SOME PROPERTIES

OF

PLANAR BROWNIAN MOTION

Jean-Francois LE GALL

SOME PROPERTIES OF PLANAR BROWNIAN MOTION

Jean-François LE GALL

Université Pierre et Marie Curie
Laboratoire de Probabilités
4, Place Jussieu
Tour 56

75230 PARIS Cedex 05, France

A planar Brownian path.

CHAPTER I

Introduction

1. Historical sketch and program of the course.

The physical Brownian motion attracted the interest of the British botanist Robert Brown in 1828 . The mathematical study of the Brownian motion started in 1900 when the French mathematician Louis Bachelier guessed several important properties of this process, including a weak form of the Markov property, and the Gaussian distribution of Brownian motion at a fixed time. A more rigorous derivation of the Gaussian character of the one-dimensional marginals was provided by Albert Einstein in 1905. The first complete construction of Brownian motion as a continuous stochastic process is due to Norbert Wiener in 1923. Later, in collaboration with Paley and Zygmund, Wiener proved the non-differentiability of the Brownian paths, which had been conjectured by the French physicist Perrin.

Much of what we know about Brownian motion is due to Paul Lévy. Lévy discovered many remarkable sample path properties, as well as several important distributions connected with Brownian motion. Lévy also introduced the local times of linear Brownian motion, which have given rise to many important developments.

Since Lévy's work, linear Brownian motion has been studied extensively, sometimes with the help of Itô's stochastic calculus, which among other applications yields a very simple construction of local times. The books of Knight [Kn], Revuz and Yor [ReY] and Rogers and Williams [RoW] contain much information about properties of one-dimensional Brownian motion.

Multidimensional Brownian was not neglected after Lévy : see in particular Chapter 7 of Itô and Mc Kean [IM]. However several questions raised by Lévy were left aside until very recently.

In the last few years, there has been much interest in properties of planar Brownian motion : e.g. geometric properties of sample path (Burdzy, Mountford, Shimura,...), asymptotic distributions (Pitman, Yor,...) or multiple points and intersection problems (Dynkin, Rosen,...). The purpose of these lectures is to provide a detailed account of a number of these recent developments. We will mainly consider sample path (that is, almost sure) properties, although for instance Chapter II presents a proof of the celebrated Spitzer theorem on the winding number of planar Brownian motion.

We also restrict our attention to the two-dimensional case (with the important exception of Chapter VI). Sometimes the extension to higher dimensions is possible if not straightforward (this is the case for Chapters III, IV), sometimes on the contrary the extension is impossible or has no meaning (this is the case for most of the results concerning multiple points). Generally speaking, planar Brownian motion has several very nice properties, which disappear in higher dimensions. This can be explained by the relationship between Brownian motion and holomorphic functions, and also by the fact that the dimension 2 is critical for Brownian motion, meaning that a planar Brownian path, on the time interval $[0,\infty)$, is dense in the plane although it does not hit a given point.

In Chapter II, we recall the conformal invariance of planar Brownian paths and we use it to derive their first basic properties. The main topics of the next chapters are :

- the existence of the exceptional points of the path called cone points (Chapters III-IV);

- the smoothness of the convex hull of the planar Brownian path, on the time interval $[0,1]$ (Chapter III) ;

- the connected components of the complement of the Brownian path (Chapter VII);

- the shape of the Brownian path near a typical point of the boundary of one such component (Chapter V);

- the area of a tubular neighborhood of the path, and more generally of the so-called Wiener sausage (Chapters VI-VIII);

- the existence of points of finite and infinite multiplicity (Chapters VIII-IX);

- the associated "self-intersection local times" (Chapter VIII);

- the renormalization of self-intersections and its application to asymptotic expansions of the area of the Wiener sausage (Chapters X-XI).

It is worth noting that most of the previous topics are related to questions raised by Lévy. The non-existence of angular points on the boundary the convex hull of a Brownian path was stated without proof in [Lé4, p. 240]. As for the boundary of the complement of the Brownian path, it is interesting to compare the twist points theorem of Chapter V with Lévy's assertion that "la plupart des points de cette frontière ne sont accessibles que par des chemins très compliqués, le long desquels l'angle polaire n'est pas borné" [Lé4, p. 239]. Lévy [Lé4, p. 325-329] also raised many questions about multiple points, and most of them can be answered using the modern notion of intersection local time. See in particular Chapter IX for a rigorous version of Lévy's heuristic assertion that "un point double choisi sur la courbe n'a aucune chance d'être triple" [Lé4, p. 325].

2. Some comments about the proofs.

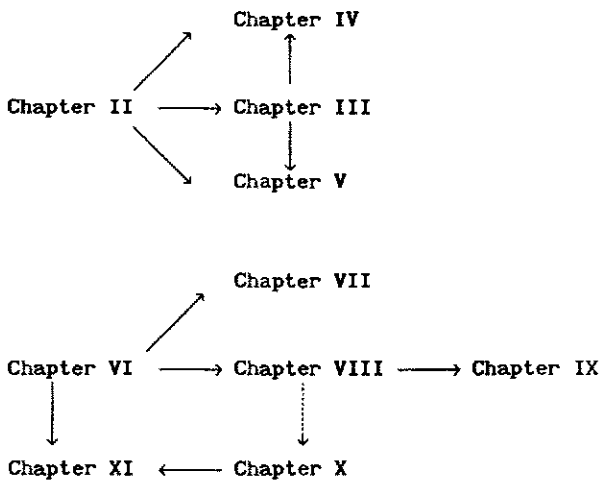
Generally speaking, we do not assume much from the reader, except for some well-known facts such as the strong Markov property or the Brownian scaling property, which both play an essential role throughout this work. We use stochastic calculus only to derive the conformal invariance of planar Brownian paths in Chapter II. Some classical results of probabilistic potential theory are used in Chapters VI and XI. They are recalled in detail at the beginning of Chapter VI. In Chapter V, we use a rather involved result of complex analysis, namely Mc Millan's theorem. It is presumably possible, and it would have been more satisfactory in a sense, to prove at least part of this result using Brownian motion. This however would have taken us too far, and we simply recall this theorem without proof in Chapter V.

An important concept in this work is the notion of local time. We use the term local time in a very wide sense. A local time is a random measure supported either on the state space $\mathbb{R}^d$ of the process or on $\mathbb{R}_+$, the set of times (or even on $\mathbb{R}_+ \times \mathbb{R}_+$, when we consider double points, etc...). This random measure is supported on a certain class of exceptional points of the path (in the first case) or on a set of exceptional times. For instance, the local time at 0 of a linear Brownian motion B may be viewed as a random measure supported on the zero set of B. Note that the zero set is a set of exceptional times since for every $t > 0$, $B_t \neq 0$ w.p. 1. As is well-known, this local time is very useful when investigating various properties of the zero set. In these lectures, we do not use the local times of linear Brownian motion (except in a remark of Chapter II and, up to some extent, in Chapters X, XI). However, in Chapters IV, VIII, we construct local times associated with certain classes of random sets, and we then apply these local times to various sample path properties. A typical example is provided by Chapter IX, where we use intersection local times (associated with points of finite multiplicity) to get the existence of points of infinite multiplicity.

The previously mentioned topics are related to various problems in probability theory or in other branches of mathematics and physics. The shortest proof of the existence of cone points uses the notion of reflected Brownian motion in a wedge, which has been studied extensively in the last few years. Many properties of planar Brownian motion can be proved from complex analysis via the conformal invariance theorem (Theorem II-1). A typical example is the twist points theorem of Chapter V, whose proof uses both McMillan's theorem and a weak form of Makarov's theorem on the support of harmonic measure. The asymptotics of the volume of the Wiener sausage give information about certain problems connected with the heat equation. The mathematical notion of intersection local time was motivated by the models of polymer physics, as well as by Symanzik's approach to quantum field theory. In the same connection, Dynkin's renormalization for multiple self-intersections of planar Brownian motion was inspired by the renormalization techniques of field theory. It was not possible in these notes to explain all the connections between Brownian motion and various problems of mathematics or physics. It should however be kept in mind that these connections often motivated the proof of the results that are presented below.

Remerciements. Je tiens ici à remercier l'ensemble des participants de l'Ecole d'Eté de Probabilités de St-Flour pour l'intérêt qu'ils ont porté à ce cours et leurs remarques souvent pertinentes. Je remercie tout particulièrement Paul-Louis Hennequin pour l'excellente organisation de l'école d'été. Je veux aussi remercier Chris Burdzy et Jay Rosen pour leurs commentaires sur la première version de ce travail. Enfin, je remercie Nicolas Bouleau pour la simulation de trajectoire brownienne plane, réalisée au CERMA, qui illustre ce cours.

Interconnections between chapters.



Main notation.

The complex Brownian motion is denoted by $(B_t)_{t\geq 0}$, or $(Z_t)_{t\geq 0}$. As usual, B_0 (or Z_0) = z under the probability P_z .

$B[u,v] = \{ B_s ; u \leq s \leq v \}$.

m denotes the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$ (or on $\mathbb{R}^d$ in Chapter VI).

The Brownian transition density is denoted by:

$$p_t(x,y) = (2\pi t)^{-1} \exp(-|y-x|^2/2t)$$

For $z \in \mathbb{C}$ and $\varepsilon > 0$,

$$D(z,\varepsilon) = \{ y \in \mathbb{C} ; \ |z - y| < \varepsilon \} , \quad D = D(0,1)$$

$$T_\varepsilon(z) = \inf\{ t \geq 0 ; \ |B_t - z| \leq \varepsilon \}$$

If K is a subset of $\mathbb{C}$ and $\varepsilon > 0$,

$\varepsilon K = \{ \varepsilon y ; y \in K \}$,

$z - K = \{ z - y ; y \in K \}$,

$T_K = \inf\{ t \geq 0 ; B_t \in K \}$ ($\inf \emptyset = +\infty$)

$T_K(z) = T_{z-K} = \inf\{ t \geq 0 ; B_t \in z - K \}$

dim A denotes the Hausdorff dimension of a subset A of $\mathbb{R}^d$ and diam A is the diameter of A .

CHAPTER II

Basic properties of planar Brownian motion.

1. Conformal invariance and the skew-product representation.

Throughout this chapter, $Z = (Z_t, t \geq 0)$ denotes a complex-valued Brownian motion started at $z_0 \in \mathbb{C}$. This simply means that the real and imaginary parts of Z are two independent linear Brownian motions. The rotational invariance property of planar Brownian motion states that for any $\theta \in \mathbb{R}$, the process $e^{i\theta}Z_t$ is again a complex Brownian motion, which starts at $e^{i\theta}z_0$. This is easily proved by checking that the real and imaginary parts of $e^{i\theta}\, Z_t$ are independent linear Brownian motions. This result is slightly extended by considering mappings of the type $\phi(z) = az + b$, with $a \neq 0$. Then using the scaling property of Brownian motion, we obtain that:

$$\phi(Z_t) = Z'_{\lambda^2 t}$$

where $\lambda = |a|$ and Z' is a complex Brownian motion started at $\phi(z_0)$. In particular, the image of a Brownian path under ϕ is a (time-changed) Brownian path. A very important theorem of Lévy shows that the latter property still holds if we only assume that ϕ is locally tangent to a mapping of the type $z \longrightarrow az + b$, that is if ϕ is conformal. More precisely, we have the following result.

Theorem 1. *Let* U *be an open subset of* $\mathbb{C}$, *such that* $z_0 \in U$, *and let* $\phi : U \longrightarrow \mathbb{C}$ *be holomorphic. Set*

$$\tau_U = \inf\{\, t \geq 0\ ;\ Z_t \notin U \,\} \leq +\infty$$

Then there exists a complex Brownian motion Z' *such that, for any* $t \in [0,\tau_U)$,

$$\phi(Z_t) = Z'_{C_t}$$

where

$$C_t = \int_0^t |\phi'(Z_s)|^2\, ds\ .$$

Remark. At any point $z_1 \in U$, the holomorphic function ϕ is "locally tangent" to the mapping $z \longrightarrow \phi(z_1) + \phi'(z_1)\,(z - z_1)$. Notice that the derivative of $t \longrightarrow C_t$ is precisely $|\phi'(Z_t)|^2$.

Proof : Set $\phi = g + ih$, so that g and h are harmonic on U . By the Itô formula applied to $g(Z_t^1 + iZ_t^2)$, we get for $t < \tau_U$,

$$g(Z_t) = g(z_0) + \int_0^t \frac{\partial g}{\partial x}(Z_s)\, dZ_s^1 + \int_0^t \frac{\partial g}{\partial y}(Z_s)\, dZ_s^2$$

and similarly,

$$h(Z_t) = h(z_0) + \int_0^t \frac{\partial h}{\partial x}(Z_s)\, dZ_s^1 + \int_0^t \frac{\partial h}{\partial y}(Z_s)\, dZ_s^2 \quad .$$

This shows that $M_t = g(Z_t)$, $N_t = h(Z_t)$ are two continuous local martingales on the stochastic interval $[0,\tau_U)$.

By the Cauchy-Riemann equations, $\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}$, $\frac{\partial g}{\partial y} = - \frac{\partial h}{\partial x}$. It follows that:

$$\langle M\rangle_t = \langle N\rangle_t = \int_0^t |\phi'(Z_s)|^2\, ds = C_t$$

$$\langle M,N\rangle_t = 0 \ .$$

By a standard result of stochastic calculus, the last two properties imply the existence of two *independent* linear Brownian motions such that, for $t \in [0,\tau_U)$

$$M_t = B^1_{C_t} \ , \quad N_t = B^2_{C_t} \quad .$$

The desired result follows, with $Z'_t = B_t^1 + iB_t^2$. □

Notice that, in the previous proof, the Brownian motion Z' is determined from Z only on the time interval $[0,\tau_U)$. As a matter of fact, when $P[\ \tau_U < \infty\ ] > 0$, in order to define Z'_s for $s \geq C_{\tau_U}$, it may be necessary to enlarge the underlying probability space. This fact is unimportant in applications of Theorem 1.

The proof of Theorem 1 also yields the formula:

$$\phi(Z_t) = \phi(z_0) + \int_0^t \phi'(Z_s)\, dZ_s \ .$$

Here the "complex stochastic integral" $\int_0^t \phi'(Z_s)\, dZ_s$ is obviously defined by:

$$\int_0^t \phi'(Z_s) dZ_s = \int_0^t (\mathrm{Re}\ \phi'(Z_s)\, dZ_s^1 - \mathrm{Im}\ \phi'(Z_s)\, dZ_s^2)$$

$$+ i\int_0^t (\mathrm{Re}\ \phi'(Z_s)\, dZ_s^2 + \mathrm{Im}\ \phi'(Z_s) dZ_s^1) \ .$$

Theorem 1 can be used to interpret (and sometimes to prove) many results of complex analysis in terms of planar Brownian motion. On the other hand, it allows one to prove properties of Brownian motion using holomorphic functions. We shall be interested in this second type of applications. We first use Theorem 1 to establish the polarity of single points for planar Brownian motion.

Corollary 2 : *Let* $z_1 \in \mathbb{C} \setminus \{z_0\}$. *Then,*

$$P[\ Z_t = z_1 \ \textit{ for some } \ t \geq 0\ ] = 0\ .$$

Let m *denote Lebesgue measure on* $\mathbb{C}$. *Then,*

$$m(\{\ Z_t\ ;\ t \geq 0\ \}) = 0 \quad , \quad a.s.$$

Proof : We may assume that $z_0 = 1$, $z_1 = 0$. Let $\Gamma = (\Gamma_t,\ t \geq 0)$ be a planar Brownian motion started at 0 . By Theorem 1,

$$\exp(\Gamma_t) = Z'_{C_t}$$

where

$$C_t = \int_0^t \exp(2\ \mathrm{Re}\ \Gamma_s)\ ds\ ,$$

and Z' is a complex Brownian motion started at 1 . It is immediate that $\lim_{t\to\infty} C_t = +\infty$, a.s. , so that

$$\{\ Z'_t\ ;\ t \geq 0\ \} = \{\ \exp \Gamma_t\ ;\ t \geq 0\ \} \quad \text{a.s.}$$

Obviously, $0 \notin \{\ \exp \Gamma_t\ ;\ t \geq 0\ \}$, and we get the desired result, with Z replaced by Z' . This however makes no difference since the two processes Z, Z' have the same distribution.

To get the second assertion, write

$$E[\ m(\{\ Z_t\ ;\ t \geq 0\})\ ] = E\Big[\int dy\ 1_{(Z_t = z \text{ for some } t \geq 0)} \Big]$$
$$= \int dy\ P[\ Z_t = z \text{ for some } t \geq 0\ ] = 0\ . \ \square$$

As a second consequence of Theorem 1, we get the skew-product representation of planar Brownian motion.

Theorem 3 : *Suppose that* $z_0 \neq 0$, *and write* $z_0 = \exp(r + i\theta)$, *with* $r \in \mathbb{R}$ *and* $\theta = \arg(z) \in (-\pi,\pi]$. *There exist two independent linear Brownian motions* β, γ , *started respectively at* r, θ , *such that, for every* $t \geq 0$,

$$Z_t = \exp(\ \beta_{H_t} + i\ \gamma_{H_t}\ )$$

where

$$H_t = \int_0^t \frac{ds}{|Z_s|^2} = \inf\{\ u \geq 0\ ,\ \int_0^u \exp(2\beta_v)\ dv > t\ \}\ .$$

Remark. Corollary 2 shows that H_t is well-defined for any $t \geq 0$.

Proof : The "natural" method would be to apply Theorem 1 to $\phi(z) = \mathrm{Log}\, z$, that is to some determination of the complex logarithm. This approach however leads to certain minor technical difficulties (due to the fact that one cannot take $U = \mathbb{C} \setminus \{0\}$!). Therefore we will use another method, similar to the proof of Corollary 2.

We may assume that $z_0 = 1$ and thus $r = \theta = 0$. Let $\Gamma_t = \Gamma^1_t + i\Gamma^2_t$ be a complex Brownian motion started at 0. By Theorem 1,

(1) $$\exp \Gamma_t = Z'_{C_t} ,$$

where

$$C_t = \int_0^t \exp(2\, \Gamma^1_s)\, ds \; ;$$

Let $(H_t,\ t \geq 0)$ be the inverse function of C_t :

$$H_s = \int_0^s \exp(-2\, \Gamma^1_{H_u})\, du = \int_0^s \frac{du}{|Z'_u|^2} ,$$

since $\exp(\Gamma^1_{H_u}) = |Z'_u|$. By **(1)** with $t = H_s$,

$$Z'_s = \exp(\, \Gamma^1_{H_s} + i\, \Gamma^2_{H_s}\,) .$$

This is the desired result, with $\beta = \Gamma^1$, $\gamma = \Gamma^2$, except that we have replaced Z by Z'.

To complete the proof, we argue as follows. Theorem 1 is equivalent to saying that, if

$$\beta_t = (\log |Z|)_{\inf\{ s \,;\, \int_0^s |Z_u|^{-2} du > t \}} ,$$

$$\gamma_t = (\arg Z)_{\inf\{ s \,;\, \int_0^s |Z_u|^{-2} du > t \}} ,$$

then β, γ are two independent linear Brownian motions. Observe that β, γ are deterministic functions of the process Z. Therefore their joint distribution depends only on that of Z. □

Another approach to Theorem 3, avoiding the use of Theorem 1, would be to check that:

$$\log |Z_t| = r + \int_0^t \frac{Z^1_s\, dZ^1_s + Z^2_s\, dZ^2_s}{|Z_s|^2} , \quad \arg Z_t = \theta + \int_0^t \frac{Z^1_s\, dZ^2_s - Z^2_s\, dZ^1_s}{|Z_s|^2}$$

and then to use the same argument as in the proof of Theorem 1 in order to write $\log |Z_t|$, $\arg Z_t$ as time-changed independent linear Brownian motions.

Notice the intuitive contents of Theorem 3. When $|Z_t|$ is large, then H_t increases slowly, so that $\arg Z_t$ also varies slowly.

The formula

$$\log|Z_t| = \beta_{\inf\{ u \geq 0 \,,\, \int_0^u \exp(2\beta_v)\, dv > t \}}$$

shows that $|Z|$ is completely determined by the linear Brownian motion β (and conversely). This is related to the fact that $|Z|$ is a Markov process, namely a two-dimensional Bessel process. On the other hand, $\arg Z_t = \gamma_{H_t}$ is a linear Brownian motion time-changed by an independent increasing process. The independence of γ and H_t is especially important in applications of the skew-product representation.

2. Some applications of the skew-product representation.

We start by proving that planar Brownian motion is recurrent.

Theorem 4 : *For any open subset* U *of* $\mathbb{C}$ *,*

$$P[\limsup_{t\to\infty} \{ Z_t \in U \}] = 1 .$$

Proof : We may take $z_0 = 1$, $U = D(0,\varepsilon)$ for $\varepsilon \in (0,1)$. Theorem 3 gives:

$$\log|Z_t| = \beta_{H_t}$$

and the obvious facts: $\lim_{t\to\infty} H_t = +\infty$, $\liminf_{s\to\infty} \beta_s = -\infty$ imply

$$\liminf_{t\to\infty} \log|Z_t| = -\infty \quad \text{a.s.} \ \square$$

From now on, we assume $z_0 \neq 0$. Let $(\theta_t, t \geq 0)$ be the continuous determination of $\arg Z_t$ such that $\theta_0 = \arg z_0 \in (-\pi,\pi]$.

Proposition 5 : *With probability* 1 *,*

$$\limsup_{t\to\infty} \theta_t = +\infty \quad , \quad \liminf_{t\to\infty} \theta_t = -\infty .$$

Proof : Similar to that of Theorem 4 , using now $\theta_t = \gamma_{H_t}$. $\square$

Remark. We can use the previous results to prove the conformal invariance of harmonic measure, in a special case that will be used in Chapter V . Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere and let V be a simply connected subset of $\hat{\mathbb{C}}$, such that $\hat{\mathbb{C}} \setminus V$ has a nonempty interior. Let D be the open unit disk of $\mathbb{C}$. The Riemann mapping theorem yields a one-to-one conformal mapping f from D onto V . The value of f at 0 may be chosen arbitrarily, so that we can impose $f(0) \neq \infty$. By Fatou's theorem, the radial limits

$$\lim_{r\to 1, r<1} f(re^{i\theta}) =: \tilde{f}(e^{i\theta})$$

exist for $d\theta$-a.a. $\theta \in [0,2\pi)$. We take $U = D$ if $\infty \notin V$, $U = D \setminus \{f^{-1}(\infty)\}$ if $\infty \in V$ and we will apply Theorem 1 to $\phi = f_{|U}$, with $z_0 = 0$. Notice that the distribution of Z_{τ_U} is the uniform distribution $\sigma(d\zeta)$ over ∂D (by rotational invariance, and Corollary 2 if $\infty \in V$).

Let Z' be another Brownian motion started at $z'_0 = f(0)$ and let $\tau'_V = \inf\{ t ; Z'_t \notin V \}$ ($\tau'_V < \infty$ a.s. by Theorem 4). Theorem 1 implies that the limit

$$\lim_{t\to\tau_U} f(Z_t)$$

exists a.s. and is distributed as $Z'_{\tau'_V}$. Furthermore, using the skew-product representation and some well-known properties of linear Brownian motion, it is easy to check that this limit coincides a.s. with $\tilde{f}(Z_{\tau_U})$. We conclude that $\tilde{f}(Z_{\tau_U}) \overset{(d)}{=} Z'_{\tau'_V}$. In other words, the harmonic measure in V relative to z'_0 is the image of $\sigma(d\zeta)$ under $\tilde{f}$.

Theorem 4 and Proposition 5 are straightforward applications of the skew-product representation. The idea of these applications is that many properties of Z can be derived by looking at the independent Brownian motions β, γ and taking account of the time-change H_t. Until now, we only used the simple fact $\lim_{t\to\infty} H_t = +\infty$. For further applications, it is important to control the asymptotic behavior of H_t. We know that H_t has a simple expression in terms of the Brownian motion β. The next lemma relates the asymptotic behavior of H_t to that of an even simpler functional of β. This result is a key ingredient in the proof of several asymptotic theorems for planar Brownian motion.

Lemma 6 : *For every* $\lambda > 0$, *set:*

$$\beta_t^{(\lambda)} = \frac{1}{\lambda}\beta_{\lambda^2 t} \quad (t \geq 0) \quad , \quad T_1^{(\lambda)} = \inf\{ t \geq 0 , \beta_t^{(\lambda)} = 1 \} .$$

Then,

$$\frac{4}{(\log t)^2} H_t - T_1^{(\frac{1}{2}\log t)} \xrightarrow[t\to\infty]{\text{Probability}} 0 .$$

Remark. For every $\lambda > 0$, $\beta^{(\lambda)}$ is a linear Brownian motion started at $\lambda^{-1}\log|z_0|$. Therefore, Lemma 6 entails in particular that $4(\log t)^{-2} H_t$ converges in distribution towards the hitting time of 1 by a linear Brownian motion started at 0.

Proof : By scaling, we may assume $|z_0| = 1$, so that $\beta_0 = 0$. To simplify notation, we write

$$\lambda = \lambda(t) = \frac{1}{2} \log t \quad .$$

Let $\varepsilon > 0$ and $T^{(\lambda)}_{1+\varepsilon} = \inf\{ t \geq 0 , \beta^{(\lambda)}_t = 1 + \varepsilon\}$. We first prove that:

$$\text{(2)} \qquad P[\lambda^{-2} H_t > T^{(\lambda)}_{1+\varepsilon}] \xrightarrow[t\to\infty]{} 0 \quad .$$

Since

$$H_t = \inf\{ u \geq 0 , \int_0^u \exp(2\beta_v)\, dv > t \} .$$

we have

$$\{ \lambda^{-2} H_t > T^{(\lambda)}_{1+\varepsilon} \} = \left\{ \int_0^{\lambda^2 T^{(\lambda)}_{1+\varepsilon}} \exp(2\beta_v)\, dv < t \right\}$$

$$= \left\{ \frac{1}{2\lambda} \log \int_0^{\lambda^2 T^{(\lambda)}_{1+\varepsilon}} \exp(2\beta_v)\, dv < 1 \right\}$$

(recall that $2\lambda = \log t$). However,

$$\frac{1}{2\lambda} \log \int_0^{\lambda^2 T^{(\lambda)}_{1+\varepsilon}} \exp(2\beta_v)\, dv = \frac{\log \lambda}{\lambda} + \frac{1}{2\lambda} \log \int_0^{T^{(\lambda)}_{1+\varepsilon}} \exp(2\lambda\, \beta^{(\lambda)}_v)\, dv$$

$$\stackrel{(d)}{=} \frac{\log \lambda}{\lambda} + \frac{1}{2\lambda} \log \int_0^{T^{(1)}_{1+\varepsilon}} \exp(2\lambda\, \beta^{(1)}_v)\, dv$$

since the processes $\beta^{(\lambda)}$ are identically distributed. We now use the fact that, for any continuous function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$, for any $t > 0$,

$$\frac{1}{2\lambda} \log \int_0^t \exp(2\lambda\, f(v))\, dv \xrightarrow[\lambda\to\infty]{} \sup_{[0,t]} f(s) .$$

It follows that

$$\frac{1}{2\lambda} \log \int_0^{T^{(1)}_{1+\varepsilon}} \exp(2\lambda\, \beta_v)\, dv \xrightarrow[\lambda\to\infty]{} \sup_{[0,T^{(1)}_{1+\varepsilon}]} \beta_s = 1 + \varepsilon \quad , \quad \text{a.s.}$$

and thus

$$\frac{1}{2\lambda} \log \int_0^{\lambda^2 T^{(\lambda)}_{1+\varepsilon}} \exp(2\lambda\, \beta^{(\lambda)}_v)\, dv \xrightarrow[\lambda\to\infty]{\text{Probability}} 1 + \varepsilon .$$

This completes the proof of (2). Exactly the same arguments give:

$$P[\lambda^{-2} H_t < T^{(\lambda)}_{1-\varepsilon}] \xrightarrow[t\to\infty]{} 0$$

which completes the proof of Lemma 6. □

The next theorem, due to Spitzer (1958), gives precise information on the order of θ_t when t is large.

__Theorem 7__ : *As* $t \to \infty$, $\frac{2}{\log t}\ \theta_t$ *converges in distribution towards the standard symmetric Cauchy distribution. Equivalently, for any* $x \in \mathbb{R}$,

$$\lim_{t\to\infty} P[\ \frac{2}{\log t}\ \theta_t \le x\ ] = \int_{-\infty}^{x} \frac{dy}{\pi(1+y^2)} .$$

Proof : For $\lambda > 0$, write

$$\gamma_t^{(\lambda)} = \frac{1}{\lambda}\,\gamma_{\lambda^2 t} \quad (t \ge 0) .$$

Take $\lambda = \lambda(t) = \frac{1}{2}\log t$ as previously. Then,

$$\lambda^{-1}\,\theta_t = \lambda^{-1}\,\gamma_{H_t} = \gamma^{(\lambda)}_{\lambda^{-2}H_t} .$$

Hence, by Lemma 6,

$$\lambda^{-1}\,\theta_t - \gamma^{(\lambda)}(T_1^{(\lambda)}) \xrightarrow[\lambda\to\infty]{\text{Probability}} 0 .$$

To complete the proof, note that the variable $\gamma^{(\lambda)}(T_1^{(\lambda)})$ obviously converges in distribution towards $\gamma^{(\infty)}(T_1^{(\infty)})$ where $T_1^{(\infty)} = \inf\{\ t \ge 0\ ,\ \beta_1^{(\infty)} = 1\}$, and $\beta^{(\infty)}$, $\gamma^{(\infty)}$ are two independent linear Brownian motions started at 0 . It is easy to compute the characteristic function of $\gamma^{(\infty)}(T_1^{(\infty)})$ and to check that it is that of a standard symmetric Cauchy distribution (an alternative method is to observe that the process $(\gamma^{(\infty)}(T_a^{(\infty)}),\ a \ge 0)$ is a symmetric Lévy process, stable with index 1 , hence must be a symmetric Cauchy process: this is Spitzer's construction of the Cauchy process). □

Lemma 6 can be applied to the proof of other asymptotic theorems for planar Brownian motion. Let us mention the following result of Kallianpur and Robbins, which yields information on the time spent by the Brownian path in domains of the plane. Let $f : \mathbb{C} \longrightarrow \mathbb{R}_+$ be a bounded measurable function with compact support. Then,

$$\frac{2}{\log t}\int_0^t f(Z_s)\,ds \xrightarrow[t\to\infty]{(d)} \left(\frac{1}{\pi}\int f(y)\,dy\right)\varepsilon$$

where ε denotes an exponential variable with parameter 1 . When f is radial, that is $f(z) = f(|z|)$, Lemma 6 yields a simple proof of this convergence. Denote by $L_t^a(\beta)$ the local time of the Brownian motion β at level a , at time t . Then,

$$\frac{2}{\log t}\int_0^t f(Z_s)\,ds = \frac{1}{\lambda}\int_0^t f(\exp\beta_{H_s})\,ds$$

$$= \frac{1}{\lambda}\int_0^{H_t} f(\exp\beta_u)\exp 2\beta_u\,du$$

$$= \lambda\int_0^{\lambda^{-2}H_t} f(\exp\lambda\,\beta_v^{(\lambda)})\exp(2\lambda\,\beta_v^{(\lambda)})\,dv$$

$$= \lambda\int_{\mathbb{R}} f(\exp\lambda a)\exp(2\lambda a)\,L^a_{\lambda^{-2}H_t}(\beta^{(\lambda)})\,da$$

$$= \int_0^\infty r\,f(r)\,L^{\lambda^{-1}\log r}_{\lambda^{-2}H_t}(\beta^{(\lambda)})\,dr$$

$$\xrightarrow[\lambda\to\infty]{(d)} \left(\int_0^\infty r\,f(r)\,dr\right) L^0_{T_1^{(\infty)}}(\beta^{(\infty)})$$

by Lemma 6. To complete the proof, note that $L^0_{T_1^{(\infty)}}(\beta^{(\infty)}) \overset{(d)}{=} 2\,e$. The general case (f non radial) can then be handled using the Chacon-Ornstein ergodic theorem.

Via a scaling argument, we can use Theorem 7 to get information about the behavior of the process Z in small time. Suppose now that $z_0 = 0$. Of course we can no longer define θ_t . However, by Corollary 2 and the Markov property, we know that $Z_t \neq 0$ for every $t > 0$, a.s. Hence, for every $\varepsilon>0$, we may consider $\theta_{[\varepsilon,1]}$, defined as the variation of (a continuous determination of) $\arg Z_t$ between times ε and 1 . By scaling,

$$\theta_{[\varepsilon,1]} \overset{(d)}{=} \theta_{[1,1/\varepsilon]} .$$

Therefore, Theorem 7 and the Markov property imply the convergence in distribution of $2|\log\varepsilon|^{-1}\,\theta_{[\varepsilon,1]}$ towards a standard Cauchy distribution. An application of the zero-one law also gives, a.s. for any $\delta > 0$,

$$\limsup_{\varepsilon\to 0}\ \theta_{[\varepsilon,\delta]} = +\infty \quad , \quad \liminf_{\varepsilon\to 0}\ \theta_{[\varepsilon,\delta]} = -\infty$$

Informally, on any interval $[0,\delta]$, the Brownian path performs an infinite number of windings around its starting point.

3. The Hausdorff dimension of the Brownian curve.

In this section, which is independent of the previous two ones, we propose to compute the Hausdorff dimension of the Brownian path. We have already noticed that the Lebesgue measure of the path is zero a.s. Nonetheless we will check that its Hausdorff dimension is 2 , which shows that in a sense the Brownian path is not far from having positive Lebesgue measure (see Lévy [Lé4, p. 242-243] for comments about the area of the planar Brownian curve).

We first recall the definitions of Hausdorff measure and Hausdorff dimension. Let h be a continuous monotone increasing function from $\mathbb{R}_+$ into $\mathbb{R}_+$. For any Borel subset A of $\mathbb{R}^d$, the Hausdorff measure $h\text{-}m(A)$ is defined by:

$$h\text{-}m(A) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \left(\inf_{(R_i)\in\mathcal{R}_\varepsilon(A)} \sum_i h(\operatorname{diam} R_i) \right)$$

where $\operatorname{diam}(R_i)$ denotes the diameter of the set R_i , and, for $\varepsilon > 0$, $\mathcal{R}_\varepsilon(A)$ is the collection of all countable coverings of A by subsets of $\mathbb{R}^d$ of diameter less than ε . Notice that the limit exists in $[0,\infty]$ since the infimum is a nonincreasing function of ε .

In what follows, we only consider functions h such that $h(2x) \le C\, h(x)$ for some constant C , and we are interested in knowing whether $h\text{-}m(A) > 0$, or $h\text{-}m(A) < \infty$. To this end, we may restrict our attention to coverings by balls, or cubes, or rectangles (notice for instance that any bounded subset R of $\mathbb{R}^d$ is contained in a ball of diameter $2 \operatorname{diam} R$).

For any $\alpha > 0$, we set $h_\alpha(x) = x^\alpha$. It can be proved that

$$h_d\text{-}m(A) = C_d\, m(A)$$

for some universal constant $C_d > 0$. It is also easy to check that, for any Borel subset A of $\mathbb{R}^d$, there exists a number $\dim A \in [0,d]$ such that:

$$h_\alpha\text{-}m(A) = \begin{cases} +\infty & \text{if } \alpha < \dim A , \\ 0 & \text{if } \alpha > \dim A . \end{cases}$$

The number $\dim A$ is the Hausdorff dimension of A . If $A = \bigcup_{n\in\mathbb{N}} A_n$, we have

$$\dim A = \sup_n \dim A_n .$$

Theorem 8. *With probability* 1 *, for every* $t > 0$ *,*

$$\dim(\{ Z_s ; 0\le s\le t \}) = 2 .$$

Proof. We need only check that

$$\dim(\{ Z_s ; 0\le s\le t \}) \ge 2 - \delta ,$$

for any $\delta > 0$. We introduce the random measure

$$\mu(A) = \int_0^t 1_A(Z_s)\, ds .$$

Fix $\delta > 0$. We will prove that, w.p. 1 , there exists a constant $\rho(\omega) > 0$ such that, for any subset A of $\mathbb{C}$ with $\operatorname{diam} A < \rho(\omega)$,

$$\mu(A) \le 16\ (\operatorname{diam} A\ )^{2-\delta} . \tag{3}$$

Then, if (R_i) is a countable covering of $\{ Z_s \ ; \ 0 \le s \le t \}$ by sets of diameter less than $\rho(\omega)$, we have

$$\sum_i\ (\operatorname{diam} R_i)^{2-\delta} \ \ge \frac{1}{16} \sum_i \ \mu(R_i) \ \ge \ \frac{1}{16}\, \mu(\{ Z_s \ ; \ 0 \le s \le t \}) = \frac{1}{16}\, t \ .$$

Therefore, $h_{2-\delta}\text{-}m(\{ Z_s \ ; \ 0 \le s \le t \}) > 0$ and $\dim(\{ Z_s \ ; \ 0 \le s \le t \}) \ge 2-\delta$.

It remains to prove (3). Suppose first that A is a square of the type $A = [u,u+r] \times [v,v+r]$. For every integer $p \ge 1$, we evaluate

$$E[\mu(A)^p] = E\left[\left(\int_0^t 1_A(Z_s)\ ds\right)^p\right] = E\Big[\int_{[0,t]^p} ds_1 \dots ds_p \ \ 1_A(Z_{s_1}) \ \dots \ 1_A(Z_{s_p}) \ \Big]$$

$$= p! \int_{A^p} dy_1 \dots dy_p \int_{0 \le s_1 \le \dots \le s_p \le t} ds_1 \dots ds_p \ p_{s_1}(z_0,y_1) p_{s_2-s_1}(y_1,y_2) \dots p_{s_p-s_{p-1}}(y_{p-1},y_p)$$

$$\le p! \ \left(\sup_{z \in \mathbb{C}} \int_A dy \int_0^t ds \ p_s(z,y) \ \right)^p .$$

At this point, we use the easy bound:

$$\int_0^t ds \ p_s(z,y) \ \ \le \ \ C \left(\ 1 + \log_+ \frac{1}{|z-y|} \ \right) \ \ e^{-|z-y|}$$

and after integration over A we get

$$E[\mu(A)^p] \le p! \ C'^{\,p} \ \varphi(m(A))$$

where $\varphi(x) = x\ (1 + \log_+ 1/x)$. It follows that, for $\lambda > 0$ small enough, for any square A ,

$$E[\ \exp \lambda \ \frac{\mu(A)}{\varphi(m(A))} \ ] \ \ \le \ \ 2 \ .$$

Then, by the Tchebicheff inequality, for every $r > 0$,

$$P[\ \mu(A) \ge r \ \varphi(m(A)) \ ] \le 2 \exp -\lambda r \ . \tag{4}$$

The proof of (3) is now easily completed. Denote by $A(n,j,k)$ the dyadic square $[j2^{-n},(j+1)2^{-n}] \times [k^{-n},(k+1)2^{-n}]$. By (4) ,

$$\sum_{n=1}^{\infty} \ \ \sum_{j=-2^{2n}}^{2^{2n}} \ \sum_{k=-2^{2n}}^{2^{2n}} P[\ \mu(A(n,j,k) \ge 2^{-(2-\delta)n} \ ] \ \ < \ \ \infty \ .$$

Therefore, by the Borel-Cantelli lemma, we may w.p. 1 find $n_0(\omega)$ such that:

$$\mu(A(n,j,k)) \le 2^{-(2-\delta)n}$$

for every $n \geq n_o(\omega)$, $j, k \in \{-2^{2n},\dots,2^{2n}\}$. The bound (3) now follows: use the fact that any set A such that $\text{diam } A < 1/2$ is contained in the union of 4 dyadic squares $A(n,j,k)$, with n such that $\text{diam } A \leq 2^{-n} \leq 2 \text{ diam } A$.□

The previous proof is certainly not the shortest one (in particular, the connection between Hausdorff measures and capacities can be used to give a very short proof). It is however interesting as it serves as a prototype for the evaluation of the Hausdorff dimension of random sets. The upper bound is usually easy (here it was trivial) because it suffices to construct good coverings. The lower bound requires the introduction of an auxiliary measure (a "local time") which is in a sense uniformly distributed over the random set. See Chapter IV for an application of this technique to cone points and [L9] for an application to multiple points of the Brownian path. In the latter case, the auxiliary measure is provided by the intersection local time introduced in Chapter VIII.

Bibliographical notes. The conformal invariance of Brownian paths was stated by Lévy (see [Lé4, p. 254]*), with a heuristic proof. A (succinct) proof using stochastic calculus was provided by McKean* [MK, p. 109] *(see also* [IMK, p. 279-280] *for a different approach). Several results related to Theorem 1, and a detailed proof of the needed arguments of stochastic calculus, may be found in Getoor and Sharpe* [GS] *(see also the Chapter 5 of Revuz and Yor* [ReY]*). Applications of Theorem 1 to complex analysis are given in Davis* [Da] *and Durrett* [Du2]. *Corollary 2 and Theorem 4 are due to Lévy (Lévy's proof of Corollary 2 is elementary, it uses only the scaling properties of Brownian motion, see* [Lé4, p. 240-241]*). The skew-product representation is stated in Itô and McKean* [IMK, p. 265] *, in the more general setting of d-dimensional Brownian motion. Theorem 7 was first proved by Spitzer* [Sp1], *using explicit calculations of the Fourier transform of* θ_t *. The basic idea of our proof is due to Durrett* [Du1] *(see also Pitman and Yor* [PY1] *and Le Gall and Yor* [LY], *the latter paper dealing with diffusions more general than Brownian motion). See* [KR] *for the original proof of the Kallianpur-Robbins law. Pitman and Yor* [PY1,PY3] *(see also the Chapter 13 of* [ReY]*) have obtained limit theorems which extend Spitzer's result and the Kallianpur-Robbins law in many respects. A typical example is the determination of the asymptotic joint distribution of the winding numbers around several points of the plane* [PY1]. *Theorem 8 is only a weak form of Taylor's result on the exact Hausdorff measure of the sample path of planar Brownian motion* [T1] . *See Lévy* [Lé2] *and Ciesielski and Taylor* [CT] *for the analogous theorem in higher dimensions, and* [L1] *for a unified approach to these results.*

CHAPTER III

Two-sided cone points and the convex hull of planar Brownian motion.

