 - r_{f-1})qp^{f-1} + r_0 + a(q+1)
\end{aligned}$$

for some integer a. Hence $\omega_{2f}^{\sum_{j=0}^{f-1}(r_j+1)p^j}\eta = \omega_{2f}^{\sum_{j=0}^{f-1}(r_j'+1)q_j}\eta'$ where $q_j = p^j$, $r_j' = r_j$ if $j \notin \{i, i+1\}$ (resp. $j \notin \{f-1, 0\}$), $q_i = qp^i$, $r_i' = p - 2 - r_i$, $q_{i+1} = p^{i+1}$, $r_{i+1}' = r_{i+1} + 1$ (resp. $q_{f-1} = qp^{f-1}$, $r_{f-1}' = p - 2 - r_{f-1}$, $q_0 = 1$, $r_0' = r_0 - 1$) and where $\eta' = \eta\omega_{2f}^{a(q+1)}$. Note that η' still extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Moreover, we see from the genericity of ρ that if $i = f - 1$, then $(q_{f-1}, q_0) = (qp^{f-1}, 1)$ and $r_0' = r_0 - 1 \in \{0, \cdots, p-3\}$ and if $i < f - 1$, then $(q_i, q_{i+1}) = (qp^i, p^{i+1})$ and $r_{i+1}' = r_{i+1} + 1 \in \{1, \cdots, p-2\}$. Iterating this process gives the proposition. $\qquad\square$

From what happens for $F = \mathbb{Q}_p$, one is tempted to associate the weight $(r_0', \cdots, r_{f-1}') \otimes \eta'$ to $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$.

Definition 8.2. *Let* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ *be a continuous irreducible generic representation. The set of Diamond weights* $\mathcal{D}(\rho)$ *associated to* ρ *(in fact to* $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$*) is the set of weights* $\{(r_0', \cdots, r_{f-1}') \otimes \eta'\}$ *for all possible writings:*

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i'+1)q_i} & 0 \\ 0 & \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i'+1)q_i} \end{pmatrix} \otimes \eta'$$

as in Lemma 8.1 for all choices of q_i.

One should of course read $(r'_0, \cdots, r'_{f-1}) \otimes (\eta' \circ r_F^{-1} \circ \det)$. It turns out one can describe explicitly the set $\mathcal{D}(\rho)$ in the same way we described irreducible constituents of principal series and injective envelopes in §7.

Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{D}(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p-1-x_0\}$. If $f > 1$, then:

(i) $\lambda_0(x_0) \in \{x_0, x_0 - 1, p-2-x_0, p-1-x_0\}$ and $\lambda_i(x_i) \in \{x_i, x_i+1, p-2-x_i, p-3-x_i\}$ if $i > 0$

(ii) if $i > 0$ and $\lambda_i(x_i) \in \{x_i, x_i + 1\}$ (resp. $\lambda_0(x_0) \in \{x_0, x_0 - 1\}$), then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p-2-x_{i+1}\}$

(iii) if $0 < i < f-1$ and $\lambda_i(x_i) \in \{p-2-x_i, p-3-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-3-x_{i+1}, x_{i+1}+1\}$

(iv) if $\lambda_0(x_0) \in \{p-1-x_0, p-2-x_0\}$, then $\lambda_1(x_1) \in \{p-3-x_1, x_1+1\}$

(v) if $\lambda_{f-1}(x_{f-1}) \in \{p-2-x_{f-1}, p-3-x_{f-1}\}$, then $\lambda_0(x_0) \in \{p-1-x_0, x_0-1\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. Concretely, we see that $(\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ is a succession of sequences like $p-2-x_j, p-3-x_{j+1}, \cdots, p-3-x_{j+l}, x_{j+l+1}+1$ among the x_i with the caveat that $p-3-x_0$ (resp. $x_0 + 1$) has to be replaced by $p-1-x_0$ (resp. $x_0 - 1$).

For $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$, define if $f > 1$:

$$e(\lambda) := \frac{1}{2}\left(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\right) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1}+1\}$$

$$e(\lambda) := \frac{1}{2}\left(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i))\right) \text{ otherwise,}$$

and, if $f = 1$, $e(\lambda) := 0$ if $\lambda_0(x_0) = x_0$, $e(\lambda) := x_0$ if $\lambda_0(x_0) = p-1-x_0$. As previously, one checks that $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$.

Proposition 8.3. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous irreducible generic representation, that is:*

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_{2f}^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & \omega_{2f}^{q\sum_{i=0}^{f-1}(r_i+1)p^i} \end{pmatrix} \otimes \eta$$

with $1 \leq r_0 \leq p - 2$ and $0 \leq r_i \leq p - 3$ for $i > 0$. Then $\mathcal{D}(\rho)$ is the set of (all distinct) weights:

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0, \cdots, r_{f-1})} \eta$$

for $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$.

When $F = \mathbb{Q}_p$ and $\rho|_{I(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \omega_2^{r_0+1} \oplus \omega_2^{p(r_0+1)}$, we just recover $\mathcal{D}(\rho) = \{\mathrm{Sym}^{r_0} E^2, \mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}\}$. Let us examine the case $f = 2$:

Example 8.4. Assume $f = 2$ and consider ρ as in Proposition 8.3 with $\eta = 1$ (for simplicity). Then $\mathcal{D}(\rho)$ is the following list:

$$\{(r_0, r_1), (r_0 - 1, p - 2 - r_1) \otimes \det^{p(r_1+1)}$$
$$(p - 1 - r_0, p - 3 - r_1) \otimes \det^{r_0+p(r_1+1)}, (p - 2 - r_0, r_1 + 1) \otimes \det^{r_0+p(p-1)}\}$$

(look at Example 5.8!). Indeed, one can check the following equalities:

$$\begin{pmatrix} \omega_4^{(r_0+1)+(r_1+1)p} & 0 \\ 0 & \omega_4^{(r_0+1)p^2+(r_1+1)p^3} \end{pmatrix} =$$

$$\begin{pmatrix} \omega_4^{r_0+(p-1-r_1)p^3} & 0 \\ 0 & \omega_4^{r_0 p^2+(p-1-r_1)p} \end{pmatrix} \otimes \omega_2^{p(r_1+1)} =$$

$$\begin{pmatrix} \omega_4^{(p-r_0)p^2+(p-2-r_1)p^3} & 0 \\ 0 & \omega_4^{p-r_0+(p-2-r_1)p} \end{pmatrix} \otimes \omega_2^{r_0+p(r_1+1)} =$$

$$\begin{pmatrix} \omega_4^{(p-1-r_0)p^2+(r_1+2)p} & 0 \\ 0 & \omega_4^{(p-1-r_0)+(r_1+2)p^3} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)}.$$

We don't prove Proposition 8.3 as it is technical and doesn't present any difficulty from the analysis of the possible r_i' in Lemma 8.1. Note that one can replace $(r_i)_{0 \leq i \leq f-1}$ by $(p - 1 - r_0, p - 3 - r_1, \cdots, p - 3 - r_{f-1})$ in the way we start writing ρ (see the comment after Corollary 2.9), hence $\mathcal{D}(\rho)$ shouldn't be affected by this "change of variables". Indeed, one can check that $\mathcal{D}(x_0, \cdots, x_{f-1})$ is symmetrical with respect to $x_0 \mapsto p - 1 - x_0$ and $x_i \mapsto p - 3 - x_i$, $i > 0$.

The set $\mathcal{D}(x_0, \cdots, x_{f-1})$ can be naturally identified with the set of subsets $\mathcal{S}$ of $\{0, \cdots, f - 1\}$ as follows. Fix $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$. If $i > 0$ set $i \in \mathcal{S}(\lambda)$ if and only if $\lambda_i(x_i) \in \{p - 3 - x_i, x_i + 1\}$ and set $0 \in \mathcal{S}(\lambda)$ if and only if $\lambda_0(x_0) \in \{p - 1 - x_0, x_0 - 1\}$. One checks that, given $\mathcal{S}$, there is only

one possible $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$ such that $\mathcal{S} = \mathcal{S}(\lambda)$. By Proposition 8.3 we can thus identify $\mathcal{D}(\rho)$ with the subsets of $\{0, \cdots, f-1\}$. In particular we see that $|\mathcal{D}(\rho)| = 2^f$.

Let us finish with a few easy definitions. If $\sigma \in \mathcal{D}(\rho)$ corresponds to $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$, we set $\ell(\sigma) = \ell(\lambda) := |\mathcal{S}(\lambda)|$. If $\lambda, \lambda' \in \mathcal{D}(x_0, \cdots, x_{f-1})$ define $\lambda \cap \lambda' \in \mathcal{D}(x_0, \cdots, x_{f-1})$ as the element corresponding to $\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda')$ and $\lambda \cup \lambda' \in \mathcal{D}(x_0, \cdots, x_{f-1})$ as the element corresponding to $\mathcal{S}(\lambda) \cup \mathcal{S}(\lambda')$. We have a partial order on the elements of $\mathcal{D}(x_0, \cdots, x_{f-1})$ by declaring that $\lambda' \leq \lambda$ if and only if $\mathcal{S}(\lambda') \subseteq \mathcal{S}(\lambda)$. If $\sigma, \sigma' \in \mathcal{D}(\rho)$ correspond to λ, λ', we let $\sigma \cap \sigma'$ (resp. $\sigma \cup \sigma'$) be the unique weight in $\mathcal{D}(\rho)$ which corresponds to $\lambda \cap \lambda'$ (resp. $\lambda \cup \lambda'$). We also write $\sigma \leq \sigma'$ if $\lambda \leq \lambda'$.

8.2 Diamond diagrams I

We fix a continuous generic irreducible representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$. We define basic diagrams with K-socle being the direct sum of the weights in $\mathcal{D}(\rho)$.

Again, let us go back to $F = \mathbb{Q}_p$. Remember that to $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \omega_2^{r_0+1} \oplus \omega_2^{p(r_0+1)}$ we associate $\pi(r_0, 0, 1)$ and that:

$$\mathrm{soc}_K \pi(r_0, 0, 1) = \mathrm{Sym}^{r_0} E^2 \oplus \mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0} = \oplus_{\sigma \in \mathcal{D}(\sigma)} \sigma.$$

Hence it is natural when F is arbitrary (unramified) to look for smooth admissible and (hopefully) irreducible representations π of $\mathrm{GL}_2(F)$ over E such that $\mathrm{soc}_K \pi = \oplus_{\sigma \in \mathcal{D}(\sigma)} \sigma$. Inspired by our machinery of §5.2 (Theorem 5.10), we first wish to construct natural basic diagrams (D_0, D_1, r) such that $\mathrm{soc}_K D_0 = \oplus_{\sigma \in \mathcal{D}(\sigma)} \sigma$. We saw in §5.1 that for $f = 1$ one could indeed make a unique basic diagram such that $D_0 = \mathrm{Sym}^{r_0} E^2 \oplus \mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}$ and that this diagram could be used to characterize $\pi(r_0, 0, 1)$. However, when $f > 1$, it is impossible to proceed in the same easy way:

Example 8.5. Assume $f = 2$ and go back to Example 8.4. We know that $\mathcal{D}(\rho)$ is the list:

$$\{(r_0, r_1), (r_0 - 1, p - 2 - r_1) \otimes \det^{p(r_1+1)}$$
$$(p - 1 - r_0, p - 3 - r_1) \otimes \det^{r_0+p(r_1+1)}, (p - 2 - r_0, r_1 + 1) \otimes \det^{r_0+p(p-1)}\}.$$

If we take D_0 to be the direct sum of these four weights, then we can't put an action of N on $D_0^{I_1}$. Indeed, the I-representation $D_0^{I_1}$ (which is a sum

of characters) is not stable under $\chi \mapsto \chi^s$ contrary to what happened for $f = 1$. For instance if χ is the action of I on $(r_0, r_1)^{I_1}$ then χ^s is missing as the weight $(p - 1 - r_0, p - 1 - r_1) \otimes \det^{r_0 + pr_1}$ is not in $\mathcal{D}(\rho)$. The same phenomena happens for $f > 2$.

Therefore we will have to "enlarge" D_0 to get stability by $\chi \mapsto \chi^s$. We already saw such an enlargement in Example 5.8, but actually there is a bigger one. To see how to get it, let us again have a look at $F = \mathbb{Q}_p$. Although the basic diagram with $D_0 = \operatorname{Sym}^{r_0} E^2 \oplus \operatorname{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}$ was sufficient to characterize $\pi(r_0, 0, 1)$, it is not the biggest basic diagram that is actually contained in $\pi(r_0, 0, 1)$:

Theorem 8.6. *With the previous notations, let $\sigma_1 := \operatorname{Sym}^{r_0} E^2$ and $\sigma_2 := \operatorname{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}$ with $1 \le r_0 \le p - 2$. Then $\pi(r_0, 0, 1)^{K_1} = D_{0, \sigma_1} \oplus D_{0, \sigma_2}$ where:*

$$
\begin{aligned}
D_{0, \sigma_1} &= \operatorname{Sym}^{r_0} E^2 \text{——} \operatorname{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1} \\
D_{0, \sigma_2} &= \operatorname{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0} \text{——} \operatorname{Sym}^{r_0-2} E^2 \otimes \det
\end{aligned}
$$

(with the notations of Example 7.10) forgetting $\operatorname{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1}$ (resp. $\operatorname{Sym}^{r_0-2} E^2 \otimes \det$) when $r_0 = p - 2$ (resp. $r_0 = 1$).

