On 4D split-conformal structures with G_2-symmetric twistor distribution

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SCREAM = Symmetry, Curvature Reduction, & EquivAlence Methods

Themes in my recent projects: Homogeneous structures / classification, ODE geometry, parabolic geometry, supergeometry.

A tale of three (parabolic) geometries

twistor XXO-structure (5D) $(N^5 = \mathbb{T}^+(M); \ell, \mathcal{D})$

$\mathcal{D}$ non-int. on $U \subset N$ $\rightsquigarrow$ $(2,3,5)$-distribution (5D) $(U^5; \mathcal{D})$

$\downarrow$

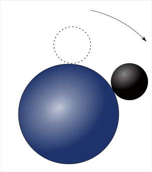
(oriented) split-conformal (4D) $(M^4; [\mathrm{g}])$

Here, $\mathcal{D}$ (coming from $[\mathrm{g}]$) is the twistor distribution. We'll describe its construction in more detail later.

> Q: Which $[\mathrm{g}]$ lead to $\mathcal{D}$ with maximal symmetry (G_2)?

Geometric properties: Half-flat? Symmetries? Conformal holonomy? Almost-Einstein?

Rolling examples $\rightsquigarrow G_2$



Rolling without twisting or slipping:

- (Σ_i, g_i): Riemannian surfaces;
- $M = \Sigma_1 \times \Sigma_2$ with $\mathrm{g} = \mathrm{g}_1 \oplus (-\mathrm{g}_2)$.

Famous example: Two 2-spheres with ratio of radii $1:3$.

An–Nurowski (2014) – new examples:

surface of revolution: $\mathrm{g}_1 = (\rho^2+\epsilon)^2 d\rho^2 + \rho^2 d\varphi^2, \quad \epsilon \in \{0, \pm 1\}$
plane: $\mathrm{g}_2 = dx^2 + dy^2$

Symmetry algebra $\mathfrak{f}$:

- $\epsilon = \pm 1$: $\langle \partial_\varphi, \partial_x, \partial_y, -y\partial_x + x\partial_y \rangle$ (inhomogeneous)
- $\epsilon = 0$: Add $\langle \rho\partial_\rho + 2\varphi\partial_\phi + 3x\partial_x + 3y\partial_y \rangle$ to above. (hom.)

$(2,3,5)$-geometry

$(N^5, \mathcal{D} \subset TN)$ is a $(2,3,5)$-geometry if

$$\mathrm{rank}(\mathcal{D}) = 2, \quad \mathrm{rank}([\mathcal{D},\mathcal{D}]) = 3, \quad [\mathcal{D},[\mathcal{D},\mathcal{D}]] = TN.$$

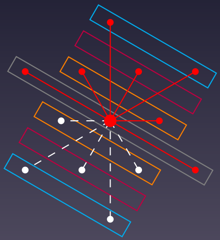
Goursat (1896): Locally, $\mathcal{D} = \mathcal{D}_f$ with $\mathcal{D}_f$ spanned by

$$D_x := \partial_x + p\partial_y + q\partial_p + f(x,y,p,q,z)\partial_z, \qquad \partial_q.$$

This is $(2,3,5)$ iff $f_{qq} \neq 0$.

Theorem (Cartan 1910)

Any $(2,3,5)$-distribution has at most 14-dim symmetry. Locally, $\exists!$ maximally symmetric model, and this has G_2-symmetry.



Example (Cartan 1893)

$\mathcal{D}_{q^2}$ has G_2-symmetry.

XXO-geometry

... consists of a 5-mfld N with rank 3 dist. $\mathcal{H} \subset TN$ satisfying:

- $[\mathcal{H}, \mathcal{H}] = TN$ and $\mathcal{H} = \ell \oplus \mathcal{D}$ (of ranks 1 and 2);
- $[\mathcal{D}, \mathcal{D}] \subseteq \mathcal{H}$. : $[\mathcal{D}, \mathcal{D}]$ may have non-constant rank.