1. The definition of cone points.

We consider a standard complex-valued Brownian motion $(B_t, t \geq 0)$ started at 0. As was noticed in Chapter II, for every fixed $t > 0$, with probability 1, the curve $(B_{t+s}, 0 < s \leq 1)$ performs an infinite number of windings around the point B_t. The same is true for the curve $(B_{t-s}, 0 < s \leq t)$. These results hold for any fixed t with probability 1. It is natural to ask whether there can be exceptional times (depending on ω) for which these properties fail to hold. A simple geometric argument shows that there must exist such times. Write

$$B_t = B_t^1 + i\, B_t^2$$

and set :

$$T = \inf\{t \geq 0 \; ; \; B_t^1 = \sup_{0 \leq s \leq 1} B_s^1\}.$$

It is very easy to see that $0 < T < 1$ a.s. Furthermore the definition of T shows that both curves $(B_{T-s}, 0 \leq s \leq T)$ and $(B_{T+s}, 0 < s \leq 1-T)$ lie in the hyperplane $\{x \leq B_T^1\}$. Therefore the previous properties cannot hold for $t = T$.

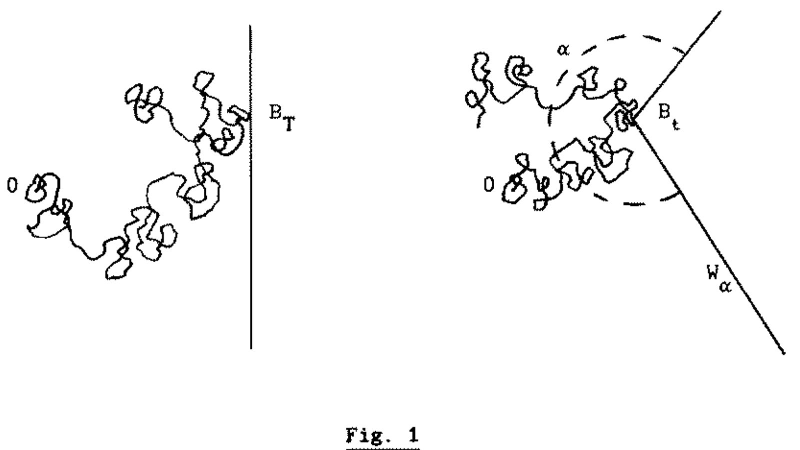


Fig. 1

Definition : *Let* $\alpha \in (0,2\pi)$ *and* $t > 0$. *We say that* B_t *is a two-sided cone point with angle* α *if there exist* $\delta > 0$ *and a closed wedge* W_α *with vertex* B_t *and angle* α, *such that the two curves* $(B_{t+s}, 0 < s \le \delta)$ *and* $(B_{t-s}, 0 < s \le \delta)$ *lie inside the wedge* W_α . *We say that* B_t *is a one-sided cone point with angle* α *if the same property holds for one of the two curves* $(B_{t+s}, 0 \le s \le \delta)$, $(B_{t-s}, 0 \le s \le \delta)$.

The point B_T constructed above is with probability 1 a two-sided cone point with angle π. One may ask whether there exist two-sided cone points with angle less than π. We shall see that the answer is no and that this fact is closely related to the non-existence of "corners" on the boundary of the convex hull of $(B_s, 0 \le s \le 1)$. One-sided cone points will be studied in the next chapter.

2. Estimates for two-sided cone points.

As we have already observed, for any fixed $t > 0$, B_t is w.p. 1 not a cone point. It will therefore be convenient to introduce a weaker notion of "approximate cone point". Fix $A > 0$ and let $z \in \mathbb{C} \setminus \{0\}$. Write the skew-product decomposition of the Brownian motion $z - B_t$:

$$z - B_t = R_t \exp(i\ \theta_t) \qquad (\theta_o = \arg(z) \in (-\pi\ ;\ \pi]).$$

Set :

$$T_\varepsilon(z) = \inf\{s \ge 0\ ;\ R_s \le \varepsilon\},$$

$$S_\varepsilon(z) = \inf\{s \ge T_\varepsilon(z)\ ;\ R_s \ge A\}.$$

For $\varepsilon < |z|$, we say that z is an ε-approximate (two-sided) cone point with angle α if :

$$\forall\ s \le S_\varepsilon(z), \quad |\theta_s| \le \frac{\alpha}{2}\ .$$

Note that we do not require z to belong to the Brownian curve. We will discuss later the connection between cone points and approximate cone points. Notice that z is an ε-approximate cone point iff the curve $\{B_s, 0 \le s \le S_\varepsilon(z)\}$ lies inside the wedge $\{z - r\, e^{i\gamma}\ ;\ r \ge 0,\ |\gamma| \le \frac{\alpha}{2}\}$.

We will now get upper bounds on the probability that z is an ε-approximate cone point. Clearly, the only non-trivial case is when $\theta_o = \arg(z) \in (-\frac{\alpha}{2}, \frac{\alpha}{2})$, which we assume now. The basic idea is to split the interval $[0,\ S_\varepsilon(z)]$ as $[0,\ T_\varepsilon(z)] \cup [T_\varepsilon(z),\ S_\varepsilon(z)]$ and to bound separately the corresponding probabilities, making use of the Markov property at time $T_\varepsilon(z)$.

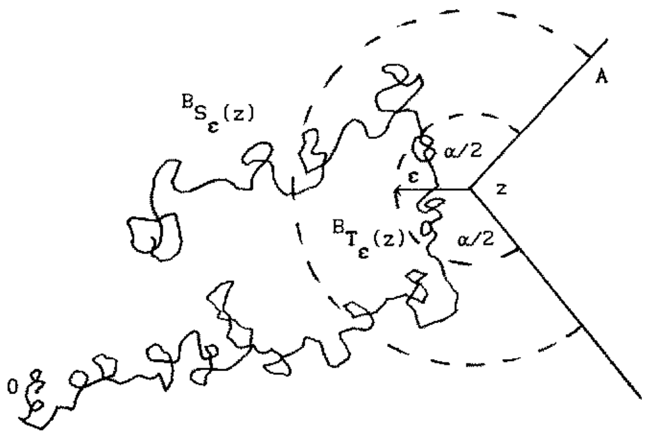


Fig. 2

The skew-product representation gives us

$$\log R_t = \beta_{H_t} \ , \ \theta_t = \gamma_{H_t}$$

where $H_t = \int_0^t R_s^{-2}\, ds$ and β, γ are two independent linear Brownian motions, with $\beta_0 = \log|z|$ and $\gamma_0 = \arg(z)$. Clearly,

$$H_{T_\varepsilon(z)} = \inf\{u, \ \beta_u \le \log \varepsilon\} =: \sigma_{\log \varepsilon}.$$

Therefore,

$$\{\forall\ s \le T_\varepsilon(z) \ , \ |\theta_s| \le \tfrac{\alpha}{2}\} = \{\forall\ u \le \sigma_{\log \varepsilon} \ , \ |\gamma_u| \le \tfrac{\alpha}{2}\}.$$

The probability of the last event is easy to estimate. We note that $\sigma_{\log\varepsilon}$ and γ are independent and we make use of the following classical lemma.

Lemma 1 : *Let* $(W_t, t \ge 0)$ *be a standard linear Brownian motion started at* 0, *and let* $a < 0 < b$. *Then for every* $t > 0$,

$$P[\forall s \le t, \ a \le W_s \le b] = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi b}{b-a}\right) \exp - \frac{(2k+1)^2\pi^2}{2(b-a)^2}\, t.$$

Proof : (see e.g. Feller (1971), p. 342) The function

$$\varphi(t,x) = P[\forall s \le t, \ a \le x + W_s \le b]$$

solves $\frac{\partial\varphi}{\partial t} = \frac{1}{2}\Delta\varphi$ in $(0,\infty) \times (a,b)$, with Dirichlet boundary conditions and initial value 1. This equation is solved by the usual eigenfunction expansion. □

It follows that :

$$P[\forall\ u \le \sigma_{\log\varepsilon};\, |\gamma_u| \le \tfrac{\alpha}{2}]$$

$$= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi(\alpha/2-\arg(z))}{\alpha}\right) E\left[\exp - \frac{(2k+1)^2\pi^2}{2\alpha^2}\sigma_{\log\varepsilon}\right]$$

$$= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi(\alpha/2-\arg(z))}{\alpha}\right)\left(\frac{\varepsilon}{|z|}\right)^{(2k+1)\pi/\alpha},$$

using the well-known formula for the Laplace transform of hitting times of points for linear Brownian motion:

$$E[\exp - \lambda\, \sigma_{r+\log|z|}\ ] = \exp - |r|\sqrt{2\lambda}\ .$$

In this chapter, we will only need the following simple consequence of the previous explicit formula. There exists a constant C, independent of $z \in \mathbb{C}$, $\varepsilon \in (0,1)$, such that :

$$\text{(1)} \qquad P\left[\forall\ s \le T_\varepsilon(z),\ |\theta_s| \le \frac{\alpha}{2}\right] \le C\ \left(\frac{\varepsilon}{|z|}\right)^{\pi/\alpha}.$$

Formula (1) is trivial when $|z| \le 2\varepsilon$ and follows from the previous expansion when $|z| > 2\varepsilon$.

Let $(\mathcal{F}_t)$ be the canonical filtration of B. Our next goal is to bound

$$P\left[\forall\ s \in [T_\varepsilon(z),S_\varepsilon(z)],\ |\theta_s| \le \frac{\alpha}{2}\ \middle|\ \mathcal{F}_{T_\varepsilon(z)}\right].$$

The Markov property at time $T_\varepsilon(z)$ leads us to consider a Brownian motion started at some point of $D(z,\varepsilon) := \{y, |z-y| \le \varepsilon\}$, and to bound the probability that it exits $D(z,A)$ before exiting the wedge $\{z-re^{iu}\ ;\ r \ge 0, |u| \le \frac{\alpha}{2}\}$. However the previous calculations apply as well to this situation. Therefore we get the bound :

$$\text{(2)} \qquad P\left[\forall\ s \in [T_\varepsilon(z),S_\varepsilon(z)],\ |\theta_s| \le \frac{\alpha}{2}\ |\ \mathcal{F}_{T_\varepsilon(z)}\right] \le C\ \left(\frac{\varepsilon}{A}\right)^{\pi/\alpha}.$$

Let $\Theta_\varepsilon^{\alpha,A}$ denote the set of all ε-approximate two-sided cone points with angle α. The next lemma follows readily from (1) and (2).

<u>Lemma 2</u> : *There exists a constant* C_α *such that :*

$$P[z \in \Theta_\varepsilon^{\alpha,A}] \le C_\alpha\ |z|^{-\pi/\alpha}\ A^{-\pi/\alpha}\ \varepsilon^{2\pi/\alpha}.$$

As a simple consequence of Lemma 2, we get that for any compact subset K of $\mathbb{C} \setminus \{0\}$, for $\varepsilon \in (0,1)$,

$$E[m(K \cap \Theta_\varepsilon^{\alpha,A})] = \int_K dz\ P[z \in \Theta_\varepsilon^{\alpha,A}] \le C'_{\alpha,A,K}\ \varepsilon^{2\pi/\alpha}$$

so that, by Fatou's lemma,

(3) $$\liminf_{\varepsilon \to 0} \varepsilon^{-2\pi/\alpha}\, m(K \cap \Theta_\varepsilon^{\alpha,A}) < \infty, \text{ a.s.}$$

This fact will be the main ingredient in the proof of the following theorem.

Theorem 3 : *Let* Γ_α *denote the set of all two-sided cone points with angle* α. *Then, with probability* 1,

(i) *if* $\alpha \in (0,\pi)$, $\Gamma_\alpha = \emptyset$;

(ii) *if* $\alpha \in [\pi,2\pi)$, $\dim \Gamma_\alpha \le 2 - \frac{2\pi}{\alpha}$

($\dim\ \Gamma_\alpha$ *denotes the Hausdorff dimension of* Γ_α).

Remark : In case (ii), it can in fact be proved that $\dim \Gamma_\alpha = 2 - \frac{2\pi}{\alpha}$ (see Evans [Ev1]).

Proof : We set

$$\Theta^{\alpha,A} = \bigcap_{\varepsilon>0} \Theta_\varepsilon^{\alpha,A}$$

and

$$\Theta^\alpha = \bigcup_{A>0} \Theta^{\alpha,A}.$$

It is easy to check that $z \in \Theta^\alpha$ iff $z = B_t$ for some $t > 0$, and, for some $\delta > 0$, the curves $(B_s, 0 \le s \le t)$, $(B_{t+s}, 0 \le s \le \delta)$ lie inside the wedge

$$W_\alpha(z) = \{y = z - re^{iu}\ ;\ r \ge 0,\ |u| \le \tfrac{\alpha}{2}\}.$$

In particular, Θ^α is contained in Γ^α.

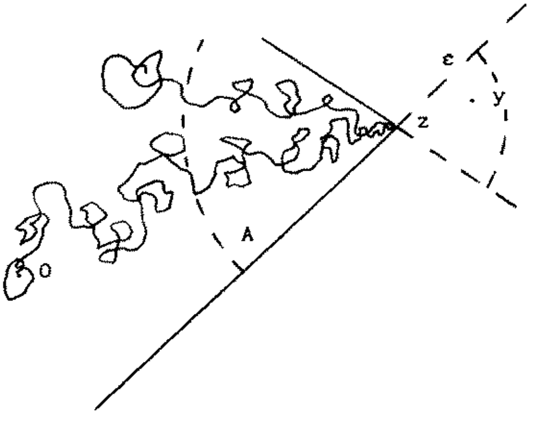


Fig. 3

Consider first the case $\alpha \in (0,\pi)$. We make use of the following simple observation. Fix $A > 0$, then for ε small, if $z \in \Theta^{\alpha,A}$, any point y of the form $y = z + re^{iu}$ with $0 \le r \le \varepsilon$, $|u| < \frac{\alpha}{2}$ belongs to $\Theta_{\varepsilon}^{\alpha,A/2}$ (see fig. 3).

It follows that for any compact subset K of $\mathbb{C} \setminus \{0\}$,

$$(4) \qquad m(\Theta_{\varepsilon}^{\alpha,A/2} \cap K_{\varepsilon}) \ge c_{\alpha}\, \varepsilon^2\, 1_{(\Theta^{\alpha,A} \cap K \ne \emptyset)}$$

for some $c_{\alpha} > 0$ (here K_{ε} denotes the ε-neighborhood of K). Since $2\pi/\alpha > 2$, (3) and (4) give

$$\Theta^{\alpha,A} \cap K = \emptyset \quad , \quad \text{a.s.}$$

Since this is true for any $A > 0$ and any compact subset K we conclude that

$$\Theta^{\alpha} = \emptyset \quad , \quad \text{a.s.}$$

It is then quite easy to show also that $\Gamma^{\alpha} = \emptyset$ a.s. First observe that we may replace B by any of the Brownian motions $B_t^{(p)} = B_{p+t} - B_p$, for all rational p. Then choose $\alpha' \in (\alpha,\pi)$ and notice that we may find a finite number of wedges with vertex 0 and angle α' such that any wedge with vertex 0 and angle α is contained in one of these. From the fact that $\Theta^{\alpha'} = \emptyset$ a.s. and the rotational invariance of Brownian motion it is easy to deduce that $\Gamma^{\alpha} = \emptyset$ a.s.

We now turn to the case $\alpha > \pi$ (we may forget about the case $\alpha = \pi$). It will be enough to show that for any $A > 0$,

$$\dim \Theta^{\alpha,A} \le 2 - \frac{2\pi}{\alpha} \quad \text{a.s.}$$

Indeed the previous arguments then show that, for any $\alpha' > \alpha$, Γ_{α} is contained in a countable union of sets of the type $\Theta^{\alpha',A}$, hence has dimension less than $2 - 2\pi/\alpha'$.

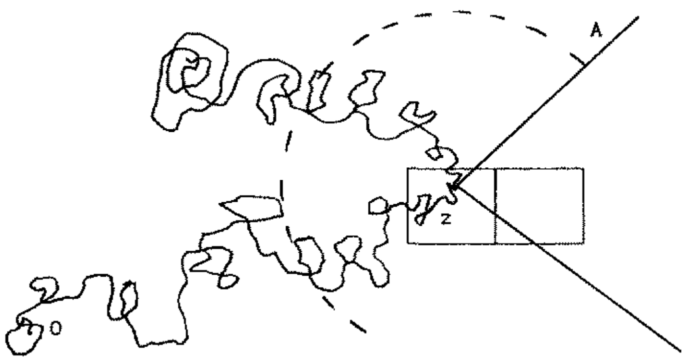


Fig. 4

Let K be a compact subset of $\mathbb{C} \setminus \{0\}$. For every $n \geq 1$, denote by $\mathscr{C}_n$ the collection of all squares $Q^n_{i,j} = [i\ 2^{-n}, (i+1)2^{-n}] \times [j\ 2^{-n}, (j+1)2^{-n}]$ for $i, j \in \mathbb{Z}$. Let

$$N_n = \sum_{Q \in \mathscr{C}_n} 1_{(Q \cap K \cap \Theta^{\alpha,A} \neq \emptyset)}$$

be the number of squares in $\mathscr{C}_n$ that intersect $K \cap \Theta^{\alpha,A}$. We observe that for n large, for every square $Q^n_{i,j}$ which intersects $\Theta^{\alpha,A}$, we may find a subset of $Q^n_{i+i,j}$ of measure larger than $C_\alpha 2^{-2n}$, which is contained in $\Theta^{\alpha,A/2}_{4 \cdot 2^{-n}}$ (here C_α is some positive constant depending on α). See fig. 4.

This shows that

$$C_\alpha\, 2^{-2n}\, N_n \leq m(\Theta^{\alpha,A/2}_{4\cdot 2^{-n}} \cap K_{4\cdot 2^{-n}}). \tag{5}$$

Then (3) and (5) imply :

$$\liminf_{n \to \infty} 2^{n(2\pi/\alpha - 2)}\, N_n < \infty\ , \quad \text{a.s.}$$

From the definition of Hausdorff measures we conclude that

$$\dim(\Theta^{\alpha,A} \cap K) \leq \frac{2\pi}{\alpha}\ , \quad \text{a.s.} \quad \square$$

3. Application to the convex hull of planar Brownian motion.

Let H be a compact convex subset of $\mathbb{C}$. We say that H has a corner at $z \in \partial H$ if H is contained in a wedge with vertex z and opening $\alpha < \pi$.

Theorem 4 : *Let* $t > 0$. *With probability* 1, *the convex hull of* $\{B_s, 0 \leq s \leq t\}$ *has no corners.*

Proof : Denote by H_t the convex hull of $\{B_s, 0 \leq s \leq t\}$. Spitzer's theorem implies that w.p. 1, B_0 and B_t belong to the interior of H_t. Suppose that H_t has a corner at z. It is then clear that z must belong to $\{B_s, 0 \leq s \leq t\}$, and therefore $z = B_s$ for some $s \in (0,t)$. But then z would be a two-sided cone point with angle $\alpha < \pi$, which contradicts Theorem 3. $\square$

Remark : We will see in the next chapter that, at certain exceptional times t, the convex hull of $\{B_s, 0 \leq s \leq t\}$ will have a corner at $z = B_t$. This fact is closely related to the existence of one-sided cone points with angle $\alpha < \pi$.

As a consequence of Theorem 4 we get that the boundary of the convex hull of $\{B_s, 0 \le s \le t\}$, parametrized by the argument, is with probability one a C^1-curve. We also get the following result.

Theorem 5 : *With probability one, the convex hull of* $\{B_s, 0 \le s \le t\}$ *has no isolated extreme points, and the set of all extreme points has dimension* 0.

Proof : Let H_t be as above. It is easy to check that any extreme point of H_t must belong to $\{B_s, 0 \le s \le t\}$ (this is true for the convex hull of any continuous curve). It follows that the set of extreme points is contained in the set of two-sided cone points with angle π, hence has dimension 0 by Theorem 3. Finally, if $z = B_s$ is an isolated extreme point, the set H_t must also be the convex hull of $\{z\} \cup (H_t \setminus D(z,\delta))$ for some $\delta > 0$ (use the Krein - Milman theorem). However this implies that H_t has a corner at z (otherwise z would not be extremal) and so the desired result follows from Theorem 4. □

4. The first intersection of a line with the Brownian path.

For any $y \in \mathbb{R}$ let D_y be the horizontal line $D_y = \{x + iy \ ; \ x \in \mathbb{R}\}$. Fix $t > 0$ and set $B[0,t] = \{B_s, 0 \le s \le t\}$, and

$$x(y) = \sup\{x \ ; \ x + iy \in B[0,t]\}$$

(by convention $\sup \varnothing = -\infty$).

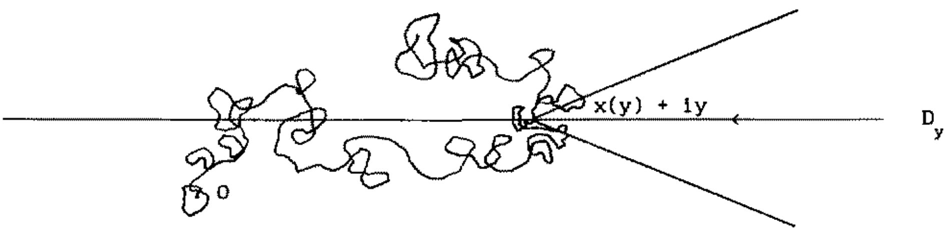


Fig. 5

If we imagine a particle coming from infinity along the line D_y, the point $x(y) + iy$ is the first hitting point of the Brownian path by this particle. One might expect this point to be a two-sided cone point. The next result shows that this is usually not the case.

Theorem 6 : *With probability one, for* dy-*almost all* $y \in \mathbb{R}$, *either* $x(y) = -\infty$ *or, for any* $\theta > 0$

$$\{x(y) + iy + r\,e^{iu} \ ; \ r > 0, \ |u| < \theta\} \cap B[0,t] \neq \varnothing .$$

Proof : Theorem 3 shows that for every $\theta > 0$

$$\dim \Gamma_{2\pi-\theta} < 1, \quad \text{a.s.}$$

Let p denote the projection $p(x + iy) = y$. It follows that

$$\dim p(\Gamma_{2\pi-\theta}) < 1 , \quad \text{a.s.}$$

and so

$$m(p(\Gamma_{2\pi-\theta})) = 0 , \quad \text{a.s.}$$

where m denotes Lebesgue measure on $\mathbb{R}$. Taking a sequence (θ_n) decreasing to 0, we get

$$m\left(\bigcup_{\theta>0} p(\Gamma_{2\pi-\theta})\right) = 0 , \quad \text{a.s.}$$

which gives the statement of Theorem 6. □

Remark : The previous proof shows that a statement analogous to Theorem 6 holds simultaneously for all directions, for (almost) all lines of the chosen direction.

The result of Theorem 6 can be stated in a slightly different form as follows. With probability one, for any $\theta > 0$

$$\{x(0) + re^{iu} ; r > 0, |u| < \theta\} \cap B[0,t] \neq \emptyset.$$

To check that this property holds, apply the Markov property at time $\delta > 0$ small, and use the fact that the law of B^2_δ is absolutely continuous w.r.t. Lebesgue measure.

Bibliographical notes. The non-existence of angular points on the convex hull of planar Brownian motion was already stated in Lévy [Lé4, p. 239-240], *but without a convincing proof. Detailed proofs were given by Adelman* [A1], *El Bachir* [EB] *and more recently by Cranston, Hsu and March* [CHM]. *The latter paper also discusses the smoothness of the boundary of the convex hull. Further results in this direction have been obtained by Burdzy and San Martin* [BSM]. *The approach taken here is inspired from* [L7], *although this paper deals with one-sided cone points. Theorem 5 is from Evans* [Ev1], *who has also obtained precise estimates on the Hausdorff dimension of cone points (Theorem 3 is only a very weak form of Evans' results). Finally, Burdzy* [B3] *contains many interesting results along the lines of Theorem 6 and Shimura* [Sh3] *treats a problem closely related to two-sided cone points with angle* π .

CHAPTER IV

One-sided cone points and a two-dimensional version of Lévy's theorem on the Brownian supremum process

1. A local time for one-sided cone points.

In this chapter, $B = (B_t, t \geq 0)$ is again a standard complex-valued Brownian motion started at 0. Let $\alpha \in (0,\pi]$. We shall be interested in a special class of one-sided cone points with angle α. We set

$$W_\alpha = \{z = r\, e^{i\theta} \; ; \; r \geq 0,\ |\theta| \leq \tfrac{\alpha}{2}\}.$$

Observe that W_α is convex since $\alpha \leq \pi$. Set

$$H_\alpha = \{t \geq 0 \; ; \; \forall s \leq t,\ B_t - B_s \in W_\alpha\}$$

$$\Delta^\alpha = \{B_t \; ; \; t \in H_\alpha\}$$

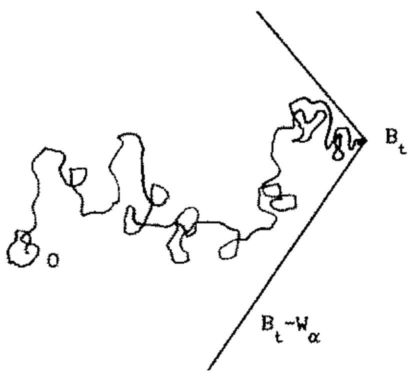


Fig. 1

Notice that $0 \in \Delta^\alpha$. According to the definitions of the previous chapter any $z \in \Delta^\alpha \setminus \{0\}$ is a one-sided cone point with angle α. This gives only a rather special class of one-sided cone points. However it is easy to see that much useful information (such as existence or non-existence, Hausdorff measure properties...) can be derived from the consideration of this special class.

We intend to show that Δ^α (or H_α) $\neq \{0\}$ if $\alpha > \pi/2$. To this end, we will construct a non-trivial measure supported on $\Delta^\alpha \setminus \{0\}$. This measure, the so-called local time of cone points, will also be extremely useful when investigating various properties of the cone points. The local time is constructed by approximation from the (suitably normalized) Lebesgue measure on a class of approximate cone points similar to the one used in Chapter III.

For $\varepsilon > 0$ we set

$$\Delta_\varepsilon^\alpha = \{z \in \mathbb{C} \ ; \ \forall s \le T_\varepsilon(z),\ z - B_s \in W_\alpha\}$$

where $T_\varepsilon(z) = \inf\{s \ ; \ |B_s - z| \le \varepsilon\}$.

Lemma 1 : *(i) For* $z \neq 0$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-\pi/\alpha} P[z \in \Delta_\varepsilon^\alpha] = h_\alpha(z),$$

where

$$h_\alpha(re^{i\theta}) = \begin{cases} \frac{4}{\pi} \cos(\frac{\pi\theta}{\alpha})\ r^{-\pi/\alpha} & \text{if} \quad \theta \in (-\frac{\alpha}{2}, \frac{\alpha}{2}) , \\ 0 & \text{if} \quad \theta \in [-\pi, \pi] \setminus (-\frac{\alpha}{2}, \frac{\alpha}{2}) . \end{cases}$$

The convergence is uniform when z *varies outside a neighborhood of* 0.

(ii) There exists a constant C_α *such that for any* $z \neq 0$, *for any* $\varepsilon \in (0,1]$,

$$\varepsilon^{-\pi/\alpha} P[z \in \Delta_\varepsilon^\alpha] \le C_\alpha\ |z|^{-\pi/\alpha}.$$

Proof : Clearly $P[z \in \Delta_\varepsilon^\alpha] = 0$ if $z \notin W_\alpha$. Suppose $z \in W_\alpha$ and let (θ_s) be the continuous determination of $\arg(z-B_s)$ such that $\theta_0 = \arg(z)$. In the previous chapter we have obtained the expansion, valid for $|z| > \varepsilon$,

$$P[z \in \Delta_\varepsilon^\alpha] = P[\forall\ s \le T_\varepsilon(z)\ ,\ |\theta_s| \le \alpha\ ]$$

$$= \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin\left(\frac{(2k+1)\pi(\frac{\alpha}{2} - \arg(z))}{\alpha}\right) \left(\frac{\varepsilon}{|z|}\right)^{(2k+1)\pi/\alpha} .$$

Both assertions of Lemma 1 are immediate consequences of this formula. □

We shall also need estimates for the probability that two or more given points belong to $\Delta_\varepsilon^\alpha$.

Lemma 2 : *(i) For* $z, z' \in \mathbb{C} \setminus \{0\}$, $z \neq z'$

$$\lim_{\varepsilon, \varepsilon' \to 0} (\varepsilon\varepsilon')^{-\pi/\alpha} P[z \in \Delta_\varepsilon^\alpha\ ,\ z' \in \Delta_{\varepsilon'}^\alpha] = h_\alpha(z)\ h_\alpha(z'-z) + h_\alpha(z')\ h_\alpha(z-z').$$

(ii) There exists a constant C'_α *such that for any* $n \geq 1$, $z_1,\dots,z_n$ *distinct points of* $\mathbb{C} \setminus \{0\}$ *and* $\varepsilon_1,\dots,\varepsilon_n \in (0,1]$,

$$(\varepsilon_1\dots\varepsilon_n)^{-\pi/\alpha}\, P[z_1 \in \Delta^\alpha_{\varepsilon_1},\dots,z_n \in \Delta^\alpha_{\varepsilon_n}] \leq (C'_\alpha)^n \sum_{\sigma\in\Sigma_n} \prod_{i=1}^{n} |z_{\sigma(i)} - z_{\sigma(i-1)}|^{-\pi/\alpha}.$$

Here Σ_n *denotes the set of all permutations of* $\{1,\dots,n\}$ *and for* $\sigma \in \Sigma_n$, $z_{\sigma(0)} = 0$ *by convention.*

Proof : (i) We may assume that $z \in W_\alpha$, $z'-z \in W_\alpha$, or $z' \in W_\alpha$, $z-z' \in W_\alpha$. Indeed, if not the case, $P[z \in \Delta^\alpha_\varepsilon, z' \in \Delta^\alpha_{\varepsilon'}]$ will be zero for ε,ε' small enough. Suppose $z \in W_\alpha$, $z'-z \in W_\alpha$. For ε,ε' small, the conditions $z \in \Delta^\alpha_\varepsilon$, $z' \in \Delta^\alpha_{\varepsilon'}$ force $T_\varepsilon(z) < T_{\varepsilon'}(z')$. Also, if $z \in \Delta^\alpha_\varepsilon$, we have automatically $B[0,T_\varepsilon(z)] \subset z - W_\alpha \subset z' - W_\alpha$ (because W_α is a convex cone !) and it is then enough to check that $B[T_\varepsilon(z),T_{\varepsilon'}(z')] \subset z' - W_\alpha$. The desired result follows from Lemma 1 (i) by using the Markov property at time $T_\varepsilon(z)$.

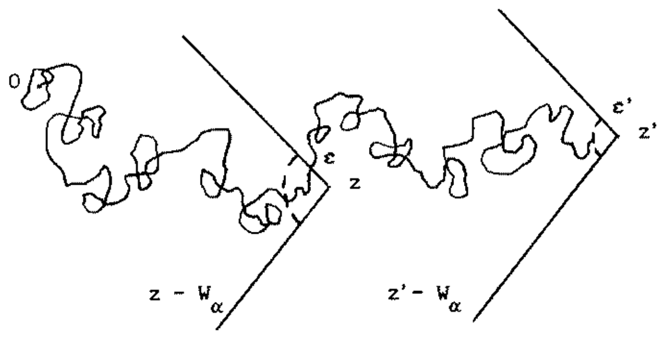


Fig. 2

(ii) We only treat the case $n = 2$. The idea is to deal separately with the cases $T_{\varepsilon_1}(z_1) \leq T_{\varepsilon_2}(z_2)$ and $T_{\varepsilon_1}(z_2) \leq T_{\varepsilon_1}(z_1)$. Suppose first that $|z_2 - z_1| \geq 2(\varepsilon_1 + \varepsilon_2)$. Then the Markov property at time $T_{\varepsilon_1}(z_1)$ and Lemma 1 (ii) give the bound

$$P[z_1 \in \Delta^\alpha_{\varepsilon_1},\ z_2 \in \Delta^\alpha_{\varepsilon_2},\ T_{\varepsilon_1}(z_1) \leq T_{\varepsilon_2}(z_2)] \leq P[z_1 \in \Delta^\alpha_{\varepsilon_1}]\, C_\alpha \left(\frac{|z_2-z_1|}{2}\right)^{-\pi/\alpha} (\varepsilon_2)^{\pi/\alpha}$$
$$\leq C_\alpha^2\, 2^{\pi/\alpha}\, (|z_1||z_2-z_1|)^{-\pi/\alpha}\, (\varepsilon_1\varepsilon_2)^{\pi/\alpha}.$$

If $|z_2 - z_1| \leq 2(\varepsilon_1 + \varepsilon_2)$ we can directly bound $P[z_1 \in \Delta^\alpha_{\varepsilon_1},\ z_2 \in \Delta^\alpha_{\varepsilon_2}]$. Supposing for instance $\varepsilon_1 \leq \varepsilon_2$ we write

$$P[z_1 \in \Delta^\alpha_{\varepsilon_1} \ , \ z_2 \in \Delta^\alpha_{\varepsilon_2}] \le P[z_1 \in \Delta^\alpha_{\varepsilon_1}] \le C_\alpha \ |z_1|^{-\pi/\alpha} \ \varepsilon_1^{\pi/\alpha}$$

$$\le C_\alpha \ 4^{\pi/\alpha} (|z_1||z_2 - z_1|)^{-\pi/\alpha} \ (\varepsilon_1 \varepsilon_2)^{\pi/\alpha}$$

since in this case $|z_2 - z_1| \le 4 \ \varepsilon_2$. □

<u>**Theorem 3**</u> : *Suppose* $\alpha \in (\pi/2,\pi]$. *With probability* 1 *there exists a (unique) Radon measure* μ_α *on* $\mathbb{C}$ *such that, for any compact subset* K *of* $\mathbb{C}$,

$$\mu_\alpha(K) = L^2 - \lim_{\varepsilon \to 0} \varepsilon^{-\pi/\alpha} \ m(\Delta^\alpha_\varepsilon \cap K).$$

Moreover, for any $\varepsilon, M > 0$, *there exists w.p.* 1 *a constant* $C_{\varepsilon,M}(\omega)$ *such that, for any square* $[u,u+r] \times [v,v+r]$ *contained in* $[-M,M]^2$,

(1) $$\mu_\alpha([u,u+r] \times [v,v+r]) \le C_{\varepsilon,M} \ r^{2-\frac{\pi}{\alpha}-\varepsilon} \ .$$

The measure μ_α *is w.p.* 1 *supported on* Δ^α. *Furthermore* $\mu_\alpha(D(0,\varepsilon)) > 0$ *for any* $\varepsilon > 0$, *a.s.*

<u>**Corollary 4**</u> : *If* $\alpha \in (\pi/2,\pi]$, $\Delta^\alpha \neq \{0\}$ *a.s. More precisely,* dim $\Delta^\alpha = 2 - \frac{\pi}{\alpha}$.

<u>Proof of Corollary 4</u> : Let $\alpha \in (\pi/2,\pi]$. Notice that $\mu_\alpha(\{0\}) = 0$ by **(1)**. Therefore μ_α is a non-trivial measure supported on $\Delta_\alpha \setminus \{0\}$, which implies $\Delta_\alpha \neq \{0\}$.

The upper bound on dim Δ_α follows from arguments exactly similar to those used in the proof of Theorem III.3. The key ingredient is now the fact that

$$\liminf_{\varepsilon \to 0} \varepsilon^{-\pi/\alpha} \ m(\Delta^\alpha_\varepsilon \cap K) < \infty \quad , \quad \text{a.s.}$$

The lower bound follows from **(1)**. Let (R_i) be a covering of $\Delta^\alpha \cap [-M,M]^2$ by squares contained in $[-M,M]^2$. Then, a.s.,

$$\sum_i (\text{diam}(R_i))^{2-\frac{\pi}{\alpha}-\varepsilon} \ge (C_{\varepsilon,M})^{-1} \sum_i \mu_\alpha(R_i) \ge (C_{\varepsilon,M})^{-1} \ \mu_\alpha(\Delta^\alpha \cap [-M,M]^2) \ ,$$

since μ_α is supported on Δ^α. Using the last assertion of Theorem 3 we get that dim $\Delta^\alpha \ge 2 - \frac{\pi}{\alpha} - \varepsilon$ a.s. □

<u>Remark</u> : Since $\Delta^\alpha = \{B_s, s \in H_\alpha\}$, a result of Kaufman [Ka] implies that

$$\dim H_\alpha = 1 - \frac{\pi}{2\alpha} \qquad \text{a.s.}$$

Proof of Theorem 3 : Set $\mu_{\alpha,\varepsilon}(K) = \varepsilon^{-\pi/\alpha}\, m(\Delta^{\alpha}_{\varepsilon} \cap K)$. Then,

$$E[\mu_{\alpha,\varepsilon}(K)\ \mu_{\alpha,\varepsilon'}(K)] = \int_{K\times K} dz\ dz'\ (\varepsilon\varepsilon')^{-\pi/\alpha}\ P[z \in \Delta^{\alpha}_{\varepsilon}\ , z' \in \Delta^{\alpha}_{\varepsilon'}].$$

Lemma 2 and the dominated convergence theorem imply that

$$\lim_{\varepsilon,\varepsilon'\to 0} E[\mu_{\alpha,\varepsilon}(K)\mu_{\alpha,\varepsilon'}(K)] = 2\int_{K\times K} dz\ dz'\ h_{\alpha}(z)h_{\alpha}(z'-z)$$

(notice that the function $|z|^{-\pi/\alpha}$ is locally integrable since $\alpha > \pi/2$). It follows that $(\mu_{\alpha,\varepsilon}(K))_{\varepsilon>0}$ is Cauchy in L^2, so that we may set :

$$\bar{\mu}_{\alpha}(K) = L^2\text{-}\lim_{\varepsilon\to 0} \mu_{\alpha,\varepsilon}(K).$$

Lemma 2 (ii) and Fatou's lemma give the bound

$$E[(\bar{\mu}_{\alpha}(K))^n] \le n!\,(C'_{\alpha})^n \int_{K^n} dz_1\ldots dz_n \prod_{i=1}^{n} |z_i - z_{i-1}|^{-\pi/\alpha} \le n!\,(C''_{\alpha})^n\ m(K)^{n(1-\pi/2\alpha)}$$

(notice that, if $m(K)$ is fixed, $\int_K |z - y|^{-\pi/\alpha}\, dz$ is maximal when K is a disk centered at y, with radius $\pi^{-1/2}\ m(K)^{1/2}$). This bound and the multidimensional version of the Kolmogorov lemma imply the existence of a continuous version of the mapping $(a,b,c,d) \longrightarrow \bar{\mu}_{\alpha}([a,b] \times [c,d])$ (for $a\le b$, $c\le d$). Denote by $\mu_{\alpha}([a,b] \times [c,d])$ this continuous version. Obviously $\mu_{\alpha}([a,b] \times [c,d])$ is a nondecreasing function of $[a,b] \times [c,d]$. Standard measure-theoretic arguments show that $\mu_{\alpha}(\cdot)$ can be extended to a Radon measure on $\mathbb{C}$. Furthermore, the monotone class theorem gives $\mu_{\alpha}(K) = \bar{\mu}_{\alpha}(K)$ a.s. for any compact K.

It remains to prove **(1)**. The previous bound on the moments of $\mu_{\alpha}(K)$ and the arguments of the proof of Theorem II-8 give **(1)** for any dyadic square contained in $[-M,M]^2$. A simple covering argument completes the proof of **(1)**.

Let us check that μ_{α} is a.s. supported on Δ . Let R be a compact rectangle with rational coordinates. We have

$$\mu_{\alpha}(R) \le \liminf_{\varepsilon \to 0} \mu_{\alpha,\varepsilon}(R) \qquad \text{a.s.}$$

Note that $\Delta^{\alpha} = \bigcap_{\varepsilon>0} \Delta^{\alpha}_{\varepsilon}$ and that every $\Delta^{\alpha}_{\varepsilon}$ is closed. It follows that, on $\{R \cap \Delta^{\alpha} = \emptyset\}$ we have for ε small $R \cap \Delta^{\alpha}_{\varepsilon} = \emptyset$ so that $\mu_{\alpha,\varepsilon}(R) = 0$ and $\mu_{\alpha}(R) = 0$.

Finally a scaling argument gives

$$P[\mu_{\alpha}(D(0,1)) > 0] = P[\mu_{\alpha}(D(0,\varepsilon)) > 0] = P[\forall\varepsilon > 0,\ \mu_{\alpha}(D(0,\varepsilon)) > 0].$$

However $P[\mu_{\alpha}(D(0,1) > 0] > 0$ since

$$E[\mu_\alpha(D(0,1))] = \lim_{\varepsilon\to 0} E[\mu_{\alpha,\varepsilon}(D(0,1))] = \int_{D(0,1)} h_\alpha(z)dz$$

by Lemma 1. It is easy to check from the construction of μ_α that the event $\{\forall\varepsilon > 0,\ \mu_\alpha(D(0,\varepsilon)) > 0\}$ is asymptotic. The 0-1 law then gives the desired result. □

Remark : For any $t \in H_\alpha \setminus \{0\}$, the convex hull of $\{B_s, 0 \le s \le t\}$ has a corner at B_t, with opening (less than) α. This comes in contrast to the fact (Theorem III-4) that for a fixed t, w.p. 1 the convex hull of $\{B_s, 0 \le s \le t\}$ has no corners.