Proof. We actually only give here a sketch of the proof (the proof will be finished later, see below). Since $\operatorname{soc}_K \pi(r_0, 0, 1) = \operatorname{soc}_{\operatorname{GL}_2(\mathbb{F}_q)} \pi(r_0, 0, 1)^{K_1}$, we have:
$$
\pi(r_0, 0, 1)^{K_1} \hookrightarrow \operatorname{inj}_{\operatorname{GL}_2(\mathbb{F}_q)}(\sigma_1) \oplus \operatorname{inj}_{\operatorname{GL}_2(\mathbb{F}_q)}(\sigma_2)
$$
by the universal property of injective envelopes. Note that we have canonical injections $D_{0, \sigma_1} \subset \operatorname{inj}_{\operatorname{GL}_2(\mathbb{F}_q)} \sigma_1$ and $D_{0, \sigma_2} \subset \operatorname{inj}_{\operatorname{GL}_2(\mathbb{F}_q)} \sigma_2$, in fact we even have $D_{0, \sigma_1} \subseteq V_{\sigma_1}$ and $D_{0, \sigma_2} \subseteq V_{\sigma_2}$ as D_{0, σ_1} and D_{0, σ_2} are multiplicity free. Let us first explain why one has $\pi(r_0, 0, 1)^{K_1} \subseteq D_{0, \sigma_1} \oplus D_{0, \sigma_2}$. Assume this is not the case. Then $\pi(r_0, 0, 1)^{K_1}$ contains as a subquotient one of the constituents of $\operatorname{inj}_{\operatorname{GL}_2(\mathbb{F}_q)}(\sigma_1 \oplus \sigma_2)$ which is not in $D_{0, \sigma_1} \oplus D_{0, \sigma_2}$. Looking at this injective envelope (Example 7.10), we see that this constituent must be either σ_1 or σ_2. Then a close examination of all the possibilities yields that $\pi(r_0, 0, 1)^{K_1}$ must necessarily contain either the unique non-split extension $\sigma_1 \text{——} \sigma_2$ or the unique non-split extension $\sigma_2 \text{——} \sigma_1$. But we have already met this kind of extension in the proof Proposition 4.7 (see the exact sequence (9)) and by Lemma 7.3 we know it has a 2-dimensional space of I_1-invariants. Together with the other weight in the socle, we see that either $\pi(r_0, 0, 1)$ contains $\sigma_2 \oplus (\sigma_1 \text{——} \sigma_2)$ or it contains $\sigma_1 \oplus (\sigma_2 \text{——} \sigma_1)$. In all cases we have $\dim_E \pi(r_0, 0, 1)^{I_1} \ge 3$. This contradicts Remark 5.3 (although we have not proven the latter). To prove that we have exactly $D_{0, \sigma_1} \oplus D_{0, \sigma_2}$, I will exhibit

elements in $\pi(r_0, 0, 1)^{K_1}$ that generate the 2 "extra" weights $\mathrm{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1}$ and $\mathrm{Sym}^{r_0-2} E^2 \otimes \det$. By the entertwining $\pi(r_0, 0, 1) \simeq \pi(p - 1 - r_0, 0, \omega_1^{r_0})$ (see §4.1) it is enough to find an element generating $\mathrm{Sym}^{r_0-2} E^2 \otimes \det$ and we can thus assume $r_0 \geq 2$ (otherwise there is nothing to prove). Such an element is (using the notations of §3.2 and §4.2) $\sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), x^{r_0-1}y]$ if $r_0 \geq 3$ and $\sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), xy] - [\Pi, xy]$ if $r_0 = 2$ (to check this requires a certain amount of computation, see Lemma 11.8 and Proposition 11.6). $\quad\square$

Now, comparing Theorem 8.6 with Example 7.10, we see that $D_{0,\sigma_1} \oplus D_{0,\sigma_2}$ is the maximal subrepresentation of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}(\sigma_1 \oplus \sigma_2)$ such that σ_1 and σ_2, that is the weights of the socle, appear only once. What happens for $\mathbb{Q}_p$ is that when one considers I_1-invariants, then they actually coincide with the I_1-invariants of the socle.

Since considering just the socle doesn't work when $f > 1$, Theorem 8.6 suggests another approach: we could instead try to consider the maximal subrepresentation of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)}(\oplus_{\sigma \in \mathcal{D}(\rho)}\sigma)$ such that the weights σ of the socle appear only once (assuming it exists).

This indeed will lead to beautiful basic diagrams, except that this time we will have (for $f > 1$) an infinite family of basic diagrams associated to one given ρ!

Theorem 8.7. *Fix a continuous irreducible generic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$.*

(i) There exists a unique finite dimensional representation $D_0(\rho)$ of $\mathrm{GL}_2(\mathbb{F}_q)$ over E such that:

    (a) $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)} D_0(\rho) \simeq \oplus_{\sigma \in \mathcal{D}(\rho)}\sigma$

    (b) each irreducible σ in $\mathcal{D}(\rho)$ only occurs once as a Jordan-Hölder factor of $D_0(\rho)$ (hence in the socle)

    (c) $D_0(\rho)$ is maximal (for inclusion) for properties (a) and (b).

(ii) Each Jordan-Hölder factor of $D_0(\rho)$ only occurs once in $D_0(\rho)$.

(iii) As an I-representation, one has:

$$D_0(\rho)^{I_1} \simeq \bigoplus_{\substack{certain \ (\chi, \chi^s) \\ \chi \neq \chi^s}} \chi \oplus \chi^s$$

(in particular $D_0(\rho)^{I_1}$ is stable under $\chi \mapsto \chi^s$).

We will prove (part of) this theorem in the next lecture.

Example 8.8. Assume $f = 2$ as in Example 5.8 or Example 8.4 or Example 8.5. Then one can check using Theorem 7.14 that $D_0(\rho) = D_{0,\sigma_1} \oplus D_{0,\sigma_2} \oplus D_{0,\sigma_3} \oplus D_{0,\sigma_4}$ with the D_{0,σ_i} as follows (forgetting the twists in the weights and forgetting weights with negative entries):

$$
\begin{aligned}
D_{0,\sigma_1} &= (r_0, r_1) \longrightarrow S_1 \longrightarrow (p - 3 - r_0, p - 1 - r_1) \\
D_{0,\sigma_2} &= (r_0 - 1, p - 2 - r_1) \longrightarrow S_2 \longrightarrow (p - r_0, r_1 - 1) \\
D_{0,\sigma_3} &= (p - 1 - r_0, p - 3 - r_1) \longrightarrow S_3 \longrightarrow (r_0 - 2, r_1 + 2) \\
D_{0,\sigma_4} &= (p - 2 - r_0, r_1 + 1) \longrightarrow S_4 \longrightarrow (r_0 + 1, p - 4 - r_1)
\end{aligned}
$$

where:

$$
\begin{aligned}
S_1 &= (p - 2 - r_0, r_1 - 1) \oplus (r_0 + 1, p - 2 - r_1) \\
S_2 &= (r_0 - 2, r_1) \oplus (p - 1 - r_0, p - 1 - r_1) \\
S_3 &= (r_0 - 1, p - 4 - r_1) \oplus (p - r_0, r_1 + 1) \\
S_4 &= (p - 3 - r_0, p - 3 - r_1) \oplus (r_0, r_1 + 2).
\end{aligned}
$$

Indeed, as the socle appears in $D_0(\rho)$ with multiplicity 1, we have *a fortiori* $D_0(\rho) \subseteq V_{\sigma_1} \oplus V_{\sigma_2} \oplus V_{\sigma_3} \oplus V_{\sigma_4}$ (denoting as in Example 5.8 by σ_i, $1 \le i \le 4$ the four weights of the socle). Theorem 7.14 together with Theorem 7.11 and Theorem 7.9 give us the complete structure of each V_{σ_i} and an explicit computation yields then the above result. In particular, $D_0(\rho)$ contains the representation D_0 of Example 5.8. In fact, one can prove that $D_0 = \langle K D_0(\rho)^{I_1} \rangle \subset D_0(\rho)$.

One can manufacture many basic zero diagrams from $D_0(\rho)$ as follows. First, we extend the action of K on $D_0(\rho)$ to an action of $KF^\times$ by sending ϖ_F to the identity. Then, because of (iii) of Theorem 8.7 above, there is up to isomorphism a unique action of Π on $D_0(\rho)^{I_1}$ such that $\Pi^2 = \varpi_F$ acts trivially. Let us denote by $D_1(\rho)$ the resulting N-representation. Now, to make a basic diagram, one needs an $IF^\times$-equivariant injection $r : D_1(\rho) \hookrightarrow D_0(\rho)$. Up to isomorphism of basic diagrams, there are plenty of such injections when $f > 1$. For instance when $f = 2$, one can check that the resulting diagrams $D(\rho, r) := (D_0(\rho), D_1(\rho), r)$ are parametrized by $\lambda \in E^\times$ as is shown in Example 5.8. When f grows, things get worse, and the basic diagrams $D(\rho, r)$ form a family which depends on more and more parameters. The meaning of those parameters in terms of the Galois representation ρ (if any) is mysterious so far. Note that the basic diagrams $D(\rho, r)$ will never be irreducible. We have seen this for $f = 1$ in this lecture and for $f = 2$, compare Examples 5.8 and 8.8. The cases $f > 2$ are similar. Thus, we cannot apply Proposition 5.11 to the diagrams $D(\rho)$.

9 Week 9

Diamond diagrams II: We prove (i) and (iii) of Theorem 8.7 and give hints for the proof of (ii).

We start with a general proposition:

Proposition 9.1. *Let $\mathcal{D}$ be a finite set of distinct weights. Then there exists a unique (up to isomorphism) finite dimensional smooth representation D_0 of $\mathrm{GL}_2(\mathbb{F}_q)$ over E such that:*

(i) $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)} D_0 = \bigoplus_{\sigma \in \mathcal{D}} \sigma$

(ii) any weight of $\mathcal{D}$ appears at most once (as a subquotient) in D_0

(iii) D_0 is maximal with respect to properties (i), (ii).

Proof. Note first that condition (iii) means that, if D_0' is any finite dimensional representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over E that strictly contains D_0 as a subrepresentation, then (ii) is not satisfied for D_0'. Set $\tau := \oplus_{\sigma \in \mathcal{D}} \sigma$ and let τ' be a representation of $\mathrm{GL}_2(\mathbb{F}_q)$ satisfying (i). Then τ' satisfies (ii) if and only if $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau'/\tau, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) = 0$ for all $\sigma \in \mathcal{D}$. Since $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ is an injective representation, we have an exact sequence of $\mathrm{GL}_2(\mathbb{F}_q)$-representations:

$$0 \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau'/\tau, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau', \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) \to$$
$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) \to 0$$

and hence τ' satisfies (ii) if and only if:

$$\dim_E \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau', \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) = 1 \quad \text{for all } \sigma \in \mathcal{D}.$$

Let τ_1 and τ_2 be two $\mathrm{GL}_2(\mathbb{F}_q)$-invariant subspaces of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \tau$ containing τ and satisfying (ii). We are going to prove that $\tau_1 + \tau_2 \subseteq \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \tau$ still satisfies (ii), that is, satisfies $\dim_E \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 + \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) = 1$. Since $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ is injective, we again have an exact sequence:

$$0 \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 + \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 \oplus \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) \to$$
$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 \cap \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) \to 0.$$

Since $\tau_1 + \tau_2$ and $\tau_1 \cap \tau_2$ both contain τ as a subobject, they also contain σ. By the injectivity property of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ we get that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 + \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma)$ and $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 \cap \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma)$ are non-zero. Moreover, since the term in the middle has dimension $1 + 1 = 2$ (by what we have proven

above), we obtain in particular $\dim_E \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\tau_1 + \tau_2, \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma) = 1$. All this implies there exists a maximal subspace D_0 of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \tau$ satisfying (i) and (ii). Since any representation τ' of $\mathrm{GL}_2(\mathbb{F}_q)$ with $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)} \tau' \cong \tau$ can be embedded into $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \tau$, we obtain unicity and thus the proposition. $\square$

Note that if $\mathcal{D} = \{\sigma\}$, then $D_0 = V_\sigma$ where V_σ is as in Theorem 7.11. Indeed, by the universal property of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$, one has an embedding $D_0 \subseteq \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ and it then follows from the maximality that $D_0 = V_\sigma$.

Lemma 9.2. *Keep the notations of Proposition 9.1. Then we have an isomorphism of $\mathrm{GL}_2(\mathbb{F}_q)$-representations:*

$$D_0 \cong \bigoplus_{\sigma \in \mathcal{D}} D_{0,\sigma}$$

where $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{F}_q)} D_{0,\sigma} \cong \sigma$.

Proof. We keep the notations of the previous proof. Since $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \tau \simeq \bigoplus_{\sigma \in \mathcal{D}} \mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$, let us denote by e_σ the projector onto $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$ (note that this is well defined as $\mathcal{D}$ is multiplicity free). Now consider $\bigoplus_{\sigma \in \mathcal{D}} e_\sigma(D_0)$. It obviously satisfies (i). It also satisfies (ii) because if $\sigma \in \mathcal{D}$ appears elsewhere than as the socle of $e_\sigma(D_0)$, it *a fortiori* appears elsewhere in D_0 than in the socle and this is impossible. By maximality of D_0 the natural injection:

$$D_0 \hookrightarrow \bigoplus_{\sigma \in \mathcal{D}} e_\sigma(D_0)$$

has to be an isomorphism. Thus we have $D_{0,\sigma} = e_\sigma(D_0)$. $\square$

Exercise 9.3. Prove that if $\mathcal{D}$ and D_0 are as above, then $\mathrm{End}_{\mathrm{GL}_2(\mathbb{F}_q)}(D_0) \cong E^{|\mathcal{D}|}$.