Model: $\mathrm{Flag}_{1,2}(\mathbb{R}^4)$ $(\mathrm{SL}_4/P_{1,2})$

$$\begin{pmatrix} 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ -2 & -1 & 0 & 0 \end{pmatrix}$$

Example (Pairs of 2nd order ODE as integrable XXO-str.)

$$\begin{cases} \ddot{x} = F(t, x, y, \dot{x}, \dot{y}) \\ \ddot{y} = G(t, x, y, \dot{x}, \dot{y}) \end{cases} \qquad \begin{cases} N^5 : (t, x, y, \dot{x}, \dot{y}) \\ \ell = \langle \partial_t + \dot{x}\partial_x + \dot{y}\partial_y + F\partial_{\dot{x}} + G\partial_{\dot{y}} \rangle \\ \mathcal{D} = \langle \partial_{\dot{x}}, \partial_{\dot{y}} \rangle \quad \text{(integrable: } [\mathcal{D}, \mathcal{D}] = \mathcal{D}) \end{cases}$$

Example (Enhancing $(2, 3, 5)$ to a non-integrable XXO-str.)

For $\mathcal{D} = \mathcal{D}_{q^2} = \langle D_x, \partial_q \rangle$, we have $[\mathcal{D}, \mathcal{D}]/\mathcal{D} = \langle \partial_p + 2q\partial_z \rangle$. We can define an XXO geometry via a choice of ℓ:

$$\ell = \langle \partial_p + 2q\partial_z + AD_x + B\partial_q \rangle \quad \text{(non-int: } [\mathcal{D}, \mathcal{D}] = \mathcal{H} := \ell \oplus \mathcal{D})$$

An–Nurowski construction

General construction:

- Input: $(M^4, [\mathrm{g}])$, with g a split-signature $(2,2)$-metric.
- Output: On "circle-twistor bundle" $N = \mathbb{T}^+(M) \to M$ (of SD totally null 2-planes), get twistor XXO structure $\mathcal{H} = \ell \oplus \mathcal{D}$.
  - $\ell = \ker(TN \to TM)$;
  - $\mathcal{D}$ = "twistor distribution": distinguished via the 1-dim kernel of $\bigwedge^2 \mathcal{H}^* \to TN/\mathcal{H}$.
- Locally, $\mathrm{g} = \theta^1\theta^2 + \theta^3\theta^4$, $\mathrm{vol}_{\mathrm{g}} = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$, SD totally null 2-planes look like $\langle e_1 + \xi e_3, e_4 - \xi e_2 \rangle$. Lift to XXO:

$$\ell = \langle \partial_\xi \rangle, \quad \mathcal{D} = \langle e_1 + \xi e_3 + A\partial_\xi, e_4 - \xi e_2 + B\partial_\xi \rangle.$$

- Efficiently compute $\mathcal{W}^+$ via: $\bigwedge^2 \mathcal{D} \xrightarrow{[\cdot,\cdot]} \mathcal{H} = \ell \oplus \mathcal{D} \xrightarrow{\mathrm{pr}_\ell} \ell$. (Locally, $\mathcal{W}^+(\xi)$ is a quartic polynomial in $\xi \rightsquigarrow$ Petrov type.)
- $\mathcal{D}$ is $(2,3,5)$ where $\mathcal{W}^+ \neq 0$.

Q: Which $[\mathrm{g}]$ lead to $\mathcal{D}$ with G_2 symmetry?

A classification theorem

Theorem (Nurowski–Sagerschnig–T. 2024)

We have a complete classification of those locally homogeneous 4D split-conformal structures with:

1. *multiply-transitive twistor XXO-structure, and*
2. *G_2-symmetric twistor distribution.*

Complexified summary:

Label	Petrov type	Comments
M9	N.O	$\mathfrak{p}_1^{\mathrm{op}}$
M8	D.O	$\mathfrak{sl}(3,\mathbb{R}), \mathfrak{su}(1,2)$
$\mathrm{M7}_a$	$\begin{cases} \text{N.N}, & a^2 \neq \frac{4}{3}; \\ \text{N.O}, & a^2 = \frac{4}{3} \end{cases}$	new : $\mathbb{R}^2 \ltimes \mathfrak{heis}_5$
M6S	D.D	1:3 rolling spheres + variants
M6N	III.O	new : $\mathfrak{aff}(2)$

