Geometry of quantum correlations

Adam Sawicki

Quantum Systems (Q.S)

- Any Q. S. is described by a complex Hilbert space, $\mathcal{H}$. We will assume $\mathcal{H}$ is finite dimensional, that is, $\mathcal{H} \simeq \mathbb{C}^d$ $\longrightarrow$ we call it QUDIT

$\rightarrow$ For $v, w \in \mathcal{H}$ the inner product is $\langle v | w \rangle$.

$\langle \cdot | \cdot \rangle$ is conjugate linear in first arg. and linear in the second

$\rightarrow$ Riesz Lemma: Any linear functional $\varphi : \mathcal{H} \to \mathbb{C}$ is given by inner product

$$\exists v_\varphi \text{ s.t. } \forall u \in \mathcal{H} \qquad \varphi(u) = \langle v_\varphi | u \rangle$$

bra

$\rightarrow$ Linear functionals from $\mathcal{H}^*$ will be denoted by $\langle \varphi |$, $\langle \psi |$, etc

ket

$\rightarrow$ vectors in $\mathcal{H}$ will be denoted by $|\varphi\rangle$, $|\psi\rangle$, etc

$\rightarrow$ Action of $\langle \varphi | \in \mathcal{H}^*$ on $|\psi\rangle \in \mathcal{H}$ is

$$\langle \varphi | (|\psi\rangle) = \underbrace{\langle \varphi}_{\text{bra}} | \underbrace{\psi \rangle}_{\text{ket}}$$

- Pure states of Q. S.

→ In $\mathcal{H}$ we introduce equivalence relation:

$$|\psi\rangle \sim |\varphi\rangle \iff |\psi\rangle = \alpha |\varphi\rangle \qquad \alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$

→ Equivalence classes of $\mathcal{H}/\sim$ are pure states of Q. S.

→ Pure states are points in $\mathbb{P}(\mathcal{H})$

→ Physicists say pure states are normalized to 1 vectors where we neglect the 'global phase factor' $e^{i\alpha}$

- Observables

→ To learn properties of Q. S. we measure values of physically relevant quantities, for example: spin, momentum, energy etc. They are called observables

→ Observables are represented by selfadjoint (Hermitian) operators on $\mathcal{H}$

$$F: \mathcal{H} \to \mathcal{H}, \quad F^* = F \quad \text{that is } \langle \varphi | F\psi \rangle = \langle F\varphi | \psi \rangle \quad \forall |\varphi\rangle, |\psi\rangle \in \mathcal{H}$$

$\sigma(F)$ - spectrum of F, $\quad F = \sum_{\lambda \in \sigma(F)} \lambda P_\lambda$, $\qquad \mathcal{H} = \bigoplus_{\lambda \in \sigma(F)} \mathcal{H}_\lambda \qquad \mathcal{H}_\lambda \perp \mathcal{H}_\mu$

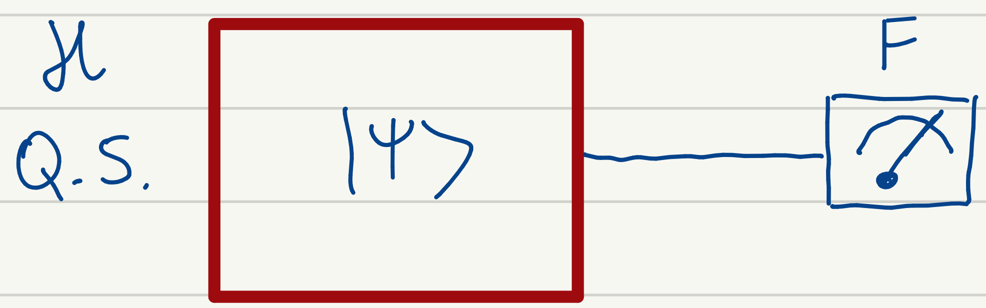
P_λ - orthogonal projection onto $\mathcal{H}_\lambda = \operatorname{Ker}(F - \lambda I)$

- Observables

→ To learn properties of Q.S. we measure values of physically relevant quantities, for example: spin, momentum, energy etc. They are called observables

→ Observables are represented by selfadjoint (Hermitian) operators on $\mathcal{H}$