Lemma 9.4. *Keep the notations of Proposition 9.1. If χ appears in $D_0^{U(\mathbb{F}_q)}$ (as a character of $B(\mathbb{F}_q)$) then so does χ^s.*

Proof. By Frobenius reciprocity:

$$\mathrm{Hom}_{B(\mathbb{F}_q)}(\chi, D_0|_{B(\mathbb{F}_q)}) = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{F}_q)}(\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi, D_0),$$

we have a non-zero map $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi \to D_0$ and hence there is a quotient of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ with K-socle contained in $\bigoplus_{\sigma \in \mathcal{D}} \sigma$. As $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$ and $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ have the same irreducible constituents (Theorem 7.6), we get that $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ contains irreducible constituents that are in $\mathcal{D}$. Consider all the non-zero quotients of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ with irreducible $\mathrm{GL}_2(\mathbb{F}_q)$-socle which is a weight of

$\mathcal{D}$. Then at least one of these quotients Q satisfies (ii) of Proposition 9.1, i.e. doesn't contain any other weight of $\mathcal{D}$. For instance, among all quotients of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ with irreducible $\mathrm{GL}_2(\mathbb{F}_q)$-socle which is a weight of $\mathcal{D}$, take one which has a minimal number of constituents. By maximality of D_0, we deduce an embedding:

$$Q \oplus (\oplus_{\substack{\sigma \in \mathcal{D} \\ \sigma \neq \mathrm{soc}\, Q}} \sigma) \hookrightarrow D_0$$

hence a non-zero map $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s \to D_0$ which implies χ^s appears in $D_0^{U(\mathbb{F}_q)}$ by Frobenius reciprocity. □

Note that Lemma 9.4 doesn't say that χ appears as many times as χ^s (and indeed this is wrong in general), it just says that if one appears, then so does the other. For instance consider the case $\mathcal{D} = \{(r_0, r_1), (p - 2 - r_0, r_1 - 1) \otimes \det^{r_0+1}, (r_0 - 1, p - 2 - r_1) \otimes \det^{p(r_1+1)}\}$. Let χ be the action of I on $(r_0, r_1)^{I_1}$, then one can check that χ appears only once in D_0 (as must be) but χ^s appears twice.

Now I would like to sketch the proof of (ii) of Theorem 8.7. This statement is again specific to the set of weights $\mathcal{D}(\rho)$ and is not true for an arbitrary set of distinct weights $\mathcal{D}$ (take e.g. the example $\mathcal{D}$ above). It is based on the following lemma:

Lemma 9.5. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous irreducible generic Galois representation. Let τ be an arbitrary weight and assume there exist two distinct weights $\sigma, \sigma' \in \mathcal{D}(\rho)$ such that τ is a subquotient of both V_σ and $V_{\sigma'}$ (see Theorem 7.11). Let $I(\sigma, \tau)$ (resp. $I(\sigma', \tau)$) be the unique subrepresentation of V_σ (resp. $V_{\sigma'}$) with socle σ (resp. σ') and co-socle τ. Then there exists $\sigma'' \in \mathcal{D}(\rho)$ such that either $\sigma'' \neq \sigma$ and σ'' occurs in $I(\sigma, \tau)$ or $\sigma'' \neq \sigma'$ and σ'' occurs in $I(\sigma', \tau)$.*

Note that the genericity of ρ implies that each $\sigma \in \mathcal{D}(\rho)$ satisfies the assumption in (iii) of Theorem 7.11 and hence V_σ is multiplicity free. This implies that $I(\sigma, \tau)$ and $I(\sigma', \tau)$ are well defined. Note also that if $\sigma'' \neq \sigma$ and σ'' occurs in $I(\sigma, \tau)$, it is possible that $\sigma'' = \sigma'$, and likewise in the other case. The proof of this lemma is a combinatorial computation. Here is the idea: writing $\tau = (s_0, \cdots, s_{f-1}) \otimes \theta$ we can find $\lambda, \lambda' \in \mathcal{I}(x_0, \cdots, x_{f-1})$ such that:

$$\sigma = (\lambda_0(s_0), \cdots, \lambda_{f-1}(s_{f-1})) \otimes \det^{e(\lambda)(r_0,\cdots,r_{f-1})}\theta$$
$$\sigma' = (\lambda'_0(s_0), \cdots, \lambda'_{f-1}(s_{f-1})) \otimes \det^{e(\lambda')(r_0,\cdots,r_{f-1})}\theta$$

(using that τ occurs in V_σ and $V_{\sigma'}$ and using Theorem 7.9 "backwards"). Then, using the fact that both σ and σ' are in $\mathcal{D}(\rho)$, one can compute that

λ and λ' must be compatible in the sense of Definition 7.13. We can thus define $\lambda'' := \lambda \cap \lambda' \in \mathcal{I}(x_0, \cdots, x_{f-1})$ in a way analogous to what we did in §8.1 for Diamond weights. The weight σ'' is then $(\lambda_0''(s_0), \cdots, \lambda_{f-1}''(s_{f-1})) \otimes \det^{e(\lambda'')(r_0, \cdots, r_{f-1})}\theta$ which, by another computation, is proved to be again in $\mathcal{D}(\rho)$ and satisfy the property of Lemma 9.5.

It is then straightforward to deduce from this that $D_0(\rho) = \oplus_{\sigma \in \mathcal{D}(\rho)} D_{0,\sigma}(\rho)$ is multiplicity free. Note first that one has an embedding $D_{0,\sigma}(\rho) \hookrightarrow V_\sigma$. As V_σ is multiplicity free, so is $D_{0,\sigma}(\rho)$. Now, assume that some weight τ appears twice in $D_0(\rho)$. Since each $D_{0,\sigma}(\rho)$ is multiplicity free, it must appear in $D_{0,\sigma}(\rho)$ and $D_{0,\sigma'}(\rho)$ for two distinct weights $\sigma, \sigma' \in \mathcal{D}(\rho)$. By Lemma 9.5 we obtain that $D_{0,\sigma}(\rho)$, say, contains a weight $\sigma'' \in \mathcal{D}(\rho)$ distinct from σ (as it contains $I(\sigma, \tau)$). This is impossible by definition of $D_0(\rho)$.

Finally, once we have that $D_0(\rho)$ is multiplicity free, we get that the $B(\mathbb{F}_q)$-representation $D_0(\rho)^{U(\mathbb{F}_q)}$ is also multiplicity free. Indeed, if $\chi \neq \chi^s$, then it is clear that χ appears at most once in $D_0(\rho)^{U(\mathbb{F}_q)}$. If $\chi = \chi^s$ appears, then by Frobenius reciprocity and Lemma 7.4, either a twist of the trivial representation or a twist of $(p-1, \cdots, p-1)$ must appear in the socle of $D_0(\rho)$, that is in $\mathcal{D}(\rho)$ and this can't happen as we have assumed ρ generic (easy check). Together with Lemma 9.4, this finishes the proof of (iii) of Theorem 8.7.

Next week, we will study in more details the basic diagrams $D(\rho, r)$ made out of $D_0(\rho)$.

10 Week 10

10.1 Diamond diagrams III

We study more closely $D_0(\rho)$ and the "indecomposability" of the diagrams $D(\rho, r)$.

Let us first completely determine the representation $D_0(\rho)$, that is to say the representations $D_{0,\sigma}(\rho)$ for $\sigma \in \mathcal{D}(\rho)$. Fix $\sigma \in \mathcal{D}(\rho)$ and write $\sigma = (\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0, \cdots, r_{f-1})}\eta$ with $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$ as in Proposition 8.3. One defines $\mu_\lambda \in \mathcal{I}(y_0, \cdots, y_{f-1})$ as follows:

(i) $\mu_{\lambda,i}(y_i) := p - 1 - y_i$ if $\lambda_i(x_i) \in \{p - 3 - x_i, x_i\}$ and $i > 0$ or if $\lambda_0(x_0) \in \{p - 2 - x_0, x_0 - 1\}$

(ii) $\mu_{\lambda,i}(y_i) := p - 3 - y_i$ if $\lambda_i(x_i) \in \{p - 2 - x_i, x_i + 1\}$ and $i > 0$ or if $\lambda_0(x_0) \in \{p - 1 - x_0, x_0\}$.

For $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$, define $\mu \circ \lambda := (\mu_i(\lambda_i(x_i))_i$ and $e(\mu \circ \lambda) \in \bigoplus_{i=0}^{f-1} \mathbb{Z} x_i$ as in Lemma 7.8 according to whether $\mu_{f-1}(\lambda_{f-1}(x_{f-1})) \in \mathbb{Z} + x_{f-1}$ or $\mathbb{Z} - x_{f-1}$.

Theorem 10.1. *Let* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ *be a continuous irreducible generic Galois representation. Fix* $\sigma \in \mathcal{D}(\rho)$ *and* λ *the corresponding* f*-tuple.*

(i) *The irreducible subquotients of* $D_{0,\sigma}(\rho)$ *are exactly the (all distinct) weights:*

$$(\mu_0(\lambda_0(r_0)), \cdots, \mu_{f-1}(\lambda_{f-1}(r_{f-1}))) \otimes \det{}^{e(\mu \circ \lambda)(r_0, \cdots, r_{f-1})} \eta \qquad (13)$$

for $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$ *such that* μ *and* μ_λ *are compatible (see Definition 7.13) forgetting the weights such that* $\mu_i(\lambda_i(r_i)) < 0$ *or* $\mu_i(\lambda_i(r_i)) > p - 1$ *for some* i.

(ii) *The graded pieces of the socle filtration on* $D_{0,\sigma}(\rho)$ *are:*

$$D_{0,\sigma}(\rho)_i = \bigoplus_{\ell(\mu)=i} \tau$$

for $0 \le i \le f - 1$ *and weights* τ *as in (13) with* $\ell(\mu)$ *as in §7.2.*

Proof. We may embed $D_{0,\sigma}(\rho)$ into $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_q)} \sigma$, and actually even into the subspace V_σ. By Lemma 7.9, all weights of $D_{0,\sigma}(\rho)$ are of type (13) for certain $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$. Take $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$ which is not compatible with μ_λ, assume $0 \le \mu_i(\lambda_i(r_i)) \le p - 1$ for all i and let τ be the corresponding weight (13). Two possibilities can occur: (1) there is $j \in \{1, \cdots, f - 1\}$ such that either $\lambda_j(x_j) \in \{p - 3 - x_j, x_j\}$ and $\mu_j(y_j) \in \{p - 3 - y_j, y_j + 1\}$ or $\lambda_j(x_j) \in \{p - 2 - x_j, x_j + 1\}$ and $\mu_j(y_j) \in \{p - 1 - y_j, y_j - 1\}$ or (2) $\lambda_0(x_0) \in \{p - 2 - x_0, x_0 - 1\}$ and $\mu_0(y_0) \in \{p - 3 - y_0, y_0 + 1\}$ or $\lambda_0(x_0) \in \{p - 1 - x_0, x_0\}$ and $\mu_0(y_0) \in \{p - 1 - y_0, y_0 - 1\}$. We give the proof for (1) as (2) is completely similar. In the first case of (1), define $\mu' = (\mu_i'(y_i))_i$ by $\mu_i'(y_i) := y_i$ if $i \notin \{j - 1, j\}$, $\mu_{j-1}'(y_{j-1}) := p - 2 - y_{j-1}$ and $\mu_j'(y_j) := y_j + 1$. In the second case, define $\mu' = (\mu_i'(y_i))_i$ by $\mu_i'(y_i) := y_i$ if $i \notin \{j - 1, j\}$, $\mu_{j-1}'(y_{j-1}) := p - 2 - y_{j-1}$ and $\mu_j'(y_j) := y_j - 1$. Let τ' be the corresponding weight (13). Then one checks that in both cases $\tau' \in \mathcal{D}(\rho)$ and $\tau' \ne \sigma$ (this is straightforward). Moreover, one has that τ' is a subquotient of the representation $I(\sigma, \tau)$ of Lemma 9.5. This follows from (ii) of Theorem 7.14 using that $\mathcal{S}(\mu') \subseteq \mathcal{S}(\mu)$ and that μ and μ' are compatible. Hence τ cannot

appear in $D_{0,\sigma}(\rho)$ by multiplicity 1. Conversely, if μ is compatible with μ_λ and $\mu \neq (y_0, \cdots, y_{f-1})$, then the weight (13) is never in $\mathcal{D}(\rho)$ as is immediately checked. By maximality of $D_{0,\sigma}(\rho)$ together with Theorem 7.14, it is then easy to derive (i). (ii) follows also from Theorem 7.14. $\qquad\square$

One can work out $D_{0,\sigma}(\rho)$ in a few examples. For $f = 1$, this is in the statement of Theorem 8.6. For $f = 2$, this is Example 8.8. One can work out $f = 3$ (one instance of $D_{0,\sigma}(\rho)$ given completely). Here is a non-trivial exercise:

Exercise 10.2. Prove that $\dim_E D_{0,\sigma}(\rho) = q - 1$ for all $\sigma \in \mathcal{D}(\sigma)$ and hence $\dim_E D_0(\rho) = 2^f(q - 1)$.

The basic diagrams $D(\rho, r)$ are not irreducible. But there is in some sense a trace of the irreducibility of ρ in the following statement:

Theorem 10.3. *The basic diagrams $D(\rho, r)$ are indecomposable, that is one cannot write $D(\rho, r) = D_1 \oplus D_2$ where D_1, D_2 are non-zero basic subdiagrams.*

Of course, for this, you have to use the action of Π on I_1-invariants. This can be immediately checked on the above examples $f = 1, 2$. I shall indicate now how one can prove Theorem 10.3. Let V be a finite dimensional representation of $\mathrm{GL}_2(\mathbb{F}_q)$ over E and assume that V is multiplicity free. Let τ be an irreducible constituent of V. We say that τ^{I_1} has a lift in V^{I_1} if the map $V(\tau)^{I_1} \to \tau^{I_1}$ is surjective where $V(\tau) \subseteq V$ is the unique subrepresentation with co-socle τ.

Proposition 10.4. *Let $\sigma = (r_0, \cdots, r_{f-1})$ with $0 \leq r_i \leq p - 2$ for all i and let V_σ be the representation in §7.2. The irreducible subquotients τ of V_σ such that τ^{I_1} has a lift in $V_\sigma^{I_1}$ are exactly the weights of V_σ such that $\mu_i(y_i) \in \{p - 2 - y_i, p - 1 - y_i, y_i, y_i + 1\}$.*

You can check that this statement is consistent with what we did for $f = 2$ in Examples 5.8 and 8.8. To describe the action of Π on $D_0^{I_1}$, we have to introduce some more notations.

Let $\mathcal{S}$ be a subset of $\{0, \cdots, f - 1\}$ and define $\delta(\mathcal{S})$ as follows (with the convention $f - 1 + 1 = 0$): if $i \neq 0$, $i \in \delta(\mathcal{S})$ if and only if $i + 1 \in \mathcal{S}$ and $0 \in \delta(\mathcal{S})$ if and only if $1 \notin \mathcal{S}$. If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ is a continuous irreducible generic Galois representation and $\sigma \in \mathcal{D}(\rho)$ corresponds to $\mathcal{S}$ (see §8.1), we write $\delta(\sigma)$ for the unique weight in $\mathcal{D}(\rho)$ corresponding to $\delta(\mathcal{S})$.