Example: 9-dim symmetry

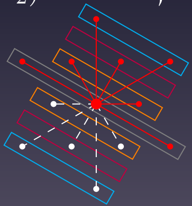
Plebanski: $\mathrm{g} = dwdx + dydz - \Theta_{xx}dz^2 - \Theta_{yy}dw^2 + 2\Theta_{xy}dwdz$

$$\rightsquigarrow \quad \mathcal{W}^+(\xi) = (\partial_x + \xi\partial_y)^4\Theta.$$

Example

Let $\Theta = -\frac{y^4}{12}$. Then $\mathrm{g} = y^2dw^2 + dwdx + dydz$ has 9 CKV's.

- Lift to XXO-geometry $\rightsquigarrow (\ell, \mathcal{D})$ admits 9-dim. sym. alg. $\mathfrak{f}$.
- $\mathcal{W}^+ = -2\xi^4$, so $\mathcal{D}$ is $(2,3,5)$ when $\xi \neq 0$.
- Cartan (1910): Submax. sym. dim. for $(2,3,5)$ is 7.
- Thus, $\mathfrak{f} \hookrightarrow \mathfrak{g} :=$ Lie(G_2). Indeed $\mathfrak{f} \cong \mathfrak{p}_1^{op} \subset \mathfrak{g}$.



Harmonic curvatures

Structure	Hieroglyphic	Harmonic curvatures	Hom.
$(2,3,5)$	$-8 \quad 4$ 	Cartan quartic: $\mathcal{Q}$	$+4$
4D split-conf	$0 \quad -4 \quad 4$ 	ASD Weyl: $\mathcal{W}^-$	$+2$
	$4 \quad -4 \quad 0$ 	SD Weyl: $\mathcal{W}^+$	$+2$
XXO	$0 \quad -4 \quad 4$ 	$\mathcal{S}$	$+3$
	$-4 \quad 1 \quad 2$ 	$\mathcal{T}$	$+2$
	$4 \quad -4 \quad 0$ 	$\mathcal{I}$	$+1$

Remarks:

- $\mathcal{I}$ precisely obstructs integrability of D.
- Čap (2006) $\rightsquigarrow$ 4D split-conf. $\leftrightarrow$ XXO geometry with $\mathcal{T} \equiv 0$.

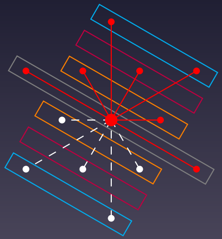
How to systematically classify?

- Via coordinate methods: not sure how to start.
- Cartan reduction on XXO with $\mathcal{T} \equiv 0$ and $\mathcal{I} \neq 0$: get a list, but need to impose $\mathcal{Q} \equiv 0$ to remove extraneous items. (This is cumbersome, especially for the 6D case.)

Strategy:

- Build in $\mathcal{Q} \equiv 0$ and $\mathcal{I} \neq 0$ from the outset, but need $\mathcal{T} \equiv 0$;
- Work inside $\mathfrak{g} = \mathrm{Lie}(G_2)$, filtered by the parabolic $P = P_1$:

$$\mathfrak{g} = \mathfrak{g}^{-3} \supset \ldots \supset \mathfrak{g}^{0} = \mathfrak{p} \supset \ldots \supset \mathfrak{g}^{3}.$$



What should we classify?

Suppose we have a Lie-theoretic XXO model $(\mathfrak{f}, \mathfrak{f}^0; \ell, \mathcal{D})$.