2. A stable process embedded in two-dimensional Brownian motion.

At this point, we have proved that, for $\alpha > \pi/2$, $\Delta_\alpha \ne \{0\}$ so that in particular there exist one-sided cone points with angle α. We will prove in the next section that $\Delta_\alpha = \{0\}$ for $\alpha \le \frac{\pi}{2}$. In the present section we will use the local time constructed in Theorem 3 to get certain interesting probabilistic properties of the sets Δ_α and H_α.

Let $(\mathcal{F}_t)$ denote the canonical filtration of B. A random closed subset H of $\mathbb{R}_+$ is called $(\mathcal{F}_t)$-regenerative if $0 \in H$ and :

(i) $\forall t \ge 0$, $\{(s,\omega)\ ;\ s \le t,\ s \in H(\omega)\}$ is $\mathcal{B}_{[0,t]} \otimes \mathcal{F}_t$ measurable ($\mathcal{B}_{[0,t]}$ denotes the Borel σ-field on $[0,t]$)

(ii) For any $(\mathcal{F}_t)$ stopping time T such that $T \in H$ a.s., the set $\{(t-T)_+, t \in H\}$ is independent of $\mathcal{F}_T$ and distributed as H.

With every regenerative set H we can associate its local time process (ℓ_t), defined up to a multiplicative constant. The process (ℓ_t) is càdlàg, non decreasing and $(\mathcal{F}_t)$-adapted. It is characterized (up to a multiplicative constant) by the following two properties :

(i) $\ell_0 = 0$ and ℓ_t increases only on H.

(ii) For any stopping time T such that $T \in H$ a.s., the process $\ell_t^T = \ell_{T+t} - \ell_T$ is independent of $\mathcal{F}_T$ and distributed as (ℓ_t).

Theorem 5 : *Let $\alpha \in (\frac{\pi}{2},\pi]$. The set Δ^α is an $(\mathcal{F}_t)$-regenerative set. Its local time may be defined by :*

$$\ell_t^\alpha = \mu_\alpha(\{B_s, 0 \le s \le t\}).$$

Set

$$\tau_t^\alpha = \inf\{s,\ \ell_s^\alpha > t\} < \infty \qquad a.s.$$

The process (τ_t^α) *is a stable subordinator with index* $1 - \pi/2\alpha$. *The process* $(B(\tau_t^\alpha))$ *is a two-dimensional stable process with index* $2 - \pi/\alpha$. *In particular,* $(B^1(\tau_t^\alpha))$ *is a stable subordinator and* $(B^2(\tau_t^\alpha))$ *is a symmetric stable process. Finally,* H_α *coincides with the closure of the range of* τ^α *and* Δ^α *coincides with the closure of the range of* $B \circ \tau^\alpha$.

Before proving Theorem 5 let us discuss the limiting case $\alpha = \pi$. In this case it is easy to check that

$$H_\pi = \{t\ ;\ B_t^1 = \sup_{s\le t} B_s^1\},$$

so that H_π coincides with the zero set of the process $\sup_{s\le t} B_s^1 - B_t^1$, which by a famous theorem of Lévy is a (one-dimensional) reflecting Brownian motion. Therefore, H_π is distributed as the zero set of a linear Brownian motion, which is the typical example of a regenerative set. Moreover, ℓ_t^π $(= C \sup_{s\le t} B_s^1)$ is distributed as (C times) the local time process at 0 of a linear Brownian motion, so that τ_t^π is a stable subordinator with index 1/2. Finaly, $B^1(\tau_t^\pi) = C^{-1}t$ (the stable subordinator with index 1 !) and $B^2(\tau_t^\pi)$ is a symmetric Cauchy process. The latter fact was first discovered by Spitzer [Sp1] and has been used since by many authors.

In conclusion, when $\alpha = \pi$, the different assertions of Theorem 5 are well-know facts. It turns out that all of them carry over to the general case $\alpha \in (\frac{\pi}{2},\pi]$. It is interesting to note that the last assertions of Theorem 5 give a probabilistic description of the random sets H_α and Δ^α.

Proof of Theorem 5 : Let T be a stopping time such that $T \in H_\alpha$ a.s. Let $B^{(T)}$ denote the Brownian motion $B_t^{(T)} = B_{T+t} - B_T$ $(t \ge 0)$.

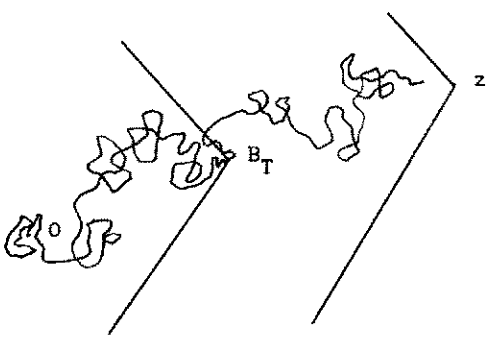


Fig. 3

Then $B^{(T)}$ is independent of $\mathcal{F}_T$. Moreover, a simple geometric argument shows that :

$$\{(t-T)_+ \ , t \in H_\alpha\} = H_\alpha^{(T)} \ ,$$

with an obvious notation. The fact that H_α is an $\mathcal{F}_t$-regenerative set follows at once.

Note that $\ell_t^\alpha = \mu_\alpha(B[0,t])$ increases only on H_α. Furthermore, the construction of μ_α easily gives :

$$\mu_\alpha(B[0,T+t]) = \mu_\alpha(B[0,T]) + \mu_\alpha(B[T,T+t]) = \mu_\alpha(B[0,T]) + \mu_\alpha^{(T)}(B^{(T)}[0,T]).$$

Therefore $(\ell_{T+t}^\alpha - \ell_T^\alpha)$ is independent of $\mathcal{F}_T$ and distributed as (ℓ_t^α). It follows that (ℓ_t^α) is a local time for H_α.

By the general theory of regenerative sets, (τ_t^α) is an $\mathcal{F}_{\tau_t^\alpha}$-subordinator (this also follows from the previous arguments) so that $(B(\tau_t^\alpha))$ is also an $\mathcal{F}_{\tau_t^\alpha}$-Lévy process. Next, fix $\lambda > 0$ and set

$$\tilde{B}_t = \lambda \, B_{t/\lambda^2} \, .$$

Then, for any $\varepsilon > 0$,

$$\tilde{\Delta}_\varepsilon^\alpha = \lambda \, \Delta_{\varepsilon/\lambda}^\alpha \ ,$$

and after some easy manipulations,

$$\tilde{\tau}_t^\alpha = \lambda^2 \, \tau_{t/\lambda^{2-\pi/\alpha}}^\alpha \ , \qquad \tilde{B}(\tilde{\tau}_t^\alpha) = \lambda \, B(\tau_{t/\lambda^{2-\pi/\alpha}}^\alpha).$$

It follows that τ^α is stable with index $1 - \pi/2\alpha$ and $B \circ \tau^\alpha$ is stable with index $2 - \pi/\alpha$.

Geometric considerations entail that $B^1 \circ \tau^\alpha$ is a subordinator and $B^2 \circ \tau^\alpha$ is symmetric. Finally, the general theory of regenerative sets shows that H_α is the closure of the range of τ^α. □

3. A two-dimensional version of Lévy's theorem on the Brownian supremum process.

Let $X = (X_t, t \geq 0)$ be a standard linear Brownian motion started at 0 and $S_t = \sup_{s \leq t} X_s$. A theorem of Lévy states that the process $S - X$ is a (one-dimensional) reflecting Brownian motion, i.e. is distributed as $|X|$. As we noticed in the previous section, this theorem is closely related to the structure of H_π, which coincides with the zero set of $\sup_{s \leq t} B_s^1 - B_t^1$. We will now prove that for any $\alpha \in (\pi/2, \pi)$, H can also be interpreted as the zero set of a two-dimensional reflecting Brownian motion in the wedge W_α. This result is related to a two-dimensional version of Lévy's theorem.

We first recall a few basic facts about reflecting Brownian motion in a wedge.

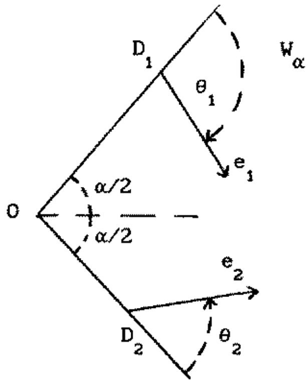


Fig. 4

We set $D_1 = \{r\ e^{i\alpha/2}\ ;\ r \ge 0\}$, $D_2 = \{r\ e^{-i\alpha/2}\ ;\ r \ge 0\}$. Let $\theta_1, \theta_2 \in (0,\pi)$, and

$$e_1 = e^{i(\alpha/2-\theta_1)}\ ,\ e_2 = e^{-i(\alpha/2-\theta_2)}.$$

A process $Z = (Z_t\ ;\ t \ge 0)$ with valued in W_α is called reflecting Brownian motion with angles of reflection θ_1, θ_2 if :

$$Z_t = Y_t + A^1_t\ e_1 + A^2_t\ e_2$$

where

- Y is a two-dimensional Brownian motion
- A^1, A^2 are two continuous non-decreasing processes adapted to the filtration of Y, and A^1 (resp. A^2) increases only when $Z_t \in D_1$ (resp. $Z_t \in D_2$).

This is not the most general presentation of reflecting Brownian motion in a wedge. It will however be sufficient to our purposes. Notice that it is far from obvious (and in fact not true) that such a process Z exists for all values of θ_1, θ_2. Assuming that Z exists, it can be proved that $\{t,\ Z_t = 0\}$ contains non-zero times iff $\theta_1 + \theta_2 > \pi$. To check the sufficiency of this condition, one introduces the function: For $r > 0$, and $|\theta| \le \alpha/2$,

$$\psi(re^{i\theta}) = r^\xi \sin(\ \xi\theta + \frac{\theta_1-\theta_2}{2}\ )\quad ,\quad \text{where}\quad \xi = \frac{\theta_1+\theta_2-\pi}{\alpha} > 0\ .$$

An application of Itô's formula shows that $\psi(Z_t)$ is a local martingale on the time interval $[0,\tau)$, where $\tau = \inf\{\ s\ ;\ Z_s = 0\ \}$. The proof can then be completed by standard arguments (see [VW] for details).

If K is a compact subset of $\mathbb{C}$, the intersection of all cones of the type $z - W_\alpha$ that contain K is again a cone of the same type, which is the smallest one that contains K.

Theorem 6 : *Let* $\alpha \in (0,\pi)$. *For every* $t \geq 0$, *let* S_t *be the vertex of the smallest cone of the type* $z - W_\alpha$ *that contains* $B[0,t]$. *The process* $S - B$ *is a reflecting Brownian motion in* W_α *with angles of reflection* $\theta_1 = \theta_2 = \alpha$.

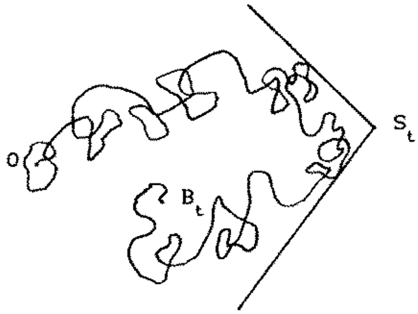


Fig. 5

Corolllary 7 : $\Delta_\alpha \neq \{0\}$ *iff* $\alpha > \pi/2$.

Proof of Corollary 7 : Note that H_α is exactly the zero set of $S-B$. Then we may apply the previous criterion observing that $\theta_1 + \theta_2 = 2\alpha > \pi$ iff $\alpha > \pi/2$. In fact, we do not need this criterion. The case $\alpha > \pi/2$ was treated in Corollary 4. Then it suffices to check that $\Delta_{\pi/2} = \{0\}$. However when $\alpha = \pi/2$, W_α is a quadrant and the directions of reflection are normal. It follows that $S_t - B_t = |\beta_t| \, e^{i\pi/4} + |\gamma_t| \, e^{-i\pi/4}$, where β, γ are two independent linear Brownian motions. By Corollary II-2, $\{t \; ; \; S_t - B_t = 0\} = \{0\}$. □

Proof of Theorem 6 : Set $f_1 = e^{i\alpha/2}$, $f_2 = e^{-i\alpha/2}$. We have :

$$B_t = U_t \, f_1 + V_t \, f_2 \, ,$$

where U, V are two (correlated) linear Brownian motions. It is easy to check that

$$S_t = \hat{U}_t \, f_1 + \hat{V}_t \, f_2 \, ,$$

where $\hat{U}_t = \sup_{s \leq t} U_s$, $\hat{V}_t = \sup_{s \leq t} V_s$. Then,

$$S_t - B_t = -B_t + \hat{U}_t \, f_1 + \hat{V}_t \, f_2 .$$

Now notice that $\hat{U}_t$ increases only when $U_t = \hat{U}_t$ that is when $S_t - B_t \in D_2$, and similarly for $\hat{V}_t$. This gives the desired representation with $e_1 = f_2$, $e = f$, hence $\theta = \theta = \alpha$. □

If we combine Theorem 6 and Theorem 5 we get that, for a certain class of reflecting Brownian motions in a wedge, the zero set is exactly the closure of the range of a stable subordinator. This result in fact holds in great generality (see Williams [Wi3]).

As a by-product of the previous statements, we get the following result. Suppose that β, γ are two linear Brownian motions started at 0, correlated in the sense that $\langle\beta,\gamma\rangle_t = \rho\, t$, for some constant ρ. If $\rho > 0$, the set $\{ t \geq 0,\ \beta_t = \sup_{s\leq t} \beta_s$ and $\gamma_t = \sup_{s\leq t} \gamma_s \}$ is non-empty and is distributed as the range of a stable subordinator. If $\rho \leq 0$, this set is empty.

Using the arguments of the proof of Theorem III.3 it is easy to deduce from Corollary 7 that there are no one-sided cone points with angle $\alpha < \pi/2$. The problem of the existence of one-sided cone points with angle $\pi/2$ remains open.

4. More about the first intersection of a line with the Brownian path.

At the end of Chapter III we obtained the following result. For $t \geq 0$ set

$$x_t = x_t(0) = \sup(B[0,t] \cap \mathbb{R}).$$

Then, for a fixed $t > 0$, with probability 1 for any $\beta > 0$,

$$(x_t + \mathring{W}_\beta) \cap B[0,t] \neq \emptyset \tag{2}$$

($\mathring{W}_\beta$ denotes the interior of W_β).

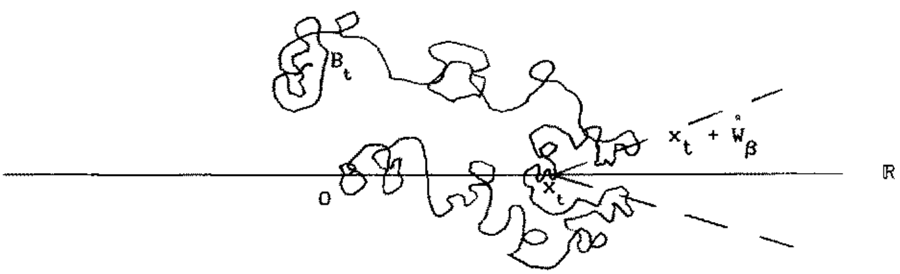


Fig. 6

We will now show that this property fails to hold at certain exceptional times t : these exceptional times will be such that $B_t \in \mathbb{R}$ and B_t is a one-sided cone point with angle $\alpha \in (\pi,2\pi)$.

Fix $\alpha \in (0,2\pi)$. If $\beta = 2\pi-\alpha$, property (2) is equivalent to the fact that $B[0,t]$ is not contained in $x_t - W_\alpha$.

For any $t \geq 0$ denote by $R_t \in \mathbb{R}$ the vertex of the smallest cone of the type $r - W$ $(r \in \mathbb{R})$ that contains $B[0,t]$.

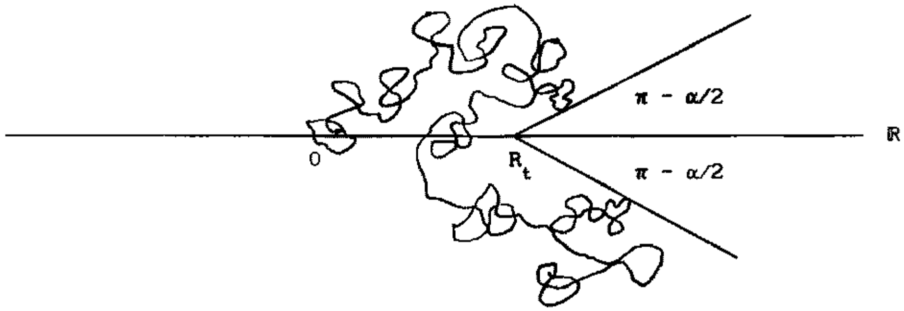


Fig. 7

Theorem 8 : *The process* $R_t - B_t$ *is a reflecting Brownian motion in the wedge* W_α *with angles of reflection* $\theta_1 = \theta_2 = \alpha/2$ *(equivalently* $e_1 = e_2 = 1$*). In particular, the zero set* of $R - B$ *contains non-zero* times iff $\alpha > \pi$.

Observe that if $R_t - B_t = 0$ for $t > 0$ then obviously $x_t = R_t$ and $B[0,t] \subset x_t - W_\alpha$, so that property (2) does not hold for $\beta \le 2\pi - \alpha$.

Proof of Theorem 8 : The proof is similar to that of Theorem 6. It is easy to check that $R_t = \sup_{s\le t} (B^1_s + |B^2_s| \operatorname{cotg} \alpha/2)$ where $\operatorname{cotg} x = \frac{\cos x}{\sin x}$. Next observe that R_t increases only when

$$B^1_t + |B^2_t| \operatorname{cotg} \alpha/2 = \sup_{s\le t} (B^1_s + |B^2_s| \operatorname{cotg} \alpha/2)$$

and this condition is clearly equivalent to $R_t - B_t \in D_1 \cup D_2$.

The last assertion follows from the general criterion given in Section 3 (when $\alpha = \pi$, the given result is equivalent to the polarity of simple points for the symmetric Cauchy process : recall Spitzer's construction of the Cauchy process...). □

It is again possible to avoid the use of the general criterion. One possibility is to contruct the local time of the set $\{t, R_t = B_t\}$ in a way similar to what we did in Section 1. The analogues of the sets $\Delta^\alpha_\varepsilon$ are then subsets of $\mathbb{R}_+$ and the key technical ingredient is the fact that the function $|x|^{-\pi/\alpha}$ is locally integrable on $\mathbb{R}$ if $\alpha > \pi$.

Still another method would be to extend Corollary 4 to the case $\alpha > \pi$ (this can be done but is non-trivial). We get that $\dim \Delta^\alpha > 1$ if $\alpha > \pi$. Then some Hausdorff measure arguments show that $\Delta^\alpha - \{0\}$ must intersect any fixed horizontal line with positive probability. Finally the zero-one law entails that $\Delta^{\alpha} - \{0\}$ intersects $\mathbf{R}$ w.p. 1.

Bibliographical Notes. *One-sided cone points with angle less than* π *were discovered simultaneously by Burdzy* [B1] *and Shimura* [Sh2] *(see also* [Sh1] *for a related work). A very simple proof of their existence has been given by Adelman* [A2]. *The approach developed in this chapter follows closely* [L7], *with the important simplification that we deal only with the case* $\alpha < \pi$. *This aproach is certainly not the shortest one, but it leads to the local time of cone points, which plays an important role in many applications. In particular, the local time allows one to understand how the process behaves just before arriving at a cone point (see* [L7] *and also* [B3] *for certain related results). Sharp results about the Hausdorff measure of cone points are given in Evans* [Ev1]. *The construction of the symmetric Cauchy process recalled in Section 2 was given by Spitzer* [S1]. *The idea of Theorems 6 and 8 was discovered independently in El Bachir* [EB] *and in* [L7]. *However, it seems that this idea was incorrectly applied in* [EB], *where the oblique reflection property of the process* $S - B$ *was unnoticed. See also Burdzy* [B3] *for applications of this idea and for many results related to Theorem 8. Information about reflected Brownian motion in a wedge may be found in Varadhan and Williams* [VW] *and in Williams* [Wi1], [Wi2]. *The fact that the inverse local time at the vertex is a stable subordinator is proved in great generality in Williams* [Wi3].

CHAPTER V

Burdzy's theorem on twist points.

1. Twist points of the planar Brownian motion.

We consider a standard complex-valued Brownian motion $(B_t, t \geq 0)$ started at 0. We denote by F the unbounded connected component of $\mathbb{C} \setminus B[0,1]$ ($B[0,1] = \{B_s, 0 \leq s \leq 1\}$). Then ∂F consists of all points of $B[0,1]$ that can be reached from the "exterior" of $B[0,1]$ along a continuous curve. In other words, $z \in B[0,1]$ is in ∂F iff there exists a continuous function $\varphi : [0,1] \longrightarrow \mathbb{C}$ such that :

$$(1) \quad \varphi(s) \in F \quad , \quad \forall s \in [0,1) ,$$

$$(2) \quad \varphi(1) = z .$$

Let $z \in \partial F$. We say that z is a twist point of ∂F if there exists a continuous function φ satisfying (1) and (2) and such that:

$$\limsup_{s\to 1, s<1} \arg(\varphi(s) - z) = +\infty ,$$

$$\liminf_{s\to 1, s<1} \arg(\varphi(s) - z) = -\infty .$$

Here and in what follows $\arg(\varphi(s)-z)$ denotes a continuous determination of the argument of $\varphi(s) - z$. Fig. 1 gives a very crude idea of the shape of the boundary near a twist point.

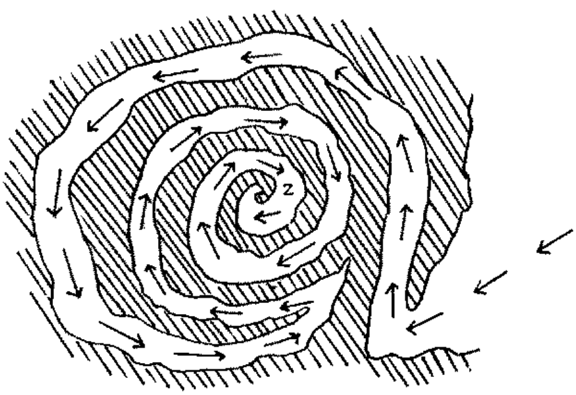


Fig. 1

It is a simple exercise to check that any point of ∂F that is also a two-sided cone point is not a twist point. Two-sided cone points form a dense subset of ∂F (indeed this is true for two-sided cone points with angle π). Nonetheless the next theorem shows that, in a sense, most of the points of ∂F are twist points.

Theorem 1 : *With probability 1, in the sense of harmonic measure almost all points of* ∂F *are twist points.*

We can rephrase Theorem 1 as follows. Let B' be another complex Brownian motion, independent of B and started at $z_1 \neq 0$. Let

$$T = \inf\{ t \geq 0,\ B'_t \in B[0,1] \}.$$

Then the point B'_T is w.p. 1 a twist point of ∂F.

The proof of Theorem 1 uses the following three ingredients.

(a) A theorem of McMilan in complex analysis.

(b) Certain estimates on harmonic measure.

(c) The bounds on the Hausdorff dimension of two-sided cone points derived in Chapter III.

2. Some results in complex analysis.

It will be convenient to work on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$. Then, $\hat{F} := F \cup \{ \infty \}$ is a simply connected open subset of $\hat{\mathbb{C}}$. By the Riemann mapping theorem, we may find a one-to-one analytic mapping f from the open unit disk D onto $\hat{F}$. By Fatou's theorem, for $d\theta$ a.a. $\theta \in [0,2\pi]$, the radial limit

$$\lim_{\substack{r\to 1\\ r<1}} f(re^{i\theta})$$

exists. This limit is simply denoted by $f(e^{i\theta})$.

In our situation, $\mathbb{C} \setminus F$ is locally connected and it can be shown (see Pommerenke [Po, Chapter IX]) that the radial limit exists for every $\theta \in [0,2\pi)$, and that the extended mapping $f : \bar{D} \longrightarrow \hat{F} \cup \partial F$ is continuous and onto. Notice that this extended mapping needs not be one-to-one (in fact, in the present setting, f will not be one-to-one : it can be shown that $f_{|\partial D}$ is one-to-one iff ∂F has no cut points, and two-dimensional Brownian paths do have cut points, as was recently shown by Burdzy).

For any $\zeta \in \partial D$ and $r \in (0,1)$, we define the Stolz angle $S(\zeta,r)$ as the interior of the convex hull of $\{\zeta\} \cup D(0,r)$. We say that f has angular derivative ω at ζ if, for any $r \in (0,1)$,

$$\lim_{\substack{z \to \zeta \\ z \in S(\zeta,r)}} \frac{f(z) - f(\zeta)}{z - \zeta} = \omega$$

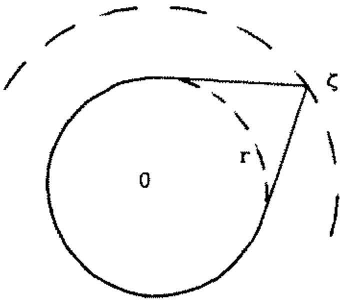


Fig. 2

Finally, we say that $\zeta \in \partial D$ is an f-twist point if $\arg(f(z) - f(\zeta))$ is unbounded above and below along every curve in D ending at ζ . Clearly, if f has a nonzero angular derivative at ζ , ζ cannot be an f-twist point.

The following theorem due to McMillan [MM] plays a basic role in the proof of Theorem 1.

Theorem. *For a.a. $\zeta \in \partial D$, either f has a non-zero angular derivative at ζ or ζ is an f-twist point.*

Let us turn to the proof of Theorem 1 , using McMillan's theorem. We denote by T_f the set of all f-twist points and by A_f the set of all points $\zeta \in \partial D$ such that f has a non-zero angular derivative at ζ . We also denote by $T_{\partial F}$ the set of all twist points of ∂F . We observe that:

$$f^{-1}(\partial F \setminus T_{\partial F}) \subset (\partial D \setminus T_f) \quad .$$

Indeed, let $\zeta \in \partial D$ be such that $f(\zeta)$ is not a twist point of ∂F . Then, for any curve $(\varphi(t),\ 0 \le t < 1)$ in D ending at ζ , $(f(\varphi(t)),\ 0 \le t < 1)$ is a curve in F ending at $f(\zeta)$, so that $\arg(f(\varphi(t)) - f(\zeta))$ must be bounded above or below.

Using the conformal invariance of harmonic measure (see Section II-2), it is then enough to check that $\partial D \setminus T_f$ has Lebesgue measure 0 . However, McMillan's theorem states that $\partial D \setminus (T_f \cup A_f)$ has measure zero. To complete the proof, it suffices to prove that A_f has measure 0 , or, by the conformal invariance of harmonic measure again, that $f(A_f)$ is contained in a set of harmonic measure zero. We need the following elementary lemma.

Lemma 2 : *Suppose that* $z = f(\zeta)$ *for some* $\zeta \in A_f$. *Then, for any* $\alpha < \pi$, *there exist* $\varepsilon > 0$ *and an open wedge* W_α *with vertex* z *and angle* α *such that* $(W_\alpha \cap D(z,\varepsilon)) \subset F$.

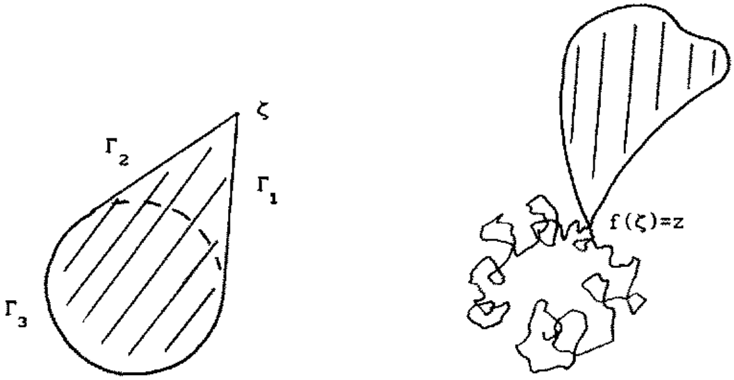


Fig. 3

Proof: Fix $r \in (0,1)$ and denote by $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ the boundary of the Stolz angle $S(\zeta,r)$, as on Figure 3. Then $f(S(\zeta,r))$ is a simply connected subset of F with boundary $f(\Gamma_1) \cup f(\Gamma_2) \cup f(\Gamma_3)$. The fact that f has a non-zero angular derivative at ζ implies that $f(S(\zeta,r))$ contains $D(z,\varepsilon_r) \cap W_{\alpha_r}$ for some $\varepsilon_r > 0$ and some open wedge W_{α_r} with vertex z and angle α_r. Moreover by choosing r close to one, we can get α_r as close to π as desired. □

It follows from Lemma 2 that $f(A_f)$ is contained in the set of two-sided cone points with angle β of the Brownian path B, for any $\beta < \pi$. The results of Chapter III give

$$\dim f(A_f) = 0.$$

To complete the proof of Theorem 1, it suffices to prove that, for any subset H of ∂F such that $\dim H = 0$, the harmonic measure of H is zero. This follows from Makarov's theorem, which states (in particular) that the harmonic measure of H is 0 as soon as $\dim H < 1$. Clearly, we do not need the full strength of Makarov's theorem. In the next section, we will give a probabilistic proof of a much weaker statement, which nonetheless suffices to complete the proof of Theorem 1.

3. An estimate for harmonic measure

The results of this section apply to any simply connected open set $F \subset \hat{\mathbb{C}}$ such that $\hat{\mathbb{C}} \setminus F$ contains more than one point. We fix $z_0 \in F$ and we let $\mu = \mu_{z_0}$ be the associated harmonic measure on ∂F. In probabilistic terms

$$\mu(A) = P_{z_0}[B_T \in A]$$

where $T = \inf\{ t \geq 0 , B_t \notin F \}$.

Proposition 3: *There exists* $\alpha > 0$ *such that* $\mu(H) = 0$ *as soon as* $\dim H < \alpha$.

Proof : Without loss of generality we may assume that $\infty \in F$, $\infty > d(z_0, \partial F) > 1$ and $\operatorname{diam}(\partial F) > 2$. Let $z \in \partial F$. We write P for P_{z_0} and we first look for a bound on :

$$P[B_T \in D(z,\varepsilon)]$$

Set

$$T_\varepsilon = T_\varepsilon(z) = \inf\{ t \geq 0 \; ; \; |B_t - z| \leq \varepsilon\}$$

and

$$L_1 = \sup\{ t \leq T_\varepsilon \; ; \; |B_t - z| = 1\}$$

We claim that, on $\{ B_T \in D(z,\varepsilon)\}$, z belongs to the unbounded component of $\mathbb{C} \setminus B[L_1, T_\varepsilon]$. Indeed, if this were not the case, the component of z would be contained in $D(z,1)$, and so would be the connected set $\mathbb{C} \setminus F$ (which is contained in $\mathbb{C} \setminus B[L_1, T_\varepsilon]$ on $\{ B_T \in D(z,\varepsilon)\}$). This gives a contradiction since we have assumed $\operatorname{diam}(\partial F) > 2$.

For every integer $m \geq 1$, set

$$T_{(m)} = \inf\{ t \geq 0 \; ; \; |B_t - z| \leq 2^{-m} \}$$

$$L_{(m)} = \sup\{ t < T_{(m)} \; ; \; |B_t - z| = 2^{-m+1} \}$$

By the previous arguments,

$$P[B_T \in D(z, 2^{-m})] \leq P[\bigcap_{k=1}^{m} A_k]$$

where

$$A_k = \{ z \text{ belongs to the unbounded component of } \mathbb{C} \setminus B[L_{(k)}, T_{(k)}]\}.$$

However, the strong Markov property implies that the events A_k, $k = 1,2,\ldots$ are independent, and a scaling argument shows that they have the same probability $c < 1$ (use the skew-product representation to check that $c < 1$). Therefore,

$$P[B_T \in D(z, 2^{-m})] \leq c^m$$

and also for $\varepsilon \in (0,1/2)$,

$$P[B_T \in D(z,\varepsilon)] \le \varepsilon^a ,$$

for some constant $a > 0$.

It is then easy to check that Proposition 3 holds with $\alpha = a$. Indeed if $\dim H < a$ we may find a covering of H by disks $D(z_i,\varepsilon_i)$ with $z_i \in H$, $\varepsilon_i \in (0,1/2)$, in such a way that

$$\sum_i (\varepsilon_i)^a \le \delta$$

where δ is any fixed positive number. Then

$$P_{z_0}[B_T \in H] \le \sum_i P_{z_0}[B_T \in D(z_i,\varepsilon_i)] \le \sum_i (\varepsilon_i)^a \le \delta$$

and so $P_{z_0}[B_T \in H] = 0$, since δ was arbitrary. □

Bibliographical notes. Theorem 1 is due to Burdzy [B3] . *The idea of this result was already present, in a heuristic form, in Lévy* [Lé, p.239] *(see Chapter I). Our proof is somewhat different from Burdzy's one and perhaps simpler. The needed results of complex analysis, including the proof of McMillan's theorem, may be found in Pommerenke* [Po]. *Burdzy* [B4] *proves the existence of cut points on two-dimensional Brownian paths. Proposition 3 is a first step towards a probabilistic proof of Makarov's theorem* [Ma]. *K. Burdzy has pointed out that his recent work with G.F. Lawler* [BL1,BL2] *allows one to prove Proposition 3 with* $\alpha = 1/\pi^2$. *See also Bishop* [Bi] *for some recent related work. An interesting problem is to determine the Hausdorff dimension of* ∂F *(in the notation of Section 1). Mandelbrot has conjectured that the dimension of* ∂F *is* $4/3$. *See Burdzy and Lawler* [BL2] *for some recent progress on this problem.*

CHAPTER VI

Asymptotics for the Wiener sausage.

1. The definition of the Wiener sausage.

In this chapter, B is a Brownian motion in $\mathbb{R}^d$. As usual we make the convention that B starts from y under the probability P_y, and we write P for P_0.

Definition : *Let* K *be a compact subset of* $\mathbb{R}^d$ *and* $a, b \in \mathbb{R}_+$, $a \leq b$. *The Wiener sausage* $S_K(a,b)$ *is defined by*

$$S_K(a,b) = \{y \in \mathbb{R}^d \ ; \ y - B_s \in K \ \textit{for some} \ s \in [a,b]\} = \bigcup_{a\leq s\leq b} (B_s + K)$$

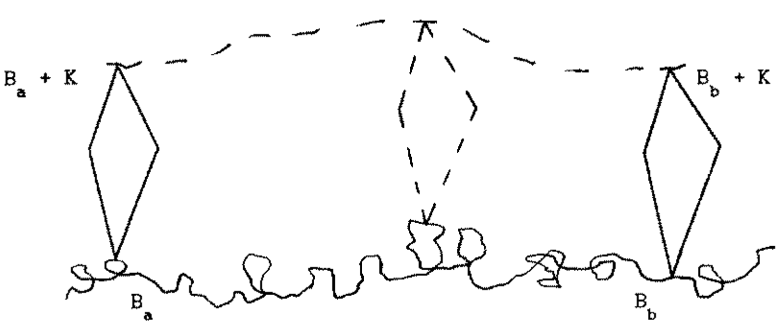


Fig. 1

When K is a closed ball centered at 0, $S_K(a,b)$ is a tubular neighborhood of $B[a,b]$.

We shall be interested in the following two problems:

(i) What is the asymptotic behavior of $m(S_K(0,t))$ as $t \to \infty$?

(ii) What is the asymptotic behavior of $m(S_{\varepsilon K}(0,1))$ as $\varepsilon \to 0$?

Notice that a scaling transformation gives

$$m(S_K(0,t)) \overset{(d)}{=} t^{d/2}\, m(S_{t^{-1/2}K}(0,1))$$

so that, up to some extent, questions (i) and (ii) are equivalent.

Let us briefly discuss question (i). The process $m(S_K(0,t))$ is sub-additive, meaning that

$$m(S_K(0,t+s)) \le m(S_K(0,t)) + m(S_K(0,s)) \circ \theta_t$$

where θ_t is the usual shift on Brownian paths. This property is obvious since $m(S_K(0,s)) \circ \theta_t = m(S_K(t,t+s))$ and $S_K(0,t+s) = S_K(0,t) \cup S_K(t,t+s)$. Then Kingman's subadditive ergodic theorem gives :

$$\frac{1}{t}\, m(S_K(0,t)) \xrightarrow{a.s.,L^1} C_K \tag{1}$$

for some constant $C_K \ge 0$. If $d \ge 3$, C_K can be identified as the Newtonian capacity of K. However, if $d = 1$ or 2 ($d = 2$ is the only interesting case) $C_K = 0$ for any compact set K, so that (1) does not give much information on the limiting behavior of $m(S_K(0,t))$.

In this chapter we will put the emphasis on question (ii). Our approach is independent of Kingman's theorem and applies as well to any dimension $d \ge 2$. Furthermore, it may be extended to diffusion processes more general than Brownian motion.

For simplicity we write $S_{\varepsilon K} = S_{\varepsilon K}(0,1)$. Our approach consists of two steps of independent interest:

1. Estimation of the mean value $E[m(S_{\varepsilon K})]$.
2. Bounds on the fluctuations of $m(S_{\varepsilon K})$.

The proofs make use of certain results of probabilistic potential theory that are recalled in the next section.

2. Potential-theoretic preliminaries.

Let ζ denote an exponential time with parameter $\lambda > 0$, independent of B. It will be convenient to work with the process B killed at time ζ, which is a symmetric Markov process with Green function :

$$G_\lambda(x,y) = G_\lambda(y-x) = \int_0^\infty ds\; e^{-\lambda s}\, p_s(x,y)$$

where $p_s(x,y) = (2\pi s)^{-d/2} \exp - |y-x|^2/2s$. It is easily checked that :

- if $d \ge 3$,

$$G_\lambda(x,y) \underset{|y-x|\to 0}{\sim} G_0(x,y) = C_d |y-x|^{2-d} ; \tag{2}$$

- if $d = 2$,

$$G_\lambda(x,y) \underset{|y-x|\to 0}{\sim} \frac{1}{\pi} \log \frac{1}{|y-x|} . \tag{3}$$

Let K be a compact subset of $\mathbb{R}^d$. Assume that K is non-polar and set

$$T_K = \inf\{t \; ; \; B_t \in K\} \le +\infty .$$

A basic formula of probabilistic potential theory gives the hitting probability of K for the process B killed at time ζ. For any $y \in \mathbb{R}^d \setminus K$,

$$P_y(T_K < \zeta) = \int_K G_\lambda(y,z)\, \mu_K^\lambda(dz) \tag{4}$$

where μ_K^λ is a finite measure supported on K, the λ-equilibrium measure of K. The total mass of μ_K^λ is denoted by $C_\lambda(K)$ and called the λ-capacity of K. The fact that K is non-polar is equivalent to $C_\lambda(K) > 0$ for some (or for any) $\lambda > 0$. Finally,

$$C_\lambda(K) = \left(\inf_{\mu\in\mathcal{P}(K)} \int \mu(dy)\, \mu(dz)\, G_\lambda(y,z)\right)^{-1} \tag{5}$$

where $\mathcal{P}(K)$ denotes the set of all probability measures supported on K.

The previous results also hold for $\lambda = 0$, i.e. $\zeta \equiv +\infty$, when $d \ge 3$. The quantity $C_0(K)$ is the Newtonian capacity of K.

We now observe that $y \in S_{\varepsilon K}$ iff $T_{y-\varepsilon K} \le 1$. It will therefore be important to get information on the distribution function of $T_{y-\varepsilon K}$.

Lemma 1 : *Suppose that* K *is non-polar.*

(i) If $d \ge 3$,

$$C_\lambda(\varepsilon K) \underset{\varepsilon\to 0}{\sim} \varepsilon^{d-2}\, C_0(K)$$

and, for any $y \ne 0$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\, P[T_{y-\varepsilon K} < \zeta] = C_0(K)\, G_\lambda(0,y)$$

(ii) If $d = 2$,

$$C_\lambda(\varepsilon K) \underset{\varepsilon\to 0}{\sim} \pi\, (\log 1/\varepsilon)^{-1}$$

and for any $y \ne 0$

$$\lim_{\varepsilon\to 0} (\log 1/\varepsilon)\, P[T_{y-\varepsilon K} < \zeta] = \pi\, G_\lambda(0,y)$$

(iii) There exists a constant $C_{\lambda,K,d}$ *such that, for any* $\varepsilon \in (0,1/2)$, $y \in \mathbb{R}^d$,

$$P[T_{y-\varepsilon K} < \zeta] \le C_{\lambda,K,d}\, G_\lambda(0,y/2) \times \begin{cases} (\log 1/\varepsilon)^{-1} & \text{if } d = 2 , \\ \varepsilon^{d-2} & \text{if } d \ge 3 . \end{cases}$$

Proof : First notice that by (5)

$$C_\lambda(\varepsilon K) = \left(\inf_{\mu\in\mathcal{P}(K)} \int \mu(dy)\,\mu(dz)\,G_\lambda(\varepsilon y,\varepsilon z)\right)^{-1}.$$

If $d \geq 3$, the desired result follows from (2). If $d = 2$, (3) gives

$$C_\lambda(\varepsilon K) \underset{\varepsilon\to 0}{\sim} \left(\frac{1}{\pi}\log\frac{1}{\varepsilon} + \inf_{\mu\in\mathcal{P}(K)}\int\mu(dy)\mu(dz)\log\frac{1}{|y-z|}\right)^{-1} = \left(\frac{1}{\pi}\log\frac{1}{\varepsilon} + \text{const.}\right)^{-1}.$$

To get the other assertions of (i), (ii), simply write

$$P[T_{y-\varepsilon K} < \zeta] = \int_{y-\varepsilon K} G_\lambda(0,z)\,\mu^\lambda_{y-\varepsilon K}(dz) \underset{\varepsilon\to 0}{\sim} G_\lambda(0,y)\,\mu^\lambda_{y-\varepsilon K}(y-\varepsilon K),$$

and note that $\mu^\lambda_{y-\varepsilon K}(y-\varepsilon K) = C_\lambda(y-\varepsilon K) = C_\lambda(\varepsilon K)$.