Remark 10.5. Going back to the case $f = 2$ (Examples 5.8 and 8.5), we have that σ_1 corresponds to $\mathcal{S}_1 := \emptyset$, σ_2 to $\mathcal{S}_2 := \{0\}$, σ_3 to $\mathcal{S}_3 := \{0, 1\}$ and

σ_4 to $\mathcal{S}_4 := \{1\}$. Recall that σ_i^s is a constituent of $D_{0,\sigma_{i+1}}$ (with "$5 = 1$"). Now, we see that $\mathcal{S}_{i+1} = \delta(\mathcal{S}_i)$.

Let ρ, σ be as above and let τ be an irreducible subquotient of $D_{0,\sigma}(\rho)$ such that τ^{I_1} has a lift in $D_{0,\sigma}(\rho)^{I_1}$. Then $\Pi\tau^{I_1} = (\tau^s)^{I_1}$ for a unique weight τ^s which is a constituent of a unique $D_{0,\sigma'}$, $\sigma' \in \mathcal{D}(\rho)$. I want to give the formula for σ'.

Let $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$ correspond to σ and $\mathcal{S} \subseteq \{0, \cdots, f-1\}$ correspond to λ. Write τ as in (13) for a $\mu \in \mathcal{I}(y_0, \cdots, y_{f-1})$. Note that by Proposition 10.4, one has $\mu_i(y_i) \in \{p-2-y_i, p-1-y_i, y_i, y_i+1\}$. Define $\mathcal{S}^- \subseteq \{0, \cdots, f-1\}$ as follows:

- If $i \neq 1$, $i \in \mathcal{S}^-$ if and only if $\big(\lambda_{i-1}(x_{i-1}) \in \{p-3-x_{i-1}, p-2-x_{i-1}\}$ and $\mu_{i-1}(y_{i-1}) = p-2-y_{i-1}\big)$ or $\big(\lambda_{i-1}(x_{i-1}) = p-2-x_{i-1}$ and $\mu_{i-1}(y_{i-1}) = y_{i-1}+1\big)$ (with "$-1 = f-1$")

- $1 \in \mathcal{S}^-$ if and only if $\big(\lambda_0(x_0) \in \{p-1-x_0, p-2-x_0\}$ and $\mu_0(y_0) = p-2-y_0\big)$ or $\big(\lambda_0(x_0) = p-1-x_0$ and $\mu_0(y_0) = y_0+1\big)$.

Likewise, define $\mathcal{S}^+ \subseteq \{0, \cdots, f-1\}$ as follows:

- If $i \neq 1$, $i \in \mathcal{S}^+$ if and only if $\big(\lambda_{i-1}(x_{i-1}) \in \{x_{i-1}, x_{i-1}+1\}$ and $\mu_{i-1}(y_{i-1}) = p-2-y_{i-1}\big)$ or $\big(\lambda_{i-1}(x_{i-1}) = x_{i-1}+1$ and $\mu_{i-1}(y_{i-1}) = y_{i-1}+1\big)$ (with "$-1 = f-1$")

- $1 \in \mathcal{S}^+$ if and only if $\big(\lambda_0(x_0) \in \{x_0-1, x_0\}$ and $\mu_0(y_0) = p-2-y_0\big)$ or $\big(\lambda_0(x_0) = x_0$ and $\mu_0(y_0) = y_0+1\big)$.

Note that we have $\mathcal{S}^- \subseteq \mathcal{S}$ and $\mathcal{S}^+ \cap \mathcal{S} = \emptyset$ (this just follows from $\lambda \in \mathcal{D}(x_0, \cdots, x_{f-1})$). In Remark 10.5, we always have $\mathcal{S}^- = \mathcal{S}^+ = \emptyset$. Remark 10.5 generalizes as follows:

Lemma 10.6. *With the previous notations, the weight $\sigma' \in \mathcal{D}(\rho)$ such that τ^s is a constituent of $D_{0,\sigma'}(\rho)$ is the weight of $\mathcal{D}(\rho)$ that corresponds to the subset $\delta((\mathcal{S} \setminus \mathcal{S}^-) \cup \mathcal{S}^+)$.*

In particular if we take $\tau = \sigma$ we check that $\mathcal{S}^- = \mathcal{S}^+ = \emptyset$ and thus Lemma 10.5 tells us that σ^s is a constituent of $D_{0,\delta(\sigma)}(\rho)$. This lemma is the main ingredient in the (very much combinatorial!) proof of Theorem 10.3 as we now know where is χ^s starting from χ (drawings for $f = 2$, $f = 3$ and $f = 4$).

Counting the dimension of the subspace of I_1-invariants in $D_0(\rho)$ (using Proposition 10.4) yields the nice result:

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Proposition 10.7. *With the previous notations, we have* $\dim_E D_0(\rho)^{I_1} = 3^f - 1$.

For $f = 1$, we recover $3 - 1 = 2$ and for $f = 2$, $3^2 - 1 = 8$.

10.2 Back to representations of $\mathrm{GL}_2(F)$

We associate supercuspidal representations of $\mathrm{GL}_2(F)$ to irreducible generic ρ (F unramified).

Fix a continuous irreducible generic Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ and assume that $\varpi_F = p$ acts trivially on $\det(\rho)$ (via r_F^{-1}). The two main results of the course are the following two theorems:

Theorem 10.8.   (i) *Let $D(\rho, r)$ be one of the basic diagrams associated to ρ in §8.2. There exists a smooth admissible representation π of $\mathrm{GL}_2(F)$ over E such that:*
(a) $\mathrm{soc}_K \pi = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma$
(b) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *contains* $D(\rho, r)$
(c) π *is generated by* $D_0(\rho)$
where can *is the canonical injection* $\pi^{I_1} \subset \pi^{K_1}$.

(ii) *If $D(\rho, r)$ and $D(\rho, r')$ are non-isomorphic and π, π' satisfy (a), (b), (c) for $D(\rho, r)$ and $D(\rho, r')$ respectively, then π and π' are non-isomorphic.*

Proof. (i) is exactly Theorem 5.10 applied to $D(\rho, r)$. Let us prove (ii). Let π satisfy (a), (b) and $D_0(\pi)$ be the unique maximal K-subrepresentation of π^{K_1} such that each $\sigma \in \mathrm{soc}_K \pi$ occurs in $D_0(\pi)$ only once. The existence of $D_0(\pi)$ is proved exactly by the same argument as in (i) of Proposition 9.1: if τ_1 and τ_2 are two subspaces of π^{K_1} such that $\mathrm{soc}_K \tau_1 = \mathrm{soc}_K \tau_2 = \mathrm{soc}_K \pi^{K_1} = \mathrm{soc}_K \pi$ and each $\sigma \in \mathrm{soc}_K \pi$ occurs in each τ_i exactly once, then $\tau_1 + \tau_2 \subseteq \pi^{K_1}$ satisfies the same property. Then it is clear from (b) and the definition of $D_0(\rho)$ that we have $D_0(\pi) = D_0(\rho)$ (as $D_0(\rho)$ is already as maximal as can be) and even $(D_0(\rho), D_1(\rho), r) \simeq (D_0(\pi), D_0(\pi) \cap \pi^{I_1}, \mathrm{can})$. Now if π and π' are as in (ii) and $\pi \cong \pi'$, then we certainly also have $(D_0(\pi), D_0(\pi) \cap \pi^{I_1}, \mathrm{can}) \simeq (D_0(\pi'), D_0(\pi') \cap \pi'^{I_1}, \mathrm{can})$ and thus $D(\rho, r) \simeq D(\rho, r')$ which is impossible. Thus $\pi \not\cong \pi'$. $\square$

Note that any π satisfying (a), (b), (c) above is such that p acts trivially (because this is so on $D_0(\rho)$).

Theorem 10.9. *Any smooth admissible π satisfying (a), (b), (c) in Theorem 10.8 is irreducible and is a supercuspidal representation.*

Assuming irreducibility, let us prove supercuspidability. We can assume $f > 1$ as we know the result for $f = 1$ by Lemma 5.2. But the K-socle of any such π contains strictly more than 2 irreducible constituents (as $2^f > 2$ if $f > 1$). If π is a subquotient of a principal series, we know from Remark 4.9 that the K-socle of π has at most two components. Thus this can't happen here and therefore π is supercuspidal.

When $F = \mathbb{Q}_p$, it follows from Lemma 5.2 and Theorem 8.6 that there exists a unique (up to isomorphism) smooth admissible representation $\pi(\rho, r)$ of $\mathrm{GL}_2(F)$ over E satisfying (a), (b), (c) as in (i) of Theorem 10.8. Moreover, this representation is then such that $(\pi(\rho, r)^{K_1}, \pi(\rho, r)^{I_1}, \mathrm{can}) \cong D(\rho, r)$. However, this results seems to be wrong when $f > 1$ (a counter-example was recently found by Yongquan Hu).

Let us sum up what we have done so far (postponing the proof of irreducibility in Theorem 10.9 for below). By the above two theorems, to each continuous irreducible generic representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ such that p acts trivially on $\det(\rho)$ we associate a non-empty family of smooth irreducible admissible supercuspidal representations with K-socle made out of the weights of $\mathcal{D}(\rho)$ (those representations satisfying (a), (b), (c) above for some basic diagram associated to ρ). Note however that we are still far from a complete understanding of this family (which is may-be too big for instance).

We now start the proof of irreducibility in Theorem 10.9. We will deduce it from Theorem 10.3 and from another theorem that I want now to state. Fix $\sigma \in \mathcal{D}(\rho)$ and let $\delta(\sigma) \in \mathcal{D}(\rho)$ be the weight defined in §10.1. From Lemma 10.6, recall we can characterize $\delta(\sigma)$ as being the unique weight of $\mathcal{D}(\rho)$ such that $D_{0,\delta(\sigma)}(\rho)$ contains the weight σ^s as a subquotient (the notation $\delta(\sigma)$ is actually quite bad since this weight depends on σ *and* on ρ). Denote by $I(\delta(\sigma), \sigma^s)$ the unique subrepresentation of $D_{0,\delta(\sigma)}(\rho)$ with co-socle σ^s (and socle $\delta(\sigma)$). Equivalently, it is the unique subrepresentation of $V_{\delta(\sigma)}$ with co-socle σ^s.

Theorem 10.10. *With the previous notations, let π be a smooth representation of $\mathrm{GL}_2(F)$ over E such that p acts trivially and $\psi : \mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma \to \pi$ a $\mathrm{GL}_2(F)$-equivariant morphism. Assume that π has a K-socle which is contained in $\oplus_{\sigma \in \mathcal{D}(\rho)} \sigma$ and that the K-subrepresentation of π generated by the image of $[\Pi, \sigma^{I_1}]$ is isomorphic to $I(\delta(\sigma), \sigma^s)$. Then the embedding $I(\delta(\sigma), \sigma^s) \hookrightarrow \pi$ extends uniquely to an embedding $D_{0,\delta(\sigma)}(\rho) \hookrightarrow \pi$.*

Let us first prove a preliminary lemma:

Lemma 10.11. *Keep the previous notations and let $\chi : I \to E$ be the character giving the action of I on σ^{I_1}. Then both $I(\delta(\sigma), \sigma^s)$ and the K-subrepresentation of π generated by the image of $[\Pi, \sigma^{I_1}]$ are non-zero quotients of $\mathrm{ind}_I^K \chi^s$.*

Proof. Denote by τ the K-subrepresentation of π generated by the image of $[\Pi, \sigma^{I_1}]$. As I acts on $[\Pi, \sigma^{I_1}]$ via χ^s, by Frobenius reciprocity, we have $\mathrm{Hom}_I(\chi^s, \tau|_I) = \mathrm{Hom}_K(\mathrm{ind}_I^K \chi^s, \tau)$ and the non-zero embedding $\chi^s \hookrightarrow \tau$ induces a surjection $\mathrm{ind}_I^K \chi^s \twoheadrightarrow \tau$. Hence τ is a quotient of $\mathrm{ind}_I^K \chi^s$. By construction, $(\sigma^s)^{I_1}$ has a lift in $I(\delta(\sigma), \sigma^s)^{I_1} \subset D_{0,\delta(\sigma)}(\rho)^{I_1}$ and thus we have again a non-zero I-equivariant embedding $\chi^s \hookrightarrow I(\delta(\sigma), \sigma^s)|_I$ which by Frobenius reciprocity gives a non-zero map $\mathrm{ind}_I^K \chi^s \to I(\delta(\sigma), \sigma^s)$. This map must be surjective as σ^s is the co-socle of $I(\delta(\sigma), \sigma^s)$ and doesn't appear elsewhere inside $I(\delta(\sigma), \sigma^s)$. $\qquad\square$

So we see that the above lemma is consistent with Theorem 10.10 in the sense that it can *a priori* happen that the two quotients $I(\delta(\sigma), \sigma^s)$ and τ of $\mathrm{ind}_I^K \chi^s$ are isomorphic.