→ Assume Q.S. is in a state $|\psi\rangle$ and we measure F



- Possible measurment outcomes are $\sigma(F)$

- The probability of getting $\lambda \in \sigma(F)$ is $\mathbb{P}_{|\psi\rangle}(\lambda) = \langle \psi | P_\lambda \psi \rangle$

- If the result is λ the state changes to

$$|\psi\rangle \longrightarrow \frac{P_\lambda |\psi\rangle}{\| P_\lambda |\psi\rangle \|} \in \mathcal{H}_\lambda$$

- The expected value of F: $\langle F \rangle_{|\psi\rangle} = \sum_{\lambda \in \sigma(F)} \lambda \cdot \mathbb{P}_{|\psi\rangle}(\lambda) = \langle \psi | F \psi \rangle$

- Free time evolution

  → Assume Q.S. is in a state $|\psi_0\rangle$ at $t = t_0$ and we do not measure anything

  → Every Q.S. has one special observable $H : \mathcal{H} \to \mathcal{H}$ called Hamiltonian

  → Free time evolution is given by time evolution operator $\mathcal{U}(t, t_0) : \mathcal{H} \to \mathcal{H}$

$$|\psi(t)\rangle = \mathcal{U}(t, t_0)\, |\psi_0\rangle$$

that satisfies

$$i \frac{d}{dt} \mathcal{U}(t, t_0) = H\, \mathcal{U}(t, t_0), \qquad \mathcal{U}(t_0, t_0) = \mathbb{1}$$

$$\Updownarrow$$

$$i \frac{d}{dt} |\psi(t)\rangle = H\, |\psi(t)\rangle, \qquad |\psi(t_0)\rangle = |\psi_0\rangle$$

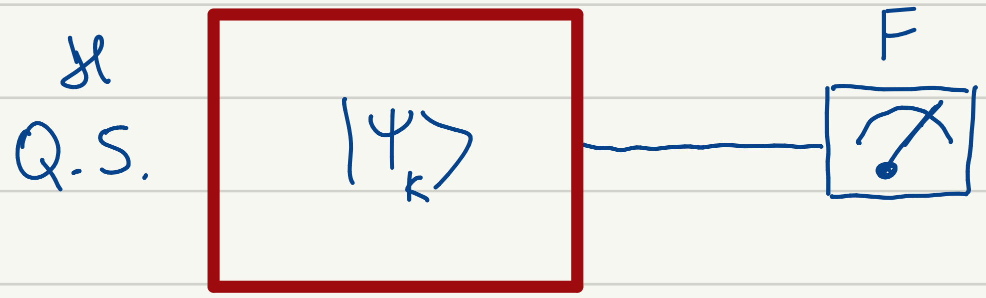
We assume a full control over Q.S. that is we can use any H

- $\mathcal{U}(t, t_0)$ is unitary
- $\mathcal{U}(t, t_0) = e^{i(t-t_0) H}$. Putting $t_0 = 0$ and $\mathcal{U}(t, 0) = \mathcal{U}(t) = e^{itH}$

$\mathcal{U}(t)$ is a 1-parameter subgroup of $\mathcal{U}(\mathcal{H})$ generated by H

- Mixed states

↦ So far we assumed that our Q.S. is prepared in a state $|\psi\rangle \in \mathcal{H}$
What if the preparing procedure is also probabilistic



The probability that system $\mathcal{H}$ is prepared in state $|\psi_k\rangle$ is p_k