Assume $\mathcal{D}$ is non-int. & $\mathcal{Q} \equiv 0$. Define admissible $(\mathfrak{f}, \mathfrak{f}^0; \ell, \mathcal{D})$:

(X.1) $\mathfrak{f} \hookrightarrow \mathfrak{g} = \mathrm{Lie}(G_2)$ as a filtered (wrt P) Lie subalgebra;

(X.2) $\mathrm{gr}_-(\mathfrak{f}) = \mathfrak{g}_-$ & $\dim(\mathfrak{f}^0) \geq 1$ ("multiply-transitive");

(X.3) On $\mathfrak{f}/\mathfrak{f}^0$, we have the data

$$\mathfrak{f}^{-1}/\mathfrak{f}^0 = \mathcal{D}, \quad \mathfrak{f}^{-2}/\mathfrak{f}^0 = \ell \oplus \mathcal{D};$$

(X.4) $\mathcal{T} \equiv 0$;

(X.5) maximal wrt natural partial order: $\mathfrak{f} \leq \widetilde{\mathfrak{f}}$ iff $\mathfrak{f} \hookrightarrow \widetilde{\mathfrak{f}}$.

Classify admissible $(\mathfrak{f}, \mathfrak{f}^0; \ell, \mathcal{D})$ (up to the natural P-action).

Classification steps

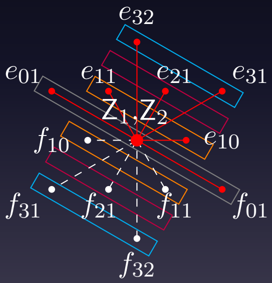
1. Classify complex admissible $(\mathfrak{f}, \mathfrak{f}^0; \ell, D)$ + real forms.
2. Check Petrov types.
3. Integrate structure eqns to get local coordinate models. For half-flat ones, write 2nd order ODE pairs.
4. Find “Cartan-theoretic” embeddings into $(\mathfrak{sl}_4, \mathfrak{p}_{1,2})$. (This also determines Petrov types, as well as holonomy.)

We will:

1. Sketch key ideas relating to the first step;
2. Show an example demonstrating: Petrov type, holonomy, Cartan-theoretic embedding, assessing almost-Einstein

Examples

Let $\mathsf{Z}_1, \mathsf{Z}_2 \in \mathfrak{h}$ be dual to the simple roots $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ of G_2. Grading element: Z_1. For $i, j \geq 0$, write $e_{ij} \in \mathfrak{g}_{i\alpha_1 + j\alpha_2}$ and $f_{ij} \in \mathfrak{g}_{-i\alpha_1 - j\alpha_2}$. Define $\mathsf{H} := [e_{01}, f_{01}] = -\mathsf{Z}_1 + 2\mathsf{Z}_2$.



	M9	M6S
$\mathfrak{f}^0$	$f_{01}, \mathsf{Z}_1, \mathsf{H}, e_{01}$	H
$\mathcal{D} = \mathfrak{f}^{-1}/\mathfrak{f}^0$	f_{10}	$f_{10} + e_{11}$
	f_{11}	$f_{11} + e_{10}$
$\mathfrak{f}^{-2}/\mathfrak{f}^{-1}$	f_{21}	$f_{21} + e_{21}$
$\mathfrak{f}^{-3}/\mathfrak{f}^{-2}$	f_{31}	$f_{31} + e_{32}$
	f_{32}	$f_{32} + e_{31}$
ℓ	f_{21}	$f_{21} + e_{21}$

Leading parts

The graded subalgebra $\mathfrak{s} := \mathrm{gr}(\mathfrak{f}) \subset \mathfrak{g}$ determines leading parts of $\mathfrak{f}$. (Also need to determine "tails".) Start with the isotropy $\mathfrak{f}^0$.

Lemma

1. $\mathfrak{f}^i = 0$ *for* $i \geq 1$
2. $\mathrm{gr}(\mathfrak{f}^0) \subseteq \mathfrak{g}_0 \cong \mathfrak{gl}(\mathfrak{g}_{-1}) \cong \mathfrak{gl}_2$, *so* $\dim(\mathfrak{f}^0) \leq 4$.