Let us now explain how to derive irreducibility from Theorems 10.3 and 10.10. Let $\pi' \subseteq \pi$ be a non-zero subrepresentation and pick up a weight σ in $\mathrm{soc}_K \pi'$. We prove that $D_{0,\delta(\sigma)}(\rho) \subseteq \pi'$ (inside π). By Frobenius reciprocity we have a non-zero map $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma \to \pi'$. Let τ be the K-subrepresentation of $\pi' \subseteq \pi$ generated by the image of $[\Pi, \sigma^{I_1}]$. From Lemma 10.11, τ has co-socle σ^s. By definition of $\delta(\sigma)$, we have $\tau \subseteq D_{0,\delta(\sigma)}(\rho)$ inside π. By definition of $I(\delta(\sigma), \sigma^s)$, we thus have $\tau \simeq I(\delta(\sigma), \sigma^s)$. By Theorem 10.10 applied to π', the embedding $\tau \subseteq \pi'$ extends to an embedding $D_{0,\delta(\sigma)}(\rho) \subseteq \pi'$. Starting again with $\delta(\sigma)$ instead of σ, we obtain that π' contains $D_{0,\delta^2(\sigma)}(\rho)$ etc. As $\delta^n(\sigma) = \sigma$ for some $n > 0$ (this is easily checked from the definition of $\sigma \mapsto \delta(\sigma)$), we finally get $D_{0,\sigma}(\rho) \subset \pi'$. As this is true for all $\sigma \in \mathrm{soc}_K \pi'$, we deduce:

$$\bigoplus_{\sigma \in \mathrm{soc}_K \pi'} D_{0,\sigma}(\rho) = \pi' \cap \bigoplus_{\sigma \in \mathcal{D}(\rho)} D_{0,\sigma}(\rho),$$

the intersection being taken inside π. Indeed, we have just proven the inclusion of the left hand side into the right hand side and if R is a subrepresentation of $\bigoplus_{\sigma \in \mathcal{D}(\rho)} D_{0,\sigma}(\rho)$ then we certainly have $R \subseteq \bigoplus_{\sigma \in \mathrm{soc}_K R} D_{0,\sigma}(\rho)$ (this follows from the fact that all weights of $\mathcal{D}(\rho)$ are distinct) which implies inclusion of the right hand side into the left hand side. All this implies that $\bigoplus_{\sigma \in \mathrm{soc}_K \pi'} D_{0,\sigma}(\rho)^{I_1} = \pi'^{I_1} \cap D_0(\rho)^{I_1}$ is preserved by the action of Π inside π' (inside π). As $\bigoplus_{\sigma \in \mathrm{soc}_K \pi'} D_{0,\sigma}(\rho)$ is a non-zero direct factor of $D_0(\rho)$, Theorem 10.3 tells us that it must be the whole of $D_0(\rho)$, that is, we have $D_0(\rho) \subset \pi'$.

As π is generated by $D_0(\rho)$, this implies $\pi' = \pi$. Thus π is irreducible.

Next week, we prove the crucial Theorem 10.10.

11 Week 11

11.1 Irreducibility theorem I

We sketch the proof of Theorem 10.10 using as input a theorem on extensions of weights that will be proved in the next lecture.

We will actually work in a small K-subrepresentation R_σ of $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma$ that will be sufficient for our purpose as it will contain (as subquotient) all the representations $D_{0,\delta(\sigma)}(\rho)$ for varying ρ (and fixed σ). We fix $\sigma = (r_0, \cdots, r_{f-1}) \otimes \eta$ a weight such that $\sigma \neq \sigma^s$ (this always holds if $\sigma \in \mathcal{D}(\rho)$) and let χ be the character giving the action of I on σ^{I_1}. Let $r := r_0 + pr_1 + \cdots + p^{f-1}r_{f-1}$ and, for any $t = t_0 + pi_1 + \cdots + p^{f-1}t_{f-1}$ with $0 \leq t_j \leq r_j$, $J_t := \{i = i_0 + pi_1 + \cdots + p^{f-1}i_{f-1},\ 0 \leq i_j \leq t_j\}$. Recall that any element of σ can be seen as a polynomial over E in the variables $x^{r-i}y^i$ for $i \in J_r$. We first define $\widetilde{R}_\sigma$ to be the K-subrepresentation of $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma$ generated by the elements:
$$\left[\Pi, x^{r-i}y^i\right],\ i \in J_r.$$

An easy calculation shows that this is the E-subvector space of $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma$ generated by the elements:
$$\left[\left(\begin{smallmatrix} p & [\lambda_0] \\ 0 & 1 \end{smallmatrix}\right), x^{r-i}y^i\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r-i}y^i\right],\ i \in J_r, \lambda_0 \in \mathbb{F}_q.$$

For $J_t \subseteq J_r$, we define $\mathrm{Fil}^t \widetilde{R}_\sigma$ to be the K-subrepresentation of $\widetilde{R}_\sigma$ generated by the elements $[\Pi, x^{r-i}y^i], i \in J_t$. We obviously have $\mathrm{Fil}^{t'} \widetilde{R}_\sigma \subseteq \mathrm{Fil}^t \widetilde{R}_\sigma$ whenever $J_{t'} \subseteq J_t$. Let $\alpha : I \to E, \left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \mapsto \overline{a}\overline{d}^{-1}$.

Lemma 11.1. *We have:*
$$\frac{\mathrm{Fil}^t \widetilde{R}_\sigma}{\sum_{J_{t'} \subsetneq J_t} \mathrm{Fil}^{t'} \widetilde{R}_\sigma} = \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s \alpha^{\sum_{j=0}^{f-1} t_j p^j}. \tag{14}$$

Proof. Let us call Gr^t the quotient on the left. By definition, Gr^t is generated under K by the image of $\left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r-t}y^t\right]$. By an easy calculation, we find that Gr^t is exactly the E-vector space generated by
$$\left[\left(\begin{smallmatrix} p & [\lambda_0] \\ 0 & 1 \end{smallmatrix}\right), x^{r-t}y^t\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r-t}y^t\right], \lambda_0 \in \mathbb{F}_q$$

(more precisely by the image of these elements in Gr^t). But I acts on the image of $\left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r-t}y^t\right]$ by the character $\chi^s \alpha^{\sum_{j=0}^{f-1} t_j p^j}$ as is immediately checked (recall we work modulo the $x^{r-i}y^i$ with $i < t$). Therefore we have by Frobenius reciprocity $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s \alpha^{\sum_{j=0}^{f-1} t_j p^j} \twoheadrightarrow \mathrm{Gr}^t$. This is an isomorphism as it is surjective and both spaces have dimension $q + 1$. $\qquad\square$

So we see that $\widetilde{R}_\sigma$ is a successive extension of principal series of type (14), although we *a priori* know nothing about these extensions (i.e. we don't know in which subquotient they become split, or just if some of them are already split). The only thing we know for sure is what happens *inside* the principal series thanks to Theorem 7.6. It turns out there exists a subrepresentation of $\widetilde{R}_\sigma$ that has exactly the same constituents as V_{σ^s} without multiplicities (see §7.2):

Proposition 11.2. *There exists a K-subrepresentation R_σ of $\widetilde{R}_\sigma$ containing $\left[\left(\begin{smallmatrix} p & [\lambda_0] \\ 0 & 1 \end{smallmatrix}\right), x^r\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^r\right] (\lambda_0 \in \mathbb{F}_q)$ and such that its irreducible constituents are exactly the (all distinct) weights:*

$$(\mu_0(r_0), \cdots, \mu_{f-1}(r_{f-1})) \otimes \det{}^{e(\mu)(r_0, \cdots, r_{f-1})} \eta$$

for $\mu_i(x_i) := \lambda_i(p - 1 - x_i)$ with $\lambda \in \mathcal{I}(x_0, \cdots, x_{f-1})$ (see §7.2) and $e(\mu)$ defined in the usual way (forgetting the weights such that $\mu_i(r_i) < 0$ or $\mu_i(r_i) > p - 1$ for some i).

In particular, R_σ is multiplicity free. Note that in general R_σ is not isomorphic to V_{σ^s}. For example, the action of K_1 on R_σ is not trivial in general contrary to what happens by definition on V_{σ^s}. Since the $q + 1$ elements $\left[\left(\begin{smallmatrix} p & [\lambda_0] \\ 0 & 1 \end{smallmatrix}\right), x^r\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^r\right] (\lambda_0 \in \mathbb{F}_q)$ form a basis of the K-representation $\mathrm{Fil}^0 \widetilde{R}_\sigma = \mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$, Proposition 11.2 tells us that R_σ contains $\mathrm{Fil}^0 \widetilde{R}_\sigma$ (which is consistent with the fact that the constituents of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ are symmetric with respect to $\lambda \leftrightarrow \lambda(p - 1 - \cdot)$ by Theorem 7.6). Of course, Proposition 11.2 tells us nothing about extensions between the weights or about the order in which the weights might appear in R_σ (the composition series). I won't prove at all Proposition 11.2 as it is technical and doesn't really bring anything enlightening. Suffice it here to give the example for $f = 1$:

Example 11.3. Assume $f = 1$. If $r_0 = 1$ (and $r_0 \neq p - 1$), we have $R_\sigma = \mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s$. If $2 \leq r_0 < p - 1$, R_σ is an extension:

$$0 \to \mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s \to R_\sigma \to \mathrm{Sym}^{r_0-2} E^2 \otimes \det\eta \to 0$$

where the weight on the right hand side comes from the socle of $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s \alpha$ (compare with Example 7.12). Here, R_σ is a $\mathrm{GL}_2(\mathbb{F}_p)$-representation only if $r_0 = 1$.

We now work inside the K-representation R_σ (inside $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma$) and wish to prove that it contains as subquotients all the $\mathrm{GL}_2(\mathbb{F}_q)$-representations $D_{0,\delta(\sigma)}$ for varying ρ (σ being fixed). To do this, we will need to prove that R_σ contains the same non-split extensions between weights as those which appear in $D_{0,\delta(\sigma)}$ (and in the same order). We first need to recall a lemma about extensions for representations of $\mathrm{GL}_2(\mathbb{F}_q)$:

Lemma 11.4. *Let $\tau := (t_0, \cdots, t_{f-1}) \otimes \eta_\tau$ and $\tau' := (t'_0, \cdots, t'_{f-1}) \otimes \eta_{\tau'}$ be two distinct weights. Then there exists a non-split $\mathrm{GL}_2(\mathbb{F}_q)$-extension between these two weights if and only if we are in one of the following cases:*

(i) $f = 1$, $t_0 = p - 2 - t'_0 \pm 1$ and $\eta_\tau = \eta_{\tau'}\det^{t'_0 + 1 - 1/2(1\pm 1)p}$ (unless $t'_0 = 0$ in which case only $t_0 = p - 3$ can occur)

(ii) $f > 1$ and there is $i \in \{0, \cdots, f-1\}$ such that $t_j = t'_j$ if $j \neq i, i+1$, $t_i = p - 2 - t'_i$, $t_{i+1} = t'_{i+1} \pm 1$, $\eta_\tau = \eta_{\tau'}\det^{p^i(t'_i + 1) - 1/2(1\pm 1)p^{i+1}}$ (with $i + 1 = 0$ if $i = f - 1$).

Moreover, there is then a unique non-split $\mathrm{GL}_2(\mathbb{F}_q)$-extension $0 \to \tau \to * \to \tau' \to 0$ and a unique non-split $\mathrm{GL}_2(\mathbb{F}_q)$-extension $0 \to \tau' \to * \to \tau \to 0$.

Note that there can also exist non-split K-extensions between two weights that are not $\mathrm{GL}_2(\mathbb{F}_q)$-extensions but I won't describe them here (any K-extension between τ and τ' as in (i) or (ii) of Lemma 11.4 is necessarily a $\mathrm{GL}_2(\mathbb{F}_q)$-extension).

The most important result about R_σ is the following:

Theorem 11.5. *Let $\tau := (t_0, \cdots, t_{f-1}) \otimes \eta_\tau$ and $\tau' := (t'_0, \cdots, t'_{f-1}) \otimes \eta_{\tau'}$ be two irreducible subquotients of R_σ and assume we are in one of the situations (i), (ii) of Lemma 11.4. Then either the unique non-split $\mathrm{GL}_2(\mathbb{F}_q)$-extension $0 \to \tau \to \epsilon \to \tau' \to 0$ or the unique non-split $\mathrm{GL}_2(\mathbb{F}_q)$-extension $0 \to \tau' \to \epsilon \to \tau \to 0$ occurs as a subquotient of R_σ.*

That is, for any pair of distinct weights (τ, τ') of R_σ such that there can exist a non-split $\mathrm{GL}_2(\mathbb{F}_q)$-extension between the 2 weights, such a non-split extension does occur in R_σ. I will indicate next time the proof of this important theorem in the case $f = 1$. Although technically more involved, the proof for $f > 1$ ultimately relies on the same computation. I should mention

that this computation, although "just" a computation, is somewhat critical.

I now indicate how one can use it to finish the proof of Theorem 10.10.

Proposition 11.6. *Fix* $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ *a continuous irreducible generic Galois representation and assume that p acts trivially on* $\det(\rho)$. *Let* $\sigma \in \mathcal{D}(\rho)$ *and define σ^s and $\delta(\sigma)$ as in §10.2. Then the representation R_σ contains $D_{0,\delta(\sigma)}(\rho)$ as a subquotient.*

Proof. (rough sketch) We first check that any irreducible constituent τ of $D_{0,\delta(\sigma)}(\rho)$ is also a constituent of R_σ. Write $\sigma^s = (s_0, \cdots, s_{f-1}) \otimes \theta$ and $\delta(\sigma) = (s'_0, \cdots, s'_{f-1}) \otimes \theta'$. Equivalently by Proposition 11.2, it is enough to prove there is $\lambda \in \mathcal{I}(y_0, \cdots, y_{f-1})$ such that $\tau = (\lambda_i(s_i)) \otimes \det^{e(\lambda)(s_i)}\theta$. By (i) of Theorem 10.1 we have:

$$\tau = (\nu_0(s'_0), \cdots, \nu_{f-1}(s'_{f-1})) \otimes \det^{e(\nu)(s'_0, \cdots, s'_{f-1})}\theta'$$
$$\sigma^s = (\nu'_0(s'_0), \cdots, \nu'_{f-1}(s'_{f-1})) \otimes \det^{e(\nu')(s'_0, \cdots, s'_{f-1})}\theta'$$

for compatible $\nu, \nu' \in \mathcal{I}(y_0, \cdots, y_{f-1})$ in the sense of Definition 7.13. Let $\nu'^{-1} \in \mathcal{I}(y_0, \cdots, y_{f-1})$ be the unique f-tuple such that $\nu'(\nu'^{-1}(y_i)) = y_i$. From the compatibility of ν and ν', one checks that the unique f-tuple $(\lambda_i(y_i))_i$ such that $\lambda_i(y_i) := \nu_i(\nu_i'^{-1}(y_i))$ is in $\mathcal{I}(y_0, \cdots, y_{f-1})$. This λ gives the result. Now pick up two weights τ, τ' of $D_{0,\delta(\sigma)}(\rho)$. Lemma 11.4 and (ii) of Theorem 10.1 tell us that each time there can be a $\mathrm{GL}_2(\mathbb{F}_q)$-extension between these two weights, then one of the two non-split extensions of Lemma 11.4 occurs in $D_{0,\delta(\sigma)}(\rho)$. Analogously, Lemma 11.4 and Theorem 11.5 tell us that each time there can be a $\mathrm{GL}_2(\mathbb{F}_q)$-extension between these two weights, then one of the two non-split extensions of Lemma 11.4 occurs in R_σ. More work shows that these extensions are actually the same in both cases (that is, they are in the same sense). From this, we deduce that we exactly have a copy of $D_{0,\delta(\sigma)}$ appearing "inside" R_σ. $\square$

The interested reader can check Proposition 11.6 on Example 11.3 in the case $f = 1$.