↦ Possible outcomes are still $\sigma(F)$

↦ But we need to modify $\mathbb{P}(\lambda)$, $\lambda \in \sigma(F)$

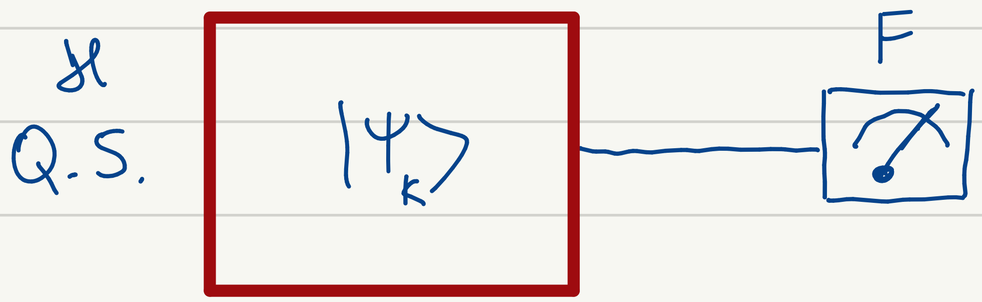
↦ Total law of probability

$$\mathbb{P}(\lambda) = \sum_k{}' \underbrace{\mathbb{P}(\lambda \mid \psi_k)}_{} \cdot p_k = \sum_k{}' \mathbb{P}_{|\psi_k\rangle}(\lambda) \cdot p_k = \sum_k{}' \langle \psi_k | P_\lambda \psi_k \rangle \cdot p_k \overset{①}{=} tr(\rho P_\lambda)$$

probability that the outcome is λ
under the condition Q.S. is in the state $|\psi_k\rangle$

$$\rho = \sum_k{}' p_k \underbrace{|\psi_k\rangle\langle\psi_k|}_{}$$

① $tr\left(\sum_k{}' p_k |\psi_k\rangle\langle\psi_k| P_\lambda\right) = \sum_k{}' p_k \, tr\left(|\psi_k\rangle\langle\psi_k| P_\lambda\right) = \sum_k{}' \langle \psi_k | P_\lambda \psi_k \rangle \cdot p_k$



The probability that system $\mathcal{H}$ is prepared in state $|\psi_k\rangle$ is p_k

$\longmapsto$ $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ – mixed state

$\longmapsto$ $\mathbb{P}_\rho(\lambda) = tr(\rho P_\lambda) = \langle P_\lambda \rangle_\rho$

$\longmapsto$ $\langle F \rangle_\rho = \sum_{\lambda \in \sigma(F)} \lambda \cdot \mathbb{P}_\rho(\lambda) = \sum_{\lambda \in \sigma(F)} \lambda \cdot tr(\rho P_\lambda) = tr(\rho F)$

$\longmapsto$ If the result of measurement is $\lambda \in \sigma(F)$ the state is

$$\frac{P_\lambda \rho P_\lambda}{tr(P_\lambda \rho)}$$

- Properties of mixed states ρ :

$\longmapsto$ $tr(\rho) = \sum_k p_k = 1$

$\longmapsto$ $\rho \geq 0$, $\quad |\varphi\rangle \in \mathcal{H}$ $\quad \langle \varphi | \rho \varphi \rangle = \sum_k p_k |\langle \varphi | \psi_k \rangle|^2 \geq 0$

$\longmapsto$ If $\forall F \in L(\mathcal{H})$ $\quad tr(\rho_1 F) = tr(\rho_2 F) \Rightarrow \rho_1 = \rho_2$

$\langle F \rangle_{\rho_1} = \langle F \rangle_{\rho_2}$

- Pure state is a mixed state that is given by $|\psi\rangle\langle\psi|$ - orthogonal projection onto $\mathbb{C}|\psi\rangle$

- For a fixed $\rho \geq 0$, $\mathrm{tr}(\rho) = 1$ there are many ensembles $\{|\psi_k\rangle, p_k\}_k$ for which

$$\rho = \sum_i p_k |\psi_k\rangle\langle\psi_k|$$

- A canonical choice is to use spectral decomposition of $\rho = \sum_{k=1}^{d} \lambda_k |e_k\rangle\langle e_k|$

  - von Neumann entropy of state ρ is

$$S(\rho) = -\mathrm{tr}(\rho \log(\rho)) = -\sum_{k=1}^{d} \lambda_k \log \lambda_k$$