P-action $\rightsquigarrow$ G_0-action on $\mathfrak{s}_0 \subseteq \mathfrak{g}_0$. Classification over $\mathbb{C}$:

$\dim$	$\mathfrak{gl}_2$ subalgebra
4	$\mathfrak{gl}_2$
3	$\mathfrak{sl}_2$, $\left\langle \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\rangle$
2	$\left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} \lambda_1 & 0 \\ * & \lambda_2 \end{pmatrix} \right\rangle$
1	$\left\langle \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix} \right\rangle$

(Here, identify $\mathsf{Z}_1 \leftrightarrow -\mathrm{id}$ and $\mathsf{H} \leftrightarrow \mathrm{diag}(1,-1)$.)

Tails

- Suppose $\exists S \in \mathfrak{f}^0$ with $S_0 = \mathrm{gr}_0(S) \in \mathfrak{s}_0$ semisimple.
- Try to use P-action to normalize tails so that $S = S_0$.
- Pick S-invariant decomp. $\mathfrak{s} \oplus \mathfrak{s}^\perp = \mathfrak{g}$.
- Tails $\rightsquigarrow$ deform. map $\mathfrak{d} \in (\mathfrak{s}^* \otimes \mathfrak{s}^\perp)_+$ is highly constrained:

$$S \cdot \mathfrak{d} = 0. \quad (\rightsquigarrow \text{ eigenvalue restrictions!})$$

All cases with such semisimple elements in (normalized) $\mathfrak{s}_0$:

$S_0 \in \mathfrak{s}_0$	Constraints	Admissible models
Z_1	–	M9
H	$\mathsf{Z}_1 \notin \mathfrak{s}_0$	M8 & M6S
$\mathsf{Z}_1 + c\mathsf{H}$	$c \neq 0$, $\mathfrak{s}_0 \subseteq \langle \mathsf{Z}_1 + c\mathsf{H}, f_{01} \rangle$	$c \neq 1$: none $c = 1$: M7$_a$

Remaining cases with no such: $\mathfrak{s}_0 = \langle f_{01} + r\mathsf{Z}_1 \rangle$ (Jordan case).

- $r \neq 0$: No admissible model. (Easy.)
- $r = 0$: M6N. (Most challenging case; $\mathcal{T} \equiv 0$ is essential.)

Running example: M6S

Filtrand	M6S
$\mathfrak{f}^0$	H
$\mathcal{D} = \mathfrak{f}^{-1}/\mathfrak{f}^0$	$X_1 = f_{10} + e_{11}$ $X_2 = f_{11} + e_{10}$
$\mathfrak{f}^{-2}/\mathfrak{f}^{-1}$	$X_3 = f_{21} + e_{21}$ (also spans ℓ)
$\mathfrak{f}^{-3}/\mathfrak{f}^{-2}$	$X_4 = f_{31} + e_{32}$ $X_5 = f_{32} + e_{31}$

Let $\mathfrak{k} := \mathfrak{f}^0 + \ell$. On $\mathfrak{f}/\mathfrak{k}$ (4D), $\exists!$ $\mathfrak{k}$-inv. conf. str. $[\mathrm{g}]$. Define

$$\mathfrak{f} \ni \quad e_1 = X_1, \quad e_2 = X_5 + \tfrac{2}{3}X_2, \quad e_3 = X_4, \quad e_4 = X_2.$$

Let $\{\theta^i\}$ be dual to $\{e_i\} \pmod{\mathfrak{k}}$. Then $\mathrm{g} = 2(\theta^1\theta^2 + \theta^3\theta^4)$.
SD totally null 2-planes:

$$\Pi_\xi := \langle e_1 + \xi e_3, e_4 - \xi e_2 \rangle \pmod{\mathfrak{k}}.$$

M6S: Isotropy

$\mathfrak{k} = \langle \mathsf{H}, X_3 \rangle$ acts on SD family. FACTS: For any $\xi \in \mathbb{R}$,

1. Π_ξ has isotropy $\mathfrak{k}_\xi = \langle \mathsf{H} \rangle$.
2. On $\mathfrak{f}/\mathfrak{k}_\xi$ (5D), have $\mathfrak{k}/\mathfrak{k}_\xi$ modelling ℓ.
3. X_3 acts via $(\xi + 3)(\xi - 1)\partial_\xi$.