We can now deduce the proof of Theorem 10.10. Let $\overline{R}_\sigma$ be the image of R_σ in π. From the assumptions, we have that $\overline{R}_\sigma$ contains $I(\delta(\sigma), \sigma^s)$ as induced quotient of $\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s \subseteq R_\sigma$. But it is not difficult using Proposition 11.2 to check that none of the constituents of $R_\sigma/\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ are in $\mathcal{D}(\rho)$, and hence in $\mathrm{soc}_K \pi$. If there is a weight $\tau \neq \delta(\sigma)$ in $\mathrm{soc}_K \overline{R}_\sigma \subseteq \mathrm{soc}_K \pi$, then τ is necessarily a constituent of $R_\sigma/\mathrm{ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi^s$ which is thus impossible.

Hence we have $\operatorname{soc}_K \overline{R}_\sigma = \delta(\sigma)$. By Proposition 11.6 and the fact R_σ is multiplicity free, this implies $\overline{R}_\sigma$ contains $D_{0,\delta(\sigma)}$. Since $D_{0,\delta(\sigma)}$ is indecomposable, the injection $\overline{R}_\sigma \to \pi$ necessarily induces an injection $D_{0,\delta(\sigma)} \hookrightarrow \pi$. Finally this injection is unique up to scalar otherwise $\operatorname{soc}_K \pi$ would contain a weight that is not in $\mathcal{D}(\rho)$.

11.2   Irreducibility theorem II

We sketch the proof of Theorem 11.5 for $f = 1$.

Lemma 11.7. *Let τ, τ' be two weights and ϵ a K-extension $0 \to \tau \to \epsilon \to \tau' \to 0$. Let $F \in \epsilon$ be a non-zero eigenvector for the diagonal matrices with eigencharacter χ where χ is the action of I on τ'^{I_1}. Assume that χ doesn't occur as an eigencharacter on τ (for the diagonal matrices) and that $\langle K \cdot F \rangle$ contains τ. Then ϵ is non-split.*

Proof. Note that τ and τ' are necessarily distinct because of the assumption on χ. If ϵ was split, as χ doesn't occur in τ we would have that F necessarily belongs to τ' via a splitting $\tau' \hookrightarrow \epsilon$. This would imply $\langle K \cdot F \rangle = \tau'$ which contradicts $\tau \subset \langle K \cdot F \rangle$. $\qquad\square$

When $f = 1$, recall we have described R_σ in Example 11.3. Looking at that example and using what we know on $\operatorname{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s$, we see that Theorem 11.5 boils down in that case to the following statement:

Lemma 11.8. *If $r_0 \geq 2$, the extension:*

$$0 \to \operatorname{Sym}^{p-1-r_0} E^2 \otimes \det{}^{r_0}\eta \to \epsilon \to \operatorname{Sym}^{r_0-2} E^2 \otimes \det\eta \to 0$$

induced by the push-out of the extension R_σ in Example 11.3 is a non-split $\mathrm{GL}_2(\mathbb{F}_p)$-extension.

Proof. One can prove that any such extension is actually always a $\mathrm{GL}_2(\mathbb{F}_p)$-extension (and not just a K-extension), see the comment after Lemma 11.4. Twisting if necessary, I can assume $\eta = 1$. I will work inside the K-extension:

$$0 \to \operatorname{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s \to * \to \operatorname{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s \alpha \to 0$$

of Lemma 11.1 (recall that $\operatorname{Sym}^{r_0-2} E^2 \otimes \det$ is just the socle of the right hand side). This extension is the E-subvector space of $\operatorname{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)} \operatorname{Sym}^{r_0} E^2$ generated by the elements:

$$\left[\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), x^{r_0}\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r_0}\right], \left[\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), x^{r_0-1}y\right], \left[\left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right), x^{r_0-1}y\right], \lambda \in \mathbb{F}_p.$$

Define the following element in $*$:

$$F \;:=\; \sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), x^{r_0-1}y] \text{ if } r_0 \geq 3$$

$$F \;:=\; \sum_{\lambda \in \mathbb{F}_p} [\left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix}\right), xy] - [\Pi, xy] \text{ if } r_0 = 2$$

(see end of proof of Theorem 8.6). I first prove that (i) the K-subrepresentation $\langle K \cdot F \rangle \subseteq *$ actually sits inside R_σ and (ii) the image of $\langle K \cdot F \rangle \subseteq R_\sigma$ in the above quotient ϵ of R_σ contains $\mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}$. I only give the details for $r_0 > 2$, that is $\chi^s \alpha \neq (\chi^s \alpha)^s$ (check it as an exercise when $r_0 = 2$!). Let us start with (i). By Lemma 7.3 together with Theorem 7.6, it is enough to check that I acts on the image $\overline{F}$ of F in the quotient $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s \alpha$ by the character $(\chi^s \alpha)^s = \chi \alpha^{-1}$. Let $\left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right) \in I$, we have in $\mathrm{GL}_2(F)$:

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & [\mu] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ pc' & d' \end{pmatrix}$$

where $\mu = \frac{\overline{b} + \overline{a}\lambda}{\overline{d}} \in \mathbb{F}_p$ and $a' \equiv a \bmod. p$, $d' \equiv d \bmod. p$. We compute:

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \overline{F} \;=\; \sum_{\lambda \in \mathbb{F}_p} \left[\begin{pmatrix} p & [\frac{\overline{b}+\overline{a}\lambda}{\overline{d}}] \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} x^{r_0-1}y \right]$$

$$=\; \overline{a}^{r_0-1}\overline{d} \sum_{\lambda \in \mathbb{F}_p} \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0-1}y \right]$$

$$=\; \overline{a}^{r_0-1}\overline{d}\,\overline{F}$$

$$=\; \overline{a}^{r_0}(\overline{a}\,\overline{d}^{-1})^{-1}\overline{F}$$

which is precisely $\chi \alpha^{-1}$ applied to $\left(\begin{smallmatrix} a & b \\ pc & d \end{smallmatrix}\right)$. This proves (i). Let us now prove (ii). We have in $\mathrm{GL}_2(F)$:

$$\begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & [\mu] + [\lambda] \\ 0 & 1 \end{pmatrix}.$$

Now, we use the following property of addition in $\mathbb{Z}_p = W(\mathbb{F}_p)$:

$$[\mu] + [\lambda] \equiv [\mu + \lambda] - p[X] \quad (p^2)$$

where:

$$X = X^p := \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \mu^{p-s} \lambda^s \in \mathbb{F}_p.$$

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We then compute again:

$$
\begin{aligned}
\begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} F &= \sum_{\lambda \in \mathbb{F}_p} \left[\begin{pmatrix} p & [\mu] + [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0 - 1} y \right] \\
&= \sum_{\lambda \in \mathbb{F}_p} \left[\begin{pmatrix} p & [\mu + \lambda] \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} x^{r_0 - 1} y \right] \\
&= -\sum_{\lambda \in \mathbb{F}_p} \left(\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \mu^{p-s} \lambda^s \right) \left[\begin{pmatrix} p & [\mu + \lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] + F \\
&= -\sum_{\lambda \in \mathbb{F}_p} \left(\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \mu^{p-s} (\lambda - \mu)^s \right) \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] + F.
\end{aligned}
$$

One checks that $\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} \mu^{p-s} (\lambda - \mu)^s = -\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} (-\mu)^{p-s} \lambda^s \in \mathbb{F}_p$, therefore we have:

$$
\begin{aligned}
\begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} F &= F + \sum_{\lambda \in \mathbb{F}_p} \left(\sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} (-\mu)^{p-s} \lambda^s \right) \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] \\
&= F + \sum_{s=1}^{p-1} \frac{\binom{p}{s}}{p} (-\mu)^{p-s} \left(\sum_{\lambda \in \mathbb{F}_p} \lambda^s \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] \right).
\end{aligned}
$$

Varying μ in $\mathbb{F}_p$ and using that the E-vector space generated by $\sum_{s=0}^{p-1} \mu^s v_s$ for all $\mu \in \mathbb{F}_p$ is the E-vector space generated by v_s for $0 \le s \le p-1$ (as in the proof of Lemma 2.16), we get that $\langle K \cdot F \rangle$ contains F and all the vectors $\sum_{\lambda \in \mathbb{F}_p} \lambda^s \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right]$ for $1 \le s \le p-1$. In particular, it contains the vector:

$$
\begin{aligned}
\sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1} \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] &= \sum_{\lambda \in \mathbb{F}_p^\times} \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] \\
&= \sum_{\lambda \in \mathbb{F}_p} \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right] - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} [\Pi, x^{r_0}].
\end{aligned}
$$

But by Lemma 7.3 together with Theorem 7.6, we know that the element $\sum_{\lambda \in \mathbb{F}_p} \left[\begin{pmatrix} p & [\lambda] \\ 0 & 1 \end{pmatrix}, x^{r_0} \right]$ generates the K-socle of $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s$. Therefore the image of $\langle K \cdot F \rangle \subseteq R_\sigma$ in the quotient ϵ contains the image of $[\Pi, x^{r_0}]$ in the quotient $\mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}$ of $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s$. Since $[\Pi, x^{r_0}]$ is a generator of $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)} \chi^s$, this image is non-zero and thus we get that the image of $\langle K \cdot F \rangle$ in the quotient ϵ contains $\mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0}$. This proves (ii). Now we

finish the proof of the lemma. The characters of the diagonal matrices of $\mathrm{GL}_2(\mathbb{F}_p)$ acting on the various eigenvectors of $\mathrm{Sym}^{p-1-r_0}E^2 \otimes \det^{r_0}$ are:

$$a^{r_0}, a^{r_0+1}d^{p-2}, a^{r_0+2}d^{p-3}, \cdots, a^{p-3}d^{r_0+2}, a^{p-2}d^{r_0+1}, d^{r_0}$$

and are thus distinct from the character $a^{r_0-1}d$ on the eigenvector F. We can thus apply Lemma 11.7 to ϵ and the image of F in ϵ and deduce that ϵ is non-split. $\qquad\square$

An analogous (although more involved) computation using ultimately the addition law on Witt vectors $W(\mathbb{F}_q)$ provides a proof of Theorem 11.5 for an arbitrary f.

I would like now to show that Lemma 11.8 collapses if, say, $F = \mathbb{F}_p((t))$, that is, the extension obtained by push-out from Example 11.3 actually *splits* in that case. Indeed, set $F = \sum_{\lambda \in \mathbb{F}_p}[\left(\begin{smallmatrix} t & \lambda \\ 0 & 1 \end{smallmatrix}\right), x^{r_0-1}y]$ with $r_0 > 2$ as in the above proof and let $\left(\begin{smallmatrix} a & b \\ tc & d \end{smallmatrix}\right) \in I_1$. A computation analogous to the one in the proof of Lemma 11.8 yields in the subextension R_σ (which is also defined in that case and looks like Example 11.3):

$$\left(\begin{smallmatrix} a & b \\ tc & d \end{smallmatrix}\right)F = F + \alpha \sum_{\lambda \in \mathbb{F}_p}[\left(\begin{smallmatrix} t & \lambda \\ 0 & 1 \end{smallmatrix}\right), x^{r_0}] + \beta \sum_{\lambda \in \mathbb{F}_p}\lambda[\left(\begin{smallmatrix} t & \lambda \\ 0 & 1 \end{smallmatrix}\right), x^{r_0}] + \gamma \sum_{\lambda \in \mathbb{F}_p}\lambda^2[\left(\begin{smallmatrix} t & \lambda \\ 0 & 1 \end{smallmatrix}\right), x^{r_0}]$$

where $\alpha, \beta, \gamma \in E$. But one can check that the K-socle of $\mathrm{ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_2(\mathbb{F}_p)}\chi^s$ is actually generated as an E-vector space by the elements $\sum_{\lambda \in \mathbb{F}_p}\lambda^s[\left(\begin{smallmatrix} t & \lambda \\ 0 & 1 \end{smallmatrix}\right), x^{r_0}]$ for $0 \leq s \leq r_0-1$ and $[\Pi, x^{r_0}]+(-1)^{r_0}\sum_{\lambda \in \mathbb{F}_p}\lambda^{r_0}[\left(\begin{smallmatrix} t & \lambda \\ 0 & 1 \end{smallmatrix}\right), x^{r_0}]$ where $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$. We thus see that, for $r_0 > 2$, the image of F in the quotient ϵ is I_1-invariant as the above 3 elements vanish in this quotient. Now assume that ϵ is non-split, then from Proposition 10.4, we get that $(\mathrm{Sym}^{r_0-2}E^2 \otimes \det)^{I_1}$ can't have a lift in ϵ^{I_1}. This contradicts the computations we have just done since F is such a lift. Therefore ϵ splits when $F = \mathbb{F}_p((t))$. This phenomena will actually extend to F being totally ramified over $\mathbb{Q}_p$ with e sufficiently big.

12 Week 12

12.1 The split Galois case I

In the coming two lectures, I study the case where ρ is reducible split (with F unramified over $\mathbb{Q}_p$). The reason is (i) it is quite similar to the irreducible case and (ii) eventhough ρ is reducible, it turns out this case involves new supercuspidal representations of $\mathrm{GL}_2(F)$ (when $F \neq \mathbb{Q}_p$) which don't appear

in the irreducible case!