Finally (iii) follows easily from (4) and (i), (ii) when $|y| > 2\varepsilon$, and is trivial if $|y| \leq 2\varepsilon$. □

3. Estimates for $E[m(S_{\varepsilon K})]$.

We have

$$E[m(S_{\varepsilon K})] = E\left[\int dy\ 1_{S_{\varepsilon K}}(y)\right] = \int dy\ P[T_{y-\varepsilon K} \leq 1].$$

Therefore we need estimates for $P[T_{y-\varepsilon K} \leq 1]$ as $\varepsilon \to 0$. However these estimates are easily derived from Lemma 1. In this section and the next ones, K is a non-polar subset of $\mathbb{R}^d$, $d \geq 2$ (when K is polar, it is immediate that $E[m(S_K)] = \int dy\ P[T_{y-\varepsilon K} \leq 1] = 0$, so that $m(S_K) = 0$ a.s.).

Lemma 2 : *Let* $t > 0$, $y \in \mathbb{R}^d \setminus \{0\}$.

(i) *If* $d \geq 3$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\,P[T_{y-\varepsilon K} \leq t] = C(K)\int_0^t p_s(0,y)\,ds.$$

(ii) *If* $d = 2$,

$$\lim_{\varepsilon\to 0} \left(\log\frac{1}{\varepsilon}\right) P[T_{y-\varepsilon K} \leq t] = \pi\int_0^t p_s(0,y)\,ds.$$

Proof : Let us concentrate on the case $d = 2$ (the case $d \geq 3$ is similar). Denote by $\gamma_\varepsilon(ds)$ the law of $T_{y-\varepsilon K}$. Lemma 1 (ii) gives

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)\int_0^\infty e^{-\lambda s}\,\gamma_\varepsilon(ds) = \pi\,G_\lambda(0,y) = \pi\int_0^\infty e^{-\lambda s}\,p_s(0,y)ds.$$

Since this result holds for any $\lambda > 0$ it follows that the sequence of measures $|\log \varepsilon| \gamma_\varepsilon(ds)$ converges weakly towards the measure $\pi\, p_s(0,y)ds$. In particular,

$$\lim_{\varepsilon\to 0} (\log \frac{1}{\varepsilon})\, \gamma_\varepsilon([0,t]) = \pi \int_0^t p_s(0,y)ds. \quad \square$$

<u>Theorem 3</u> : *(i) If* $d \geq 3$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\, E[m(S_{\varepsilon K})] = C(K)$$

(ii) If $d = 2$,

$$\lim_{\varepsilon\to 0} (\log \frac{1}{\varepsilon})\, E[m(S_{\varepsilon K})] = \pi.$$

<u>Proof</u> : Consider the case $d \geq 3$ (the case $d = 2$ is exactly similar). Then

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\, E[m(S_{\varepsilon K})] = \lim_{\varepsilon\to 0} \varepsilon^{2-d} \int dy\, P[T_{y-\varepsilon K} \leq 1] = \int dy\, C(K) \int_0^1 p_s(0,y)ds = C(K).$$

Note that the use of dominated convergence is justified by Lemma 1 (iii), the bound

$$P[T_{y-\varepsilon K} \leq 1] \leq e^{\lambda}\, P[T_{y-\varepsilon K} < \zeta]$$

and the fact that the function $G_\lambda(0,y/2)$ is integrable over $\mathbb{R}^d$. $\square$

The previous arguments yield as well the following slightly stronger result. Take $d \geq 3$ for instance. Let f be a bounded Borel function on $\mathbb{R}^d$. Then

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\, E[\int dy\, f(y)\, 1_{S_{\varepsilon K}}(y)] = C(K) \int dy\, f(y) \int_0^1 ds\, p_s(0,y) = C(K)\, E\Big[\int_0^1 ds\, f(B_s)\Big].$$

<u>Remark</u> : The previous proofs, as well as those of the next section, depend heavily on the tools of probabilitic potential theory that we have recalled in Section 2. When K is a ball, say when K is the unit ball of $\mathbb{R}^d$, it is possible to give elementary proofs of all the previous results. Note that in this case

$$T_{y-\varepsilon K} = T_\varepsilon(y) := \inf\{t\ ;\ |B_t - y| \leq \varepsilon\}.$$

The idea is then to compute the expected time spent in the ball of radius ε centered at y, in two different ways. Take $d = 2$ for definiteness. Then,

$$E\Big[\int_0^\zeta 1_{(|B_s-y|\leq\varepsilon)}ds\Big] = \int_0^\infty ds\, e^{-\lambda s} \int_{|z-y|\leq\varepsilon} dz\, p_s(0,z) \underset{\varepsilon\to 0}{\sim} \pi\, \varepsilon^2\, G_\lambda(0,y).$$

On the other hand, assuming that $|y| \geq \varepsilon$, we have by the Markov property at time $T_\varepsilon(y)$,

$$E\left[\int_0^\zeta 1_{(|B_s-y|\leq\varepsilon)}ds\right] = P[T_\varepsilon(y) < \zeta]\ E_{y_\varepsilon}\left[\int_0^\zeta 1_{(|B_s-y|\leq\varepsilon)}ds\right],$$

where y_ε is such that $|y_\varepsilon - y| = \varepsilon$. Easy calculations show that

$$E_{y_\varepsilon}\left[\int_0^\zeta 1_{(|B_s-y|\leq\varepsilon)}ds\right] = \int_{|z-y|\leq\varepsilon} dz\ G_\lambda(y_\varepsilon,z) \underset{\varepsilon\to 0}{\sim} \varepsilon^2 \log\frac{1}{\varepsilon}$$

and we recover Lemma 1 (ii) in this special case.

4. Bounds on $\mathrm{var}(m(S_{\varepsilon K}))$.

It turns out that, in order to get bounds on $\mathrm{var}(m(S_{\varepsilon K}))$, it is important to estimate the volume of the intersection of the Wiener sausages corresponding to two disjoint time intervals. We start with a lemma which gives bounds on the volume of the intersection of two independent Wiener sausages. We denote by B' another Brownian motion independent of B and also started at 0 under P. The associated Wiener sausage is denoted by $S'_{\varepsilon K}$.

Lemma 4 : *There exists a constant* $c = c_{d,K}$ *such that, for* $\varepsilon \in (0,1/2)$,

$$E[m(S_{\varepsilon K} \cap S'_{\varepsilon K})^2]^{1/2} \leq \begin{cases} c\ (\log 1/\varepsilon)^{-2} & \text{if } d = 2\ , \\ c\ \varepsilon^2 & \text{if } d = 3\ , \\ c\ \varepsilon^4 \log 1/\varepsilon & \text{if } d = 4\ , \\ c\ \varepsilon^d & \text{if } d \geq 5\ . \end{cases}$$

Proof : We have :

$$\begin{aligned} E[m(S_{\varepsilon K} \cap S'_{\varepsilon K})^2] &= \int dy\ dz\ P[y \in S_{\varepsilon K} \cap S'_{\varepsilon K},\ z \in S_{\varepsilon K} \cap S'_{\varepsilon K}] \\ &= \int dy\ dz\ P[y \in S_{\varepsilon K},\ z \in S_{\varepsilon K}]^2 \\ &= \int dy\ dz\ P[T_{y-\varepsilon K} \leq 1,\ T_{z-\varepsilon K} \leq 1]^2 \end{aligned}$$

However

$$P[T_{y-\varepsilon K} \leq 1, T_{z-\varepsilon K} \leq 1] = P[T_{y-\varepsilon K} \leq T_{z-\varepsilon K} \leq 1] + P[T_{z-\varepsilon K} < T_{y-\varepsilon K} \leq 1].$$

The Markov property gives the bound

$$P[T_{y-\varepsilon K} \le T_{z-\varepsilon K} \le 1] \le E\Big[1_{\{T_{y-\varepsilon K}\le 1\}}\ E_{B(T_{y-\varepsilon K})}\ [T_{z-\varepsilon K} \le 1]\Big]$$

$$\le c\ s(\varepsilon)^2\ G_\lambda(0,\tfrac{y}{2})\ G_\lambda(0,\tfrac{z-y}{2})$$

where $s(\varepsilon) = (\log 1/\varepsilon)^{-1}$ if $d = 2$, ε^{d-2} if $d \ge 3$. The last bound follows from Lemma 1(iii) by dealing separately with the cases $|z-y| > 4\varepsilon$, $|z-y| \le 4\varepsilon$. Then,

$$E[m(S_{\varepsilon K} \cap S'_{\varepsilon K})^2] \le \int dydz\Big(\Big(c\ s(\varepsilon)^2\ (G_\lambda(0,\tfrac{y}{2})G_\lambda(0,\tfrac{z-y}{2}) + G_\lambda(0,\tfrac{z}{2})G_\lambda(0,\tfrac{y-z}{2}))\Big)^2 \wedge 1\Big)$$

and after some easy calculations we get the desired bounds (note that for $d\ge 4$, $G_\lambda(0,y/2)$ is not square-integrable). □

__Theorem 5__ : *There exists a constant* $c = c_{d,K}$ *such that, for* $\varepsilon \in (0,1/2)$

$$(\text{var } m(S_{\varepsilon K}))^{1/2} \le \begin{cases} c(\log 1/\varepsilon)^{-2} & \text{if } d = 2\ , \\ c\ \varepsilon^2 \log 1/\varepsilon & \text{if } d = 3\ , \\ c\ \varepsilon^{d-1} & \text{if } d \ge 4\ . \end{cases}$$

__Proof__ : Set $h(\varepsilon) = (\text{var } m(S_{\varepsilon K}))^{1/2}$. Crude bounds show that h is bounded over $[0,1]$. The basic idea of the proof is to get a bound for $h(\varepsilon)$ in terms of $h(\varepsilon\sqrt{2})$. Our starting point is the trivial identity

$$m(S_{\varepsilon K}) = m(S_{\varepsilon K}(0,1/2)) + m(S_{\varepsilon K}(1/2,1)) - m(S_{\varepsilon K}(0,1/2) \cap S_{\varepsilon K}(1/2,1)).$$

Set $B'_t = B_{1/2-t} - B_{1/2}$, $B''_t = B_{1/2+t} - B_{1/2}$ for $0 \le t \le 1/2$. Then B', B'' are two independent Brownian motions started at 0, run on the time interval $[0,1/2]$. Furthermore, with an obvious notation,

$$m(S_{\varepsilon K}(0,1/2)) \cap S_{\varepsilon K}(1/2,1)) = m(S'_{\varepsilon K}(0,1/2) \cap S''_{\varepsilon K}(0,1/2))$$

and we can apply the bounds of Lemma 4 to the latter quantity.

On the other hand, the variables $m(S_{\varepsilon K}(0,1/2))$, $m(S_{\varepsilon K}(1/2,1))$ are independent and identically distributed, and a scaling argument gives :

$$m(S_{\varepsilon K}(0,1/2)) \overset{(d)}{=} 2^{-d/2}\ m(S_{\varepsilon\sqrt{2}K}).$$

Then, by the triangle inequality,

$$(\text{var } m(S_{\varepsilon K}))^{1/2} \le (2 \text{ var } m(S_{\varepsilon K}(0,1/2)))^{1/2} + (\text{var } m(S_{\varepsilon K}(0,1/2) \cap S_{\varepsilon K}(1/2,1)))^{1/2}$$

so that:

$$h(\varepsilon) \le 2^{(1-d)/2}\ h(\varepsilon\sqrt{2}) + E[m(S'_{\varepsilon K}(0,1/2) \cap S''_{\varepsilon K}(0,1/2))^2]^{1/2}.$$

It remains to apply the bounds of Lemma 4 and to discuss according to the value of d.

If $d = 2$, we get :

$$h(\varepsilon) \le 2^{-1/2}\, h(\varepsilon\sqrt{2}) + c(\log 1/\varepsilon)^{-2}.$$

Set $k(\varepsilon) = (\log 1/\varepsilon)^2\, h(\varepsilon)$. For any $\rho \in (2^{-1/2},1)$, for ε small, we have

$$k(\varepsilon) \le \rho\, k(\varepsilon\sqrt{2}) + c.$$

This implies that k is bounded over $(0,1/2)$.

If $d = 3$,

$$h(\varepsilon) \le \frac{1}{2}\, h(\varepsilon\sqrt{2}) + c\, \varepsilon^2.$$

Set $k(\varepsilon) = \varepsilon^{-2}\, h(\varepsilon)$. Then

$$k(\varepsilon) \le k(\varepsilon\sqrt{2}) + c,$$

which implies

$$k(\varepsilon) \le c' \log 1/\varepsilon.$$

The case $d \ge 4$ is similar. □

5. The main results.

Theorem 6 : *If* $d = 2$,

$$\lim_{\varepsilon\to 0} (\log 1/\varepsilon)\, m(S_{\varepsilon K}) = \pi.$$

If $d \ge 3$,

$$\lim_{\varepsilon\to 0} \varepsilon^{2-d}\, m(S_{\varepsilon K}) = C_o(K).$$

In both cases, the convergence holds in the L^2*-norm, and a.s. if* K *is star-shaped, that is if* $\varepsilon K \subset K$ *for* $\varepsilon \in (0,1)$.

Proof : The L^2-convergence is easy from Theorem 3 and Theorem 5. Simply observe that :

$$\lim_{\varepsilon\to 0} E\left[\left(\frac{m(S_{\varepsilon K})}{E[m(S_{\varepsilon K})]} - 1\right)^2\right] = 0.$$

When K is star-shaped, $m(S_{\varepsilon K})$ is a monotone increasing function of ε. We may therefore use a monotonicity argument to restrict our attention to a suitable sequence (ε_p). For instance, if $d = 2$, we take $\varepsilon_p = \exp - p^2$. Theorems 3 and 5 then imply that :

$$\sum_{p=1}^{\infty} E\left[\left(\frac{m(S_{\varepsilon_p K})}{E[m(S_{\varepsilon_p K})]} - 1\right)^2\right] < \infty$$

which gives

$$\lim_{p\to\infty} \frac{m(S_{\varepsilon_p K})}{E[m(S_{\varepsilon_p K})]} = 1, \quad \text{a.s.} \quad \square$$

The limiting behavior of $m(S_K(0,t))$ as $t \longrightarrow \infty$ can be deduced from Theorem 6 by the usual scaling transformation . The results are even better since $m(S_K(0,t))$ is always a monotone function of t.

Theorem 6' : *If* $d = 2$,

$$\lim_{t\to\infty} \frac{\log t}{t}\, m(S_K(0,t)) = 2\pi.$$

If $d \geq 3$,

$$\lim_{t\to\infty} \frac{1}{t}\, m(S_K(0,t)) = C_o(K).$$

In both cases the convergence holds a.s. and in the L^2*-norm.*

Remarks : It is interesting to observe that, when $d = 2$, the limiting behavior of $m(S_{\varepsilon K})$ as ε tends to 0 does not depend on K (provided K is non-polar). This fact is closely related to the recurrence properties of planar Brownian motion. It can be explained as follows. Let H, K be two compact subsets of $\mathbb{R}^2$ such that $H \subset K$ and H is non-polar. Then the conditional probability of the event $\{T_{y-\varepsilon H} \leq 1\}$ knowing that $\{T_{y-\varepsilon K} \leq 1\}$ tends to 1 as ε tends to 0. This can be checked by applying the Markov property at time $T_{y-\varepsilon K}$ and then using a suitable scaling argument and the recurrence of planar Brownian motion.

Theorem 6 is also related to the fact that the Hausdorff dimension of the Brownian curve is 2. In particular, for $d \geq 3$, the order of magnitude of the volume of a tubular neighborhood of the Brownian path is the same as would be that of a portion of plane. Note that for a C^1 curve the volume of a tubular neighborhood is of order ε^{d-1}.

6. A heat conduction problem.

The previous results are closely related to the following heat conduction problem. Assume that the compact set K is held at the temperature 1 from time $t = 0$ to $+\infty$, whereas the surrounding medium $\mathbb{R}^d \setminus K$ is at the temperature 0 at time $t = 0$. Clearly the temperature in the surrounding medium will increase, and one is interested in the total energy flow in time t from K to the surrounding medium. More precisely, the temperature at time t, at $x \in \mathbb{R}^d \setminus K$ solves the heat equation :

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u$$

with boundary conditions

$$u(0,x) = 0,$$

$$\lim_{x \to x_o} u(t,x) = 1,$$

for any $t > 0$, x_o regular point of ∂K.

Then $u(t,x)$ has the following probabilistic interpretation :

$$u(t,x) = P_x[T_K \le t] .$$

The quantity of interest is

$$E_K(t) = \int_{\mathbb{R}^d \setminus K} u(t,x)dx .$$

Now observe that :

$$m(K) + E_K(t) = \int_{\mathbb{R}^d} P_x[T_K \le t]dx = \int_{\mathbb{R}^d} P[T_{x-K} \le t]dx = E[m(S_K(0,t))].$$

Therefore the limiting behavior of $E_K(t)$ is given by that of $m(S_K(0,t))$:

- if $d = 2$, $E_K(t) \underset{t\to\infty}{\sim} 2\pi \frac{t}{\log t}$;

- if $d \ge 3$, $E_K(t) \underset{t\to\infty}{\sim} C(K)\, t$.

Bibliographical notes. The strong law of large numbers for the Wiener sausage in $\mathbb{R}^d$, $d \ge 3$, *was first derived by Kesten, Spitzer and Whitman (cf* [IMK, p.252-253] , [S3, p.40]*). See Spitzer* [S4] *for a derivation using Kingman's subadditive ergodic theorem. Our approach is inspired from* [L3], [L10] *and Sznitman* [Sz] *. The relevant results of probabilistic potential theory may be found in the book of Port and Stone* [PS] *. Sharp estimates for the expected volume of the Wiener sausage were first derived by Spitzer* [S2]*. These estimates are refined in* [L11] *for* $d \ge 3$ *and in* [L12] *for* $d = 2$ *(see also Chapter XI of the present work). The bounds of Lemma 4 and Theorem 5 are* *sharp: see* [L3], [L5] *(and Chapter VIII) for additional information about intersections of independent Wiener sausages. The application developed in Section 6 is taken from Spitzer* [S2]. *Other applications may be found in Kac* [K]. *Certain large deviations results for the volume of the Wiener sausage, also motivated by physical applications, are proved in Donsker and Varadhan* [DV]. *The results of this chapter can be extended to processes more general than Brownian motion in* $\mathbb{R}^d$ *. See Chavel and Feldman* [CF1], [CF2] *for the case of Brownian motion on a Riemannian manifold. Sznitman* [Sz] *deals with elliptic diffusion processes in* $\mathbb{R}^d$ *: roughly speaking, the behavior of the sausage of small radius remains the same as for Brownian motion. However, if one considers hypoelliptic diffusion processes (that is diffusion processes whose generator satisfies the strong Hörmander condition), then the volume of a tubular neighborhood of the path may become much smaller: see Chaleyat-Maurel and Le Gall* [CML]. *Hawkes* [H] *considers the sausage associated with Lévy processes: Kingman's theorem can still be applied to the behavior in large time of the volume of the sausage. Weinryb* [W1] *extends Theorem 6 by considering* $\mu(m(S_{\varepsilon K}))$ *for certain measures* μ *such as the Lebesgue measure on a hyperplane. Finally, a discrete analogue of the volume of the Wiener sausage is the number of distinct sites (or the range) visited by a random walk. Discrete versions of Theorem 6' are proved in Dvoretzky and Erdös* [DE] *(see also Spitzer* [S3, p. 38-40] *and Jain and Pruitt* [JP]).

CHAPTER VII

Connected components of the complement of a planar Brownian path.

Let $B = (B_t, t \geq 0)$ be a complex-valued Brownian motion, and $B[0,1] = \{B_s; 0 \leq s \leq 1\}$. It seems very likely, and can be proved rigorously, that with probability 1 the open set $\mathbb{C} \setminus B[0,1]$ has an infinite number of connected components. The following question was raised by Mandelbrot. Let N_ε denote the number of connected components whose area is greater than $\varepsilon > 0$. What is the limiting behavior of N_ε as ε goes to 0 ? Mandelbrot conjectured that $N_\varepsilon \sim \varepsilon^{-1} L(\varepsilon)$ for some slowly varying function L such that $\int_0^1 u^{-1}L(u)du < \infty$. The goal of this chapter is to prove that Mandelbrot's conjecture holds with $L(\varepsilon) = 2\pi(\log \varepsilon)^{-2}$.

The problem of determining the asymptotics of N_ε is closely related to the study of the planar Wiener sausage. To explain this, denote by W_ε the union of all connected components whose area is smaller than $\pi\varepsilon^2$. It is obvious that W_ε is contained in $S_\varepsilon(0,1)$ (the Wiener sausage of radius ε associated with the unit disk). The converse inclusion is also "almost true". Precisely, if $y \in \mathbb{C}$ belongs to $S_\varepsilon(0,1)$, then, with a probability close to 1, y will also belong to W_ε. This fact is explained by the recurrence properties of planar Brownian motion: if B comes within a distance ε of y before time 1 then B will come much closer with great probability, and it will be very likely that the connected component of y is contained in $D(y,\varepsilon)$ (these heuristic arguments can easily be made rigorous). We conclude that

$$m(W_\varepsilon) \sim m(S_\varepsilon(0,1)) \sim \frac{\pi}{|\log \varepsilon|} ,$$

by Theorem VI-6.

It turns out, but is now non-trivial, that much more is true. For any fixed $\lambda \in (0,1)$,

$$(1) \qquad m(W_\varepsilon) - m(W_{\lambda\varepsilon}) \underset{\varepsilon \to 0}{\sim} \frac{\pi}{|\log \varepsilon|} - \frac{\pi}{|\log \lambda\varepsilon|} \sim \frac{\pi|\log \lambda|}{|\log \varepsilon|^2} .$$

Notice that $m(W_\varepsilon) - m(W_{\lambda\varepsilon})$ is closely related to the number of connected components with area between $\pi(\lambda\varepsilon)^2$ and $\pi\varepsilon^2$. Most of this chapter is devoted to a rigorous formulation and proof of (1) . The asymptotics of N_ε then follow rather easily.

1. Estimates for the probability distribution of the area of the connected component of a given point.

Throughout this section, we assume that the Brownian motion B starts at 1 and, for any $R > 0$, we set : $T_R = \inf\{t, |B_t| = R\}$. We denote by $\mathcal{C}_R$ the connected component of $\mathbb{C} \setminus B[0,T_R]$ that contains 0.

Lemma 1 : *There exists a positive constant* α *such that, for any* $R \geq 2$,

$$P[\mathcal{C}_R \text{ is unbounded}] \leq R^{-\alpha}.$$

Proof : Fix $\rho > 1$ and denote by $\mathcal{C}_{(n)}$ the connected component of $\mathbb{C} \setminus B[T_{\rho^n}, T_{\rho^{n+1}}]$ that contains 0. For any $n \geq 1$,

$$P[\mathcal{C}_{\rho^n} \text{ is unbounded}] \leq P\left[\bigcap_{k=0}^{n-1} \{\mathcal{C}_{(k)} \text{ is unbounded}\}\right].$$

The strong Markov property shows that the events $\{\mathcal{C}_{(k)}$ is unbounded$\}$ are independent. By scaling they also have the same probability $c < 1$. Hence,

$$P[\mathcal{C}_{\rho^n} \text{ is unbounded}] \leq c^n.$$

Lemma 1 now follows easily. □

Remark. If $\mathcal{C}_R$ is bounded then obviously it is contained in $D(0,R)$. Using a scaling argument we obtain the following result. Suppose now that $|B_0| \leq \varepsilon$ and let $\delta \geq 2\varepsilon$, then, with a probability greater than $1 - (\varepsilon/\delta)^{\alpha}$, the connected component of $\mathbb{C} \setminus B[0,T_\delta]$ that contains 0 is contained in $D(0,\delta)$. This form of Lemma 1 will be used on several occasions.

For any $r \in (0,1)$, set

$$P(R,r) = P[m(\mathcal{C}_R) \leq \pi\, r^2].$$

Our first goal is to obtain good estimates for $P(R,r)$ as $R \longrightarrow \infty$ and $r \longrightarrow 0$. We follow the ideas described in the introduction. Firstly,

$$P(R,r) \leq P[T_r < T_R] = \frac{\log R}{\log R - \log r}.$$

On the other hand, we may get a lower bound on $P(R,r)$ by conditioning on $\{T_{r/2^n} < T_R\}$, applying the Markov property at time $T_{r/2^n}$ and using the remark after Lemma 1 to bound the probability that $\mathcal{C}_R$ is not contained in $D(0,r)$. For every $n \geq 1$,

$$P(R,r) \geq P[T_{r/2^n} < T_R]\,(1-2^{-n\alpha}) = \frac{\log R}{\log R - \log r + n \log 2}\,(1 - 2^{-n\alpha}).$$

If we choose $n = [K \log\log R]$ with K large enough we get :

$$P(R,r) \geq \frac{\log R}{\log R - \log r + O(\log\log R)} (1 + O((\log R)^{-M}))$$

for any $M > 0$.

In what follows, we shall be interested in estimates as $R \longrightarrow \infty$, holding uniformly in r. We will always assume that

$$(\log R)^{\gamma} \leq |\log r| \leq (\log R)^{1/2},$$

for some $\gamma > 0$. The previous bounds give

$$P(R,r) = 1 + \frac{\log r}{\log R} + O(\frac{\log\log R}{\log R}). \tag{2}$$

We now fix $\lambda \in (0,1)$ and set

$$Q(R,r) = P(R,r) - P(R,\lambda r).$$

<u>Lemma 2</u> : *As* $R \longrightarrow +\infty$, $Q(R,r) = \frac{|\log \lambda|}{\log R} + O(\frac{\log\log R}{(\log R)^{5/4}})$, *uniformly for* $(\log R)^{\gamma} \leq |\log r| \leq (\log R)^{1/2}$.

<u>Proof</u> : Notice that a brutal application of (2) gives nothing. The idea of the proof is to compare $Q(R,r)$ and $Q(R,\lambda r)$ using a scaling transformation, and only then to apply (2).

The event $A_r = \{\pi(\lambda r)^2 \leq m(\mathcal{C}_R) \leq \pi r^2\}$ is contained in $\{T_r < T_R\}$. By the remark following Lemma 1, and the previous arguments,

$$P(A_r) = P(A_r \cap \{\mathcal{C}_R \subset D(0,1)\}) + O((\log R)^{-M}).$$

(use the bound $|\log r| \geq (\log R)^{\gamma}$). We now want to estimate $Q(R,\lambda r) = P(A_{\lambda r})$. Note that $A_{\lambda r}$ is trivially contained in $\{T_\lambda < T_r\}$. We define $\mathcal{C}'_R$ as the connected component of $\mathbb{C} \setminus B[0,T_{\lambda R}]$ that contains 0, and we set :

$$A'_r = \{\pi(\lambda^2 r)^2 \leq m(\mathcal{C}'_R) \leq \pi(\lambda r)^2\}.$$

As previously,

$$P(A'_r) = P(A'_r \cap \{\mathcal{C}'_R \subset D(0,\lambda)\}) + O((\log R)^{-M}).$$

However, by the Markov property and scaling,

$$P(A'_r \cap \{\mathcal{C}'_R \subset D(0,\lambda)\}) = P(T_\lambda < T_R)\, P(A_r \cap \{\mathcal{C}_R \subset D(0,1)\}).$$

The point is that, on the set $\{\mathcal{C}'_R \subset D(0,\lambda)\}$, $\mathcal{C}'_R$ is also the connected component of $\mathbb{C} \setminus B[T_\lambda,T_{\lambda R}]$ that contains 0 .

We now want to compare the sets A'_r and $A_{\lambda r}$. The problem is that $\mathcal{C}_R$ may be smaller than $\mathcal{C}'_R$ because of the portion of the path between times $T_{\lambda R}$

and T_R. However, by Lemma 1 we may choose $K > 0$ large enough so that $\mathcal{C}'_R$ is contained in $D(0,(\log R)^K)$ except on a set of probability $O((\log R)^{-10})$. It follows that,

$$P[A'_r \setminus A_{\lambda r}] \le O((\log R)^{-10}) + P(A'_r)\, P(\inf_{[T_{\lambda R},T_R]} |B_u| < (\log R)^K)$$

$$\le O((\log R)^{-10}) + (1-P(\lambda R,\lambda^2 r))\, \frac{|\log \lambda|}{\log R - K\log\log R} = O((\log R)^{-3/2})$$

by (2) and our assumption $|\log r| \le (\log R)^{1/2}$. A similar reasoning gives

$$P(A_{\lambda r} \setminus A'_r) = O((\log R)^{-3/2}),$$

and we get:

$$P[A_{\lambda r}] = P[A'_r] + O((\log R)^{-3/2})$$

From the previous considerations, we obtain

$$P[A_{\lambda r}] = P[T_\lambda < T_R]\, P[A_r] + O((\log R)^{-3/2})$$

or equivalently

$$(3) \qquad Q(R,\lambda r) = Q(R,r)(1 + \frac{\log \lambda}{\log R}) + O((\log R)^{-3/2}).$$

Now let $N \ge 1$ be an integer such that $N \le (\log R)^{1/2}$. By (3),

$$P(R,r) - P(R,\lambda^N r) = \sum_{k=0}^{N-1} Q(R,\lambda^k r) = \frac{\log R}{|\log \lambda|}(1 - (1 + \frac{\log \lambda}{\log R})^N) + O(N^2 (\log R)^{-3/2})$$

uniformly for $(\log R)^\gamma \le |\log r| \le (\log R)^{1/2}$. Furthermore,

$$1 - (1 + \frac{\log \lambda}{\log R})^N = N\, \frac{|\log \lambda|}{\log R} + O(\frac{N^2}{(\log R)^2}),$$

which gives

$$P(R,r) - P(R,\lambda^N r) = N(1 + O(\frac{N}{\log R}))\, Q(R,r) + O(N^2 (\log R)^{-3/2}).$$

However, by (2),

$$P(R,r) - P(R,\lambda^N r) = \frac{N|\log \lambda|}{\log R} + O(\frac{\log\log R}{\log R}),$$

so that we obtain :

$$(1 + O(\frac{N}{\log R}))\, Q(R,r) = \frac{|\log \lambda|}{\log R} + O(\frac{\log\log R}{N \log R} + N(\log R)^{-3/2}).$$

We now take $N = [(\log R)^{1/4}]$ to complete the proof. □

2. Asymptotics for N_ε.

We now take $B_0 = 0$. We will apply the previous estimates to the asymptotics of N_ε. Most of this section is devoted to a rigorous proof of (1). For simplicity, we set

$$U_\varepsilon = W_\varepsilon \setminus W_{\lambda\varepsilon}$$

so that U_ε is the union of all components whose area is between $\pi(\lambda\varepsilon)^2$ and $\pi\varepsilon^2$. We will obtain the limiting behavior of $m(U_\varepsilon)$ by a method similar to the one we used for the area of the Wiener sausage in Chapter VI.

Proposition 3 : *As* $\varepsilon \longrightarrow 0$,

$$E[m(U_\varepsilon)] = \frac{\pi|\log \lambda|}{(\log \varepsilon)^2} + o(\frac{1}{(\log \varepsilon)^2}).$$

Proof : For $\varepsilon > 0$ small enough we define $\delta = \delta(\varepsilon) > \varepsilon$ by the condition

$$\frac{\delta}{\exp(|\log \delta|^{1/4})} = \varepsilon.$$

Note that $|\log \delta| \sim |\log \varepsilon|$ as $\varepsilon \longrightarrow 0$. Let $y \in \mathbb{C} \setminus D(0,\delta)$. Set

$$T_\delta(y) = \inf\{s \geq 0 \; ; \; |B_s - y| < \delta\}$$

and

$$R_\delta(y) = \inf\{s \geq T_\delta(y) \; ; \; |B_s - y| > (\log \delta)^{-4}\} .$$

Notice that $\{y \in U_\varepsilon\} \subset \{T_\delta(y) \leq 1\}$. We denote by $\mathcal{C}(y)$, resp. $\mathcal{C}_\delta(y)$, the connected component of $\mathbb{C} \setminus B[0,1]$, resp. $\mathbb{C} \setminus B[T_\delta(y),R_\delta(y)]$, that contains y. Then,

$$(4) \qquad |P[y \in U_\varepsilon] - P[T_\delta(y) \leq 1 \; ; \; \pi(\lambda\varepsilon)^2 \leq m(\mathcal{C}_\delta(y)) \leq \pi\varepsilon^2]|$$

$$\leq P[T_\delta(y) \leq 1 \; ; \; \pi(\lambda\varepsilon)^2 \leq m(\mathcal{C}_\delta(y)) \leq \pi\varepsilon^2 \; ; \; \mathcal{C}(y) \neq \mathcal{C}_\delta(y)]$$

$$+ P[T_\delta(y) \leq 1 \; ; \; \pi(\lambda\varepsilon)^2 \leq m(\mathcal{C}(y)) \leq \pi\varepsilon^2 \; ; \; \mathcal{C}(y) \neq \mathcal{C}_\delta(y)].$$

We proceed to bound the right side of (4). We have

$$(5) \quad P[T_\delta(y) \leq 1 \; ; \; \pi(\lambda\varepsilon)^2 \leq m(\mathcal{C}_\delta(y)) \leq \pi\varepsilon^2 \; ; \; \mathcal{C}(y)\neq\mathcal{C}_\delta(y)] \leq P[T_\delta(y) \leq 1 \leq R_\delta(y)]$$

$$+ P[T_\delta(y) \leq R_\delta(y) \leq 1 ; (B[0,T_\delta(y)] \cup B[R_\delta(y),1]) \cap \mathcal{C}_\delta(y) \neq \emptyset \; ; \pi(\lambda\varepsilon)^2 \leq m(\mathcal{C}_\delta(y)) \leq \pi\varepsilon^2].$$

It is very easy to check that :

$$P[T_\delta(y) \leq 1 \leq R_\delta(y)] \leq P[|B_1 - y| \leq (\log \delta)^{-4}] \leq (\log \delta)^{-3} \, \psi_1(y),$$

for some integrable function $\psi_1 : \mathbb{C} \longrightarrow \mathbb{R}_+$. Next, Lemma 1 gives

$$P[m(\mathscr{C}_\delta(y)) \le \pi\varepsilon^2 \ , \ \mathscr{C}_\delta(y) \cap (\mathbb{C} \setminus D(y,\delta)) \ne \emptyset \mid \mathscr{F}_{T_\delta(y)}] = O(|\log \delta|^{-M})$$

uniformly in $y \in \mathbb{C}$. It follows that the second term of the right side of (5) is bounded by :

$$P[T_\delta(y)\le 1]O(|\log \delta|^{-M}) + P[T_\delta(y)\le 1 \ ; \inf_{[R_\delta(y),R_\delta(y)+1]} |B_u|<\delta \ ; \pi(\lambda\varepsilon)^2 \le m(\mathscr{C}_\delta(y))\le \pi\varepsilon^2]$$

$$= P[T_\delta(y)\le 1] \left(O(|\log \delta|^{-M}) \quad + Q(\frac{(\log \delta)^{-4}}{\delta} \ , \ \frac{\varepsilon}{\delta} \ ) \ P\left[\inf_{[R_\delta(y);R_\delta(y)+1]} |B_u| < \delta\right]\right)$$

using the Markov property at $T_\delta(y)$ and at $R_\delta(y)$. It follows from Lemma VI-1 (iii) that

$$P\left[\inf_{[R_\delta(y),R_\delta(y)+1]} |B_u| < \delta\right] = O(\frac{\log|\log \delta|}{|\log \delta|}).$$

Then using Lemma 2 and Lemma VI-1 (iii) again we conclude that the right side of (5) is bounded by

$$\frac{\log|\log \delta|}{|\log \delta|^3} \ \psi_2(y)$$

for some integrable function $\psi_2 : \mathbb{C} \longrightarrow \mathbb{R}_+$.

Similar arguments show that the second term of the right side of (4) is bounded by :

$$\frac{\log|\log \delta|}{|\log \delta|^3} \ \psi_3(y)$$

for some integrable function $\psi_3 : \mathbb{C} \longrightarrow \mathbb{R}_+$. It follows that :

$$\left|\int dy \ P[y \in U_\varepsilon] - \int dy \ P[T_\delta(y) \le 1 \ ; \ \pi(\lambda\varepsilon)^2 \le m(\mathscr{C}_\delta(y)) \le \pi\varepsilon^2]\right| = O(\ \frac{\log|\log \delta|}{|\log \delta|^3}).$$

However, by the Markov property at time $T_\delta(y)$, if $|y| > \delta$,

$$P[T_\delta(y) \le 1 \ ; \ \pi(\lambda\varepsilon)^2 \le m(\mathscr{C}_\delta(y)) \le \pi\varepsilon^2] = P[T_\delta(y) \le 1] \ Q(\frac{(\log \delta)^{-4}}{\delta} \ , \ \frac{\varepsilon}{\delta})$$

so that

$$E[m(U_\varepsilon)] = \int dy \ P[y \in U_\varepsilon] = E[m(S_\delta(0,1))] \ Q(\frac{(\log \delta)^{-4}}{\delta} \ , \ \frac{\varepsilon}{\delta}) + o((\log \delta)^{-2}).$$

Proposition 3 now follows from Lemma 2 and Theorem VI-3 (ii). □

Proposition 4 : *There exists a constant* K *such that, for any* $\varepsilon \in (0,1/2)$,

$$\mathrm{var}(m(U_\varepsilon)) \le K\ |\log \varepsilon|^{-11/2} .$$

Proof : The main idea is the same as in the proof of Theorem VI-5. We let B^1, B^2 denote two independent complex-valued Brownian motions started at 0. For every $y \in \mathbb{C} \setminus (B^1[0,1/2] \cup B^2[0,1/2])$ we denote by $\mathscr{C}'(y)$, resp. $\mathscr{C}^1(y)$, $\mathscr{C}^2(y)$, the connected component of $\mathbb{C} \setminus (B^1[0,1/2] \cup B^2[0,1/2])$, resp. $\mathbb{C} \setminus B^1[0,1/2]$, $\mathbb{C} \setminus B^2[0,1/2]$, that contains y. We set :

$$U'_\varepsilon = \{y\ ;\ \pi(\lambda\varepsilon)^2 \le m(\mathscr{C}'(y)) \le \pi\varepsilon^2\},$$

$$U^i_\varepsilon = \{y\ ;\ \pi(\lambda\varepsilon)^2 \le m(\mathscr{C}^i(y)) \le \pi\varepsilon^2\} \qquad (i = 1,2).$$

Obviously $m(U'_\varepsilon)$ and $m(U_\varepsilon)$ are identically distributed, so that $\mathrm{var}\ m(U_\varepsilon) = \mathrm{var}\ m(U'_\varepsilon)$. The key step of the proof is to show that $m(U'_\varepsilon)$ is not too different from $m(U^1_\varepsilon) + m(U^2_\varepsilon)$.

