Pseudo-Riemannian Zoll manifolds

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(A) Definition: (M,g) Riemannian is Besse (Zoll), if all (non-constant) geodesics are closed (and of the same prime length).

Examples: (a) Spheres of constant curvature are Zoll
In fact: $n \neq 3$: (S^n, g) Besse $\Rightarrow$ (S^n, g) Zoll
($n=2$: Gromoll-Grove, $n>3$: Radeschi-Wilking)

(b) lens spaces $(S^3/\Gamma, g)$ are Besse

(c) rational Katok examples (non-reversible Finsler) (S^2, F); geodesic flow on $T^{1F}S^2$ lifts to the Reeb flow on certain rational ellipsoids in $\mathbb{C}^2$. (Ziller)

Facts: (1) (M,g) Besse $\Rightarrow$ M cpl, $\pi_1(M)$ finite
$H^*(M, \mathbb{Q}) \cong H^*(\text{CROSS}, \mathbb{Q})$

I.e. $\dim M = 2 \Rightarrow M \approx S^2, \mathbb{R}P^2$

(2) $(\mathbb{R}P^2, g)$ Besse $\Rightarrow$ $K_g \equiv$ const, esp. Zoll
(Green, Pries)

(3) $\gamma \subseteq M$ geodesic $\Rightarrow$ $\dot\gamma \subseteq TM$ geodesic of the Sasaki metric.

Wadsley's Theorem: A 1-dimensional foliation by circles is induced by an S^1-action iff it is geodesic w.r.t. a Riemannian metric, especially the length is locally bounded.

Remark: ⚠ Thurston-Sullivan: Not every foliation by circles is induced by an S^1-action.

Theorem (Darboux / Guillemin): There exists an infinite dimensional family of Zoll metrics on S^2 conformal to the standard metric.

Remark: Weinstein / ABHS:

(S^2, g) Zoll $\rightarrow$ $\left(T^{1,g}S^2, \lambda|_{T^{1,g}S^2}\right)$ is contactomorphic to $\left(T^{1,g_{can}}S^2, \lambda|_{\dots}\right)$, where λ is the Liouville form and g_{can} is the metric of constant curvature.

(B) Definition: (M,g) pseudo-Riemannian is Besse / Zoll with the same definition as before, just for separate causal characters (i.e. timelike, spacelike, null)

Examples: (a) $S^n_\nu := \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_\nu = +1\} \subseteq \mathbb{R}^{n+1}_\nu$

$n=2, \nu=1$: de Sitter space (spacelike Zoll)

(b) $(S^2 \times S^1, g_{can} - dt^2)$ (null Zoll $\sim$ Zollfrei)

Signature - rigidity Theorem (M/S '13)

(M, g) pseudo-Riemannian s.th. the geodesic flow (outside the zero section) can be reparameterized to an S^1-action. Then g is Riemannian or anti-Riemannian ($-g$ Riem.)

Remark: (1) Thurston-Sullivan examples are (null) geodesic for some pseudo-Riemannian metric

(2) Wadsley's Theorem holds for nowhere null geodesic foliations. (due to the "contact geometric" character (Cieliebak - Volkov))

(3) no known examples with 2 out of 3 geodesic types all closed (e.g. timelike + null or timelike + spacelike)

Theorem (M/S '13) (M, g) 2-dim. Lorentzian + spacelike Besse. Then (M, g) is finitely covered by a spacelike SC-metric on $S^1 \times \mathbb{R}$ (SC ≙ simple closed)

Theorem (M/S '13) Every Lorentzian surface contains a non-closed spacelike or timelike geodesic.

(C) Spacelike Zoll cylinders with symmetries

deSitter has 3 types of Killing vf's:

elliptic, parabolic, hyperbolic

(same as for Hyperbolic space)

Lemma (M/S '16)

A local Killing vf (i.e. ∇K antisym.) in a spacelike Zoll surface is complete, i.e. Killing.

Proposition (M/S '16)

A Killing vf of a spacelike Zoll surface is orbit equivalent to a Killing vf on a finite cover of deSitter.

Theorem (M/S '16)

There exist infinite dimensional families of spacelike Zoll cylinders with elliptic, parabolic, and hyperbolic Killing vf's.

Theorem (M/S '16)

(1) A spacelike Zoll cylinder with a Killing vf is C^0-conformal to a cover of deSitter (C^0-conformal $\cong$ coinciding null foliation)

(2) If the Killing vf is periodic $\Rightarrow$ C^∞-conformal

Remark: There exist spacelike Zoll cylinders with a Killing vf. not C^2-conformal to a cover of deSitter.

Anti-Green Theorem (M/S'16)
There exist Lorentzian spacelike Besse Möbius strips of non-constant curvature.

(D) Zollfrei 3-manifolds

Definition (Guillemin)
(M,g) pseudo-Riem. is Zollfrei, if the geodesic flow on the null vectors induces a fibration by circles.

Guillemin: "standard examples" covered by $S^2 \times S^1$ (4 classes: $S^2 \times S^1$ + $\mathbb{Z}_2$-quotients)

Conjecture (Guillemin) Every Zollfrei 3-manifold has the diffeomorphism type of one of the standard examples.

Theorem (S'13) Every nontrivial circle bundle over a closed and oriented surface admits a Zollfrei metric.

Idea of construction: Kaluza-Klein type metrics.