To start with, let us, as always, go back to the case $F = \mathbb{Q}_p$. I have stated the correspondence in Definition 4.11. In the split case and assuming $0 < r_0 < p - 3$ (we will only work in the "generic" situation as for the irreducible case), recall it gives the following:

$$
\begin{pmatrix} \omega_1^{r_0+1}\mathrm{unr}(\lambda) & 0 \\ 0 & \mathrm{unr}(\lambda^{-1}) \end{pmatrix} \otimes \eta \longleftrightarrow \left(\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^{r_0}E^2}{(T-\lambda)} \right) \otimes \eta \oplus
$$
$$
\left(\frac{\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^{p-3-r_0}E^2}{(T-\lambda^{-1})} \otimes \omega_1^{r_0+1} \right) \otimes \eta \quad (15)
$$

where η is a smooth character, $\lambda \in E^\times$ and $\mathrm{unr}(x)$ is the character $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \to E^\times$ sending $\mathrm{Fr}^{-1} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ to $x \in E^\times$. As in the irreducible case (see Theorem 8.6), let us work out the corresponding basic diagram, that is the K_1-invariants of the right hand side and the action of Π on the I_1-invariants.

Lemma 12.1. *Let* $\pi := (\mathrm{ind}_{K\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\mathrm{Sym}^{r_0}E^2)/(T-\lambda)$ *with* $0 < r_0 < p - 1$, $\lambda \in E^\times$ *and let* $\phi = [(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), 1] \in \mathrm{ind}_I^K 1 \otimes \overline{d}^{r_0}$ *and* $f_0 = \sum_{\lambda_0 \in \mathbb{F}_p} (\begin{smallmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{smallmatrix}) \phi \in \mathrm{ind}_I^K 1 \otimes \overline{d}^{r_0}$ *as in §7.1. Then* $(\pi^{K_1}, \pi^{I_1}, \mathrm{can}) \simeq (\mathrm{ind}_I^K 1 \otimes \overline{d}^{r_0}, E\phi \oplus Ef_0, \mathrm{can})$ *with* $\Pi\phi = \lambda^{-1}f_0$ *and* $\Pi f_0 = \lambda\phi$.

Proof. We have already seen that π is an irreducible principal series (Theorem 4.4) and that $\pi^{K_1} = \mathrm{ind}_I^K 1 \otimes \overline{d}^{r_0}$ (see proof of Proposition 4.6 or Example 5.7). We also know that $\pi^{I_1} = E\phi \oplus Ef_0$ by Lemma 7.3. We are thus left to check that $\Pi\phi = \lambda^{-1}f_0$. As in the proof of Proposition 4.7 (the argument is exactly the same), we have a K-equivariant map $\mathrm{ind}_I^K(1 \otimes \overline{d}^{r_0}) \to \pi$, $[g, 1] \mapsto g[\Pi, x^{r_0}] = [g\Pi, x^{r_0}]$. Moreover, this map here induces an isomorphism with $\pi^{K_1} \subset \pi$. We compute:

$$
\Pi\phi = \Pi[(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), 1] \mapsto \Pi[\Pi, x^{r_0}] = [(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), x^{r_0}].
$$

But we have as in the proof of Proposition 4.7:

$$
f_0 = \sum_{\lambda_0 \in \mathbb{F}_p} [(\begin{smallmatrix} [\lambda_0] & 1 \\ 1 & 0 \end{smallmatrix}), 1] \mapsto \sum_{\lambda_0 \in \mathbb{F}_p} [(\begin{smallmatrix} p & [\lambda_0] \\ 0 & 1 \end{smallmatrix}), x^{r_0}] = T([(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), x^{r_0}]) = \lambda[(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), x^{r_0}]
$$

therefore $\lambda^{-1}f_0 - \Pi\phi \mapsto 0$ in π and we have $\Pi\phi = \lambda^{-1}f_0$. $\qquad \square$

Applying Lemma 12.1 to the direct sum in (15) and using (9) (or Theorem 7.6), we thus see that the basic diagram $D(\rho)$ "corresponding" to $\rho = \begin{pmatrix} \omega_1^{r_0+1}\mathrm{unr}(\lambda) & 0 \\ 0 & \mathrm{unr}(\lambda^{-1}) \end{pmatrix}$ when $0 < r_0 < p-3$ is such that $D_0(\rho) = D_{0,\sigma_1} \oplus D_{0,\sigma_2}$ with $(\sigma_1, \sigma_2) = (\mathrm{Sym}^{r_0} E^2, \mathrm{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1})$ and:

$$\begin{aligned} D_{0,\sigma_1} &= \mathrm{Sym}^{r_0} E^2 \,\underline{\qquad}\, \mathrm{Sym}^{p-1-r_0} E^2 \otimes \det^{r_0} \\ D_{0,\sigma_2} &= \mathrm{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1} \,\underline{\qquad}\, \mathrm{Sym}^{r_0+2} E^2 \otimes \det^{-1} \end{aligned}$$

(notations of Example 7.10). Now comparing with Example 7.10, we easily see that $D_0(\rho)$ is the maximal subrepresentation of $\mathrm{inj}_{\mathrm{GL}_2(\mathbb{F}_p)}(\sigma_1 \oplus \sigma_2)$ such that σ_1 and σ_2 appear only once, exactly as in §8.2 for the irreducible case. Going further in the analogy with the irreducible case, we also see that:

$$(1 \oplus \omega_1^{(p-3-r_0)+1}) \otimes \omega_1^{r_0+1} = \omega_1^{r_0+1} \oplus \omega_1^{p-2-r_0+r_0+1} = $$
$$\omega_1^{r_0+1} \oplus \omega_1^{p-1} = \omega_1^{r_0+1} \oplus 1 = \rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$$

and so, as in §8.1 again, we can somehow "see" the two weights $\mathrm{Sym}^{r_0} E^2$ and $\mathrm{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1}$ from all possible "writings" of $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$.

We now generalize all this for $f > 1$ and:

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} = \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & 1 \end{pmatrix} \otimes \eta$$

for some character η that extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and some integers r_i with $0 \le r_i \le p-3$ and $(r_0, \cdots, r_{f-1}) \notin \{(0, \cdots, 0), (p-3, \cdots, p-3)\}$ (this is our genericity condition here, analogous to the one of Definition 2.11). As in §8.1, let us try to find other ways to write down $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$. Choose $\mathcal{I} \subseteq \{0, \cdots, f-1\}$. Then can one find integers r_i' such that there is, say, a twist ρ' of ρ satisfying $\rho'|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} = \omega_f^{\sum_{i \notin \mathcal{I}}(r_i'+1)p^i} \oplus \omega_f^{\sum_{i \in \mathcal{I}}(r_i'+1)p^i}$? The following lemma (which extends (i) of Corollary 2.9) gives the answer:

Lemma 12.2. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous reducible split generic representation (in the above sense). Then for each $\mathcal{I} \subseteq \{0, \cdots, f-1\}$ $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ can be written:*

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_f^{\sum_{i \notin \mathcal{I}}(r_i'+1)p^i} & 0 \\ 0 & \omega_f^{\sum_{i \in \mathcal{I}}(r_i'+1)p^i} \end{pmatrix} \otimes \eta'$$

for some character η' that extends to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and some integers r_i' which are such that:

if $i \in \mathcal{I}$ and $i - 1 \notin \mathcal{I}$ or if $i \notin \mathcal{I}$ and $i - 1 \in \mathcal{I}$ then $1 \leq r'_i \leq p - 2$

otherwise $0 \leq r'_i \leq p - 3$.

Proof. Write $\rho|_{I(\overline{\mathbb{Q}}_p/F)}$ as in (i) of Corollary 2.9 (which corresponds to $\mathcal{I} = \emptyset$). Let us explain how one can remove $(r_i + 1)p^i$ in the power of the top left entry and replace 1 by $\omega_f^{(r'_i+1)p^i}$ in the bottom right entry (which corresponds to the case $\mathcal{I} = \{i\}$). One has:

$$\omega_f^{\sum_{j \in \{0,\cdots,f-1\}} (r_j+1)p^j} = \omega_f^{\substack{\sum_{j \in \{0,\cdots,f-1\}} (r_j+1)p^j \\ j \notin \{i,i+1\}}} \omega_f^{(r_{i+1}+2)p^{i+1}} \omega_f^{(r_i+1)p^i - p^{i+1}}$$

hence since $\omega_f^{(r_i+1)p^i - p^{i+1}} = \omega_f^{-p^i(p-1-r_i)}$:

$$\omega_f^{\sum_{j \in \{0,\cdots,f-1\}} (r_j+1)p^j} \oplus 1 = (\omega_f^{\sum_{j \in \{0,\cdots,f-1\} \setminus \{i\}} (r'_j+1)p^j} \oplus \omega_f^{(r'_i+1)p^i}) \otimes \omega_f^{(r_i+1)p^i - p^{i+1}}$$

where $r'_j = r_j$ if $j \notin \{i, i+1\}$, $r'_i = p - 2 - r_i$ and $r'_{i+1} = r_{i+1} + 1$ (and as usual $i + 1 = 0$ if $i = f - 1$). One can then twist by η on both sides. From the genericity of ρ, we still have $0 \leq r'_j \leq p - 3$ if $j \notin \{i, i+1\}$ and $1 \leq r'_j \leq p - 2$ if $j \in \{i, i+1\}$. Iterating this process gives the proposition. $\square$

From what happens for $F = \mathbb{Q}_p$ and from the irreducible case, one is tempted to associate the weight $(r'_0, \cdots, r'_{f-1}) \otimes \eta'$ to $\rho|_{I(\overline{\mathbb{Q}}_p/F)}$.

Definition 12.3. *Let $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous reducible split generic representation. The set of Diamond weights $\mathcal{D}(\rho)$ associated to ρ (in fact to $\rho|_{I(\overline{\mathbb{Q}}_p/F)}$) is the set of weights $\{(r'_0, \cdots, r'_{f-1}) \otimes \eta'\}$ for all possible writings:*

$$\rho|_{I(\overline{\mathbb{Q}}_p/F)} \cong \begin{pmatrix} \omega_f^{\sum_{i \notin \mathcal{I}} (r'_i+1)p^i} & 0 \\ 0 & \omega_f^{\sum_{i \in \mathcal{I}} (r'_i+1)p^i} \end{pmatrix} \otimes \eta'$$

as in Lemma 12.2 for all choices of $\mathcal{I}$.

One should of course read $(r'_0, \cdots, r'_{f-1}) \otimes (\eta' \circ r_F^{-1} \circ \det)$. It turns out one can describe explicitly the set $\mathcal{D}(\rho)$ in a very similar way (an even simpler way) to what we did in §8.1 for ρ irreducible.

Let $(x_0, \cdots, x_{f-1})$ be f variables. We define a set $\mathcal{D}'(x_0, \cdots, x_{f-1})$ of f-tuples $\lambda := (\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ where $\lambda_i(x_i) \in \mathbb{Z} \pm x_i$ as follows. If $f = 1$, $\lambda_0(x_0) \in \{x_0, p - 3 - x_0\}$. If $f > 1$, then:

(i) $\lambda_i(x_i) \in \{x_i, x_i + 1, p - 2 - x_i, p - 3 - x_i\}$

(ii) if $\lambda_i(x_i) \in \{x_i, x_i + 1\}$ then $\lambda_{i+1}(x_{i+1}) \in \{x_{i+1}, p - 2 - x_{i+1}\}$

(iii) if $\lambda_i(x_i) \in \{p-2-x_i, p-3-x_i\}$, then $\lambda_{i+1}(x_{i+1}) \in \{p-3-x_{i+1}, x_{i+1}+1\}$

with the conventions $x_f = x_0$ and $\lambda_f(x_f) = \lambda_0(x_0)$. Concretely, we have that $(\lambda_0(x_0), \cdots, \lambda_{f-1}(x_{f-1}))$ is a succession of sequences like $p - 2 - x_j, p - 3 - x_{j+1}, \cdots, p - 3 - x_{j+l}, x_{j+l+1} + 1$ among the x_i.

For $\lambda \in \mathcal{D}'(x_0, \cdots, x_{f-1})$, define:

$$e(\lambda) \quad := \quad \frac{1}{2}\Big(\sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i)) \Big) \text{ if } \lambda_{f-1}(x_{f-1}) \in \{x_{f-1}, x_{f-1} + 1\}$$

$$e(\lambda) \quad := \quad \frac{1}{2}\Big(p^f - 1 + \sum_{i=0}^{f-1} p^i(x_i - \lambda_i(x_i)) \Big) \text{ otherwise.}$$

One has again $e(\lambda) \in \mathbb{Z} \oplus \bigoplus_{i=0}^{f-1} \mathbb{Z}x_i$. As in §8.1, we can prove the:

Proposition 12.4. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ be a continuous reducible split generic representation, that is:*

$$\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)} = \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1}(r_i+1)p^i} & 0 \\ 0 & 1 \end{pmatrix} \otimes \eta$$

with $0 \le r_i \le p - 3$ and $(r_i) \notin \{(0, \cdots, 0), (p - 1, \cdots, p - 1)\}$. Then $\mathcal{D}(\rho)$ is the set of (all distinct) weights:

$$(\lambda_0(r_0), \cdots, \lambda_{f-1}(r_{f-1})) \otimes \det^{e(\lambda)(r_0, \cdots, r_{f-1})} \eta$$

for $\lambda \in \mathcal{D}'(x_0, \cdots, x_{f-1})$.