Lemma 5 : *There exists a constant* K' *such that, for* $\varepsilon \in (0,1/2)$,

$$E[(m(U'_\varepsilon) - m(U^1_\varepsilon) - m(U^2_\varepsilon))^2] \le K'\ |\log \varepsilon|^{-11/2} .$$

Proof : We first observe that :

$$(6)\quad |m(U'_\varepsilon)-m(U^1_\varepsilon)-m(U^2_\varepsilon)| \le m(U'_\varepsilon\backslash(U^1_\varepsilon \cup U^2_\varepsilon)) + m(U^1_\varepsilon\backslash U'_\varepsilon) + m(U^2_\varepsilon\backslash U'_\varepsilon) + m(U^1_\varepsilon \cap U^2_\varepsilon).$$

Let us bound

$$E[m(U^1_\varepsilon \setminus U'_\varepsilon)^2] = E[\int dy\, dz\ 1_{U^1_\varepsilon\backslash U'_\varepsilon}(y)\ 1_{U^1_\varepsilon\backslash U'_\varepsilon}(z)] .$$

We take $\delta = \delta(\varepsilon)$ as in the proof of Proposition 3. It follows from Lemma 1 (and the remark after this lemma) that:

$$E[m(U^1_\varepsilon \setminus U'_\varepsilon)^2] = E[\int dy\, dz\ 1_{\{\mathscr{C}_\delta(y)\subset D(y,\delta);\mathscr{C}_\delta(z)\subset D(z,\delta)\}}\ 1_{U^1_\varepsilon\backslash U'_\varepsilon}(y)\ 1_{U^1_\varepsilon\backslash U'_\varepsilon}(z)]$$
$$+\ O(|\log \varepsilon|^{-M}) .$$

Now notice that, if $y \in U^1_\varepsilon \setminus U'_\varepsilon$ and $\mathscr{C}_\delta(y) \subset D(y,\delta)$, then $B^2[0,1/2]$ must intersect $D(y,\delta)$. It follows that

$$E[m(U^1_\varepsilon \setminus U'_\varepsilon)^2] \le E[\int dy\, dz\ 1_{U^1_\varepsilon}(y)\ 1_{U^1_\varepsilon}(z)\ 1_{S^2_\delta}(y)\ 1_{S^2_\delta}(z)] + O(|\log \varepsilon|^{-M})$$

$$= \int dy\, dz\ P[y \in U^1_\varepsilon,\ z \in U^1_\varepsilon]\ P[y \in S^2_\delta,\ z \in S^2_\delta] + O(|\log \varepsilon|^{-M}),$$

where $S^2_\delta = S^2_\delta(0,1/2)$ denotes the Wiener sausage associated with B^2. Recall from Chapter VI (see the proof of Lemma VI-4) the bound

(7) $P[y \in S^2_\delta,\ z \in S^2_\delta] \le C(\log \delta)^{-2}\,(G_1(0,y/2) + G_1(0,z/2))\,G_1(0,(z-y)/2).$

The problem is then to get a suitable bound on $P[y \in U^1_\varepsilon,\ z \in U^1_\varepsilon]$. Let $T^1_\delta(y)$, $R^1_\delta(y)$ be as previously, with B replaced by B^1. We suppose that $|z-y| \ge |\log \delta|^{-3}$ and we restrict our attention to the case $T^1_\delta(y) \le T^1_\delta(z)$. Lemma 2 and the Markov property at time $T^1_\delta(z)$ give :

$$P[y \in U^1_\varepsilon\ ,z \in U^1_\varepsilon\ ;\ T^1_\delta(y) \le T^1_\delta(z)]$$

$$\le P[T^1_\delta(y) \le T^1_\delta(z) \le \tfrac{1}{2}\ ;\ m(\mathscr{E}^1(y)) \ge \pi(\lambda\varepsilon)^2\ ;\ m(\mathscr{E}^1(z)) \ge \pi(\lambda\varepsilon)^2\ ]$$

$$\le P[T^1_\delta(y) \le T^1_\delta(z) \le \tfrac{1}{2};\ m(\tilde{\mathscr{E}}^1(y)) \ge \pi(\lambda\varepsilon)^2]\ (1 - P(\frac{(\log\ \delta)^{-4}}{\delta}, \frac{\lambda\varepsilon}{\delta})) + P[T^1_\delta(z) \le \tfrac{1}{2} \le R^1_\delta(z)]$$

$$\le C\ |\log\ \delta|^{-3/4}\ P[T^1_\delta(y) \le T^1_\delta(z) \le \tfrac{1}{2}\ ;\ m(\tilde{\mathscr{E}}^1(y)) \ge \pi(\lambda\varepsilon)^2] + P[T^1_\delta(z) \le \tfrac{1}{2} \le R^1_\delta(z)],$$

using (2). Here $\tilde{\mathscr{E}}^1(y)$ denotes the connected component of $\mathbb{C} \setminus B[0,R^1_\delta(y)]$ that contains y. Clearly the term $P[T^1_\delta(z) \le \frac{1}{2} \le R^1_\delta(z)]$ is bounded by $C|\log\delta|^{-8}$. Therefore it suffices to bound

$$P[T^1_\delta(y) \le T^1_\delta(z) \le \tfrac{1}{2}\ ;\ m(\tilde{\mathscr{E}}^1(y)) \ge \pi(\lambda\varepsilon)^2]$$

$$\le C\ \frac{G_1(0,(z-y)/2)}{|\log\ \delta|}\ P[T^1_\delta(y) \le \tfrac{1}{2}\ ;\ m(\tilde{\mathscr{E}}^1(y)) \ge \pi(\lambda\varepsilon)^2]$$

$$\le C'\,|\log\ \delta|^{-11/4}\ \ G_1(0,y/2)\ G_1(0,(z-y)/2).$$

This first bound uses the Markov property at time $R^1_\delta(y)$ and Lemma VI-1 (iii). The second one follows from the same lemma, the Markov property at $T^1_\delta(y)$ and (2).

We conclude that, for any y,z such that $|z-y| \ge |\log\ \delta|^{-3}$,

(8) $$P[y \in U^1_\varepsilon\ ,z \in U^1_\varepsilon\ ;\ T^1_\delta(y) \le T^1_\delta(z)]$$

$$\le C\ |\log\ \varepsilon|^{-7/2}\ G_1(0,y/2)\ G_1(0,(z-y)/2) + O(|\log\ \varepsilon|^{-8}).$$

Combining (7) and (8) gives the bound :

$$E[m(U^1_\varepsilon \setminus U'_\varepsilon)^2] \le C\ |\log\ \varepsilon|^{-11/2}.$$

Obviously the same bound holds for $E[m(U^2_\varepsilon \setminus U'_\varepsilon)^2]$. Similar arguments give even better bounds for the other two terms of the right side of (6). □

We now complete the proof of Proposition 4. We set $h(\varepsilon) = (\text{var } m(U_\varepsilon))^{1/2}$. Notice that $m(U^1_\varepsilon)$ and $m(U^2_\varepsilon)$ are independent and that

$$m(U^1_\varepsilon) \overset{(d)}{=} m(U^2_\varepsilon) \overset{(d)}{=} \frac{1}{2}\, m(U_{\varepsilon\sqrt{2}})$$

by a scaling argument. Then Lemma 5 implies

$$(\text{var } m(U_\varepsilon))^{1/2} \le (\text{var}(m(U^1_\varepsilon) + m(U^2_\varepsilon)))^{1/2} + O(|\log \varepsilon|^{-11/4})$$

$$= 2^{1/2}\,(\text{var } m(U_{\varepsilon\sqrt{2}}))^{1/2} + O(|\log \varepsilon|^{-11/4}).$$

Therefore,

$$h(\varepsilon) \le 2^{-1/2}\, h(\varepsilon\sqrt{2}) + O(|\log \varepsilon|^{-11/4})$$

and Proposition 4 follows using arguments similar to those of the proof of Theorem VI-5. □

We may now state the main result of this Chapter. For $u < v$, $N_{[u,v)}$ denotes the number of connected components of $\mathbb{C} \setminus B[0,1]$ whose area belongs to the interval $[u,v)$ (in particular $N_\varepsilon = N_{[\varepsilon,\infty)}$).

<u>Theorem 6</u> : *With probability* 1, *for any* $\delta > 0$,

$$\lim_{u\to 0} \left(\sup_{v \ge (1+\delta)u} \left| \frac{(\log u)^2\, N_{[u,v)}}{u^{-1} - v^{-1}} - 2\pi \right| \right) = 0.$$

In particular,

$$\lim_{\varepsilon\to 0} \varepsilon\, (\log \varepsilon)^2\, N_\varepsilon = 2\pi, \quad a.s.$$

<u>Proof</u> : Propositions 3 and 4 give

$$\lim_{n\to\infty} (\log \lambda^n)^2\ \ m(U_{\lambda^n}) = \pi|\log \lambda|, \quad a.s.$$

Let $\bar{N}_{[u,v)}$ be the number of connected components with area in $[\pi u^2, \pi v^2)$. Note that :

$$(\pi\lambda^{2n})^{-1}\ m(U_{\lambda^n}) \le \bar{N}_{[\lambda^{n+1},\lambda^n)} \le (\pi\lambda^{2n+2})^{-1}\ m(U_{\lambda^n}).$$

Therefore, w.p. 1,

$$(9)\qquad |\log \lambda| \le \liminf_{n\to\infty} \lambda^{2n}(\log \lambda^n)^2\, \bar{N}_{[\lambda^{n+1},\lambda^n)}$$

$$\le \limsup_{n\to\infty} \lambda^{2n}(\log \lambda^n)^2\, \bar{N}_{[\lambda^{n+1},\lambda^n)} \le \frac{|\log \lambda|}{\lambda^2}.$$

Fix an integer $p \ge 1$ and set $\lambda' = \lambda^{1/p}$. Since

$$\bar{N}_{[\lambda^{n+1},\lambda^n)} = \sum_{i=0}^{p-1} \bar{N}_{[\lambda'^{np+i+1},\lambda'^{np+i})},$$

it follows from (9) that :

$$\frac{|\log \lambda|}{p} \sum_{i=0}^{p-1} \lambda^{-2i/p} \le \liminf_{n\to\infty} \lambda^{2n}(\log \lambda^n)^2\, \bar{N}_{[\lambda^{n+1},\lambda^n)}$$

$$\le \limsup_{n\to\infty} \lambda^{2n}(\log \lambda^n)^2\, \bar{N}_{[\lambda^{n+1},\lambda^n)} \le \frac{|\log \lambda|}{p} \sum_{i=0}^{p-1} \lambda^{-(2i+2)/p}.$$

Choosing p large we conclude that

$$\lim_{n\to\infty} \lambda^{2n}(\log \lambda^n)^2\, \bar{N}_{[\lambda^{n+1},\lambda^n)} = |\log \lambda| \int_0^1 \lambda^{-2s}\, ds = \frac{1}{2}(\lambda^{-2}-1), \qquad \text{a.s.}$$

A simple monotonicity argument allows us to improve this convergence to

$$\lim_{x\to 0} x^2(\log x)^2\, \bar{N}_{[x,\alpha x)} = \frac{1}{2}(1-\alpha^{-2}) \quad \text{a.s.},$$

for any $\alpha > 1$. Equivalently,

$$\lim_{u\to 0} u\,(\log u)^2\, N_{[u,\alpha u)} = 2\pi(1-\frac{1}{\alpha}), \quad \text{a.s.}$$

Theorem 5 follows easily from this last result. □

Bibliographical notes. Mandelbrot ([Ma], Chapter 25) raises some interesting questions about the connected components of the complement of a planar Brownian path. Motivated by these questions, Mountford [Mo] has obtained a weak form of Theorem 6. The main ideas and techniques of this chapter are taken from [Mo], although the form of Theorem 6 given above is from [L13]. We refer to the latter paper for additional details in the proofs (the estimates of [L13] are somewhat sharper than those presented here).

CHAPTER VIII

Intersection local times and first applications.

1. The intersection local time of p independent Brownian paths.

Let $p \geq 2$ be an integer, and let $B^1,\dots,B^p$ denote p independent Brownian motions in $\mathbb{R}^2$, started at $x^1,\dots,x^p$ respectively. The intersection local time of $B^1,\dots,B^p$ is a random measure $\alpha(ds_1,\dots ds_p)$ on $(\mathbb{R}_+)^p$, supported on

$$\left\{ (t_1,\dots,t_p) \in (\mathbb{R}_+)^p \; ; \; B^1_{t_1} = \dots = B^p_{t_p} \right\}.$$

The fact that the latter set is non-empty with probability 1 is more or less equivalent to the existence of p-multiple points for the planar Brownian path. In our approach, the non-emptiness of this set will follow from the fact that it supports a non-trivial measure.

The measure $\alpha(ds_1\dots ds_p)$ is formally defined by :

$$\alpha(ds_1\dots ds_p) = \delta_{(0)}\left(B^1_{s_1} - B^2_{s_2}\right)\dots\delta_{(0)}\left(B^{p-1}_{s_{p-1}} - B^p_{s_p}\right) ds_1\dots ds_p$$

where $\delta_{(0)}$ denotes the Dirac measure at 0. Equivalently,

$$\alpha(ds_1\dots ds_p) = \left(\int_{\mathbb{R}^2} dy\ \delta_{(y)}\left(B^1_{s_1}\right)\dots\delta_{(y)}\left(B^p_{s_p}\right)\right) ds_1\dots ds_p.$$

We will use the latter formal expression as a starting point for our construction. The idea is to replace the Dirac measure at y by a suitable approximation. We set :

$$\delta^{\varepsilon}_{(y)}(z) = (\pi\varepsilon^2)^{-1}\ 1_{D(y,\varepsilon)}(z)\ ,$$

and

$$\alpha_{\varepsilon}(ds_1\dots ds_p) = \varphi_{\varepsilon}\left(B^1_{s_1},\dots,B^p_{s_p}\right) ds_1\dots ds_p$$

where

$$\varphi_{\varepsilon}(z_1,\dots,z_p) = \int_{\mathbb{R}^2} \prod_{j=1}^{p} \delta^{\varepsilon}_{(y)}(z_j)\ dy.$$

Notice that $\varphi_{\varepsilon}(z_1,\dots,z_p) = \varphi_{\varepsilon}(z_1+x,\dots,z_p+x)$ for every $x \in \mathbb{R}^2$.

Theorem 1 : *There exists w.p. 1 a (random) measure* $\alpha(ds_1\dots ds_p)$ *on* $(\mathbb{R}_+)^p$ *such that, for any* $A^1,\dots,A^p$ *bounded Borel subsets of* $\mathbb{R}_+$,

$$\lim_{\varepsilon\to 0} \alpha_\varepsilon(A^1 \times \dots \times A^p) = \alpha(A^1 \times \dots \times A^p)$$

in the L^n*-norm, for any* $n < \infty$.

The measure $\alpha(\cdot)$ *is a.s. supported on*

$$\left\{(s_1,\dots,s_p)\ ;\ B^1_{s_1} = \dots = B^p_{s_p}\right\}.$$

With probability 1 , *for any* $j \in \{1,\dots,p\}$ *and any* $t \ge 0$,

$$\alpha(\{s_j = t\}) = 0.$$

Finally,

$$\text{(1)}\qquad E\Big[\alpha(A^1 \times \dots \times A^p)^n\Big] = \int_{(\mathbb{R}^2)^n} dy_1\dots dy_n$$

$$\times \prod_{j=1}^{p}\left(\int_{(A^j)^n_<} ds_1\dots ds_n \sum_{\sigma\in\Sigma_n}\left(p_{s_1}(x^j,y_{\sigma(1)})\prod_{k=2}^{n} p_{s_k-s_{k-1}}(y_{\sigma(k-1)},y_{\sigma(k)})\right)\right)$$

where Σ_n *is the set of all permutations of* $\{1,\dots,n\}$ *and*

$$(A^j)^n_< = \left\{(s_1,\dots,s_n) \in (A^j)^n\ ;\ 0 \le s_1 < \dots < s_n\right\}.$$

Proof : First step. We first check the L^2-convergence of $\alpha_\varepsilon(A^1 \times\dots\times A^p)$. It suffices to prove that

$$\lim_{\varepsilon,\varepsilon'\to 0} E[\alpha_\varepsilon(A^1 \times\dots\times A^p)\ \alpha_{\varepsilon'}(A^1 \times\dots\times A^p)]$$

exists and is finite. By Fubini's theorem,

$$\text{(2)}\quad E[\alpha_\varepsilon(A^1 \times\dots\times A^p)\ \alpha_{\varepsilon'}(A^1 \times\dots\times A^p)]$$

$$= \int dydy' \prod_{j=1}^{p}\left(\int_{(A^j)^2} dsds'\quad E\Big[\delta^{\varepsilon}_{(y)}(B^j_s)\ \delta^{\varepsilon'}_{(y')}(B^j_{s'})\Big]\right)$$

$$= \int dydy' \prod_{j=1}^{p}\left(\int_{(A^j)^2_<} dsds'\quad E\Big[\delta^{\varepsilon}_{(y)}(B^j_s)\delta^{\varepsilon'}_{(y')}(B^j_{s'}) + \delta^{\varepsilon'}_{(y')}(B^j_s)\delta^{\varepsilon}_{(y)}(B^j_{s'})\Big]\right).$$

It is obvious that, for $(s,s') \in (A^j)^2_<$,

$$\lim_{\varepsilon,\varepsilon'\to 0} E\Big[\delta^{\varepsilon}_{(y)}(B^j_s)\ \delta^{\varepsilon'}_{(y')}(B^j_{s'})\Big] = p_s(x^j,y)\ p_{s'-s}(y,y').$$

The only problem is thus to justify the use of dominated convergence. We will find a function $\varphi(y,y',s,s')$ such that, for every $M > 0$,

$$(3) \qquad \int dy dy' \left(\int_{[0,M]^2_<} ds\, ds'\, \varphi(y,y',s,s') \right)^p < \infty$$

and for any $y,y' \in \mathbb{R}^2$, $(s,s') \in (0,\infty)^2_<$, $\varepsilon,\varepsilon' \in (0,1)$,

$$(4) \qquad E\left[\delta^{\varepsilon}_{(y)}(B^j_s)\, \delta^{\varepsilon'}_{(y')}(B^j_{s'})\right] \le \varphi(y-x^j,y',s,s').$$

The existence of such a function justifies the passage to the limit under the integral sign in the right side of (2).

Clearly we may assume that $x^j = 0$, and we drop the superscript j in what follows. We first consider $E[\delta^{\varepsilon}_{(y)}(B_s)]$ for $s > 0$. If $|y| \ge 2\varepsilon$, then obviously

$$E[\delta^{\varepsilon}_{(y)}(B_s)] \le p_s(0,y/2).$$

If $|y| < 2\varepsilon$, then

$$E[\delta^{\varepsilon}_{(y)}(B_s)] \le (\pi\varepsilon^2)^{-1} \wedge (2\pi s)^{-1} \le 4(|y|^{-2} \wedge s^{-1}).$$

Therefore,

$$E[\delta^{\varepsilon}_{(y)}(B_s)] \le \psi(y,s)$$

where :

$$\psi(y,s) = 4\, 1_{(|y|\le 2)}\, |y|^{-2} \wedge s^{-1} + p_s(0,y/2).$$

Notice that :

$$(5) \qquad \int_0^M \psi(y,s)\, ds \le C_M\, G_1(0,y)$$

where $G_1(x,y) = \int_0^\infty e^{-s}\, p_s(x,y)\, ds.$

We now bound $E[\delta^{\varepsilon}_{(y)}(B_s)\, \delta^{\varepsilon'}_{(y')}(B_{s'})]$. The easy case is when $|y'-y| \ge 2(\varepsilon+\varepsilon')$. Then the Markov property at time s gives :

$$E[\delta^{\varepsilon}_{(y)}(B_s)\, \delta^{\varepsilon'}_{(y')}(B_{s'})] \le E[\delta^{\varepsilon}_{(y)}(B_s)]\, p_{s'-s}(0,\frac{y'-y}{2}) \le \psi(y,s)\, p_{s'-s}(0,\frac{y'-y}{2})$$

Suppose now that $|y'-y| < 2(\varepsilon+\varepsilon')(\le 4)$. If $\varepsilon \le \varepsilon'$, the Markov property gives

$$E[\delta^{\varepsilon}_{(y)}(B_s)\, \delta^{\varepsilon'}_{(y')}(B_{s'})] \le E[\delta^{\varepsilon}_{(y)}(B_s)] \left((\pi\varepsilon'^2)^{-1} \wedge (2\pi(s'-s))^{-1}\right)$$

$$\le 16\, \psi(y,s)\, (|y'-y|^{-2} \wedge (s'-s)^{-1}).$$

If $\varepsilon' < \varepsilon$, then we discuss separately each of the cases $s'-s > |y'-y|^2$, $s'-s \le |y'-y|^2$. If $s'-s > |y'-y|^2$, then obviously

$$E[\delta^{\varepsilon}_{(y)}(B_s)\, \delta^{\varepsilon'}_{(y')}(B_{s'})] \le E[\delta^{\varepsilon}_{(y)}(B_s)]\, (2\pi(s'-s))^{-1} \le \psi(y,s)\, (|y'-y|^{-2} \wedge (s'-s)^{-1}).$$

Finally, if $s'-s \le |y'-y|^2$, $\varepsilon' < \varepsilon$,

$$E[\delta^{\varepsilon}_{(y)}(B_s)\ \delta^{\varepsilon'}_{(y')}(B_{s'})] \le (\pi\varepsilon^2)^{-1}E[\delta^{\varepsilon'}_{(y')}(B_{s'})] \le 16\ |y'-y|^{-2}\ \psi(y',s')$$

$$\le 16\ (|y'-y|^{-2}\wedge(s'-s)^{-1})\ \psi(y',s').$$

The previous estimates show that (4) holds with

$$\varphi(y,y',s,s') = (\psi(y,s) + \psi(y',s'))\Big(p_{s'-s}(0,(y'-y)/2) + 16(|y'-y|^{-2}\wedge(s'-s)^{-1})\Big).$$

Note that

$$\int_{[0,M]^2_<} dsds'\ \varphi(y,y',s,s') \le C'_M\ \Big(G_1(0,\tfrac{y}{2}) + G_1(0,\tfrac{y'}{2})\Big)\ G_1\Big(0,\ \frac{y'-y}{2}\Big)$$

so that (3) is clearly satisfied (the key ingredient is the fact that $G_1(0,y)$ is in L^p for any $p < \infty$).

<u>Second step</u> : The first step allows us to set :

$$\tilde{\alpha}(A^1 \times \ldots \times A^p) = L^2 - \lim_{\varepsilon\to 0}\ \alpha_\varepsilon(A^1 \times \ldots \times A^p).$$

We now check that the convergence holds in L^n , and that the n^{th}-moment of $\tilde{\alpha}(A^1 \times \ldots \times A^p)$ is the right side of (1). To this end, it is enough to obtain the convergence of

(6) $\qquad E[\alpha_\varepsilon(A^1 \times \ldots \times A^p)^n]$

$$= \int_{(\mathbb{R}^2)^n} dy_1\ldots dy_n \prod_{j=1}^{p} \Big(\int_{(A^j)^n} ds_1\ldots ds_n\ E\Big[\prod_{k=1}^{n} \delta^{\varepsilon}_{(y_k)}(B^j_{s_k})\Big]\Big)$$

$$= \int_{(\mathbb{R}^2)^n} dy_1\ldots dy_n \prod_{j=1}^{p} \Big(\sum_{\sigma\in\Sigma_n} \int_{(A^j)^n_<} ds_1\ldots ds_n\ E\Big[\prod_{k=1}^{n} \delta^{\varepsilon}_{(y_{\sigma(k)})}(B^j_{s_k})\Big]\Big)\ .$$

Clearly, for $(s_1,\ldots,s_n) \in (A^j)^n_<$,

$$\lim_{\varepsilon\to 0} E\Big[\prod_{k=1}^{n} \delta^{\varepsilon}_{(y_{\sigma(k)})}\ (B^j_{s_k})\Big] = p_{s_1}(x^j,y_{\sigma(1)}) \prod_{k=2}^{n} p_{s_k-s_{k-1}}(y_{\sigma(k-1)},y_{\sigma(k)}).$$

Again we have to justify dominated convergence. This is similar (in fact easier) to what we did in the first step. Indeed, the Markov property at times $s_{n-1}, s_{n-2}, \ldots, s_1$ leads to :

$$E\Big[\prod_{k=1}^{n} \delta^{\varepsilon}_{(y_{\sigma(k)})}(B^j_{s_k})\Big] \le 4\ \psi\Big(y_{\sigma(1)}-x^j,s_1\Big)\ \times \ldots \times\ 4\ \psi\Big(y_{\sigma(n)}-y_{\sigma(n-1)},s_n-s_{n-1}\Big)$$

with the same function ψ as above (consider separately the cases $|y_{\sigma(k)}-y_{\sigma(k-1)}| \ge 4\varepsilon$, and $|y_{\sigma(k)}-y_{\sigma(k-1)}| < 4\varepsilon$). Then the bound (5) justifies the passage to the limit in the right side of (6).

<u>Third step</u> : We will now construct a random measure $\alpha(\cdot)$ such that for any $A^1,\ldots, A^p$, $\alpha(A^1 \times \ldots \times A^p) = \tilde{\alpha}(A^1 \times \ldots \times A^p)$ a.s. We first consider the case $A^j = [a_j,b_j]$, $a_j \le b_j \le M$. Then by applying the (generalized) Hölder inequality to the right side of (1) we get :

$$E\left[\tilde{\alpha}(A_1 \times \ldots \times A_p)^n\right]$$

$$\le (n!)^p \prod_{j=1}^{p} \left(\int_{(\mathbb{R}^2)^n} dy_1\ldots dy_n \left(\int_{(A^j)^n_<} ds_1\ldots ds_n\, p_{s_1}(x^j,y_1) \prod_{k=2}^{n} p_{s_k-s_{k-1}}(y_{k-1},y_k)\right)^p\right)^{1/p}$$

$$\le (n!)^p \prod_{j=1}^{p} \left(\int_{(\mathbb{R}^2)^n} dy_1\ldots dy_n\, G^{a_j,b_j}(x^j,y_1)^p \prod_{k=2}^{n} G^{0,b_j-a_j}(y_{k-1},y_k)^p \right)^{1/p}$$

where :

$$G^{u,v}(x,y) = \int_u^v ds\, p_s(x,y).$$

It is easy to check that :

$$\int dy\, G^{u,v}(x,y)^p \le C_p(v-u)$$

for some constant C_p (use scaling when $u = 0$). We conclude that :

$$E\left[\tilde{\alpha}(A^1 \times \ldots \times A^p)^n\right] \le (C_p)^n\, (n!)^p \prod_{j=1}^{p} (b_j-a_j)^{n/p}. \tag{7}$$

It follows from (7) and the multidimensional version of Kolmogorov's lemma that the mapping

$$(a_1,b_1,a_2,b_2,\ldots,a_p,b_p) \longrightarrow \tilde{\alpha}\Big([a_1,b_1] \times\ldots\times [a_p,b_p]\Big)$$

has a continuous version, denoted by $\alpha([a_1,b_1] \times\ldots\times [a_p,b_p])$. Notice that w.p 1, $\alpha([a_1,b_1] \times\ldots\times [a_p,b_p])$ is a nondecreasing finitely additive function of $[a_1,b_1] \times\ldots\times [a_p,b_p]$ (consider first the case of rational a_j, b_j). Standard measure-theoretic arguments allow us to extend $\alpha(\cdot)$ to a Radon measure on $(\mathbb{R}_+)^p$.

If $A^{(n)} = A_1^{(n)} \times A_2 \times \ldots \times A_p$ increases (resp. decreases) towards $A = A_1 \times A_2 \times \ldots \times A_p$ then $\tilde{\alpha}(A^{(n)})$ converges in L^2 towards $\tilde{\alpha}(A)$, (by (1)), whereas $\alpha(A^{(n)})$ converges a.s towards $\alpha(A)$. This observation and the monotone class theorem easily give :

$$\alpha(A_1 \times\ldots\times A_p) = \tilde{\alpha}(A_1 \times\ldots\times A_p)\ , \text{ a.s}$$

for any $A_1,\ldots,A_p$ bounded Borel subsets of $\mathbb{R}_+$.

Fourth step : It remains to check that α has the desired properties. The fact that $\alpha(\{s_j = t\}) = 0$ for every $t \geq 0$, a.s, is obvious from the continuity of

$$(a_1, b_1, \ldots, a_p, b_p) \longrightarrow \alpha([a_1, b_1] \times \ldots \times [a_p, b_p]).$$

Finally, suppose that $A = [a_1, b_1] \times \ldots \times [a_p, b_p]$ is a closed rectangle with rational coordinates. Then on the set

$$\mathcal{A} = \left\{ \omega \ ; \ A \cap \{(s_1, \ldots, s_p) \ ; \ B^1_{s_2} = B^2_{s_2} = \ldots = B^p_{s_p}\} = \emptyset \right\}$$

we have $\alpha_\varepsilon(A) = 0$ for ε small, by the definition of α_ε. Therefore $\alpha(A) = 0$ a.s on $\mathcal{A}$. Since this is true w.p. 1 for any rectangle with rational coordinates, the support property of α follows at once. □

Remark. It was convenient in the previous proof to assume that the starting point of each Brownian motion was deterministic. However it is immediate that Theorem 1 still holds in the more general situation where the starting points may be random. The right side of (1) should then be integrated with respect to $\mu^1(dx^1) \ldots \mu^p(dx^p)$ where $\mu^j(dx^j)$ stands for the initial distribution of B^j.

Proposition 2 : *Suppose that* $B^1_0 = \ldots = B^p_0$.

(i) *For any* $t \geq 0$, $\lambda > 0$, $\alpha([0, \lambda t]^p) \overset{(d)}{=} \lambda\, \alpha([0, t]^p)$.

(ii) *With probability one, for every* $t > 0$, $\alpha([0, t]^p) > 0$.

Proof : Without loss of generality we may take $B^1_0 = \ldots = B^p_0 = 0$. Property (i) follows from a simple scaling argument. Set

$$\tilde{B}^j_t = \lambda^{-1/2} B^j_{\lambda t} \qquad (j = 1, \ldots, p).$$

Then $\tilde{\alpha}_\varepsilon([0, t]^p) = \lambda^{-1} \alpha_{\lambda^{1/2}\varepsilon}([0, \lambda t]^p)$, which implies

$$\tilde{\alpha}\left([0, t]^p\right) = \lambda^{-1} \alpha\left([0, \lambda t]^p\right) \quad \text{a.s.}$$

To prove (ii), notice that the events $\{\alpha([0, t]^p) > 0\}$ decrease as t decreases. It follows from (i) that

$$P[\alpha([0, 1]^p) > 0) = P(\alpha([0, t]^p) > 0) = P[\bigcap_{s>0} \{\alpha([0, s]^p) > 0\}] .$$

However $P[\alpha([0, 1]^p) > 0] > 0$ since $E[\alpha([0, 1]^p)] > 0$. The zero-one law yields the desired result. □

Proposition 2 (ii) and Theorem 1 imply that, provided $B^1_0 = \ldots = B^p_0$, for any $\varepsilon > 0$, there exist $t_1, \ldots, t_p \in (0, \varepsilon)$ such that

$$B^1_{t_1} = \dots = B^p_{t_p} .$$

In the case of arbitrary starting points, one can use a scaling argument to check that these equalities hold for some $t_1, \dots, t_p \in (0,\infty)$, w.p 1. Therefore the paths of $B^1,\dots,B^p$ have a common point (different from their starting point).

2. Intersections of independent Wiener sausages.

Our goal in this section is to provide an approximation of the intersection local time in terms of Wiener sausages. This approximation is similar to the well-known approximations of the usual Brownian local time.

We fix a non-polar compact subset K of $\mathbb{R}^2$ and for $j \in \{1,\dots,p\}$ we denote by $S^j_{\varepsilon K}(0,t)$ the Wiener sausage associated with the Brownian motion B^j and the compact set εK, on the time interval $[0,t]$:

$$S_{\varepsilon K}(0,t) = \bigcup_{0\le s\le t} (B_s + \varepsilon K) .$$

As we have seen in chapter VI,

$$\lim_{\varepsilon\to 0} (\log 1/\varepsilon)\ m(S^j_{\varepsilon K}(0,t)) = \pi t ,$$

in the L^2-norm.

Theorem 3 : *We have :*

$$\lim_{\varepsilon\to 0} (\log 1/\varepsilon)^p\ m(S^1_{\varepsilon K}(0,t) \cap \dots \cap S^p_{\varepsilon K}(0,t)) = \pi^p\ \alpha([0,t]^p).$$

in the L^2*-norm.*

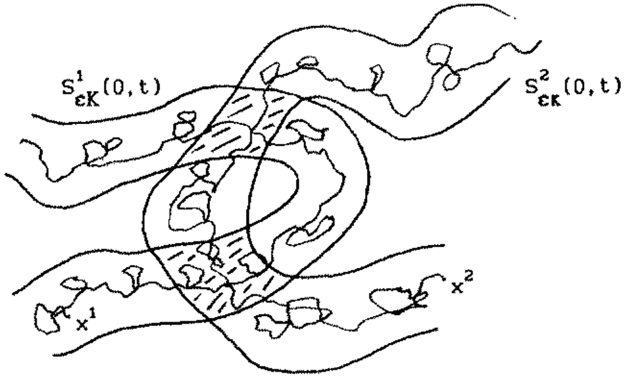


Fig. 1

Remark. The convergence of Theorem 3 holds in the L^n-norm for any $n < \infty$. However we shall restrict our attention to the L^2-convergence.

Proof : To simplify notation, we will assume that the starting points $B_0^1 = x^1$, ..., $B_0^p = x^p$ are deterministic. We fix $t > 0$. Then

$$\alpha_\varepsilon([0,t]^p) = \int_{\mathbb{R}^2} ds\ X_\varepsilon(y)$$

where

$$X_\varepsilon(y) = \prod_{j=1}^{p} \int_0^t ds\ \delta^\varepsilon_{(y)}(B^j_s).$$

Similarly,

$$\pi^{-p}(\log 1/\varepsilon)^p\ m\Big(S^1_{\varepsilon K}(0,t) \cap \dots \cap S^p_{\varepsilon K}(0,t)\Big) = \int_{\mathbb{R}^2} dy\ Y_\varepsilon(y)$$

where

$$Y_\varepsilon(y) = \prod_{j=1}^{p} \Big(\pi^{-1}(\log 1/\varepsilon)\ I(y \in S^j_{\varepsilon K}(0,t))\Big).$$

Therefore Theorem 3 is equivalent to:

$$\lim_{\varepsilon\to 0} E\left[\left(\int_{\mathbb{R}^2} dy\Big(X_\varepsilon(y) - Y_\varepsilon(y)\Big)\right)^2\right] = 0. \tag{8}$$

Write

$$X_\varepsilon(y) = \prod_{j=1}^{p} X^j_\varepsilon(y)\ , \quad Y_\varepsilon(y) = \prod_{j=1}^{p} Y^j_\varepsilon(y)$$

with an obvious notation. Then

$$(9)\quad E\left[\left(\int_{\mathbb{R}^2} dy\ (X_\varepsilon(y) - Y_\varepsilon(y))\right)^2\right]$$

$$= \int_{(\mathbb{R}^2)^2} dydz\left(\prod_{j=1}^{p} E\Big[X^j_\varepsilon(y)\ X^j_\varepsilon(z)\Big] - 2\prod_{j=1}^{p} E\Big[X^j_\varepsilon(y)\ Y^j_\varepsilon(z)\Big] + \prod_{j=1}^{p} E\Big[Y^j_\varepsilon(y)\ Y^j_\varepsilon(z)\Big]\right).$$

We will investigate the limiting behavior of each term of the right side of (9). We assume that $y \neq z$, $y,z \neq x^j$. Then,

$$(10)\quad \lim_{\varepsilon\to 0} E\Big[X^j_\varepsilon(y)\ X^j_\varepsilon(z)\Big] = \int_0^t ds \int_s^t ds'\Big(p_s(x^j,y)p_{s'-s}(y,z) + p_s(x^j,z)p_{s'-s}(z,y)\Big)$$

$$=: F_t(x^j,\ y,\ z).$$

Furthermore the bounds of the proof of Theorem 1 give :

$$E\Big[X^j_\varepsilon(y)\ X^j_\varepsilon(z)\Big] \le C\Big(G_1(0,y/2) + G_1(0,z/2)\Big)\ G_1(0,(z-y)/2).$$

Next, we have :

$$E\left[X^j_\varepsilon(y)\ Y^j_\varepsilon(z)\right] = \pi^{-2}\ \varepsilon^{-2}(\log\ 1/\varepsilon)\ E\left[\left(\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\right) I\left(z\in S_{\varepsilon K}(0,t)\right)\right].$$

Set $T^j_{\varepsilon K}(z) = \inf\{t\geq 0\ ;\ B^j_t\in z-\varepsilon K\}$. Then,

$$E\left[I(z\in S^j_{\varepsilon K}(0,t))\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\right]$$

$$= E\left[I(T^j_{\varepsilon K}(z)\leq t)\int_{T^j_{\varepsilon K}(z)}^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\right] + E\left[\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\ I\left(s< T^j_{\varepsilon K}(z)\leq t\right)\right].$$

Lemma VI.2 and the Markov property at time $T^j_{\varepsilon K}(z)$ give

$$\lim_{\varepsilon\to 0}\ \pi^{-2}\varepsilon^{-2}(\log\ 1/\varepsilon)\ E\left[I\left(T^j_{\varepsilon K}(z)\leq t\right)\int_{T^j_{\varepsilon K}(z)}^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\right]$$
$$= \int_0^t ds'\ p_{s'}(x^j,z)\int_{s'}^t ds\ p_{s-s'}(z,y).$$

Next,

$$E\left[\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\ I\left(s< T^j_{\varepsilon K}(z)\leq t\right)\right] = E\left[\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\ I\left(z\in S^j_{\varepsilon K}(s,t)\right)\right]$$
$$- E\left[\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\ I\left(z\in S^j_{\varepsilon K}(0,s)\cap S^j_{\varepsilon K}(s,t)\right)\right].$$

On one hand, Lemma VI.2 (ii) implies

$$\lim_{\varepsilon\to 0}\pi^{-2}\varepsilon^{-2}(\log\ 1/\varepsilon)\ E\left[\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\ I\left(z\in S^j_{\varepsilon K}(s,t)\right)\right]$$
$$= \int_0^t ds\ p_s(x^j,y)\int_s^t ds'\ p_{s'-s}(y,z).$$

In fact we need a little more than the convergence of Lemma VI.2 (ii): A simple compactness argument shows that this convergence holds uniformly when y varies over a compact subset of $\mathbb{R}^2\setminus\{0\}$. On the other hand, the Markov property at $T_{\varepsilon K}(z)$ and the bounds of Lemma VI.1 give :

$$\varepsilon^{-2}(\log\ 1/\varepsilon)\ E\left[\int_0^t ds\ 1_{D(y,\varepsilon)}(B^j_s)\ I\left(z\in S^j_{\varepsilon K}(0,s)\cap S^j_{\varepsilon K}(s,t)\right)\right] = o\left((\log\ 1/\varepsilon)^{-1}\right)$$

as ε tends to 0. We conclude that

$$(11) \qquad \lim_{\varepsilon\to 0} E\Big[X^j_\varepsilon(y)\ Y^j_\varepsilon(z)\Big] = F_t(x^j,y,z).$$

Moreover, Lemma VI.1 and the previous arguments show that $E\Big[X^j_\varepsilon(y)\ Y^j_\varepsilon(z)\Big]$ satisfies the same bound as $E\Big[X^j_\varepsilon(y)\ X^j_\varepsilon(z)\Big]$.

Finally we consider

$$E\Big[Y^j_\varepsilon(y)\ Y^j_\varepsilon(z)\Big] = \pi^{-2}(\log 1/\varepsilon)^2\ P\Big[T^j_{\varepsilon K}(y) \le t,\ T^j_{\varepsilon K}(z) \le t\Big].$$

We have already noticed in the proof of Lemma VI.4 that this quantity satisfies the same bound as $E\Big[X^j_\varepsilon(y)\ X^j_\varepsilon(z)\Big]$. Since

$$P\Big[T^j_{\varepsilon K}(y) \le t,\ T^j_{\varepsilon K}(z) \le t\Big]$$
$$\le P\Big[T^j_{\varepsilon K}(y) \le t,\ z \in S^j_{\varepsilon K}(T^j_{\varepsilon K}(y),t)\Big] + P\Big[T^j_{\varepsilon K}(z) \le t,\ y \in S^j_{\varepsilon K}(T^j_{\varepsilon K}(z),t)\Big],$$

Lemma VI.2 and the Markov property give

$$(12) \qquad \limsup_{\varepsilon\to 0} E\Big[Y^j_\varepsilon(y)\ Y^j_\varepsilon(z)\Big] \le F_t(x^j,y,z).$$

We now pass to the limit in the right side of (9), using (10), (11), (12). Observe that the use of dominated convergence is justified by our bounds and the fact that $G_1(0,y)$ is in L^n for any $n < \infty$. It follows that :

$$\limsup_{\varepsilon\to 0} E\Big[\Big(\int_{\mathbb{R}^2} dy\ (X_\varepsilon(y) - Y_\varepsilon(z))\Big)^2\Big] \le 0.$$

This completes the proof of (8), and that of Theorem 3. □

3. Self-intersection local times.

We now consider only one Brownian motion B started at 0. We are interested in p-multiple points of the process B. The (p-multiple) self-intersection local time of B is the Radon measure on

$$\mathcal{T}_p := \{(s_1,\ldots,s_p) \in (\mathbb{R}_+)^p\ ;\ 0 \le s_1 < s_2 < \ldots < s_p\}\ ,$$

formally defined by :

$$\beta(ds_1 \ldots ds_p) = \delta_{(0)}(B_{s_1} - B_{s_2}) \ldots \delta_{(0)}(B_{s_{p-1}} - B_{s_p})\ ds_1 \ldots ds_p.$$

To construct β rigorously we proceed as in Section 1. For $\varepsilon > 0$, we set

$$\beta_\varepsilon(ds_1\ldots ds_p) = 1_{\mathcal{T}_p}(s_1,\ldots,s_p)\ \ \varphi_\varepsilon(B_{s_1},\ldots,B_{s_p})\ ds_1\ldots ds_p$$

where

$$\varphi_\varepsilon(z_1,\ldots,z_p) = \int dy \prod_{j=1}^{p} \delta^\varepsilon_{(y)}(z_j).$$

We then have the following analogue of Theorem 1.

Theorem 4 : *There exists w.p 1 a Radon measure* $\beta(ds_1 \ldots ds_p)$ *on* $\mathcal{T}_p$ *such that, for any compact subset of* $\mathcal{T}_p$ *of the form* $A_1 \times A_2 \ldots \times A_p$,

$$\beta(A_1 \times A_2 \times \ldots \times A_p) = \lim_{\varepsilon \to 0} \beta_\varepsilon(A_1 \times A_2 \times \ldots \times A_p) ,$$

in the L^n*-norm for any* $n < \infty$.

The measure $\beta(\cdot)$ *is w.p. 1 supported on :*

$$\{(s_1, \ldots, s_p) \in \mathcal{T}_p \ ; \ B_{s_1} = \ldots = B_{s_p}\} .$$

Moreover, $\beta(\{s_j = t\}) = 0$ *for any* $j \in \{1, \ldots, p\}$ *and any* $t \ge 0$, *a.s.*

Proof : We may find a countable collection of compact rectangles

$$I^m = [a_1^m, b_1^m] \times \ldots \times [a_p^m, b_p^m] \quad (0 \le a_1^m \le b_1^m < a_2^m \le b_2^m < \ldots < a_p^m \le b_p^m)$$

such that

(i) $\mathcal{T}_p = \bigcup_{m=1}^{\infty} I^m$;

(ii) if $m \ne m'$, $I^m \cap I^{m'}$ is contained in a finite union of "hyperplanes" $\{s_j = t\}$;

(iii) any compact subset of $\mathcal{T}_p$ intersects only a finite number of the rectangles I_m .