Example

Via the isotropy rep, $\rho(X_3) = \begin{pmatrix} -2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 3 \\ -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. The local action is $\Pi_{\tilde{\xi}} = e^{t\rho(X_3)}\Pi_\xi$. This sends $e_1 + \xi e_3$ to

$$e_1 + \xi e_3 + t(-2e_1 - 3e_3 - \xi e_1) + O(t^2) \mod \{e_2, e_4\}$$

so $\widetilde{\xi} = \frac{\xi - 3t + O(t^2)}{1 - 2t - \xi t}$. Diff. at $t = 0$ to get $(\xi + 3)(\xi - 1)\partial_\xi$.

M6S: Petrov classification

Recall $\ell = \mathfrak{k}/\mathfrak{k}_\xi$. Let's determine the twistor distribution $\mathcal{D}$.

$$\begin{array}{ccc} \mathfrak{f}/\mathfrak{k}_\xi & \mathcal{D} = \langle e_1 + \xi e_3 + AX_3, e_4 - \xi e_2 + BX_3 \rangle \pmod{\mathfrak{k}_\xi} \\ \downarrow \pi & \downarrow \cong \\ \mathfrak{f}/\mathfrak{k} & \Pi_\xi \end{array}$$

The condition $[\mathcal{D}, \mathcal{D}] \subseteq \mathcal{H} := \pi^{-1}(\Pi_\xi)$ forces $A = B = 0$ for M6S.

We have $\mathcal{H} = \ell \oplus \mathcal{D}$. Then $\mathcal{W}^+$ arises $\bigwedge^2 \mathcal{D} \stackrel{[\cdot,\cdot]}{\to} \mathcal{H} \stackrel{\mathrm{pr}_\ell}{\to} \ell$. We find:

$$[e_1 + \xi e_3, e_4 - \xi e_2] \equiv \frac{2}{3}(\xi + 3)(\xi - 1) X_3 \pmod{\mathfrak{k}_\xi}.$$

Recall X_3 acts via $(\xi + 3)(\xi - 1)\partial_\xi$, so $\mathcal{W}^+$ is a multiple of $(\xi + 3)^2(\xi - 1)^2$, i.e. type D. ($\mathcal{D}$ is integrable at $\xi = -1$ or 3.)

Cartan-theoretic description

We saw Lie-theoretic XXO-models earlier, embedded in $\mathfrak{g} = \mathsf{Lie}(G_2)$. These quotient to 4D split-conformal models. Equivalent description:

Definition

A Cartan-theoretic model $(\mathfrak{f}; \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{q}})$ *is a Lie algebra* $(\mathfrak{f}, [\cdot,\cdot]_{\mathfrak{f}})$ *s.t.:*

(M1) $\mathfrak{f} \subset \widetilde{\mathfrak{g}}$ *is a filtered subspace, with filtrands* $\mathfrak{f}^i := \mathfrak{f} \cap \widetilde{\mathfrak{g}}^i$, *and* $\mathfrak{f}/\mathfrak{f}^0 \cong \widetilde{\mathfrak{g}}/\widetilde{\mathfrak{q}}$.

(M2) $\mathfrak{f}^0$ *inserts trivially into the curvature* $\kappa(x,y) := [x,y] - [x,y]_{\mathfrak{f}}$.