When $F = \mathbb{Q}_p$ and $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \omega_1^{r_0+1} \oplus 1$, we just recover $\mathcal{D}(\rho) = \{\mathrm{Sym}^{r_0} E^2, \mathrm{Sym}^{p-3-r_0} E^2 \otimes \det^{r_0+1}\}$ (corresponding to $\mathcal{I} = \emptyset$ and $\mathcal{I} = \{0\}$). When $f = 2$, we get the four weights $\mathcal{D}(\rho) = \{(r_0, r_1), (r_0 + 1, p - 2 - r_1) \otimes \det^{p-1+pr_1}, (p - 2 - r_0, r_1 + 1) \otimes \det^{r_0+p(p-1)}, (p - 3 - r_0, p - 3 - r_1) \otimes \det^{r_0+1+p(r_1+1)}\}$ corresponding to $\mathcal{I} = \emptyset, \{1\}, \{0\}, \{0, 1\}$. Indeed, one has

the following equalities:

$$\begin{pmatrix} \omega_2^{(r_0+1)+(r_1+1)p} & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \omega_2^{r_0+2} & 0 \\ 0 & \omega_2^{(p-1-r_1)p} \end{pmatrix} \otimes \omega_2^{p-1+pr_1} =$$

$$\begin{pmatrix} \omega_2^{(r_1+2)p} & 0 \\ 0 & \omega_2^{p-1-r_0} \end{pmatrix} \otimes \omega_2^{r_0+p(p-1)} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \omega_2^{(p-2-r_0)+(p-2-r_1)p} \end{pmatrix} \otimes \omega_2^{r_0+1+p(r_1+1)}.$$

As in §8.1 note that one can replace $(r_i)_{0\leq i\leq f-1}$ by $(p-3-r_i)_{0\leq i\leq f-1}$ in the way we start writing ρ (see the comment after Corollary 2.9), hence $\mathcal{D}(\rho)$ shouldn't be affected by this "change of variables". Indeed, one can check that $\mathcal{D}'(x_0,\cdots,x_{f-1})$ is symmetrical with respect to $x_i \mapsto p-3-x_i$.

12.2 The split Galois case II

We finish the description of the reducible split case.

We fix $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ a continuous reducible split generic representation and we assume that $\varpi_F = p$ acts trivially on $\det(\rho)$ via r_F^{-1}. We thus have:

$$\rho = \begin{pmatrix} \omega_f^{\sum_{i=0}^{f-1}(r_i+1)p^i}\mathrm{unr}(\lambda) & 0 \\ 0 & \mathrm{unr}(\lambda^{-1}) \end{pmatrix} \otimes \eta$$

where $\lambda \in E^\times$. From the case $F = \mathbb{Q}_p$ (see §12.1), we define $D_0(\rho)$ to be the representation D_0 in Proposition 9.1 with $\mathcal{D} = \mathcal{D}(\rho)$ where $\mathcal{D}(\rho)$ is as in Definition 12.3. One can prove that Lemma 9.5 still holds in that case and the same proof as in the irreducible case yields the following result:

Proposition 12.5. *(i) Each irreducible factor of $D_0(\rho) = \oplus_{\sigma\in\mathcal{D}(\rho)}D_{0,\sigma}(\rho)$ only occurs once in $D_0(\rho)$.*

(ii) As an I-representation, one has:

$$D_0(\rho)^{I_1} \simeq \bigoplus_{\substack{certain\ (\chi,\chi^s)\\ \chi\neq\chi^s}} \chi \oplus \chi^s$$

(in particular $D_0(\rho)^{I_1}$ is stable under $\chi \mapsto \chi^s$).

Exactly as in §8.2, from (ii) of Proposition 12.5 we can manufacture families of basic diagrams $D(\rho, r) = (D_0(\rho), D_1(\rho), r)$. Note that this family only depends on $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$ as we have not used the unramified character $\mathrm{unr}(\lambda)$ (this extra data doesn't exist in the irreducible case). The reason is that for $f > 1$, we don't know how the parameter r in $D(\rho, r)$ will depend on λ. In the case $f = 1$, we know what to do because $D_0(\rho)$ is just the direct sum of two representations of type $\mathrm{ind}_I^K \chi$ (see §12.1). However, when $f > 1$, we will see below that $D(\rho, r)$ has other pieces and we don't know how these pieces behave with respect to λ. Let us for instance give completely the case $f = 2$, as we did for ρ irreducible in Example 8.8:

Example 12.6. Assume $f = 2$. Then we have seen that $\mathcal{D}(\rho) = \{\sigma_i, 1 \leq i \leq 4\}$ with:

$$
\begin{aligned}
\sigma_1 &:= (r_0, r_1) \\
\sigma_2 &:= (r_0 + 1, p - 2 - r_1) \otimes \det^{p-1+pr_1} \\
\sigma_3 &:= (p - 2 - r_0, r_1 + 1) \otimes \det^{r_0 + p(p-1)} \\
\sigma_4 &:= (p - 3 - r_0, p - 3 - r_1) \otimes \det^{r_0 + 1 + p(r_1 + 1)}.
\end{aligned}
$$

One can then check using Theorem 7.14 that $D_0(\rho) = D_{0,\sigma_1} \oplus D_{0,\sigma_2} \oplus D_{0,\sigma_3} \oplus D_{0,\sigma_4}$ with the D_{0,σ_i} as follows (forgetting the twists in the weights and forgetting weights with negative entries):

$$
\begin{aligned}
D_{0,\sigma_1} &= (r_0, r_1) \text{ --- } S_1 \text{ --- } (p - 1 - r_0, p - 1 - r_1) \\
D_{0,\sigma_2} &= (r_0 + 1, p - 2 - r_1) \text{ --- } S_2 \text{ --- } (p - 4 - r_0, r_1 - 1) \\
D_{0,\sigma_3} &= (p - 2 - r_0, r_1 + 1) \text{ --- } S_3 \text{ --- } (r_0 - 1, p - 4 - r_1) \\
D_{0,\sigma_4} &= (p - 3 - r_0, p - 3 - r_1) \text{ --- } S_4 \text{ --- } (r_0 + 2, r_1 + 2)
\end{aligned}
$$

where:

$$
\begin{aligned}
S_1 &= (p - 2 - r_0, r_1 - 1) \oplus (r_0 - 1, p - 2 - r_1) \\
S_2 &= (r_0 + 2, r_1) \oplus (p - 3 - r_0, p - 1 - r_1) \\
S_3 &= (r_0, r_1 + 2) \oplus (p - 1 - r_0, p - 3 - r_1) \\
S_4 &= (r_0 + 1, p - 4 - r_1) \oplus (p - 4 - r_0, r_1 + 1).
\end{aligned}
$$

Studying Example 12.6, we see using Proposition 10.4 that the action of Π necessarily preserves the 3 pieces D_{0,σ_1}, D_{0,σ_4} and $D_{0,\sigma_2} \oplus D_{0,\sigma_3}$ (drawing). Actually, using Theorem 7.6, we see that we have $D_{0,\sigma_1} \simeq \mathrm{ind}_I^K \chi_1^s$ and $D_{0,\sigma_4} \simeq \mathrm{ind}_I^K \chi_4^s$ where χ_1 (resp. χ_4) gives the action of I on $\sigma_1^{I_1}$ (resp. $\sigma_4^{I_1}$). So the situation is different from what happened in the irreducible case where

$\mathcal{D}(\rho)$ was "indecomposable" (Theorem 10.3).

All this generalizes as follows to $f > 1$. As in §8.1, if $\sigma \in \mathcal{D}(\rho)$ corresponds to $\lambda \in \mathcal{D}'(x_0, \cdots, x_{f-1})$, we set $\ell(\sigma) = \ell(\lambda) := |\mathcal{S}(\lambda)|$ where $i \in \mathcal{S}(\lambda)$ if and only if $\lambda_i(x_i) \in \{p - 3 - x_i, x_i + 1\}$. Note that there is a unique weight such that $\ell(\sigma) = 0$ (resp. $\ell(\sigma) = f$), namely $\sigma_0 := (r_0, \cdots, r_{f-1})$ (resp. $\sigma_f := (p - 3 - r_0, \cdots, p - 3 - r_{f-1}) \otimes \det^{\sum_{i=0}^{f-1}(r_i+1)p^i}$) and that $0 \le \ell(\sigma) \le f$.

Theorem 12.7. (i) The basic diagrams $D(\rho, r)$ can be written:

$$D(\rho, r) = \oplus_{\ell=0}^{f} D_\ell(\rho, r_\ell)$$

where $D_\ell(\rho, r_\ell)$ are basic diagrams such that $D_{0,\ell}(\rho) = \oplus_{\substack{\sigma \in \mathcal{D}(\rho) \\ \ell(\sigma)=\ell}} D_{0,\sigma}(\rho)$.

(ii) For each $\ell \in \{0, \cdots, f\}$, the basic diagrams $D_\ell(\rho, r_\ell)$ are indecomposable (in the sense of Theorem 10.3).

(iii) If χ_0 (resp. χ_f) denotes the action of I on $\sigma_0^{I_1}$ (resp. $\sigma_f^{I_1}$), then $D_{0,0}(\rho) = D_{0,\sigma_0}(\rho) = \operatorname{ind}_I^K \chi_0^s$ (resp. $D_{0,f}(\rho) = D_{0,\sigma_f}(\rho) = \operatorname{ind}_I^K \chi_f^s$).

This theorem is proved as in the irreducible case using Proposition 10.4 and a great deal of combinatorics. In particular, one has an analogue in this split case of Lemma 10.6 that allows us to keep track of the "dynamics" of $\tau \mapsto \tau^s$ for weights τ such that τ^{I_1} has a lift in $D_0(\rho)^{I_1}$ (drawing for $f = 3$). One can also prove (compare Proposition 10.7):

Proposition 12.8. With the previous notations, we have $\dim_E D_0(\rho)^{I_1} = 3^f + 1$.

For $f = 1$, we recover $3 + 1 = 4$ and for $f = 2$, $3^2 + 1 = 10$.

Applying Theorem 5.10 to each basic diagram $D_\ell(\rho, r_\ell)$ above yields the following result, very much similar to Theorem 10.8:

Theorem 12.9. (i) Let $\ell \in \{0, \cdots, f\}$ and $D_\ell(\rho, r_\ell)$ be one of the basic diagrams associated to ρ in Theorem 12.7. There exists a smooth admissible representation π_ℓ of $\mathrm{GL}_2(F)$ over E such that:
(a) $\operatorname{soc}_K \pi_\ell = \bigoplus_{\substack{\sigma \in \mathcal{D}(\rho) \\ \ell(\sigma)=\ell}} \sigma$
(b) $(\pi_\ell^{K_1}, \pi_\ell^{I_1}, \mathrm{can})$ contains $D_\ell(\rho, r_\ell)$
(c) π_ℓ is generated by $D_{0,\ell}(\rho)$
where can is the canonical injection $\pi_\ell^{I_1} \subset \pi_\ell^{K_1}$.

(ii) *If $D_\ell(\rho, r_\ell)$ and $D_\ell(\rho, r'_\ell)$ are non-isomorphic and π_ℓ, π'_ℓ satisfy (a), (b), (c) for $D_\ell(\rho, r_\ell)$ and $D_\ell(\rho, r'_\ell)$ respectively, then π_ℓ and π'_ℓ are non-isomorphic.*

(iii) *If $\ell \in \{0, f\}$, then a representation π_ℓ satisfying (a), (b), (c) of (i) is unique and is an irreducible principal series.*

Proof. The proof of (i) and (ii) is exactly the same as in Theorem 10.8. For (iii), we can proceed as follows (for $\ell = 0$ say): by Frobenius reciprocity we have a $\mathrm{GL}_2(F)$-equivariant map $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma_0 \to \pi_0$ inducing a non-zero K-equivariant map $\mathrm{ind}_I^K \chi_0^s \to \pi_0^{K_1}$ where $\mathrm{ind}_I^K \chi_0^s$ maps to the K-subrepresentation of $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma_0$ generated by $[\Pi, \sigma_0^{I_1}]$. As $\mathrm{soc}_K \pi_0 = \sigma_0$, Theorem 7.6 implies this map must be injective and coincide with the embedding $D_{0,\sigma_0}(\rho) \hookrightarrow \pi_0^{K_1}$ (otherwise, another weight than σ_0 would appear in $\mathrm{soc}_K \pi_0$). The argument then is as in the proof of Lemma 12.1 although "backwards". With the notations of that proof, we have $\Pi \phi = \mu f_0$ in $D_0(\rho, r_0)$ for a $\mu \in E^\times$ which depends on r_0. This implies $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), x^r] = \mu T([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), x^r])$ where $r = \sum_{i=0}^{f-1} r_i p^i$. Therefore the map $\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma_0 \to \pi_0$ factors through $(\mathrm{ind}_{KF^\times}^{\mathrm{GL}_2(F)} \sigma_0)/(T - \mu^{-1})$. As the latter is an irreducible principal series and the map to π_0 is surjective by assumption, it must be isomorphic to π_0. $\qquad \square$

And as for ρ irreducible, we have by a very similar proof:

Theorem 12.10. *Any smooth admissible π_ℓ satisfying (a), (b), (c) in Theorem 10.8 for $1 \leq \ell \leq f-1$ is irreducible and is a supercuspidal representation.*

We finally sum up what we have done in the split reducible case. To each continuous reducible split generic representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{GL}_2(E)$ such that p acts trivially on $\det(\rho)$ (actually, to its restriction to $\mathrm{I}(\overline{\mathbb{Q}}_p/F)$), we associate a non-empty family of smooth admissible semi-simple representations with K-socle made out of the weights of $\mathcal{D}(\rho)$: those representations $\pi = \oplus_{\ell=0}^{f-1} \pi_\ell$ for π_ℓ satisfying (a), (b), (c) above for some basic diagram associated to ρ. Each representation in this family has f irreducible direct summands, 2 of which are principal series (π_0 and π_ℓ) and the rest are supercuspidal representations. When $F = \mathbb{Q}_p$, this family is exactly parametrized by the basic diagrams $D(\rho, r)$ associated to ρ. Moreover, when $F = \mathbb{Q}_p$, we also know how to take into account the unramified character $\mathrm{unr}(\lambda)$ in ρ and associate to ρ (and not just $\rho|_{\mathrm{I}(\overline{\mathbb{Q}}_p/F)}$) one single well-defined semi-simple representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ which is a direct sum of 2 principal series.

Note that the supercuspidal representations π_ℓ of Theorem 12.9 when $\ell \notin \{0, f\}$ are necessarily different from the supercuspidal representations π

of Theorem 10.8 (for instance, they have different K-socles). Therefore, it is possible that there exists a hierarchy among supercuspidal representations of $\mathrm{GL}_2(F)$ when F is not $\mathbb{Q}_p$, some being "more supercuspidal" than others. For instance, it is natural to expect that the π of Theorem 10.8 are more supercuspidal than the π_ℓ of Theorem 12.9.