Fix one of these rectangles $I = [a_1, b_1] \times \ldots \times [a_p, b_p]$. For every $j \in \{1, \ldots, p\}$, define a process $(\Gamma_t^j,\ 0 \le t \le b_j - a_j)$ by :

$$\Gamma_t^j = B_{a_j + t} \ , \text{ for } \ t \in [0,\ b_j - a_j] .$$

Of course the processes $\Gamma^1, \ldots, \Gamma^p$ are not independent. However, the distribution of $(\Gamma^1, \ldots, \Gamma^p)$ is absolutely continuous with respect to that of p independent Brownian motions. More precisely, define p probability measures $\mu_1, \ldots, \mu_p$ on $\mathbb{R}^2$ by

$$\mu_1(dy) = \begin{cases} \delta_{(0)}(dy) & \text{if } a_1 = 0 \\ (2\pi)^{-1} \exp -|y| \ dy & \text{if } a_1 > 0 \end{cases}$$

and for $j \ge 2$,

$$\mu_j(dy) = (2\pi)^{-1} \exp -|y| \ dy \quad .$$

Denote by $\mathcal{W}(dw_1 \ldots dw_p)$ the joint distribution of $(B^1, \ldots, B^p)$, where $B^1, \ldots, B^p$ are independent and each B^j is a planar Brownian motion defined on the time interval $[0,\ b_j - a_j]$, with initial distribution μ_j. The distribution of $(\Gamma^1, \ldots, \Gamma^p)$ is then absolutely continuous w.r.t. W. The associated Radon-Nikodym density can be written explicitly, and straightforward estimates show that it belongs to $L^2(W)$.

It follows from this observation and Theorem 1 (use also the remark after Theorem 1) that there exists a (random) measure $\beta^I(ds_1 \ldots ds_p)$ supported on I such that, for any compact subset of I of the form $A_1 \times \ldots \times A_p$,

$$\beta^I(A_1 \times \ldots \times A_p) = \lim_{\varepsilon \to 0} \beta_\varepsilon(A_1 \times \ldots \times A_p)$$

in the L^n-norm for *any* $n < \infty$. Furthermore, β^I does not charge the hyperplanes, and is supported on $\{(s_1, \ldots, s_p) \in I \; ; \; B_{s_1} = \ldots = B_{s_p}\}$.

To complete the proof of Theorem 4, we simply set :

$$\beta = \sum_{m=1}^{\infty} \beta^{I_m}$$

It is easy to check that β has the desired properties. In particular, property (iii) ensures that β is a Radon measure. □

<u>Proposition 5</u>. *With probability 1, for any* $0 \leq a < b$,

$$\beta\left(\mathcal{T}_p \cap [a,b]^p \right) = +\infty .$$

<u>Proof</u> : We take $a = 0$, $b = 1$ (the extension is trivial). Set

$$I = I_0^0 = [0, \tfrac{1}{2p}] \times [\tfrac{2}{2p}, \tfrac{3}{2p}] \times \ldots \times [\tfrac{2(p-1)}{2p}, \tfrac{2p-1}{2p}]$$

and more generally for any $k \geq 0$, $\ell \in \{0, 1, \ldots, 2^k-1\}$,

$$I_\ell^k = [\ell 2^{-k}, \ell 2^{-k} + \frac{2^{-k}}{2p}] \times \ldots \times [\ell 2^{-k} + \frac{2(p-1)}{2p} 2^{-k}, \ell 2^{-k} + \frac{2p-1}{2p} 2^{-k}].$$

It is obvious that for any fixed k, the random variables $(\beta(I_\ell^k), \ell = 0, 1, \ldots, 2^k-1)$ are independent and identically distributed. Moreover, the scaling argument of the proof of Proposition 2 gives

$$\beta(I_\ell^k) \overset{(d)}{=} 2^{-k} \beta(I).$$

It follows that :

$$E\left(\sum_{\ell=1}^{2^k-1} \beta(I_\ell^k)\right) = E(\beta(I)) = C > 0$$

$$\mathrm{var}\left(\sum_{\ell=1}^{2^k-1} \beta(I_\ell^k)\right) = 2^{-k} \mathrm{var}(\beta(I)) = 2^{-k} C' .$$

Therefore,

$$\beta\left(\mathcal{T}_p \cap [0,1]^p\right) \geq \sum_{k=0}^{\infty} \left(\sum_{\ell=1}^{2^k-1} \beta(I_\ell^k)\right) = +\infty . \quad □$$

<u>Remark</u>. As a consequence of Theorem 4 and Proposition 5, we get the existence

of p-tuples $(s_1,\dots,s_p) \in \mathcal{T}_p$ such that $B_{s_1}=\dots=B_{s_p}$, that is, the existence of p-multiple self-intersections. Our derivation of this result is certainly not the shortest one. The construction of the self-intersection local time however yields much useful information about multiple points (see in particular Chapter IX).

Proposition 5 leads us to the so-called renormalization problems. For certain physical questions (especially in polymer models) it is desirable to define a random variable "measuring the number" of p-multiple self-intersections of the Brownian path, say on the time interval [0,1]. The natural candidate would be $\beta(\mathcal{T}_p \cap [0,1]^p)$ if this variable were finite. This raises the question of whether it is possible to define a "renormalized self-intersection local time" whose value on the set $\mathcal{T}_p \cap [0,1]^p$ would be finite. The answer is yes. The case $p = 2$ is easy with the tools developed up to now, and will be treated in the next section. The general case is much harder and will be considered in Chapter X.

4. Varadhan's renormalization and an application to the Wiener sausage.

In this section, we take $p = 2$ and we set

$$\mathcal{T} = \mathcal{T}_2 \cap [0,1]^2 .$$

For any $k \geq 0$ and $\ell \in \{0,\dots,2^k-1\}$ we set

$$A^k_\ell = [\frac{2\ell}{2^{k+1}}, \frac{2\ell+1}{2^{k+1}}) \times (\frac{2\ell+1}{2^{k+1}}, \frac{2\ell+2}{2^{k+1}}]$$

Notice that the sets A^k_ℓ form a partition of $\mathcal{T}$ (see fig. 2).

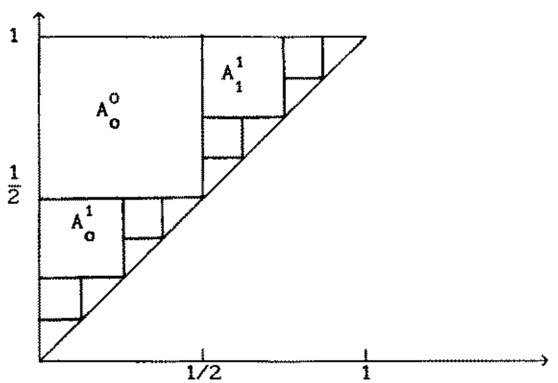


Fig. 2

Proposition 6. *For any Borel subset* A *of* $\mathcal{T}$, *the series*

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{2^k-1}\left(\beta(A\cap A_\ell^k)-E[\beta(A\cap A_\ell^k)]\right)\right)$$

converges a.s. and in L^2. *The sum of this series is denoted by* $\gamma(A)$, *and the mapping* $A\longrightarrow\gamma(A)$ *is called the renormalized self-intersection local time of* B.

Proof : Let $\alpha(\cdot)$ denote the intersection local time of two independent planar Brownian motions started at 0. We first observe that $\beta(A_\ell^k)\overset{(d)}{=}\alpha([0,2^{-k-1}]^2)$. Indeed, take $k=\ell=0$. The processes $B_t^1=B_{1/2-t}-B_{1/2}$, $B_t^2=B_{1/2+t}-B_{1/2}$ are two independent Brownian motions (defined on the time interval $[0,1/2]$) and, from our construction, it is obvious that $\beta(A_0^0)$ coincides with the intersection local time of B^1 and B^2 , on the square $[0,1/2]^2$.

Then, for any fixed k , the random variables $\beta(A\cap A_\ell^k)$, $\ell\in\{0,..,2^k-1\}$ are independent. This is clear since $\beta(A\cap A_\ell^k)$ only depends on the increments of B between times $2\ell\,2^{-k-1}$ and $(2\ell+2)2^{-k-1}$.

To complete the proof we bound

$$\mathrm{var}\left(\sum_{\ell=0}^{2^k-1}\beta(A\cap A_\ell^k)\right)=\sum_{\ell=0}^{2^k-1}\mathrm{var}\left(\beta(A\cap A_\ell^k)\right)\le\sum_{\ell=0}^{2^k-1}E\left[\beta(A_\ell^k)^2\right]=2^kE\left[\alpha([0,2^{-k-1}]^2)\right]=C2^{-k},$$

by Proposition 2. □

We will now apply Proposition 5 to a theorem concerning the fluctuations of the area of the two-dimensional Wiener sausage. By Theorem VI.6, this area is of order $\pi/(\log 1/\varepsilon)$ for ε small. The next theorem shows that the fluctuations of this area around its expected value are related to the (double) self-intersections of the process.

Theorem 7 : *Suppose that* K *is a non-polar compact subset of* $\mathbb{R}^2$. *Then,*

$$\lim_{\varepsilon\to0}(\log 1/\varepsilon)^2\left(m(S_{\varepsilon K}(0,1))-E[m(S_{\varepsilon K}(0,1))]\right)=-\pi^2\gamma(\mathcal{T})$$

in the L^2*-norm.*

Proof : To simplify notation, we write $\{U\}=U-E[U]$ for any integrable random variable U. Fix an integer $n\ge1$. We have:

$$(13)\qquad \{m(S_{\varepsilon K}(0,1))\}=\sum_{i=1}^{2^n}\{m(S_{\varepsilon K}(\tfrac{i-1}{2^n},\tfrac{i}{2^n}))\}$$
$$-\sum_{k=0}^{n-1}\sum_{\ell=0}^{2^k-1}\{m(S_{\varepsilon K}(\tfrac{2\ell}{2^{k+1}},\tfrac{2\ell+1}{2^{k+1}})\cap S_{\varepsilon K}(\tfrac{2\ell+1}{2^{k+1}},\tfrac{2\ell+2}{2^{k+1}}))\}.$$

Note that the variables $m(S_{\varepsilon\kappa}(\frac{i-1}{2^n},\frac{i}{2^n}))$, $i \in \{1,\ldots,2^n\}$ are independent. Then, by scaling and Theorem VI.5,

$$(14) \quad E[(\sum_{i=1}^{2^n} \{m(S_{\varepsilon\kappa}(\frac{i-1}{2^n}, \frac{i}{2^n}))\})^2]^{1/2} = 2^{n/2} E[\{m(S_{\varepsilon\kappa}(0, \frac{1}{2^n}))\}^2]^{1/2}$$

$$= 2^{-n/2} E[\{m(S_{\varepsilon 2^{-n/2}\kappa}(0,1))\}^2]^{1/2}$$

$$\le C\, 2^{-n/2} (\log 1/\varepsilon)^{-2} ,$$

for ε small (depending on n). On the other hand, by Theorem 3 and the arguments of the proof of Proposition 6,

$$(15) \quad L^2\text{-}\lim_{\varepsilon\to 0} (\log 1/\varepsilon)^2 \sum_{k=0}^{n-1} \sum_{\ell=0}^{2^k-1} \{m(S_{\varepsilon\kappa}(\frac{2\ell}{2^{k+1}}, \frac{2\ell+1}{2^{k+1}}) \cap S_{\varepsilon\kappa}(\frac{2\ell+1}{2^{k+1}}, \frac{2\ell+2}{2^{k+1}}))\}$$

$$= \pi^2 \sum_{k=0}^{n-1} \sum_{\ell=0}^{2^k-1} \{\beta(A_\ell^k)\} ,$$

and the latter sum is close to $\gamma(\mathcal{T})$ when n is large, by the definition of $\gamma(\mathcal{T})$.

To complete the proof, fix $\delta > 0$. We can choose n so that the right side of **(14)** is smaller than $(\delta/3)\ (\log 1/\varepsilon)^{-2}$ for ε small, and the L^2-norm of

$$\pi^2(\gamma(\mathcal{T}) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{2^k-1} \{\beta(A_\ell^k)\})$$

is less than $\delta/3$. Then by **(13)** and **(15)** the L^2-norm of

$$(\log 1/\varepsilon)^2 \{m(S_{\varepsilon\kappa}(0,1))\} - \pi^2 \gamma(\mathcal{T})$$

will be smaller than δ , for ε small. □

<u>Remark</u>. The minus sign in $-\pi^2 \gamma(\mathcal{T})$ corresponds to the intuitive idea that if there are many self-intersections then the area of the sausage will be smaller.

Spitzer [Sp2] obtains the following expansion for the expected area of the two-dimensional Wiener sausage:

$$E[m(S_{\varepsilon K}(0,1))] = \frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{(\log 1/\varepsilon)^2} (\frac{1+\kappa-\log 2}{2} + R(K)) + o(\frac{1}{(\log 1/\varepsilon)^2}),$$

where κ denotes Euler's constant, and $R(K)$ is the logarithm of the logarithmic capacity of K (see Chapter XI for a precise definition). We can combine this expansion with Theorem 7 to get:

$$m(S_{\varepsilon K}(0,1)) = \frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{(\log 1/\varepsilon)^2}\left(\frac{1+\kappa-\log 2}{2} + R(K) - \pi\,\gamma(\mathcal{J})\right) + \mathcal{R}(\varepsilon,K) ,$$

where

$$\lim_{\varepsilon\to 0} (\log 1/\varepsilon)^2\, \mathcal{R}(\varepsilon,K) = 0 ,$$

in the L^2-norm. This result will be extended in Chapter XI , where we will obtain a full asymptotic expansion of $m(S_{\varepsilon K}(0,1))$. The k^{th} term of this expansion is of order $|\log \varepsilon|^{-k}$ and involves a random variable related to the k-multiple self-intersections of B .

Bibliographical notes. *The notion of intersection local time was motivated by physical problems: see in particular Edwards* [E] *and Symanzik* [Sy]. *In appendix to Symanzik's paper* [Sy], *Varadhan gave a construction of the renormalized variable* $\gamma(\mathcal{J})$ *(in the more difficult case of the planar Brownian bridge), without introducing the intersection local time. The first work on intersection local times is probably due to Wolpert* [Wo]. *Dynkin* [Dy1] *gave a general construction of additive functionals of several independent Markov processes, which includes intersection local times as a particular case. See also* [Dy4] *for results in the special case of Brownian motion. Using a different approach, depending on the Gaussian character of Brownian motion, Geman, Horowitz and Rosen* [GHR] *derived precise information about the intersection local time of independent Brownian motions. The self-intersection local time of Brownian motion has been studied extensively by Rosen (*[R1], [R2]*) and Yor (*[Y3], [Y4]*). Rosen* [R5] *has extended some of his results to diffusion processes more general than Brownian motion. See also* [L8] *for the intersection local time of Lévy processes. In the case of Brownian motion, Rosen* [R3] *and Yor* [Y1] *prove Tanaka-like formulas for the intersection local time, and apply these formulas to the Varadhan renormalization. See also Yor* [Y2] *for a weak analogue of the Varadhan renormalization in three dimensions. The results of Section 2 are from* [L3] *(at least in the case* $K = D$ *), where they were applied to estimates concerning the Hausdorff measure of multiple points (see also Weinryb* [W2] *for some extensions). The methods of Sections 3 and 4 are taken from* [L2]. *Theorem 7 was proved in* [L2] *in the special case* $K = D$ *, and then extended in* [L10] *. The latter paper also contains fluctuation theorems for the Wiener sausage in higher dimensions. See also Chavel, Feldman and Rosen* [CFR] *for an extension of Theorem 7 to Brownian motion on Riemannian*

CHAPTER IX

Points of infinite multiplicity of the planar Brownian motion.

As a simple consequence of Theorem VIII-4 and Proposition VIII-5 we get that a planar Brownian path has p-multiple points for any integer p, w.p.1. Intersection local times certainly do not provide the shortest way of arriving at this result. Nonetheless, they can be used to get much useful information about multiple points. In this chapter we will use self-intersection local times to prove the existence of points of infinite multiplicity. The proof involves no technical estimate, mainly because the hard work has already been done in the previous chapter. The first section develops certain tools which are of independent interest.

1. The behavior of Brownian motion between the successive hitting times of a given multiple point.

Throughout this chapter, $B = (B_t, t \geq 0)$ is a planar Brownian motion started at 0. For every integer $p \geq 2$, we denote by β_p the random measure that was constructed in Theorem VIII-4.

Consider a double point $z = B_s = B_t$ for some $s < t$. One may expect that the path of B between times s and t looks like a Brownian loop with initial point z and length $t-s$. Recall that a Brownian loop with length T and initial point z is by definition a Brownian motion started at z and conditioned to be at z at time T. A simple example will show that some care is needed in order to make the previous affirmation rigorous. The easiest way of constructing a double point is to set

$$T = \inf\{t \geq 1 \; ; \; B_t \in B[0,1/2]\} ,$$

and

$$S = \sup\{s \leq 1/2 \; ; \; B_s = B_T\} .$$

Notice that $S < 1/2 < 1 < T$ a.s., and that S is certainly not a stopping time. The process $(B_{S+u}, 0 \leq u \leq T-S)$ turns out to be very different from a Brownian loop. Indeed, this process cannot perform small closed loops around its starting point as a Brownian loop would do, because this would contradict the definition of S.

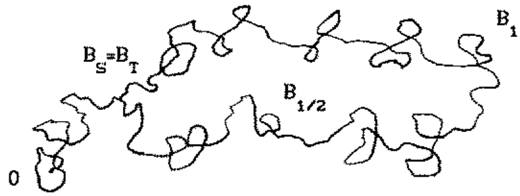


Fig. 1

This example does not mean that our previous heuristic affirmation is incorrrect. It can be explained by the fact that the double point $B_S = B_T$ is in some sense exceptional. To avoid these exceptional double points we will have to average over all double points. Averaging will simply mean integrating with respect to the self-intersection local time.

We need some notation. For $0 \le u \le v$ we set :

$$ {}_uB_v(t) = B_{(u+t)\wedge v} - B_u \,, $$
$$ {}_vB_u(t) = B_{(v-t)\vee u} - B_v \,, $$

so that ${}_uB_v, {}_vB_u$ define (random) elements of the space $C(\mathbb{R}^2)$ of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^2$.

If $L = (L(t), 0 \le t \le r)$ is a Brownian loop with initial point z and length r, we set $L(t) = z$ for $t > r$, by convention.

Finally, for any process Γ, we denote by Γ^t the process Γ stopped at time t $(\Gamma^t(s) = \Gamma(s \wedge t))$.

Theorem 1 : *Let* $p \ge 2$. *Then, for any Borel subset* A *of* $\mathcal{T}_p$ *and any non-negative measurable function* F *on* $C(\mathbb{R}^2)^{p+1}$,

$$ E\Big[\int_A \beta_p(ds_1 \ldots ds_p)\, F({}_0B_{s_1}, {}_{s_1}B_{s_2}, \ldots, {}_{s_p}B_\infty)\Big] $$

$$ = \int_A \frac{ds_1 \ldots ds_p}{(2\pi)^{p-1}(s_2-s_1)\ldots(s_p-s_{p-1})}\, E[F(\Gamma^{s_1}, L_{1,s_2-s_1}, \ldots, L_{p-1,s_p-s_{p-1}}, \Gamma')] $$

where the processes $\Gamma, \Gamma', L_{j,s_j-s_{j-1}}$ $(j \in \{2,\ldots,p\})$ *are independent,* Γ, Γ' *are two planar Brownian motions started at* 0 , *and, for* $j \in \{2,\ldots,p\}$, $L_{j,s_j-s_{j-1}}$ *is a Brownian loop with initial point* 0 *and length* $s_j - s_{j-1}$.

In particular, let H *be a Borel subset of* $C(\mathbb{R}^2)^{p+1}$ *such that, for* $ds_1 \ldots ds_p$ *a.a.* $(s_1,\ldots,s_p) \in \mathcal{T}_p$,

$$ P[(\Gamma^{s_1}, L_{1,s_2-s_1}, \ldots, L_{p-1,s_p-s_{p-1}}, \Gamma') \in H] = 1. $$

Then, w.p. 1, for $\beta_p(ds_1\ldots ds_p)$ *a.a.* $(s_1,\ldots,s_p) \in \mathcal{T}_p$,

$$({}_oB_{s_1}, {}_{s_1}B_{s_2}, \ldots, {}_{s_p}B_\infty) \in H.$$

Proof : The second assertion follows from the first one by taking for F the indicator function of the complement of H. We prove the first assertion in the case $p = 2$ (the general case is similar). We may assume that F is bounded and continuous and that A is a compact rectangle. Then it easily follows from Theorem VIII-4 that

$$E\left[\int_A \beta_2(ds_1ds_2)\, F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty)\right] = \lim_{\varepsilon\to 0} E\left[\int_A \beta_2^\varepsilon(ds_1ds_2)\, F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty)\right]$$

(we write β_p^ε instead of β_ε in Chapter VIII). However,

$$E\left[\int_A \beta_2^\varepsilon(ds_1ds_2)\, F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty)\right]$$

$$= \int_A ds_1ds_2\, (\pi\varepsilon^2)^{-2}\, E[m(D(B_{s_1},\varepsilon) \cap D(B_{s_2},\varepsilon))\, F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty)].$$

Then use the trivial observation

$$m(D(B_{s_1},\varepsilon) \cap D(B_{s_2},\varepsilon)) = m(D(0,\varepsilon) \cap D(B_{s_2}-B_{s_1},\varepsilon))$$

and condition with respect to $B_{s_2} - B_{s_1}$. It follows that :

$$E\left[\int_A \beta_2^\varepsilon(ds_1ds_2)\, F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty)\right] = \int_A ds_1ds_2 \int_{|y|\le 2\varepsilon} dy\; p_{s_2-s_1}(0,y)$$

$$\times\, (\pi\varepsilon^2)^{-2}\, m(D(0,\varepsilon) \cap D(y,\varepsilon))\, E[F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty) \mid B_{s_2} - B_{s_1} = y].$$

To complete the proof notice that

$$\lim_{y\to 0} E[F({}_oB_{s_1}, {}_{s_1}B_{s_2}, {}_{s_2}B_\infty) \mid B_{s_2} - B_{s_1} = y] = E[F(\Gamma^{s_1}, L_{1,s_2-s_1}, \Gamma')]$$

and

$$\int dy\, (\pi\varepsilon^2)^{-2}\, m(D(y,\varepsilon) \cap D(0,\varepsilon)) = 1. \;\square$$

Theorem 1 is certainly not a deep result. If we replace $\beta_p(ds_1\ldots ds_p)$ by its formal definition

$$\beta_p(ds_1\ldots ds_p) = \delta_{(o)}(B_{s_2} - B_{s_1})\ldots\delta_{(o)}(B_{s_p} - B_{s_{p-1}})\, ds_1\ldots ds_p$$

then the first assertion of Theorem 1 becomes almost obvious. The second assertion of Theorem 1 will however be useful as it provides a (very weak) form of the Markov property, at times which are typically not stopping times. Indeed, it shows that, for a typical multiple point, the behavior of the process before or after the successive hitting times of this multiple point is similar to that of a Brownian motion or a Brownian loop. Notice that the notion of intersection local time is needed to say what a "typical multiple point" is.

As a first application of Theorem 1, we state a result which shows that the points of multiplicity $p+1$ are very rare among the p-multiple points.

Proposition 2 : *With probability 1, for* β_p*-a.a.* $(s_1,\dots,s_p)$*, the point* $B_{s_1}=\dots=B_{s_p}$ *is not a* $(p+1)$*-multiple point.*

Remark : Proposition 2 is also valid for $p = 1$, in which case β_1 should be interpreted as the Lebesgue measure on $\mathbb{R}_+$. If $\ell_p(dz)$ denotes the image measure of β_p under the mapping $(s_1,\dots,s_p) \longrightarrow B_{s_1}$, then ℓ_p is in some sense the canonical measure on the set of p-multiple points, and Proposition 2 shows that the measures $\ell_p (p = 1,2,\dots)$ are singular w.r.t. each other.

Proof : For $\varphi \in C(\mathbb{R}^2)$ set

$$\zeta(\varphi) = \inf\{t > 0 \ ; \ \varphi \text{ is constant on } [t,\infty)\}$$

$(\inf \emptyset = +\infty)$ and

$$H = \{(\varphi_0,\varphi_1,\dots,\varphi_p) \in C(\mathbb{R}^2)^{p+1} \ ; \ \forall t \in [0,\zeta(\varphi_0)), \varphi_0(t) \neq \varphi_0(\zeta(\varphi_0)) \text{ and for } j = 1,\dots,p,\ \forall t \in (0,\zeta(\varphi_j)), \varphi_j(t) \neq \varphi_j(0)\}.$$

The polarity of single points for planar Brownian motion implies that H satisfies the assumption of Theorem 1. The desired result follows from Theorem 1. □

2. Points of infinite multiplicity.

We say that two compact subsets K,K' of $\mathbb{R}$ have the same order type if there exists an increasing homeomorphism φ of $\mathbb{R}$ such that $\varphi(K) = K'$.

Theorem 3 : *Let* K *be a totally disconnected compact subset of* $\mathbb{R}$*. Then with probability 1 there exists a point* z *of the plane such that* $\{t \geq 0,\ B_t = z\}$ *has the same order type as* K *.*

Note that when K is a finite set, Theorem 3 just says that there exist points of multiplicity (exactly) p for any p. As a consequence of Theorem 3 we get the existence of points of (exactly) countable multiplicity, as well as the existence of points of uncountable multiplicity.

The proof of Theorem 3 relies on a key lemma, which itself is an easy consequence of Theorem 1. Let us first explain the need for this lemma. We start with a double point $z_1 = B_r = B_s$ (with $r < s$). Choose $\varepsilon > 0$ small (at least smaller than $(s-r)/2$) and consider the 4 paths ${}_rB_{r-\varepsilon}$, ${}_rB_{r+\varepsilon}$, ${}_sB_{s-\varepsilon}$, ${}_sB_{s+\varepsilon}$. We would like to say that these 4 paths are "not too different" from those of 4 independent Brownian motions started at z_1. If this is the case, the results of Chapter VIII allow us to find a common point other than z to these 4 paths. That is, we may find $t \in (r-\varepsilon, r)$, $u \in (r, r+\varepsilon)$, $v \in (s-\varepsilon, s)$, $w \in (s, s+\varepsilon)$ such that $B_t = B_u = B_v = B_w = z_2$. We may even choose z_2 as close to z_1 as we wish. We can then by similar arguments construct a point z_3 of multiplicity 8 close to z_2. At the n^{th} step we get a point z_n of multiplicity 2^n. It should then be clear that, if the construction is performed with enough care, the point $z = \lim z_n$ will be a point of infinite multiplicity (in fact $\{t \; ; \; B_t = z\}$ will contain a Cantor set).

$B_r = B_s$
First step.

$B_t = B_u = B_v = B_w$
Second step.

Fig. 2

The only trouble in the previous arguments comes from the assertion "the 4 paths ${}_rB_{r-\varepsilon}, \ldots$ are not too different from 4 independent Brownian paths". The next lemma will demonstrate that, for most of the double points $B_r = B_s$, these 4 paths behave like 4 independent Brownian paths, at least for the properties that are of interest here.

Lemma 4 : *With probability one, for* β_p*-a.a.* $(s_1, \ldots, s_p)$ *and for any* $\delta > 0$,

$$\beta_{2p}((s_1-\delta, s_1) \times (s_1, s_1+\delta) \times \ldots \times (s_p-\delta, s_p) \times (s_p, s_p+\delta)) > 0.$$

Proof : For any compact rectangle R in $\mathcal{T}_p$ we may find a sequence (ε_k) decreasing to 0 such that

$$\beta_p(R) = \lim_{k\to\infty} \beta_p^{\varepsilon_k}(R), \quad \text{a.s.} \tag{1}$$

We may assume that the same sequence (ε_k) works for any p and for any rectangle with rational coordinates. Then a monotonicity argument shows that (1) holds simultaneously for all (compact or non-compact) rectangles of $\mathcal{T}_p$. Finally, let α_p be the intersection local time of p independent Brownian motions started at 0 (see section VIII-1). We may assume that, with the same sequence (ε_k),

$$\alpha_p(R) = \lim_{k\to\infty} \alpha_p^{\varepsilon_k}(R), \tag{2}$$

for all compact rectangles in $(\mathbb{R}_+)^p$, a.s.

Let $f_o,f_1,\ldots,f_p \in C(\mathbb{R}^2)$ and let $\zeta_i = \zeta(f_i)$ be as in the proof of Proposition 2. If $\zeta(f_i) < \infty$ for every $i \in \{0,\ldots,p-1\}$, set

$$\ell_\delta(f_o,f_1,\ldots,f_p) = \liminf_{k\to\infty} \int_{[0,\delta]^{2p}} dt_1 \ldots dt_{2p}$$
$$\varphi^{2p}_{\varepsilon_k}(f_o(\zeta_o-t_1),f_1(t_2),f_1(\zeta_1-t_3),\ldots,f_{p-1}(t_{2p-2}),f_{p-1}(\zeta_{p-1}-t_{2p-1}),f_p(t_p))$$

where

$$\varphi^{2p}_{\varepsilon}(z_1,\ldots,z_{2p}) = \int_{\mathbb{R}} dy \prod_{j=1}^{2p} \delta^{\varepsilon}_{(y)}(z_j)$$

as in Chapter VIII. Otherwise, set $\ell_\delta(f_o,\ldots,f_p) = 0$.

By looking at the finite-dimensional marginal distributions, it is very easy to check that, if L is a Brownian loop with length a, for any $\delta < a/2$, the joint distribution of $(L(t),L(a-t)\ ;\ 0 \le t \le \delta)$ is absolutely continuous with respect to that of two independent Brownian paths. It follows from this observation, (2), and Proposition VIII-2 (ii) that the set

$$H = \{(f_o,f_1,\ldots,f_p)\ ;\ \ell_\delta(f_o,f_1,\ldots,f_p) > 0,\ \forall\delta > 0\ \}$$

satisfies the assumption of Theorem 1. Therefore w.p. 1 for β_p-a.a. $(s_1,\ldots,s_p)$,

$$({}_oB_{s_1},{}_{s_1}B_{s_2},\ldots{}_{s_{p-1}}B_{s_p},{}_{s_p}B_\infty) \in H.$$

Lemma 4 follows using (1) and the definition of β^{ε}_{2p}. □

<u>Proof of Theorem 3</u> : We will show in detail how to construct a point z such that $\{t\ ;\ B_t = z\}$ contains a Cantor set. We set $t^o_1 = 1/2$, $z_o = B_{1/2}$, $\delta_o = 1/4$. We observe that for any $\delta > 0$

$$\beta_2((\tfrac{1}{2}-\delta,\tfrac{1}{2})\times(\tfrac{1}{2},\tfrac{1}{2}+\delta))>0,\quad \text{a.s.},$$

by the arguments of the proof of Proposition VIII-6. By Lemma **4** applied with $p = 2$, we may find a pair $(t_1^1, t_2^1) \in (1/4,1/2) \times (1/2,3/4)$ such that :

$$B_{t_1^1} = B_{t_2^1} =: z_1$$

and, for any $\delta > 0$,

$$\beta_4((t_1^1-\delta,t_1^1)\times(t_1^1,t_1^1+\delta)\times(t_2^1-\delta,t_2^1)\times(t_2^1,t_2^1+\delta))>0.$$

We proceed by induction on n . At the n^{th} step we have constructed

$$(t_1^n,\dots,t_{2^n}^n)\in(t_1^{n-1}-\delta_{n-1},t_1^{n-1})\times\dots\times(t_{2^{n-1}}^{n-1},t_{2^{n-1}}^{n-1}+\delta_{n-1})$$

in such a way that

$$B_{t_1^n}=\dots=B_{t_{2^n}^n}=: z_n\ ,$$

and for any $\delta > 0$,

$$\beta_{2^{n+1}}((t_1^n-\delta,t_1^n)\times(t_1^n,t_1^n+\delta)\times\dots\times(t_{2^n}^n-\delta,t_{2^n}^n)\times(t_{2^n}^n,t_{2^n}^n+\delta))>0.$$

We set $\delta_n = \frac{1}{4}(\delta_{n-1}\wedge\min(t_i^n-t_{i-1}^n\ ;\ i=2,\dots,2^n))$. By the induction hypothesis and Lemma **4** we find

$$(t_1^{n+1},\dots,t_{2^{n+1}}^{n+1})\in(t_1^n-\delta_n,t_1^n)\times\dots\times(t_{2^n}^n,t_{2^n}^n+\delta_n)$$

such that

$$B_{t_1^{n+1}}=\dots=B_{t_{2^{n+1}}^{n+1}}=: z_{n+1}$$

and for any $\delta > 0$

$$\beta_{2(n+2)}\ ((t_1^{n+1}-\delta,t_1^{n+1})\times\dots\times(t_{2^{n+1}}^{n+1},t_{2^{n+1}}^{n+1}+\delta))>0.$$

Finally the continuity of paths implies that the sequence (z_n) converges towards some $z \in \mathbb{R}^2$. Furthermore $\{t \geq 0\ ;\ B_t = z\}$ contains the closed set

$$K=\bigcap_{n=1}^{\infty}\overline{\left(\bigcup_{m=n}^{\infty}\{t_j^m\ ;\ j\in\{1,\dots,2^n\}\}\right)}.$$

Our construction (in particular the choice of the constants δ_n) ensures that K is a Cantor set.

By being a little more careful in the construction we can even get

$$K=\{t\geq 0, B_t=z\},$$

which gives Theorem 3 in the case of a Cantor set.

The general case requires some technical adjustments but no new idea. If for instance K is the union of a Cantor set and an isolated point located on the right of K, we proceed as follows. We construct t_1^1, t_2^1 as previously but in the second step we "forget" about the path during $(t_2^1-\delta_1, t_2^1)$ and we choose

$$(t_1^2, t_2^2, t_3^2) \in (t_1^1-\delta_1, t_1^1) \times (t_1^1, t_1^1+\delta_1) \times (t_2^1, t_2^1+\delta_1)$$

so that for any $\delta > 0$

$$\beta_5((t_1^2-\delta, t_1^2) \times (t_1^2, t_1^2+\delta) \times (t_2^2-\delta, t_2^2) \times (t_2^2, t_2^2+\delta) \times (t_3^2, t_3^2+\delta)) > 0$$

(this requires a new version of Lemma 4). At the $(n+1)^{th}$ step we construct

$$(t_1^{n+1}, \dots, t_{2^n+1}^{n+1}) \in (t_1^n-\delta_n, t_1^n) \times \dots \times (t_{2^{n-1}}^n, t_{2^{n-1}}^n+\delta_n) \times (t_{2^{n-1}+1}^n, t_{2^{n-1}+1}^n+\delta_n)$$

so that

$$B_{t_1^{n+1}} = \dots = B_{t_{2^n+1}^{n+1}} =: z_{n+1}$$

and, for any $\delta > 0$,

$$\beta_{2^{n+1}+1}((t_1^{n+1}-\delta, t_1^{n+1}) \times (t_1^{n+1}+\delta) \times \dots \times (t_{2^n}^{n+1}, t_{2^n}^{n+1}+\delta) \times (t_{2^n+1}^{n+1}, t_{2^n+1}^{n+1}+\delta)) > 0 .$$

The point $z = \lim z_n$ will satisfy the desired condition, again provided the construction is done with enough care.

Bibliographical notes. *The problem of the existence of points of finite multiplicity for a* d*-dimensional Brownian path was completely solved by Dvoretzky, Erdös and Kakutani* [DK1], [DK2] *and* [DKT] *in collaboration with Taylor. See Kahane* [Kh] *for an elegant modern approach. The existence of points of infinite multiplicity for a planar Brownian path was proved in* [DK3]. *However the given proof is not totally satisfactory: it seems that the authors apply the strong Markov property at certain random times that are typically not stopping times. The material of this Chapter is taken from* [L6], *to which we refer for a more detailed proof of Theorem 3. Proposition 2 is a rigorous form of Lévy's intuitive statement quoted in the introduction. See also Adelman and Dvoretzky* [AD] *for a weak form of this result. Another way of comparing the size of the sets of points of multiplicity* p *and* $p + 1$, *that was suggested by Lévy* [Lé4, p. 325-329], *is to use Hausdorff measures. The exact Hausdorff measure function for the set of* p*-multiple points is* $\varphi_p(x) = x^2 (\log 1/x \log\log\log 1/x)^p$ *(see* [L9], *for* $p = 1$, *this result is due to Taylor* [T1]*). A weaker form of this result had been conjectured by Taylor* [T2] *and proved in* [L3].

CHAPTER X

Renormalization for the powers of the occupation field of a planar Brownian motion

1. The main theorem.

Throughout this chapter $B = (B_t, t \geq 0)$ denotes a planar Brownian motion, which starts at z under the probability P_z. Let $p \geq 2$ be an integer. In chapter VIII, we introduced the (p-multiple) self-intersection local time of B as a Radon measure on

$$\mathcal{T}_p = \{(s_1,\ldots,s_p)\ ;\ 0 \leq s_1 <\ldots< s_p\},$$

supported on $\{(s_1,\ldots,s_p)\ ;\ B_{s_1} =\ldots= B_{s_p}\}$. This measure, denoted by β_p, is such that, for any compact rectangle $A \subset \mathcal{T}_p$,

$$\beta_p(A) = \lim_{\varepsilon\to 0} \int dy \int_A \varphi^y_\varepsilon(B_{s_1})\ldots\varphi^y_\varepsilon(B_{s_p})\, ds_1\ldots ds_p \tag{1}$$

in the L^2-norm. Here,

$$\varphi^y_\varepsilon(z) = (\pi\varepsilon^2)^{-1}\, 1_{D(y,\varepsilon)}(z).$$

We know that, for every $M > 0$, $\beta_p(\mathcal{T}_p \cap [0,M]^p) = \infty$ a.s. Our goal in this chapter is to define a renormalized version of $\beta_p(\mathcal{T}_p \cap [0,M]^p)$.

By (1) we have the formal expression

$$\beta_p(\mathcal{T}_p \cap [0,M]^p) = \int dy \int_{\mathcal{T}_p\cap[0,M]^p} ds_1\ldots ds_p\ \delta_{(y)}(B_{s_1})\ldots\delta_{(y)}(B_{s_p})$$

$$\text{"="}\ \frac{1}{p!}\int dy \left(\int_0^M ds\ \delta_{(y)}(B_s)\right)^p.$$

More generally, we shall introduce renormalized versions of the quantities

$$\int dy\ f(y) \left(\int_0^M ds\ \delta_{(y)}(B_s)\right)^p,$$

for $f : \mathbb{C} \longrightarrow \mathbb{R}$ bounded measurable. In this way we define what may be called the p-th power of the occupation field of B. Recall that the occupation field, or occupation measure, of B on $[0,M]$ is the measure

$$f \longrightarrow \int_0^M ds\ f(B_s),$$

whose formal density is

$$\int_0^M ds\ \delta_{(y)}(B_s).$$

As a matter of fact, the need for a renormalization of β_p is closely related to the singularity of the occupation measure with respect to Lebesgue measure.

We need some notation before stating our main result. First notice that in (1) φ_ε^y could be replaced by many other suitable approximations of the Dirac measure at y. In what follows, the most convenient approximation will be the uniform probability measure on the circle of radius ε centered at y, denoted by $C(y,\varepsilon)$. This leads us to the local time of B on $C(y,\varepsilon)$. This local time can be defined rigorously in several ways. The most elementary approach is to show that

$$(2) \qquad \lim_{\delta\to0} \frac{1}{4\pi\varepsilon\delta} \int_0^t 1_{\{\varepsilon-\delta<|B_s-y|<\varepsilon+\delta\}}\ ds =: \ell_\varepsilon^y(t)$$

exists in the L^2-norm, for any $t \geq 0$, $\varepsilon \in (0,1)$ and $y \in \mathbb{C}$. Alternatively, $\ell_\varepsilon^y(t)$ may be defined as $(2\pi\varepsilon)^{-1}$ times the usual (semi-martingale) local time of $|B_s-y|$, at level ε and at time t. Kolmogorov's lemma yields the existence of a continuous version of $(\varepsilon,y,t) \longrightarrow \ell_\varepsilon^y(t)$. From now on we shall only deal with this version.

The methods of Chapter VIII can be adapted to give :

$$(3) \qquad \beta_p(A) = \lim_{\varepsilon\to0} \int dy \int_A \ell_\varepsilon^y(ds_1)\dots\ell_\varepsilon^y(ds_p)$$

for any compact rectangle A (here $\ell_\varepsilon^y(ds)$ denotes the measure on $\mathbb{R}_+$ associated with the continuous nondecreasing function $t \longrightarrow \ell_\varepsilon^y(t)$). We shall not use (3) , except to motivate the next results, and we leave the proof as an exercise for the reader.