(M3) $\kappa \in \bigwedge^2(\mathfrak{f}/\mathfrak{f}^0)^* \otimes \widetilde{\mathfrak{g}} \cong \bigwedge^2(\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{q}})^* \otimes \widetilde{\mathfrak{g}}$ *is regular and normal, i.e.* $\kappa \in \ker(\partial^*)^1$. *(Define* $\kappa_H := \kappa \mod \operatorname{im}(\partial^*) \in H_2(\widetilde{\mathfrak{q}}_+, \widetilde{\mathfrak{g}})^1$*.)*

Example (M6S, $\widetilde{\mathfrak{g}} = \mathfrak{sl}_4$, $\widetilde{\mathfrak{q}} = \mathfrak{p}_2$)

$$\mathfrak{f} = \begin{pmatrix} s_1 & 0 & \frac{11}{6}a_2 & \frac{19}{6}a_1 \\ 0 & -s_1 & \frac{19}{6}a_4 & \frac{11}{6}a_3 \\ a_3 & a_1 & s_2 & 0 \\ a_4 & a_2 & 0 & -s_2 \end{pmatrix}, \quad \kappa = -\frac{4}{3}\phi_2 + \frac{4}{3}\psi_2, \quad \text{where}$$

$$\phi_2 = E_{31}^* \wedge E_{41}^* \otimes E_{21} + E_{42}^* \wedge E_{32}^* \otimes E_{12} + (E_{31}^* \wedge E_{42}^* + E_{32}^* \wedge E_{41}^*) \otimes H_{12}$$

$$\psi_2 = E_{42}^* \wedge E_{41}^* \otimes E_{43} + E_{31}^* \wedge E_{32}^* \otimes E_{34} + (E_{42}^* \wedge E_{31}^* + E_{32}^* \wedge E_{41}^*) \otimes H_{43}$$

Holonomy & aE scales

Given a Cartan-theoretic model $(\mathfrak{f}; \widetilde{\mathfrak{g}}, \widetilde{\mathfrak{q}})$, define:

$$\mathfrak{hol}^0 := \langle \kappa(x,y) : x, y \in \mathfrak{f} \rangle, \qquad \mathfrak{hol}^i := \mathfrak{hol}^{i-1} + [\mathfrak{f}, \mathfrak{hol}^{i-1}], \ \forall i \geq 1.$$

Then $\mathfrak{hol} := \mathfrak{hol}^\infty$ is the (infinitesimal) holonomy algebra.

Holonomy obstructs existence of almost-Einstein scales. Via the "tractor" picture, we should ask:

- Via $\mathfrak{hol} \hookrightarrow \mathfrak{sl}_4 \cong \mathfrak{so}_{3,3}$, what is $(\mathbb{R}^{3,3})^{\mathfrak{hol}}$?
- Equivalently, what is $(\bigwedge^2 \mathbb{R}^4)^{\mathfrak{hol}}$?

Example (M6S)

$\mathfrak{hol}^0 = \{E_{12}, E_{21}, H_{12}, E_{34}, E_{43}, H_{34}\}, \quad \mathfrak{hol} = \mathfrak{hol}^1 = \mathfrak{sl}_4$

There are no almost-Einstein scales here.

Holonomy & aE scales - 2

For our (complexified) models, we find:

Label	$\mathfrak{hol}$, represented on $\mathbb{C}^4 = \langle e_1, e_2, e_3, e_4 \rangle$	$(\bigwedge^2 \mathbb{C}^4)^{\mathfrak{hol}}$	Almost-Einstein?
M9	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}$	$e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$	✓
M8	$\begin{pmatrix} s_1 & s_2 & \frac{c_4}{2} & -\frac{c_2}{2} \\ s_3 & -s_1 & -\frac{c_3}{2} & \frac{c_1}{2} \\ c_1 & c_2 & t_1 & t_2 \\ c_3 & c_4 & t_3 & -t_1 \end{pmatrix}$	$e_1 \wedge e_2 - 2e_3 \wedge e_4$	✓
M7$_a$	$a^2 \neq \frac{4}{3}$: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$	$e_2 \wedge e_4$, $e_3 \wedge e_4$	✓
	$a^2 = \frac{4}{3}$: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}$	$e_2 \wedge e_3$, $e_2 \wedge e_4$, $e_3 \wedge e_4$	✓
M6S	$\mathfrak{sl}_4$	0	×
M6N	$\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$	0	×

Thanks for your attention!