For technical reasons that will appear later, it turns out to be very convenient to work with Brownian motion killed at an independent exponential time. Therefore we fix $\lambda > 0$ and we let ζ denote an exponential time with parameter λ, independent of B. For any $\varepsilon > 0$ we set :

$$h_\varepsilon = -E_\varepsilon[\ell_\varepsilon^0(\zeta)].$$

Notice that $E_z[\ell_\varepsilon^y(\zeta)] = -h_\varepsilon$ whenever $|z-y| = \varepsilon$, by the rotational invariance of planar Brownian motion. It easily follows from (2) that

$$-h_\varepsilon = \int_0^\infty ds\ e^{-\lambda s} \int_{C(0,\varepsilon)} \pi_\varepsilon(0,dy)\ p_s(\varepsilon,y) = \int_{C(0,\varepsilon)} \pi_\varepsilon(0,dy)\ G_\lambda(\varepsilon,y)$$

where $\pi_\varepsilon(0,dy)$ is the uniform probability measure on $C(0,\varepsilon)$. Recall that

$$G_\lambda(y,z) = \frac{1}{\pi} K_o(\sqrt{2\lambda}|z-y|), \tag{4}$$

where K_o is the usual modified Bessel function. It follows that

$$G_\lambda(y,z) = \frac{1}{\pi} \log \frac{1}{|z-y|} + \frac{1}{\pi}\left(\frac{\log(2/\lambda)}{2} - \kappa\right) + O\left(|z-y|^2 \log \frac{1}{|z-y|}\right) \tag{4'}$$

where κ denote Euler's constant. Hence,

$$h_\varepsilon = -\frac{1}{\pi} \log \frac{1}{\varepsilon} - \frac{1}{\pi}\left(\frac{\log(2/\lambda)}{2} - \kappa\right) + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right) \tag{5}$$

using the harmonicity of $y \longrightarrow \log|y|$. We set

$$\Delta_p = \mathcal{T}_p \cap [0,\zeta)^p.$$

It follows from (3) that :

$$\lim_{\varepsilon\to 0} \int dy\ \frac{1}{p!}\ \ell_\varepsilon^y(\zeta)^p = \lim_{\varepsilon\to 0} \int dy \int_{\Delta_p} \ell_\varepsilon^y(ds_1)\ldots\ell_\varepsilon^y(ds_p) = \beta_p(\Delta_p) = \infty,$$

in probability (in fact this limit also holds a.s.). We get a renormalized version of $\beta_p(\Delta_p)$ by the following procedure. For every $\varepsilon > 0$, we replace $\ell_\varepsilon^y(\zeta)^p/p!$ by another polynomial of $\ell_\varepsilon^y(\zeta)$, with the same leading term, and coefficients of lower degree depending on ε. A suitable choice of these coefficients allows us to get an L^2-convergence as ε goes to 0.

Theorem 1 : *For every* $\varepsilon \in (0,1)$, $p \geq 1$ *set*

$$Q_\varepsilon^p(u) = \sum_{k=1}^{p} \binom{p-1}{k-1} (h_\varepsilon)^{p-k}\ \frac{u^k}{k!}\ .$$

For any bounded Borel function $f : \mathbb{C} \longrightarrow \mathbb{R}$, *set*

$$T_p^\varepsilon f = \int dy\ f(y)\ Q_\varepsilon^p(\ell_\varepsilon^y(\zeta)).$$

Then,

$$\lim_{\varepsilon\to 0} T_p^\varepsilon f =: T_p f$$

exists in the L^2*-norm.*

Remark : For $p = 1$, it can easily be checked that

$$T_1 f = \int_0^\zeta ds\ f(B_s)$$

(simply compute $E[(T_1^\varepsilon\varphi - T_1\varphi)^2]$, etc...).

Most of the remainder of this chapter is devoted to the proof of Theorem 1. Let us briefly discuss the contents of this result. In some sense, the random variables $T_p 1$ $(p = 2,3,\ldots)$ provide the renormalized versions of $\beta_p(\Delta_p)$ that we aimed to define. In the next chapter, we will prove that these quantities appear in the different terms of a full asymptotic expansion for the area of the planar Wiener sausage. This result will also allow us to relate $T_2 1$ to the renormalized self-intersection local time (for double points) discussed in Chapter VIII. The proof of Theorem 1 for a general function f is not more difficult than in the special case $f = 1$.

The simple form of the polynomials Q_ε^p will be explained in the proof below. Notice that we could use other approximations for the Dirac measure at y : a result analogous to Theorem 1 would then hold, with (essentially) the same limiting variables $T_p f$, but the renormalization polynomials would usually be much more complicated. For instance, the approximation could be given by the function φ_ε^y , so that $\ell_\varepsilon^y(\zeta)$ should be replaced by

$$(\pi\varepsilon^{-2})^1 \int_0^\zeta 1_{D(y,\varepsilon)}(B_s)ds.$$

However, already in this simple case, the renormalization polynomials cannot be written explicitly (see [Dy3,Dy5]).

2. Preliminary estimates.

The proof of Theorem 1 depends on certain precise estimates that will be derived in this section. We start with a lemma which explains the form of the polynomials Q_ε^p.

Lemma 2 : *Set*

$$\Lambda_p = \{(s_1,\ldots,s_p) \ ; \ 0 \le s_1 \le s_2 \le \ldots \le s_p < \zeta\}.$$

Then, for every $\varepsilon > 0$, $y \in \mathbb{C}$,

$$Q_\varepsilon^p(\ell_\varepsilon^y(\zeta)) = \int_{\Lambda_p} \ell_\varepsilon^y(ds_1) \prod_{i=2}^p (\ell_\varepsilon^y(ds_i) + h_\varepsilon \delta_{(s_{i-1})}(ds_i)).$$

<u>Proof</u> : First notice that the expression

$$\ell_\varepsilon^y(ds_1) \prod_{i=2}^{p} (\ell_\varepsilon^y(ds_i) + h_\varepsilon\, \delta_{(s_{i-1})}(ds_i))$$

gives a well-defined signed measure on the set Λ_p. Furthermore we may expand the product and get terms of the form

$$(h_\varepsilon)^k\ \ell_\varepsilon^y(ds_1)\ldots\ell_\varepsilon^y(ds_{j_1-1})\ \delta_{(s_{j_1-1})}\ (ds_{j_1})\ell_\varepsilon^y(ds_{j_1+1})\ldots\ell_\varepsilon^y(ds_{j_2-1})$$

$$\delta_{(s_{j_2-1})}\ (ds_{j_2})\ \ell_\varepsilon^y(ds_{j_2+1})\ \ldots\ \delta_{(s_{j_k-1})}(ds_{j_k})\ \ell_\varepsilon^y(ds_{j_k+1})\ldots\ell_\varepsilon^y(ds_p)$$

where $k \in \{0,1,\ldots,\ p-1\}$ and $1 < j_1 < j_2 <\ldots< j_k \le p$. Next, if we integrate such a measure over Λ_p, the effect of the Dirac masses is to force $s_i = s_{i+1}$ for $i \in \{j_1,\ldots,j_k\}$, and we are left with the integral :

$$(h_\varepsilon)^k \int_{\Lambda_{p-k}} \ell_\varepsilon^y(dt_1)\ \ell_\varepsilon^y(dt_2)\ldots\ell_\varepsilon^y(dt_{p-k}) = (h_\varepsilon)^k\ \frac{\ell_\varepsilon^y(\zeta)^{p-k}}{(p-k)!}\ .$$

Finally, for every k, we have $\binom{p-1}{k-1}$ possible choices of $j_1,\ldots,j_k$. □

To simplify notation we write $G(z-y) = G_\lambda(y,z)$. Notice that $G(z)$ is a nonincreasing function of $|z|$. By **(4)** and well-known properties of the function K_o, we may find two positive constants C_*, η such that for any $y,z \ne 0$, with $|y|/2 < |z| < 2|y|$,

$$|G(z) - G(y)| \le C_*|z-y|\ \left(\frac{1}{|y|} \exp - \eta|y|\right)$$

(simply notice that the derivative of the function $r \longrightarrow K_o(r)$ is bounded by $C\ r^{-1} \exp - \alpha r$). We set $g(r) = 2\ C_*\ r^{-1} \exp(-\eta r/2)$. Note that g is nonincreasing and that, under the previous assumptions on z, y ,

$$|G(z) - G(y)| \le |z-y|\ g(2|y|) \tag{6}$$

<u>Lemma 3</u> : *For every integer* $n \ge 1$, *for* $x,y,z \in \mathbb{C}$, $\varepsilon,\varepsilon' \in (0,1/2)$,, *set*

$$H^n_{\varepsilon,\varepsilon'}(x,y,z) = \begin{cases} E_x\left[\int_{\Lambda_n} \ell_\varepsilon^y(ds_1)\ell_{\varepsilon'}^z(ds_2)\ell_\varepsilon^y(ds_3)\ldots\ell_{\varepsilon'}^z(ds_n)\right] & \textit{if}\ \ n\ \ \textit{is even,} \\ E_x\left[\int_{\Lambda_n} \ell_\varepsilon^y(ds_1)\ell_{\varepsilon'}^z(ds_2)\ell_\varepsilon^y(ds_3)\ldots\ell_\varepsilon^y(ds_n)\right] & \textit{if}\ \ n\ \ \textit{is odd.} \end{cases}$$

Then, for any $x,y,z \in \mathbb{C}$, ε, $\varepsilon' \in (0,1/2)$ *such that* $|x-y| > 4(\varepsilon \vee \varepsilon')$, $|z-y| > 8(\varepsilon \vee \varepsilon')$ *and* $\frac{\varepsilon}{2} < \varepsilon' < 2\varepsilon$,

$$H^n_{\varepsilon,\varepsilon'}(x,y,z) \le G(\tfrac{y-x}{2})\, G(\tfrac{z-y}{4})^{n-1}$$

and

$$|H^n_{\varepsilon,\varepsilon'}(x,y,z) - G(y-x)G(z-y)^{n-1}|$$

$$\le \varepsilon\, (g(2|y-x|)G(z-y)^{n-1} + 2(n-1)\, G(\tfrac{y-x}{2})G(\tfrac{z-y}{4})^{n-2}g(|z-y|)).$$

Remark. When $n = 1$, $H^n_{\varepsilon,\varepsilon'}(x,y,z) = H^1_\varepsilon(x,y)$ depends only on ε, x, y and the bounds of Lemma 3 give:

$$H^1_\varepsilon(x,y) \le G(\tfrac{y-x}{2}) ,$$

$$|H^1_\varepsilon(x,y) - G(y-x)| \le \varepsilon\, g(2|y-x|) .$$

Proof : We use induction on n. For $n = 1$,

$$H^1_{\varepsilon,\varepsilon'}(x,y,z) = H^1_\varepsilon(x,y) = E_x[\ell^y_\varepsilon(\zeta)] = \int \pi_\varepsilon(y,dw)G(w-x).$$

However, if $w \in C(y,\varepsilon)$,

$$G(w-x) \le G(\tfrac{y-x}{2}) ,$$

$$|G(w-x) - G(y-x)| \le \varepsilon\, g(2|y-x|)$$

by (6) and our assumptions on x, y, ε. The desired bounds follow.

Now let $n \ge 2$. Assume that Lemma 3 holds at the order $n-1$. Using the Markov property of Brownian motion killed at time ζ , we get

$$H^n_{\varepsilon',\varepsilon}(x,y,z) = E_x\Big[\int_0^\zeta \ell^y_\varepsilon(ds_1)\, H^{n-1}_{\varepsilon',\varepsilon}(B_{s_1},z,y)\Big].$$

Notice that $\ell^y_\varepsilon(ds_1)$ a.e., $B_{s_1} \in C(y,\varepsilon)$. The induction hypothesis gives for any $w \in C(y,\varepsilon)$,

$$H^{n-1}_{\varepsilon',\varepsilon}(w,z,y) \le G(\tfrac{z-w}{2})\, G(\tfrac{z-y}{4})^{n-2} \le G(\tfrac{z-y}{4})^{n-1} .$$

The first bound of the lemma follows by using the case $n = 1$.

Next, by the induction hypothesis again, we have for every $w \in C(y,\varepsilon)$,

$$|H^{n-1}_{\varepsilon',\varepsilon}(w,z,y) - G(z-w)\, G(z-y)^{n-2}|$$

$$\le \varepsilon\Big(g(2|z-w|)G(z-y)^{n-2} + 2(n-2)\; g(|z-y|)G(\tfrac{z-w}{2})G(\tfrac{z-y}{4})^{n-3} \Big)$$

$$\le \varepsilon\; (2(n-2)+1)\; g(|z-y|)G(\tfrac{z-y}{4})^{n-2}\ .$$

Furthermore,

$$|G(z-w) - G(z-y)| \le \varepsilon\, g(2|z-y|)\ .$$

It follows that

$$|H^{n-1}_{\varepsilon',\varepsilon}(w,z,y) - G(z-y)^{n-1}| \le \varepsilon\; (2(n-2)+2)\; g(|z-y|)G(\tfrac{z-y}{4})^{n-2}\ .$$

and, by the first bound of the lemma with $n = 1$,

$$|H^{n}_{\varepsilon,\varepsilon'}(x,y,z) - E_x[\ell^y_\varepsilon(\zeta)]\; G(z-y)^{n-1}| \le 2(n-1)\varepsilon\; G(\tfrac{y-x}{2})\; g(|z-y|)G(\tfrac{z-y}{4})^{n-2}.$$

The proof is now completed by using the second bound of the lemma with $n = 1$:

$$|E_x[\ell^y_\varepsilon(\zeta)] - G(y-x)| \le \varepsilon\; g(2|y-x|)\ .\ \square$$

3. Proof of Theorem 1.

The main step of the proof of Theorem 1 is the following key lemma.

Lemma 4 : *There exists a positive constant* C_p *such that, for any* $x,y,z \in \mathbb{C}$, $\varepsilon,\varepsilon' \in (0,1/2)$, *with* $|y-x| > 4(\varepsilon \vee \varepsilon')$, $|z-x| > 4(\varepsilon \vee \varepsilon')$, $|z-y| > 8(\varepsilon \vee \varepsilon')$ *and* $\varepsilon/2 \le \varepsilon' \le 2\varepsilon$,

$$|E_x[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\; Q^p_{\varepsilon'}(\ell^z_{\varepsilon'}(\zeta))] - (G(y-x) + G(z-x))\; G(z-y)^{2p-1}|$$

$$\le C_p\; \varepsilon|\log\, \varepsilon|^{2p-2}((g(|y-x|) + g(|z-x|))\; G(z-y) + (G(\tfrac{y-x}{2}) + G(\tfrac{z-x}{2}))\; g(|z-y|)).$$

Proof : We use Lemma 2 to write :

$$|E_x[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\;\; Q^p_{\varepsilon'}(\ell^z_\varepsilon(\zeta))] = E_x\Big[\int_{\Lambda_p} \ell^y_\varepsilon(ds_1) \prod_{j=2}^{p} (\ell^y_\varepsilon(ds_j) + h_\varepsilon\delta_{(s_{j-1})}(ds_j))$$

$$\times \int_{\Lambda_p} \ell^z_{\varepsilon'}(dt_1) \prod_{k=2}^{p} (\ell^z_{\varepsilon'}(dt_k) + h_{\varepsilon'}\delta_{(t_{k-1})}(dt_k))\Big]$$

$$= E_x\Big[\int_{\Lambda_p\times\Lambda_p} \mu^{y,z}_{\varepsilon,\varepsilon'}(ds_1\dots ds_p\; dt_1\dots dt_p)\Big].$$

Here $\mu^{y,z}_{\varepsilon,\varepsilon'}$ is a signed measure on the product $\Lambda_p \times \Lambda_p$. We now need to order $s_1,\dots,s_p$, $t_1,\dots,t_p$. Each possible order is associated with a nondecreasing function $\varphi : \{1,\dots,p\} \longrightarrow \{0,1,\dots,p\}$ in the following way. For any such function φ, let

$$\Gamma_\varphi = \{(s_1,\dots,s_p,\ t_1,\dots,t_p) \in \Lambda_p \times \Lambda_p\ ;\ \forall i \in \{1,\dots,p\},\ t_{\varphi(i)} < s_i < t_{\varphi(i)+1}\}$$

where by convention $t_0 = 0$, $t_{p+1} = \zeta$. If $\varphi \neq \varphi'$ the corresponding sets Γ_φ, $\Gamma_{\varphi'}$ are disjoint. Moreover $(\Lambda_p \times \Lambda_p) - \bigcup_\varphi \Gamma_\varphi$ is contained in

$$\bigcup_{i,j\in\{1,\dots,p\}} \{s_i = t_j\}$$

and the $\mu^{y,z}_{\varepsilon,\varepsilon'}$-measure of this set is zero because of our assumptions on $y,z,\varepsilon,\varepsilon'$ (observe that $\mu^{y,z}_{\varepsilon,\varepsilon'}$ is supported on :

$$\{(s_1,\dots,s_p,\ t_1,\dots,t_p)\ ;\ \forall i,\ |B_{s_i}-y| = \varepsilon,\ |B_{t_i}-z| = \varepsilon'\})$$

In view of the previous observations we may write

$$\text{(7)} \qquad E_x[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\ Q^p_{\varepsilon'}(\ell^z_{\varepsilon'}(\zeta))] = \sum_\varphi E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)]$$

Remark that we could as well have introduced

$$\bar\Gamma_\psi = \{(s_1,\dots,s_p,\ t_1,\dots,t_p) \in \Lambda_p \times \Lambda_p\ ;\ \forall i \in \{1,\dots,p\},\ s_{\psi(i)} < t_i < s_{\psi(i)+1}\}$$

and that $\bar\Gamma_\psi = \Gamma_\varphi$ if and only if $\psi = \bar\varphi$, where :

$$\bar\varphi(j) = \sup\{i,\ \varphi(i) < j\} \qquad (\sup \emptyset = 0).$$

We first consider the simple situation where both φ and $\bar\varphi$ are strictly monotone (in other words s_i and s_{i+1} are always separated by at least one t_j, and conversely). This can only occur in the following two cases

$$\varphi(i) = \varphi_1(i) := i - 1,$$
$$\varphi(i) = \varphi_2(i) := i.$$

We have first

$$E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_{\varphi_1})] = E_x\left[\int_0^\zeta \ell^y_\varepsilon(ds_1) \int_{s_1}^\zeta \ell^z_{\varepsilon'}(dt_1) \int_{t_1}^\zeta \dots \int_{t_{p-1}}^\zeta \ell^y_\varepsilon(ds_p) \int_{s_p}^\zeta \ell^z_{\varepsilon'}(dt_p)\right].$$

(notice that the Dirac masses give no contribution, because of the choice of φ and because of the support property of $\mu^{y,z}_{\varepsilon,\varepsilon'}$). By Lemma 3,

$$\left|E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_{\varphi_1})] - G(y-x)\ G(z-y)^{2p-1}\right|$$

$$\le \varepsilon\ \left(g(2|y-x|)\ G(z-y)^{2p-1} + 2(2p-1)G(\tfrac{y-x}{2})g(|z-y|)G(\tfrac{z-y}{4})^{2p-2}\right).$$

Similarly, Lemma 3 gives :

$$\left|E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_{\varphi_2})] - G(z-x)\ G(z-y)^{2p-1}\right|$$

$$\le \varepsilon\ \left(g(2|z-x|)\ G(z-y)^{2p-1} + 2(2p-1)G(\tfrac{z-x}{2})g(|z-y|)G(\tfrac{z-y}{4})^{2p-2}\right).$$

Our assumptions on y, z allow us to bound

$$G(z-y) \le G(\tfrac{z-y}{4}) \le G(\varepsilon) \le C\ |\log\ \varepsilon|\ .$$

The proof of Lemma 4 will be complete if we can check that the other terms of the right side of (7) give a negligible contribution. To understand why this is so, let us consider the easy case where φ is such that $\varphi(p) = \varphi(p-1) = p$, which implies that Γ_φ is contained in

$$\{(s_1,\dots,s_p,\ t_1,\dots,t_p)\ ;\ t_p < s_{p-1} < s_p\}.$$

Then,

$$E_x[\mu^{y,x}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)] = E_x\left[\int_{\Lambda_{p-1}\times\Lambda_p} \tilde\mu^{y,x}_{\varepsilon,\varepsilon'}(ds_1\dots ds_{p-1}dt_1\dots dt_p)\ \left(\ell^y_\varepsilon((s_{p-1},\zeta)) + h_\varepsilon\right)\right]$$

where $\tilde\mu^{y,x}_{\varepsilon,\varepsilon'}(ds_1\dots dt_p)$ is a signed measure on $\Lambda_{p-1}\times\Lambda_p$, supported on $\{(s_1,\dots t_p)\ ;\ t_p<s_{p-1}$ and $|B_{s_{p-1}} - y| = \varepsilon\}$ and such that the bounded variation process $t \longrightarrow \tilde\mu^{y,x}_{\varepsilon,\varepsilon'}(\{s_{p-1}\le t\})$ is predictable. Replacing $\ell^y_\varepsilon((s_{p-1},\zeta))$ by its predictable projection gives :

$$E_x[\mu^{y,x}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)] = E_x\left[\int_{\Lambda_{p-1}\times\Lambda_p} \tilde\mu^{y,x}_{\varepsilon,\varepsilon'}(ds_1\dots ds_{p-1}dt_1\dots dt_p)(E_{B_{s_{p-1}}}[\ell^y_\varepsilon(\zeta)] + h_\varepsilon)\right].$$

By the very definition of h_ε ,

$$E_{y_\varepsilon}[\ell^y_\varepsilon(\zeta)] = -\ h_\varepsilon\ \ \text{if}\ \ |y_\varepsilon - y| = \varepsilon.$$

Therefore $E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)] = 0$ in this case.

We now turn to the general case where we only assume $\varphi \ne \varphi_1$, $\varphi \ne \varphi_2$. We may restrict our attention to the case when , for some $k \in \{1,\dots,p-1\}$,

$$\Gamma_\varphi \subset \{(s_1,\dots,s_p,t_1,\dots,t_p)\ ;\ t_k< s_k< s_{k+1}< t_{k+1}< s_{k+2}< t_{k+2}<\dots< s_p< t_p\}$$

(one should also consider the case

$$\Gamma_\varphi \subset \{t_{k+1} < s_k < s_{k+1} < t_{k+2} < s_{k+2} < \ldots < t_p < s_p\}$$

and the symmetric cases where the roles of s_i and t_i are interchanged ; all these cases however are treated in the same way). Then,

$$E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)] = E_x\Big[\int_{\Lambda_k\times\Lambda_k} \tilde{\mu}^{y,z}_{\varepsilon,\varepsilon'}(ds_1..ds_k dt_1..dt_k)\int_{s_k}^{\zeta} (\ell^y_\varepsilon(ds_{k+1})+h_\varepsilon\delta_{(s_k)}(ds_{k+1}))$$

$$\times \int_{s_{k+1}}^{\zeta} \ell^z_{\varepsilon'}(dt_{k+1})\int_{t_{k+1}}^{\zeta} \ell^y_\varepsilon(ds_{k+2}) \ldots \int_{t_{p-1}}^{\zeta} \ell^y_\varepsilon(ds_p)\int_{s_p}^{\zeta} \ell^z_{\varepsilon'}(dt_p)\Big]$$

where $\tilde{\mu}^{y,z}_{\varepsilon,\varepsilon'}$ is a measure on $\Lambda_k \times \Lambda_k$, supported on $\{(s_1,\ldots,t_k)\ ;\ t_k < s_k\ ,\ |B_{s_k} - y| = \varepsilon\}$ and such that the bounded variation process $t \longrightarrow \tilde{\mu}^{y,x}_{\varepsilon,\varepsilon'}(\{s_k \le t\})$ is predictable. Crude bounds show that the total variation $|\tilde{\mu}^{y,z}_{\varepsilon,\varepsilon'}|$ of $\tilde{\mu}^{y,z}_{\varepsilon,\varepsilon'}$ satisfies :

$$E_x\Big[\int_{\Lambda_k\times\Lambda_k} |\tilde{\mu}^{y,x}_{\varepsilon,\varepsilon'}|(ds_1\ldots ds_k dt_1\ldots dt_k)\ (\ell^y_\varepsilon((s_k,\zeta)) + |h_\varepsilon|)\Big]$$

$$\le C\ |\log \varepsilon|^{2k}\ (G(\tfrac{y-x}{2}) + G(\tfrac{z-x}{2}))$$

(use the bounds $E_x[\ell^y_\varepsilon(\zeta)] \le G(\frac{y-x}{2})$ and $\sup_{w\in C} E_w[\ell^y_\varepsilon(\zeta)] \le C'\ |\log \varepsilon|$).

Next , in the previous formula for $|E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)]$ we replace $\Big(\int_{s_{k+1}}^{\zeta} \ldots\Big)$ by its predictable projection, which coincides with

$$H^{2(p-k)-1}_{\varepsilon',\varepsilon}(B_{s_{k+1}}, z, y),$$

in the notation of Lemma 3. By Lemma 3, for $w \in C(y,\varepsilon)$,

$$|H^{2(p-k)-1}_{\varepsilon',\varepsilon}(w,z,y) - G(z-y)^{2(p-k)-1}| \le C\ \varepsilon\ |\log \varepsilon|^{2(p-k)-2}\ g(|z-y|)$$

(use again the bound $G(\frac{y-x}{4}) \le C\ |\log \varepsilon|$) and it follows that

$$\Big|E_x[\mu^{y,z}_{\varepsilon,\varepsilon'}(\Gamma_\varphi)] - G(z-y)^{2(p-k)-1}\ E_x\Big[\int_{\Lambda_k\times\Lambda_k} \tilde{\mu}^{y,z}_{\varepsilon,\varepsilon'}(ds_1\ldots dt_k)\ \ (\ell^y_\varepsilon((s_k,\zeta)) + h_\varepsilon)\Big]\Big|$$

$$\le C'\ \varepsilon\ |\log \varepsilon|^{2p-2}\ (G(\tfrac{y-x}{2}) + G(\tfrac{z-x}{2}))\ g(|z-y|)$$

by our previous bound on $|\mu^{y,z}_{\varepsilon,\varepsilon'}|$. This completes the proof since

$$E_x[\int \tilde{\mu}^{y,z}_{\varepsilon,\varepsilon'}(ds_1\ldots dt_k)\ (\ell^y_\varepsilon((s_k,\zeta)) + h_\varepsilon)] = 0,$$

by the same arguments as above (that is, by replacing $\ell^y_\varepsilon((s_k,\zeta))$ by its predictable projection), using the fact that $\mu^{y,z}_{\varepsilon,\varepsilon'}$ is supported on $\{t_k < s_k\}$.□

We now need to bound the contribution of pairs (y,z) that do not satisfy the assumption of Lemma 4.

Lemma 5. *There exists a constant* C'_p *such that for every* $\varepsilon,\ \varepsilon' \in (0,1/2)$ *such that* $\varepsilon/2 \le \varepsilon' \le 2\varepsilon$ *and every* $x,\ y,\ z \in \mathbb{C}$,

$$E_x[|Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\ Q^p_{\varepsilon'}(\ell^z_{\varepsilon'}(\zeta))|] \le C'_p\ |\log\ \varepsilon|^{2p-2}\ (G(\frac{y-x}{2}) + G(\frac{z-x}{2}))^{1/2}\ G(\frac{z-y}{2})^{1/2}.$$

Proof : We use the easy bound

$$|Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))| \le \ell^y_\varepsilon(\zeta)\ (\ \ell^y_\varepsilon(\zeta) + |h_\varepsilon|\ )^{p-1}$$

and we observe that $\ell^y_\varepsilon(\zeta) = 0$ unless $T_\varepsilon(y) < \zeta$, where

$$T_\varepsilon(y) = \inf\{\ s\ ;\ |B_s - y| \le \varepsilon\ \}\ .$$

Then the Cauchy-Schwarz inequality yields:

$$E_x[|Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\ Q^p_{\varepsilon'}(\ell^z_{\varepsilon'}(\zeta))|] \le P_x[\ T_\varepsilon(y) < \zeta\ ;\ T_{\varepsilon'}(z) < \zeta\ ]^{1/2}$$

$$\times\ E_x[\ell^y_\varepsilon(\zeta)^2(\ell^y_\varepsilon(\zeta) + |h_\varepsilon|)^{2(p-1)}\ \ell^z_{\varepsilon'}(\zeta)(\ell^z_{\varepsilon'}(\zeta) + |h_{\varepsilon'}|)^{2(p-1)}]^{1/2}.$$

Next we make use of the bound

$$P_x[\ T_\varepsilon(y) < \zeta\ ;\ T_{\varepsilon'}(z) < \zeta\ ] \le C\ |\log\ \varepsilon|^{-2}\ (G(\frac{y-x}{2}) + G(\frac{z-x}{2}))\ G(\frac{z-y}{2})$$

which follows from the techniques of Chapter VI (to bound $P[T_\varepsilon(y) \le T_{\varepsilon'}(z) < \zeta]$, apply the strong Markov property at $T_\varepsilon(y)$ and use Lemma VI-1 (iii)) . Also notice that for every integer $m \ge 1$,

$$\sup_{y \in \mathbb{C}}\ E_x[\ell^y_\varepsilon(\zeta)^m] = |h_\varepsilon|^m$$

(the supremum is attained for $y \in C(x,\varepsilon)$ and in this case the distribution of $\ell^y_\varepsilon(\zeta)$ is exponential with mean $|h_\varepsilon|$).

The previous bounds and another application of the Cauchy-Schwarz inequality lead to:

$$E_x[|Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))Q^p_{\varepsilon'}(\ell^z_{\varepsilon'}(\zeta))|] \le C\ |\log\varepsilon|^{-2}\ (G(\tfrac{y-x}{2}) + G(\tfrac{z-x}{2}))^{1/2}\ G(\tfrac{z-y}{2})^{1/2}$$

$$\times\ C''_p |h_\varepsilon h_{\varepsilon'}|^p.$$

Lemma 5 follows. □

We now turn to the proof of Theorem 1. We note that:

$$E_x[T^p_\varepsilon f\ T^p_{\varepsilon'} f] = \int dy\ dz\ f(y)\ f(z)\ E_x[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))Q^p_{\varepsilon'}(\ell^z_{\varepsilon'}(\zeta))].$$

We then apply Lemma 4 and we use Lemma 5 to bound the contribution of the pairs (y,z) that do not satisfy the assumptions of Lemma 4. We get:

$$|E_x[T^p_\varepsilon f\ T^p_{\varepsilon'} f] - 2\int dy\ dz\ f(y)\ f(z)\ G(y-x)\ G(z-y)^{2p-1}| \le C\ \varepsilon\ |\log\varepsilon|^{2p-2}$$

whenever $\varepsilon' \in [\varepsilon/2,\varepsilon]$, $\varepsilon \in (0,1/2)$ (notice that both functions $G(y)$, $g(|y|)$ are integrable avec $\mathbb{C}$). The previous bound implies that for $\varepsilon' \in [\varepsilon/2,\varepsilon]$,

$$E_x[(T^p_\varepsilon f - T^p_{\varepsilon'} f)^2] \le 4C\ \varepsilon\ |\log\varepsilon|^{2p-2}.$$

It follows that the sequence $T^p_{2^{-n}} f$ converges in the L^2-norm. If $T^p f$ denotes its limit, it is then immediate that :

$$T^p f = L^2 - \lim_{\varepsilon\to 0} T^p_\varepsilon f.$$

This completes the proof of Theorem 1. □

4. Remarks.

The previous proof gives more information than is stated in Theorem 1. We get an estimate of the rate of convergence of $T^p_\varepsilon f$ towards $T^p f$:

$$E[(T^p_\varepsilon f - T^p f)^2] \le C\ \|f\|^2_\infty\ \varepsilon\ |\log\varepsilon|^{2p-2} \tag{8}$$

for some constant C independent of f. We have also obtained the second moment of $T^p f$:

$$E_x[(T^p f)^2] = 2\int dy\ dz\ f(y)\ f(z)\ G(y-x)\ G(z-y)^{2p-1}$$

and, more generally,

$$E_x[T^p f\ T^p f'] = \int dy\ dz\ f(y)\ f'(z)\ (G(y-x) + G(z-x))\ G(z-y)^{2p-1}.$$

One can also check that

$$E_x[T^{p+1}f\ T^pf'] = \int dy\, dz\, f(y)\, f'(z)\, G(y-x)\, G(z-y)^{2p},$$

and that

$$E_x[T^pf\ T^qf'] = 0$$

whenever $|q-p| \geq 2$. These results are consequences of the following bounds, which hold under the assumptions of Lemma 4,

(9) $$|E_x[Q_\varepsilon^{p+1}(\ell_\varepsilon^y(\zeta))\, Q_{\varepsilon'}^p(\ell_{\varepsilon'}^z(\zeta))] - G(y-x)G(z-y)^{2p}| \leq \varepsilon\ |\log \varepsilon|^{2p-1}\ F(y-x,z-x)$$

and

(10) $$|E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\, Q_{\varepsilon'}^q(\ell_{\varepsilon'}^z(\zeta))]| \leq \varepsilon\ |\log \varepsilon|^{p+q-2}\ F'(y-x,z-x)$$

where the functions $F(y,z)$, $F'(y,z)$ are integrable over $\mathbb{C}^2$. To prove these bounds, proceed as in the proof of Lemma 4. In the first case one needs to order $s_1,\dots,s_{p+1}$, $t_1,\dots,t_p$. The order $s_1<t_1<s_2<t_2<\dots<t_p<s_{p+1}$ is the only one that gives a nonnegligible contribution. In the second case all orders give negligible contributions.

Finally, it is easy to check that:

$$E_x[T^pf] = 0,$$

for $p \geq 2$ (indeed, $E_x[T_\varepsilon^pf] = 0$ for every $\varepsilon > 0$).

The proof of Theorem 1 can be adapted to yield L^n-convergence for any $n \geq 1$. The previous formulas have analogues for higher-order moments. For instance the n^{th}-moment of T^pf is

$$E_x[(T^pf)^n] = \int dy_1\dots dy_n\ f(y_1)\dots f(y_n) \sum_\sigma \prod_{i=1}^n G(y_{\sigma(i)} - y_{\sigma(i-1)})$$

where the summation is over all mappings $\sigma : \{1,2,\dots,np\} \longrightarrow \{1,\dots,n\}$ such that $\sigma(i) \neq \sigma(i-1)$ for any $i \geq 2$, and $\text{card}\ \sigma^{-1}(j) = p$ for $j \in \{1,\dots,n\}$ (by convention $y_{\sigma(0)} = x$).

As a final remark, one may wonder what is the role of the exponential time ζ. The estimates of the proof of Theorem 1 depend heavily on the fact that we are working with Brownian motion stopped at an exponential time. Note that changing λ would only change h_ε by an additive constant. Suppose that we replace h_ε by

$$\tilde{h}_\varepsilon = h_\varepsilon + c$$

for some constant $c \in \mathbb{R}$. Let $\tilde{T}_\varepsilon^pf$ be defined accordingly. Then it is immediately seen that :

$$\tilde{T}^p_\varepsilon f = \sum_{k=1}^{p} \binom{p-1}{k-1} c^{p-k}\, T^k_\varepsilon f$$

and therefore we may define

$$\tilde{T}^p f := \lim_{\varepsilon \to 0} \tilde{T}^p_\varepsilon f = \sum_{k=1}^{p} \binom{p-1}{k-1} c^{p-k}\, T^k f .$$

Note that $\tilde{T}^p f$ can also be considered as a renormalized version of $\beta_p(\Delta_p \cap [0,\zeta)^p)$. This corresponds to the non-uniqueness of the renormalization.

Bibliographical notes. The renormalization for self-intersections of planar Brownian motion has been inspired by renormalization in field theory: see Dynkin [Dy2] *and the references in this paper. The existence of the renormalized powers of the occupation field was derived by Dynkin* [Dy3] *(in a slightly more general setting) using his isomorphism theorem between the occupation field of a symmetric Markov process and a certain Gaussian field. Later (in* [Dy5], [Dy6], [Dy7]*), Dynkin proposed a different approach, based on a detailed combinatorial analysis. The material of this Chapter is taken from* [L12]. *It has been inspired by Dynkin's second method, but it avoids the combinatorial analysis of Dynkin's work. Our construction is however not as general as Dynkin's one in* [Dy6]. *See also Rosen* [R4] *for a different method of renormalization (whose relationship with Dynkin's work is not clear) and Rosen and Yor* [RY] *for an approach based on stochastic calculus in the case of triple self-intersections. The renormalized fields* $T^p\varphi$ *appear in certain limit theorems for planar random walks: see Dynkin* [Dy6].

CHAPTER XI

Asymptotic expansions for the planar Wiener sausage

1. A random field associated with the Wiener sausage.

Let $S_K(a,b)$ denote the Wiener sausage associated with a planar Brownian motion B and a nonpolar compact subset K of $\mathbb{R}^2$, on the time interval $[a,b]$. By definition,

$$S_K(a,b) = \bigcup_{a\le s\le b} (B_s + K).$$

Our goal in this chapter is to get a full asymptotic expansion for $m(S_{\varepsilon K}(0,t))$ as ε goes to 0. The different terms of this expansion will be the renormalized self-intersection local times introduced in Chapter X, for all multiplicity orders $p \ge 1$. Note that the expansion at the order 2 has already been derived in Chapter VIII,

$$(1) \quad m(S_{\varepsilon K}(0,t)) = \frac{\pi}{\log 1/\varepsilon} + \frac{\pi}{(\log 1/\varepsilon)^2}\left(\frac{1+\kappa-\log 2}{2} - R(K) - \pi\,\gamma(\mathcal{T})\right) + o\left(\frac{1}{(\log 1/\varepsilon)^2}\right),$$

where κ denotes Euler's constant, and $\gamma(\mathcal{T})$ is the renormalized self-intersection local time that was defined in Section VIII-3 (note however that the proof of (1) required Spitzer's expansion of $E[m(S_{\varepsilon K}(0,1))]$).

The approach of this chapter depends heavily on the estimates of Chapter X, but is independent of the results of Chapter VI and VIII (except for the potential-theoretic rersults of Section VI-2). We will recover the expansion (1), as well as Spitzer's theorem, as a special case of Theorem 5 below.

From now on, we fix a compact subset K of $\mathbb{R}^2$. We assume that K has positive logarithmic capacity, that is

$$\mathrm{cap}(K) = \exp - \left(\inf_{\mu\in\mathcal{P}(K)} \iint_{K\times K} \mu(dx)\,\mu(dy)\,\log\frac{1}{|y-x|}\right) > 0$$

where $\mathcal{P}(K)$ is the set of all probability measures supported on K. By definition, $R(K) = \log \mathrm{cap}(K)$.

As in Chapter X, it will be convenient to deal with Brownian motion killed at an independent exponential time ζ with parameter λ. As previously, we let $G(y-x) = G_\lambda(x,y)$ denote the Green function of the killed process. Set

$$T_K = \inf\{t \geq 0,\ B_t \in K\}.$$

As was recalled in Chapter VI, we have for every $x \in \mathbb{R}^2\backslash K$

$$P_x[T_K < \zeta] = \int \mu_K^\lambda(dw)\ G(w-x), \tag{2}$$

where μ_K^λ, the λ-equilibrium measure of K, is a finite measure supported on K. The λ-capacity of K is $C_\lambda(K) = \mu_K^\lambda(K)$, and we have :

$$C_\lambda(K)^{-1} = \inf_{\mu \in \mathcal{P}(K)} \iint \mu(dx)\ \mu(dy)\ G(y-x). \tag{3}$$

An important role will be played by the constants a_ε defined for $\varepsilon > 0$ by

$$a_\varepsilon = -\,C_\lambda(\varepsilon K).$$

It easily follows from (3) and formula (4') of Chapter X that, as ε goes to 0,

$$\frac{1}{a_\varepsilon} = -\frac{1}{\pi}\log\frac{1}{\varepsilon} - \frac{1}{\pi}\left(\frac{\log 2/\lambda}{2} - \kappa - R(K)\right) + O(\varepsilon^2 \log\frac{1}{\varepsilon}).$$

For any bounded Borel function f on $\mathbb{R}^2$, we set :

$$S_\varepsilon^K f = \int dy\ f(y)\ 1_{S_{\varepsilon K}(0,\zeta)}(y).$$

<u>**Theorem 1**</u> : *Let* $n \geq 1$. *Then, for any bounded Borel function* f *on* $\mathbb{R}^2$,

$$S_\varepsilon^K f = -\sum_{p=1}^{n} (a_\varepsilon)^p\ T^p f + R_n(\varepsilon,f)$$

where the remainder $R_n(\varepsilon,f)$ *satisfies :*

$$\lim_{\varepsilon\to 0} |\log\varepsilon|^{2n}\ E[R_n(\varepsilon,f)^2] = 0.$$

In the special case $f = 1$, Theorem 1 provides an asymptotic expansion of $m(S_{\varepsilon K}(0,\zeta))$ in the L^2-norm. Using scaling arguments it is then possible to check that a similar expansion holds for $m(S_{\varepsilon K}(0,t))$, for any constant time $t > 0$. In fact, one can even get an almost sure expansion of $m(S_{\varepsilon K}(0,t))$ (see the end of this chapter).

Let us briefly outline the proof of Theorem 1. Thanks to the estimate (8) of Chapter X, it is enough to check that the given statement holds with $T^p f$ replaced by $T_\varepsilon^p f$. Then,

$$E\Big[\big(S^K_\varepsilon f + \sum_{p=1}^{n} (a_\varepsilon)^p\, T^p_\varepsilon f\big)^2\Big] = E\Big[\Big(\int dy\ f(y)\ \big(1_{S_{\varepsilon K}(0,\zeta)}(y) + \sum_{p=1}^{n} (a_\varepsilon)^p\, Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\big)\Big)^2\Big]$$

$$= \iint dy\ dz\ f(y)\ f(z)$$

$$\times\ E\Big[\Big(1_{S_{\varepsilon K}(0,\zeta)}(y) + \sum_{p=1}^{n} (a_\varepsilon)^p\, Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\Big)\Big(1_{S_{\varepsilon K}(0,\zeta)}(z) + \sum_{p=1}^{n} (a_\varepsilon)^p\, Q^p_\varepsilon(\ell^z_\varepsilon(\zeta))\Big)\Big].$$

Expanding the product inside the expectation sign, we are led to study the following three quantities :

(a) $$E[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\ Q^q_\varepsilon(\ell^z_\varepsilon(\zeta))]$$

This quantity was studied in detail in Chapter X, in the special case $p = q$. The general case offers no additional difficulty.

(b) $$P[y \in S_{\varepsilon K}(0,\zeta),\ z \in S_{\varepsilon K}(0,\zeta)].$$

Sharp estimates for this probability will be derived in Section 2.

(c) $$E[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\ 1_{S_{\varepsilon K}(0,\zeta)}(z)].$$

This quantity will be studied in Section 4, after some preliminary estimates have been established in Section 3.

2. The probability of hitting two small compact sets.

From now on, we shall assume that the compact set K is contained in the closed unit disk $\bar{D}$ (this restriction can be removed by a scaling argument). To simplify notation, we set

$$T_\varepsilon(y) = T_{y-\varepsilon K} = \inf\{t \geq 0\ ;\ B_t \in y - \varepsilon K\}$$

so that

$$P[y \in S_{\varepsilon K}(0,\zeta), z \in S_{\varepsilon K}(0,\zeta)] = P[T_\varepsilon(y) < \zeta,\ T_\varepsilon(z) < \zeta].$$

Lemma 2 : *Let* $n \geq 2$. *There exists a function* $F_n \in L^1((\mathbb{R}^2)^2, dy\ dz)$, *such that, for any* $\varepsilon \in (0,1/2)$, $y,z \in \mathbb{R}^2$ *with* $|y| > 4\varepsilon$, $|z| > 4\varepsilon$, $|z-y| > 4\varepsilon$,

$$\Big|P[T_\varepsilon(y) < \zeta,\ T_\varepsilon(z) < \zeta] - \sum_{p=2}^{n} (a_\varepsilon)^p\, (G(y)+G(z))G(z-y)^{p-1}\Big| \leq |\log \varepsilon|^{-n-1}\, F_n(y,z).$$

Proof : We will give details for $n = 2,3$. It will then be clear that the proof can be continued by induction on n. We first observe that

$$P[T_\varepsilon(y) < \zeta,\ T_\varepsilon(z) < \zeta] = P[T_\varepsilon(y) \leq T_\varepsilon(z) < \zeta] + P[T_\varepsilon(z) \leq T_\varepsilon(y) < \zeta].$$

Then,

$$P[T_\varepsilon(y) \le T_\varepsilon(z) < \zeta] = P[T_\varepsilon(y) \le T'_\varepsilon(z) < \zeta] - P[T_\varepsilon(z) \le T_\varepsilon(y) \le T'_\varepsilon(z) < \zeta]$$

where :

$$T'_\varepsilon(z) = \inf\{t \ge T_\varepsilon(y) \; ; \; B_t \in z-\varepsilon K\}.$$

By the Markov property at $T_\varepsilon(y)$,

$$P\Big[T_\varepsilon(y) \le T'_\varepsilon(z) < \zeta\Big] = E\Big[(T_\varepsilon(y) < \zeta) \; P_{B_{T_\varepsilon(y)}} [T_\varepsilon(z) < \zeta]\Big].$$

Notice that $B_{T_\varepsilon(y)} \in y - \varepsilon K \subset \overline{D}(y,\varepsilon)$. By (2) and formula (6) of Chapter X, we have for any $y_\varepsilon \in \overline{D}(y,\varepsilon)$,

$$(4) \qquad |P_{y_\varepsilon}[T_\varepsilon(z) < \zeta] + a_\varepsilon \, G(z-y)| = \Big|\Big(\int \mu^\lambda_{\varepsilon K}(dw) \; G(z-w-y_\varepsilon)\Big) + a_\varepsilon \, G(z-y)\Big|$$

$$\le |a_\varepsilon| \sup_{z' \in D(z,2\varepsilon)} |G(z'-y) - G(z-y)|$$

$$\le C \; \varepsilon \; |\log \varepsilon|^{-1} \; g(|z-y|)$$

where g is as in chapter X. Similarly,

$$|P[T_\varepsilon(y) < \zeta] + a_\varepsilon \, G(z-y)| \le C \; \varepsilon \; |\log \varepsilon|^{-1} \; g(|y|)$$

and (2) also gives

$$P[T_\varepsilon(y) < \zeta] \le |a_\varepsilon| \; G(\tfrac{y}{2}).$$

It follows from these estimates that

$$(5) \qquad |P[T_\varepsilon(y) \le T'_\varepsilon(z) < \zeta] - a^2_\varepsilon \, G(y) \, G(z-y)|$$

$$\le C \; \varepsilon \; |\log \varepsilon|^{-2} \Big(G(\tfrac{y}{2}) \; g(|z-y|) + g(|y|) \; G(z-y)\Big)$$

On the other hand, by applying the Markov property at $T_\varepsilon(y)$ and then at $T_\varepsilon(z)$, one easily gets

$$(6) \qquad P[T_\varepsilon(z) \le T_\varepsilon(y) \le T'_\varepsilon(z) < \zeta] \le C \; |\log \varepsilon|^{-3} \Big(G(\tfrac{z}{2}) \; G(\tfrac{z-y}{2})^2\Big).$$

This gives the case $n = 2$ of the Lemma.

In the case $n = 3$, we again use (5) but we replace (6) by :

$$P[T_\varepsilon(z) \le T_\varepsilon(y) \le T'_\varepsilon(z) < \zeta]$$

$$= P[T_\varepsilon(z) \le T'_\varepsilon(y) \le T''_\varepsilon(z) < \zeta] - P[T_\varepsilon(y) \le T_\varepsilon(z) \le T'_\varepsilon(y) \le T''_\varepsilon(z) < \zeta]$$

where

$$T'_\varepsilon(y) = \inf\{t \geq T_\varepsilon(z),\ B_t \in y - \varepsilon K\},$$
$$T''_\varepsilon(z) = \inf\{t \geq T'_\varepsilon(y),\ B_t \in z - \varepsilon K\}.$$

The bound **(4)** and the Markov property give :

$$|P[T_\varepsilon(z) \leq T'_\varepsilon(y) \leq T''_\varepsilon(z) < \zeta] - (a_\varepsilon)^3\ G(z)\ G(z-y)^2|$$

$$\leq C\ \varepsilon\ |\log \varepsilon|^{-3} \left(G(\tfrac{z}{2})\ G(\tfrac{z-y}{2})\ g(|z-y|) + g(|z|)\ G(\tfrac{z-y}{2})^2\right),$$

whereas it is easily checked that

$$P[T_\varepsilon(y) \leq T_\varepsilon(z) \leq T'_\varepsilon(y) \leq T''_\varepsilon(z) < \zeta] \leq C\ |\log \varepsilon|^{-4}\ G(\tfrac{y}{2})\ G(\tfrac{z-y}{2})^3. \ \square$$

Remark : It immediately follows from Lemma 2 that

$$E[m(S^K_\varepsilon f)^2] = \iint dy\ dz\ f(y)\ f(z)\ P[T_\varepsilon(y) < \zeta,\ T_\varepsilon(z) < \zeta]$$

$$= 2 \sum_{p=2}^{n} (a_\varepsilon)^p \int dy\ dz\ \ f(y)\ f(z)\ G(y)\ G(z-y)^{p-1} + O(|\log \varepsilon|^{-n-1}).$$

3. A preliminary lemma.

The study of the limiting behavior of the term $E[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))1_{S_{\varepsilon K}(0,\zeta)}(z)]$ requires the following lemma, which is analogous to Lemma X-3.

Lemma 3 : *Let* $n \geq 1$ *and* $n' = n$ or $n-1$. *Set :*

$$U^{n,n'}_\varepsilon(x,y,z) = E_x\left[\int\!\!\int_{\Lambda_n} \ell^z_\varepsilon(ds_1)\ \ell^z_\varepsilon(ds_2)\ldots\ell^z_\varepsilon(ds_n) \prod_{i=0}^{n'} 1_{S_{\varepsilon K}(s_i,s_{i+1})}(y)\right],$$

$$V^{n,n'}_\varepsilon(x,y,z) = E_x\left[\int\!\!\int_{\Lambda_n} \ell^y_\varepsilon(ds_1)\ \ell^y_\varepsilon(ds_2)\ldots\ell^y_\varepsilon(ds_n) \prod_{i=1}^{n'} 1_{S_{\varepsilon K}(s_i,s_{i+1})}(z)\right],$$

where by convention $s_0 = 0$, $s_{n+1} = \zeta$.

There exists a positive constant $C_{n,n'}$ *such that, for any* $x, y, z \in \mathbb{C}$, $\varepsilon \in (0,1/2)$, *with* $|y-x| \geq 4\varepsilon$, $|z-y| \geq 8\varepsilon$,

$$|U^{n,n'}_\varepsilon(x,y,z) - |a_\varepsilon|^{n'+1}\ G(y-x)\ G(z-y)^{n+n'}|$$

$$\leq C_{n,n'}\varepsilon\ |a_\varepsilon|^{n'+1}\ (g(2|y-x|)\ G(z-y)^{n+n'} + G(\tfrac{y-x}{2})\ g(|z-y|)G(\tfrac{z-y}{4})^{n+n'-1})$$

and, if $n' \geq 1$,

$$|V_\varepsilon^{n,n'}(x,y,z) - |a_\varepsilon|^{n'} \; G(y-x) \; G(z-y)^{n+n'-1}|$$

$$\leq C_{n,n'}\varepsilon \; |a_\varepsilon|^{n'} \; (g(2|y-x|) \; G(z-y)^{n+n'-1} + G(\tfrac{y-x}{2}) \; g(|z-y|)G(\tfrac{z-y}{4})^{n+n'-2}).$$

Proof : We consider only the case of $U_\varepsilon^{n,n'}$ and we further assume that $n' = n-1$. The other cases are treated in a similar manner.

We argue by induction on n. For $n = 1$,

$$U_\varepsilon^{1,0}(x,y,z) = E_x\Big[1_{(T_\varepsilon(y)<\zeta)} \; \ell_\varepsilon^z([T_\varepsilon(y),\zeta))\Big]$$

$$= E_x\Big[1_{(T_\varepsilon(y)<\zeta)} \; E_{B_{T_\varepsilon(y)}}[\ell_\varepsilon^z(\zeta)]\Big]$$

by the Markov property. However, by Lemma X-3, for any $y_\varepsilon \in y - \varepsilon K \subset D(y,\varepsilon)$,

$$|E_{y_\varepsilon}[\ell_\varepsilon^z(\zeta)] - G(z-y)| \leq \varepsilon \; g(2|z-y)|$$

and

$$E_{y_\varepsilon}[\ell_\varepsilon^z(\zeta)] \leq G(\tfrac{z-y}{2}).$$

Moreover, by (2),

$$|P_x[T_\varepsilon(y) < \zeta] - |a_\varepsilon| \; G(y-x)| \leq \varepsilon \; |a_\varepsilon| \; g(2|y-x|)$$

and

$$|P_x[T_\varepsilon(y) < \zeta] \leq |a_\varepsilon| \; G(\tfrac{y-x}{2}).$$

The case $n = 1$ follows readily from these bounds. We also get the bound:

(7) $$U_\varepsilon^{1,0}(x,y,z) \leq |a_\varepsilon| \; G(\tfrac{y-x}{2}) \; G(\tfrac{z-y}{2})$$

Next suppose that $n \geq 2$ and that the desired result holds at the order $n-1$. We have :

$$U_\varepsilon^{n,n-1}(x,y,z) = E_x\Big[1_{(T_\varepsilon(y)<\zeta)} \int_{T_\varepsilon(y)}^{\zeta} \ell_\varepsilon^z(ds_1) \; U_\varepsilon^{n-1,n-2}(B_{s_1},y,z)\Big]$$

where we have simply replaced

$$\int_{s_1}^{\zeta} \ell_\varepsilon^z(ds_2)\ldots\int_{s_{n-1}}^{\zeta} \ell_\varepsilon^z(ds_n) \prod_{i=1}^{n-1} 1_{S_{\varepsilon K}(s_i,s_{i+1})}(y)$$

by its predictable projection $U_\varepsilon^{n-1,n-2}(B_{s_1},y,z)$.

To get the desired result at the order n , it now suffices to use the induction hypothesis, the bound (7) and the bound

$$|G(y-B_{s_1}) - G(y-z)| \le \varepsilon\, g(|y-z|)$$

which holds when $|B_{s_1}-z| \le \varepsilon$. □

4. Proof of Theorem 1.

As in Chapter X, the proof of our main result depends on a basic lemma which we now state.

Lemma 4 : *Let* $p \ge 1$. *There exists a constant* C_p *such that, for any* $y, z \in \mathbb{R}^2$, $\varepsilon \in (0,1/2)$, *with* $|y| > 4\varepsilon$, $|z| > 4\varepsilon$, $|z-y| > 8\varepsilon$,

– *if* $p \ge 2$,

$$|E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\, 1_{S_{\varepsilon K}(0,\zeta)}(z)] + a_\varepsilon^{p-1}\, G(y)\, G(z-y)^{2p-2}$$

$$+ a_\varepsilon^p\, (G(y) + G(z))\, G(z-y)^{2p-1} + a_\varepsilon^{p+1}\, G(z)\, G(z-y)^{2p}|$$

$$\le C_p\, \varepsilon\, |\log \varepsilon|^{2p}\, ((g(|y|) + g(|z|))\, G(\tfrac{z-y}{4}) + (G(\tfrac{y}{2}) + G(\tfrac{z}{2}))\, g(|z-y|))$$

– *if* $p = 1$,

$$|E[\ell_\varepsilon^y(\zeta)\, 1_{S_{\varepsilon K}(0,\zeta)}(z)] + a_\varepsilon(G(y) + G(z))\, G(z-y) + a_\varepsilon^2\, G(z)\, G(z-y)^2|$$

$$\le C_1\, \varepsilon\, |\log \varepsilon|^2\, ((g(|y|) + g(|z|))\, G(\tfrac{z-y}{4}) + (G(\tfrac{y}{2}) + G(\tfrac{z}{2}))\, g(|z-y|)) .$$

Proof : We assume that $p \ge 2$ (the case $p = 1$ is easier). By Lemma X-2,

$$E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\, 1_{S_{\varepsilon K}(0,\zeta)}(z)]$$

$$= E\Big[\int_{\Lambda_p} \ell_\varepsilon^y(ds_1) \Big(\prod_{i=2}^p (\ell_\varepsilon^y(ds_i) + h_\varepsilon\, \delta_{(s_{i-1})}(ds_i))\Big)\, 1_{S_{\varepsilon K}(0,\zeta}(z)\Big].$$

Now the key idea is to write :

$$S_{\varepsilon K}(0,\zeta) = \bigcup_{i=0}^{p} S_{\varepsilon K}(s_i, s_{i+1})$$

with the usual convention $s_o = 0$, $s_{p+1} = \zeta$. It follows that

$$1_{S_{\varepsilon K}(0,\zeta)}(z) = \sum_{L \in \mathcal{P}_p} (-1)^{|L|+1}\, 1_{(\bigcap_{i \in L} S_{\varepsilon K}(s_i, s_{i+1}))}(z)$$

where $\mathcal{P}_p$ denotes the set of all nonempty subsets of $\{0,1,\dots,p\}$, and $|L| = \mathrm{Card}(L)$. Therefore,

$$E[Q^p_\varepsilon(\ell^y_\varepsilon(\zeta))\ 1_{S_{\varepsilon K}(0,\zeta)}(z)] = \sum_{L\in\mathcal{P}_p} (-1)^{|L|+1}\ \Phi_L(\varepsilon,y,z),$$

where

$$\Phi_L(\varepsilon,y,z) = E\Big[\int_{\Lambda_p} \ell^y_\varepsilon(ds_1) \prod_{i=2}^{p} (\ell^y_\varepsilon(ds_i) + h_\varepsilon\delta_{(s_{i-1})}(ds_i)) \prod_{i\in L} 1_{S_{\varepsilon K}(s_i,s_{i+1})}(z)\Big].$$

Suppose first that $\{1,\dots,p-1\} \subset L$, which happens only in the four cases:

$$L_1 = \{0,1,\dots,p\},\quad L_2 = \{1,\dots,p\},\quad L_3 = \{0,1,\dots,p-1\},\quad L_4 = \{1,\dots,p-1\}.$$

In each of these cases, we can use Lemma 3 to analyse the behavior of $\Phi_L(\varepsilon,y,z)$. Simply notice that

$$\Phi_{L_1}(\varepsilon,y,z) = U^{p,p}_\varepsilon(0,z,y),\qquad \Phi_{L_2}(\varepsilon,y,z) = V^{p,p}_\varepsilon(0,y,z)$$

$$\Phi_{L_3}(\varepsilon,y,z) = U^{p,p-1}_\varepsilon(0,z,y),\quad \Phi_{L_4}(\varepsilon,y,z) = V^{p,p-1}_\varepsilon(0,y,z).$$

Taking account of Lemma 3, we see that the proof of Lemma 4 will be complete once we have checked that the other choices of L give a negligible contribution. This is very similar to what we did in the proof of Lemma X-4. Set

$$k = \sup\{i \in \{1,\dots,p-1\},\ i \notin L\}$$

and assume for definiteness that $p \in L$. Then we may write

$$\Phi_L(\varepsilon,y,z) = E\Big[\int_{\Lambda_k} \mu(ds_1\dots ds_k) \int_{s_k}^{\zeta} (\ell^y_\varepsilon(ds_{k+1}) + h_\varepsilon\ \delta_{(s_k)}(ds_{k+1}))$$

$$\times \int_{s_{k+1}}^{\zeta} \ell^y_\varepsilon(ds_{k+2}) \int\dots\int_{s_{p-1}}^{\zeta} \ell^y_\varepsilon(ds_p) \prod_{i=k+1}^{p} 1_{S_{\varepsilon K}(s_i,s_{i+1})}(z)\Big]$$

(notice that the Dirac measures $\delta_{(t_i)}(dt_{i+1})$, for $i > k$, have been dropped). Here the random measure $\mu(ds_1\dots ds_k)$ is such that the process $t \longrightarrow \mu(\{s_k \le t\})$ is predictable ; furthermore it is easy to get the bound

$$E\Big[\int_{\Lambda_k} |\mu|(ds_1\dots ds_k)\ (\ell^y_\varepsilon(\zeta) + |h_\varepsilon|)\Big] \le C\ |\log\varepsilon|^k\ G(\tfrac{y}{2})$$

We may replace $\Big(\int_{s_{k+1}}^{\zeta} \dots\Big)$ by its predictable projection and get :

$$\Phi_L(\varepsilon,y,z) =$$

$$= E\Big[\int_{\Lambda_k} \mu(ds_1\dots ds_k) \int_{s_k}^{\zeta} (\ell^y_\varepsilon(ds_{k+1}) + h_\varepsilon\delta_{(s_k)}(ds_{k+1}))\ U^{p-k-1,p-k-1}_\varepsilon(B_{s_{k+1}},z,y)\Big],$$

with the convention $U_\varepsilon^{0,0}(x,y,z) = P_x[y \in S_{\varepsilon K}(0,\zeta)]$. The remaining part of the proof is entirely similar to the end of the proof of Lemma X-4. Simply use Lemma 3 instead of Lemma X-3. □

We may now complete the proof of Theorem 1. Write

$$u_\varepsilon(y,z) \simeq v_\varepsilon(y,z)$$

if there exists a function $F \in L^1(\mathbb{C}^2)$ such that for $\varepsilon \in (0,1/2)$,

$$|u_\varepsilon(y,z) - v_\varepsilon(y,z)| \le |\log \varepsilon|^{-2n-2} F(y,z).$$

It is enough to prove that

$$E\Big[\Big(1_{S_{\varepsilon K}(0,\zeta)}(y) + \sum_{p=1}^{n} a_\varepsilon^p\, Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\Big)\Big(1_{S_{\varepsilon K}(0,\zeta)}(z) + \sum_{p=1}^{n} a_\varepsilon^p\, Q_\varepsilon^p(\ell_\varepsilon^z(\zeta))\Big)\Big] \simeq 0.$$

By Lemma 2,

$$E\Big[1_{S_{\varepsilon K}(0,\zeta)}(y)\ 1_{S_{\varepsilon K}(0,\zeta)}(z)\Big] \simeq \sum_{p=2}^{2n+1} a_\varepsilon^p\ (G(y) + G(z))\ G(z-y)^{p-1}.$$

By Lemma 4 (and easy bounds when $x = 0$, y, z do not satisfy the assumptions of this lemma), if $p \ge 2$,

$$E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))1_{S_{\varepsilon K}(0,\zeta)}(z)] \simeq -\ a_\varepsilon^{p-1}\ G(y)\ G(z-y)^{2p-2} - a_\varepsilon^p\ (G(y)+ G(z))\ G(z-y)^{2p-1}$$

$$- a_\varepsilon^{p+1}\ G(z)\ G(z-y)^{2p} \quad ,$$

and, if $p = 1$,

$$E[Q_\varepsilon^1(\ell_\varepsilon^y(\zeta))1_{S_{\varepsilon K}(0,\zeta)}(z)] \simeq -\ a_\varepsilon\ (G(y) + G(z))\ G(z-y) - a_\varepsilon^2\ G(z)\ G(z-y)^2.$$

Finally, Lemmas X-4, X-5 give

$$E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\ Q_\varepsilon^p(\ell_\varepsilon^z(\zeta))] \simeq (G(y) + G(z))G(z-y)^{2p-1}.$$

Furthermore, it was pointed out in Section X-4 (see formulas X-(9), X-(10)) that the proof of Lemma X-4 can be adapted to give:

$$E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\ Q_\varepsilon^{p+1}(\ell_\varepsilon^z(\zeta))] \simeq G(z)\ G(z-y)^{2p}$$

$$E[Q_\varepsilon^p(\ell_\varepsilon^y(\zeta))\ Q_\varepsilon^q(\ell_\varepsilon^z(\zeta))] \simeq 0 \quad \text{if} \quad |q-p| \ge 2.$$

The desired result follows. □

5. Further results.

Theorem 1 yields an asymptotic expansion of $m(S_{\varepsilon K}(0,\zeta))$ as ε goes to 0. A natural question is: can we replace ζ by a constant time t ? We first have to define random variables $T^p(t)$ in such a way that $T^p(t)$ coincides with T^p1 "conditionally on $\{\zeta = t\}$". The next theorem can be deduced from Theorem 1 by using the scaling properties of Brownian motion.

Theorem 5 : *There exists a sequence of processes* $T^p = (T^p(t), t \geq 0)$, *adapted to the natural filtration of* B, *such that*

$$T^p(\zeta) = T^p1 \quad a.s.$$

and the following holds. For every $n \geq 1$, $t \geq 0$,

$$m(S_{\varepsilon K}(0,t)) = -\sum_{p=1}^{n} a_\varepsilon^p \, T^p(t) + \mathcal{R}_n(\varepsilon),$$

where

$$\lim_{\varepsilon\to 0} |\log \varepsilon|^n \, \mathcal{R}_n(\varepsilon) = 0$$

in the L^2*-norm, and a.s. when* K *is star-shaped.*

Remark : Both the constants a_ε and the random variables $T^p(t)$ depend on the choice of λ (but not on the choice of ζ). Changing the value of λ leads to different equivalent expansions of $m(S_{\varepsilon K}(0,t))$. This corresponds to the non-uniqueness of the renormalization, which was already pointed out in Chapter X.

The case $n = 1$ of Theorem 5 is exactly Theorem VI-6. The case $n = 2$ is equivalent to Theorem VIII-7. If we compare these two results we get :

$$T^2(1) = \gamma(\mathcal{T}) + C_\lambda$$

for some constant C_λ depending on λ. Of course we could have proved this more directly, by comparing the approximations of $T^2 1$ and $\gamma(\mathcal{T})$.

By taking expectations in Theorem 5, one gets a full asymptotic expansion of $E[m(S_{\varepsilon K}(0,t)]$ and by scaling an asymptotic expansion of $E[m(S_K(0,t)]$ as t goes to infinity. The latter expansion refines a theorem of Spitzer in [Sp2]. These expansions involve the quantities $E[T^p(1)]$, which can be computed by induction, using the fact that $E[T^p(\zeta)] = 0$ (see Chapter X) and the scaling properties of $T^k(t)$. It is worth noting that the coefficients of the expansion for $E[m(S_{\varepsilon K}(0,t)]$ depend on K only through the constant $R(K)$. This should be compared with the similar results in higher dimensions [L11].

<u>Bibliographical notes</u>. The material of this Chapter is taken from [L12]. *In particular we refer to this paper for a detailed proof of Theorem 5. A recent paper of Rosen* [R7] *gives analogues of Theorems 1 and 5 for the sausage associated with certain stable processes. See also Feldman and Rosen* [FR] *for an extension of Theorem 1 to Brownian motion on Riemannian surfaces. The work of Le Jan* [LJ] *may provide an alternative approach to the results of this Chapter.*

REFERENCES

[A1] **O. Adelman.** Le brownien pique-t-il ? Exposé au Séminaire de Probabilités de l'Université Paris VI (1982). *Unpublished.*

[A2] **O. Adelman.** *Private communication.*

[AD] **O. Adelman, A. Dvoretzky.** Plane Brownian motion has strictly n-multiple points. *Israel J. Math.* 52, 361-364 (1985).

[Bi] **C.J. Bishop.** Brownian motion in Denjoy domains. Preprint (1990).

[B1] **K. Burdzy.** Brownian paths and cones. *Ann. Probab.* **13**, 1006-1010 (1985).

[B2] **K. Burdzy.** *Multidimensional Brownian Excursions and Potential Theory.* Longman, New York, 1987.

[B3] **K. Burdzy.** Geometric properties of two-dimensional Brownian paths. *Probab. Th. Rel. Fields* 81, 485-505 (1989).

[B4] **K. Burdzy.** Cut points on Brownian paths. *Ann. Probab.* **17**, 1012-1036.

[BL1] **K. Burdzy, G.F. Lawler.** Non-intersection exponents for Brownian paths. Part I. Existence and an invariance principle. *Probab. Th. Rel. Fields* 84, 393-410 (1990).

[BL2] **K. Burdzy, G.F. Lawler.** Non-intersection exponents for Brownian paths. Part II. Estimates and application to a random fractal. *Ann. Probab.* 18, 981-1009 (1990).

[BSM] **K. Burdzy, J. San Martin.** Curvature of the convex hull of planar Brownian motion near its minimum point. *Stoch. Process. Appl.* 33, 89-103 (1989).

[CML] **M. Chaleyat-Maurel, J.F. Le Gall.** Green function, capacity and sample path properties for a class of hypoelliptic diffusion processes. *Probab. Th. Rel. Fields* 83, 219-264 (1989).

[CF1] **I. Chavel, E.A. Feldman.** The Wiener sausage, and a theorem of Spitzer in Riemannian manifolds. *Probability and Harmonic Analysis, J. Chao, W.A. Woyczinski eds,* pp. 45-60. Dekker, New York, 1986.

[CF2] **I. Chavel, E.A. Feldman.** The Lenz shift and Wiener sausage in Riemannian manifolds. *Compositio Math.* **60**, 65-84 (1986).

[CFR] **I. Chavel, E.A. Feldman, J. Rosen.** Fluctuations of the Wiener sausage for manifolds. *To appear.*

[CT] **Z. Ciesielski, S.J. Taylor.** First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.* 103, 434-450 (1963).

[CHM] **M. Cranston, P. Hsu, P. March.** Smoothness of the convex hull of planar Brownian motion. *Ann. Probab.* **17**, 144-150 (1989).

[Da] B. **Davis.** Brownian motion and analytic functions. *Ann. Probab.* 7, 913-932 (1979).

[DV] **M.D. Donsker, S.R.S. Varadhan.** Asymptotics for the Wiener sausage. *Comm. Pure Appl. Math.* **28**, 525-565 (1975).

[Du1] **R. Durrett.** A new proof of Spitzer's result on the winding of two-dimensional Brownian motion. *Ann. Probab.* **10**, 244-246 (1982).

[Du2] **R. Durrett.** *Brownian Motion and Martingales in Analysis.* Wadsworth, Belmont Ca. , 1984.

[DE] **A. Dvoretzky, P. Erdös.** Some problems on random walk in space. *Proc. Second Berkeley Symposium on Math. Statistics and Probability,* pp 353-367. University of California Press, Berkeley, 1951.

[DK1] **A. Dvoretzky, P. Erdös, S. Kakutani.** Double points of paths of Brownian motion in *n*-space. *Acta Sci. Math. (Szeged)* **12**, 74-81 (1950).

[DK2] **A. Dvoretzky, P. Erdös, S. Kakutani.** Multiple points of Brownian motion in the plane. *Bull. Res. Council Israel Sect. F* 3, 364-371 (1954).

[DK3] **A. Dvoretzky, P. Erdös, S. Kakutani.** Points of multiplicity c of plane Brownian paths. *Bull. Res. Council Israel Sect. F* 7, 175-180 (1958).

[DKT] **A. Dvoretzky, P. Erdös, S. Kakutani, S.J. Taylor.** Triple points of Brownian motion in 3-space. *Proc. Cambridge Philos. Soc.* 53, 856-862 (1957).

[Dy1] **E.B. Dynkin.** Additive functionals of several time-reversible Markov processes. *J. Funct. Anal.* **42**, 64-101 (1981).

[Dy2] **E.B. Dynkin.** Local times and quantum fields. *Seminar on Stochastic Processes 1983*, pp 69-84. Birkhäuser, Boston, 1984.

[Dy3] **E.B. Dynkin.** Polynomials of the occupation field and related random fields. *J. Funct. Anal.* **42**, 64-101 (1984).

[Dy4] **E.B. Dynkin.** Random fields associated with multiple points of the Brownian motion. *J. Funct. Anal.* 62, 397-434 (1985).

[Dy5] **E.B. Dynkin.** Functionals associated with self-intersections of the Brownian motion. *Séminaire de Probabilités XX, Lecture Notes in Math.* **1204**, 553-571. Springer, Berlin, 1986.

[Dy6] **E.B. Dynkin.** Self-intersection gauge for random walks and for Brownian motion. *Ann. Probab.* **16**, 1-57 (1988).

[Dy7] **E.B. Dynkin.** Regularized self-intersection local times of the planar Brownian motion. *Ann. Probab.* **16**, 58-74 (1988).

[E] **S.F. Edwards.** The statistical mechanics of polymers with excluded volume. *Proc. Phys. Sci.* **85**, 613-624 (1965).

[EB] **M. El Bachir.** L'enveloppe convexe du mouvement brownien. Thèse de Troisième Cycle, Université Paul Sabatier, Toulouse, 1983.

[Ev1] **S.N. Evans.** On the Hausdorff dimension of Brownian cone points. *Math. Proc. Camb. Phil. Soc.* **98**, 343-353 (1985).

[FR] **E.A. Feldman, J. Rosen.** An asymptotic development of the area of the Wiener sausage for surfaces. *To appear.*

[F] **W.E. Feller.** *An Introduction to Probability Theory and Its Applications, vol II, 2nd edition.* Wiley, New York, 1971.

[GHR] **D. Geman, J. Horowitz, J. Rosen.** A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.* **12**, 86-107 (1984).

[GS] **R.K. Getoor, M.J. Sharpe.** Conformal martingales. *Invent. Math.* **16**, 271-308 (1972).

[H] **J. Hawkes.** Some geometric aspects of potential theory. *Stochastic Analysis and Applications. Lecture Notes in Math.* **1095**, 130-154. Springer, New York, 1984.

[IMK] **K. Itô, H.P. McKean.** *Diffusion Processes and their Sample Paths.* Springer, New York, 1965.

[JP] **N.C. Jain, W.E. Pruitt.** The range of random walk. *Proc. Sixth Berkeley Symposium on Math. Statistics and Probability, vol. 3,* pp 31-50. University of California Press, Berkeley, 1973.

[K] **M. Kac**. Probabilistic methods in some problems of scattering theory. *Rocky Mountain J. Math.* **4**, 511-537 (1974).

[Kh] **J.P. Kahane**. Points multiples des processus de Lévy symétriques stables restreints à un ensemble de valeurs du temps. Séminaire d'Analyse Harmonique. Public. Math. de l'Université d'Orsay **83-02**, p. 74-105 (1983).

[KR] **G. Kallianpur, H. Robbins**. Ergodic property of the Brownian motion process. *Proc. Nat. Acad. Sci. U.S.A.* 39, 525-533 (1953).

[Ka] **R. Kaufman**. Une propriété métrique du mouvement brownien. *C.R. Acad. Sci. Paris, Série I* , **268**, 727-728 (1969).

[Kn] **F.B. Knight**. *Essentials of Brownian Motion and Diffusion*. A.M.S., Providence, 1981.

[L1] **J.F. Le Gall**. Sur la mesure de Hausdorff de la courbe brownienne. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123**, 297-313. Springer, New York, 1985.

[L2] **J.F. Le Gall**. Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123**, 314-331. Springer, New York 1985.

[L3] **J.F. Le Gall**. Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.* **14**, 1219-1244 (1986).

[L4] **J.F. Le Gall**. Propriétés d'intersection des marches aléatoires, I. *Comm. Math. Physics* **104**, 471-507 (1986).

[L5] **J.F. Le Gall**. Propriétés d'intersection des marches aléatoires, II. *Comm. Math. Physics* **104**, 509-528 (1986).

[L6] **J.F. Le Gall**. Le comportement du mouvement brownien entre les deux instants où il passe par un point double. *J. Funct. Anal.* **71**, 246-262 (1987).

[L7] **J.F. Le Gall**. Mouvement brownien, cônes et processus stables. *Probab. Th. Rel. Fields* 76, 587-627 (1987).

[L8] **J.F. Le Gall**. Temps locaux d'intersection et points multiples des processus de Lévy. *Séminaire de Probabilités XXI. Lecture Notes in Math.* 1247, 341-375. Springer, New York, 1987.

[L9] **J.F. Le Gall**. The exact Hausdorff measure of Brownian multiple points. *Seminar on Stochastic Processes 1986*, pp. 107-137. Birkhäuser, Boston, 1987.

[L10] **J.F. Le Gall**. Fluctuation results for the Wiener sausage. *Ann. Probab.* 16, 991-1018 (1988).

[L11] **J.F. Le Gall**. Sur une conjecture de M. Kac. *Probab. Th. Rel. Fields* 78, 389-402 (1988).

[L12] **J.F. Le Gall**. Wiener sausage and self-intersection local times. *J. Funct. Anal.* **88**, 299-341 (1990).

[L13] **J.F. Le Gall**. On the connected components of the complement of a two-dimensional Brownian path. *Festschrift in Honor of Frank Spitzer*. Birkhäuser, Boston, 1991.

[LR] **J.F. Le Gall, J. Rosen**. The range of stable random walks. *Ann. Probab.*, *to appear*.

[LY] **J.F. Le Gall, M. Yor**. Etude asymptotique de certains mouvements browniens complexes avec drift. *Probab. Th. Rel. Fields* 71, 183-229 (1986).

[LJ] **Y. Le Jan**. On the Fock space representation of functionals of the occupation field and their renormalization. *J. Funct. Anal.* **80**, 88-108 (1988).

[Lé1] **P. Lévy**. Le mouvement brownien plan. *Amer. J. Math.* **62**, 487-550 (1940).

[Lé2] **P. Lévy**. La mesure de Hausdorff de la courbe du mouvement brownien. *Giornale dell'Istituto Italiano degli Attuari* 16, 1-37 (1953).

[Lé3] **P. Lévy.** Le caractère universel de la courbe du mouvement brownien et la loi du logarithme itéré. *Rendiconti del circolo matematico di Palermo, II,* **4**, 337-366 (1955).

[Lé4] **P. Lévy.** *Processus Stochastiques et Mouvement Brownien.* Gauthier-Villars, Paris, 1965.

[Ma] **N.G. Makarov.** On the distorsion of boundary sets under conformal mappings. *Proc. London Math. Soc. (3)* 51, 369-384 (1985).

[M] **B.B. Mandelbrot.** *The Fractal Geometry of Nature.* W.H. Freeman and Co., New York, 1982.

[MK] **H.P. McKean.** *Stochastic Integrals.* Academic Press, New York, 1969.

[MM] **J.E. McMillan.** Boundary behaviour of a conformal mapping. *Acta Math.* **123**, 43-68 (1969).

[Mo] **T.S. Mountford.** On the asymptotic number of small components created by planar Brownian motion. *Stochastics* **28**, 177-188 (1989).

[PY1] **J.W. Pitman, M. Yor.** Asymptotic laws of planar Brownian motion. *Ann. Probab.* **14**, 733-779 (1986).

[PY2] **J.W. Pitman, M. Yor.** Level crossings of a Cauchy process. *Ann. Probab.* **14**, 780-792 (1986).

[PY3] **J.W. Pitman, M. Yor.** Further asymptotic laws of planar Brownian motion. *Ann. Probab.* **17**, 965-1011 (1989).

[Po] **C. Pommerenke.** *Univalent Functions.* Vandenhoeck & Ruprecht, Göttingen, 1975.

[PS] **S.C. Port, C.J. Stone.** *Brownian Motion and Classical Potential Theory.* Academic Press, New York, 1978.

[ReY] **D. Revuz, M. Yor.** *Continuous Martingales and Brownian Motion.* Springer, 1991.

[RoW] **L.C.G. Rogers, D. Williams.** *Diffusions, Markov Processes, and Martingales, Vol. II.* Wiley, Chichester, 1987

[R1] **J. Rosen.** A local time approach to the self-intersections of Brownian paths in space. *Comm. Math. Physics* **88**, 327-338 (1983).

[R2] **J. Rosen.** Self-intersections of random fields. *Ann. Probab.* **12**, 108-119 (1984).

[R3] **J. Rosen.** Tanaka formula and renormalization for intersections of planar Brownian motion. *Ann. Probab.* **14**, 1245-1251 (1986).

[R4] **J. Rosen.** A renormalized local time for multiple intersections of planar Brownian motion. Séminaire de Probabilités XX. *Lecture Notes in Math.* **1204**, 515-531. Springer, Berlin, 1986.

[R5] **J. Rosen.** Joint continuity of the intersection local time of Markov processes. *Ann. Probab.* **15**, 659-675 (1987).

[R6] **J. Rosen.** Random walks and intersection local time. *Ann. Probab.* **18**, 959-977 (1990).

[RY] **J. Rosen, M. Yor.** Tanaka formulae and renormalization for triple intersections of Brownian motion in the plane. *Ann. Probab., to appear.*

[Sh1] **M. Shimura.** A limit theorem for two-dimensional conditioned random walk. *Nagoya Math. J.* 95, 105-116 (1984).

[Sh2] **M. Shimura.** Excursions in a cone for two-dimensional Brownian motion. *J. Math. Kyoto Univ.* 25, 433-443 (1985).

[Sh3] **M. Shimura.** Meandering points of two-dimensional Brownian motion. *Kodai Math. J.* **11**, 169-176 (1988).

[S1] **F. Spitzer.** Some theorems concerning two-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87**, 187-197 (1958).

[S2] **F. Spitzer.** Electrostatic capacity, heat flow and Brownian motion. *Z. Wahrsch. verw. Gebiete* 3, 110-121 (1964).

[S3] **F. Spitzer.** *Principles of Random Walk.* Princeton, Van Nostrand 1964.

[S4] **F. Spitzer.** Appendix to: Subadditive ergodic theory, by J.F.C. Kingman. *Ann. Probab.* 1, 883-909 (1973).

[Sy] **K. Symanzik.** Euclidean quantum field theory. *Local Quantum Theory, R. Jost ed.* Academic, New York, 1969.

[Sz] **A.S. Sznitman.** Some bounds and limiting results for the Wiener sausage of small radius associated with elliptic diffusions. *Stochastic Process. Appl.* 25, 1-25 (1987).

[T1] **S.J. Taylor.** The exact Hausdorff measure of the sample path for planar Brownian motion. *Proc. Cambridge Philos. Soc.* **60**, 253-258 (1964).

[T2] **S.J. Taylor.** Sample path properties of processes with stationary independent increments. *Stochastic Analysis, D. Kendall, E. Harding eds.* Wiley, London, 1973.

[T3] **S.J. Taylor.** The measure theory of random fractals. *Math. Proc. Camb. Phil. Soc.* 100, 383-406 (1986).

[V] **S.R.S. Varadhan.** Appendix to "Euclidean quantum field theory", by K. Symanzik. In *Local Quantum Theory* (R. Jost ed.). Academic, New York, 1969.

[VW] **S.R.S. Varadhan, R.J. Williams.** Brownian motion in a wedge with oblique reflection. *Commun. Pure Appl. Math.* 38, 405-443 (1985).

[W1] **S. Weinryb.** Etude asymptotique par des mesures de $\mathbb{R}^3$ de saucisses de Wiener localisées. *Probab. Th. Rel. Fields* 73, 135-148 (1986).

[W2] **S. Weinryb.** Asymptotic results for independent Wiener sausages. Application to generalized intersection local times. *Stochastics* 27, 99-127 (1989).

[Wi1] **R.J. Williams.** Recurrence classification and invariant measure for reflected Brownian motion in a wedge. *Ann. Probab.* 13, 758-778 (1985).

[Wi2] **R.J. Williams.** Reflected Brownian motion in a wedge: semi-martingale property. *Z. Wahrsch. verw. Gebiete* 69, 161-176 (1985).

[Wi3] **R.J. Williams.** Local time and excursions of reflected Brownian motion in a wedge. *Publ. Res. Inst. Math. Sci. Kyoto University* 23, 297-319 (1987).

[Wo] **R. Wolpert.** Wiener path intersections and local time. *J. Funct. Anal.* 30, 329-340 (1978).

[Y1] **M. Yor.** Compléments aux formules de Tanaka-Rosen. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123**, 332-349. Springer, Berlin, 1985.

[Y2] **M. Yor.** Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement brownien dans $\mathbb{R}^3$. *Séminaire de Probabilités XIX. Lecture Notes in Math.* **1123**, 350-365. Springer, Berlin, 1985.

[Y3] **M. Yor.** Précisions sur l'existence et la continuité des temps locaux d'intersection du mouvement brownien dans $\mathbb{R}^d$. Séminaire de Probabilités XX. *Lecture Notes in Math.* **1204**, 532-541. Springer, Berlin, 1986.

[Y4] **M. Yor.** Sur la représentation comme intégrales stochastiques des temps d'occupation du mouvement brownien dans $\mathbb{R}^d$. Séminaire de Probabilités XX. *Lecture Notes in Math.* **1204**, 543-552. Springer, Berlin, 1986.

SUBJECT INDEX