$$S(\rho) = 0 \iff \rho \text{ is a pure state}$$

- Maximally mixed state $\rho_M = \frac{1}{d} \mathbb{1}$, $S(\rho_M) = \log(d)$

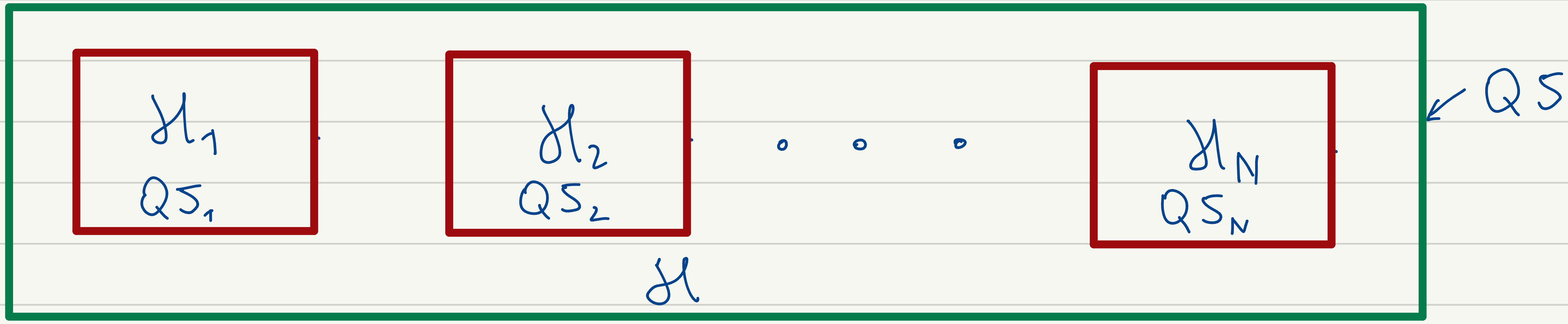
- Free time evolution of ρ is again unitary, $\{|\psi_k\rangle, p_k\}$

$$\rho(t) = \underbrace{U(t,t_0)^*}_{} \rho(t_0) U(t,t_0) \qquad U(t,t_0) = e^{i(t-t_0)H}.$$

unitary

Systems and subsystems

- Assume we have N quantum systems described by Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_N$



→ If systems $\mathcal{H}_1, \dots, \mathcal{H}_N$ are distinguishable then

$$\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$$

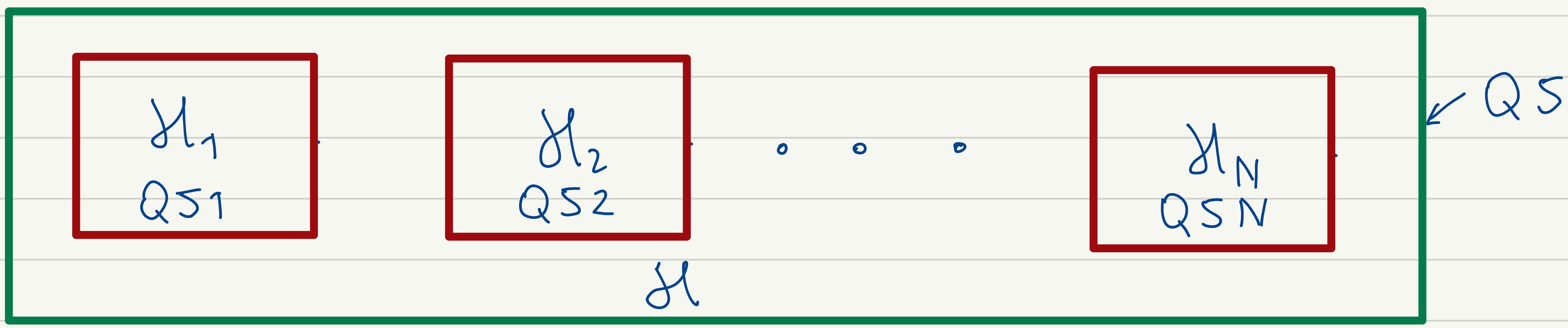
→ If systems $\mathcal{H}_1, \dots, \mathcal{H}_N$ are indistinguishable (and $\dim(\mathcal{H}_k) = d \ \forall k$)

→ Fermions : $\mathcal{H} = \Lambda^N \mathbb{C}^d$

→ Bosons : $\mathcal{H} = S^N \mathbb{C}^d$

→ We will focus mostly on distinguishable case

- Assume Q.S. is in the pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_N$



What are the states of $QS_1, \dots, QS_N$?

$\to$ $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_N\rangle \in \mathcal{H}$ a simple tensor. Then

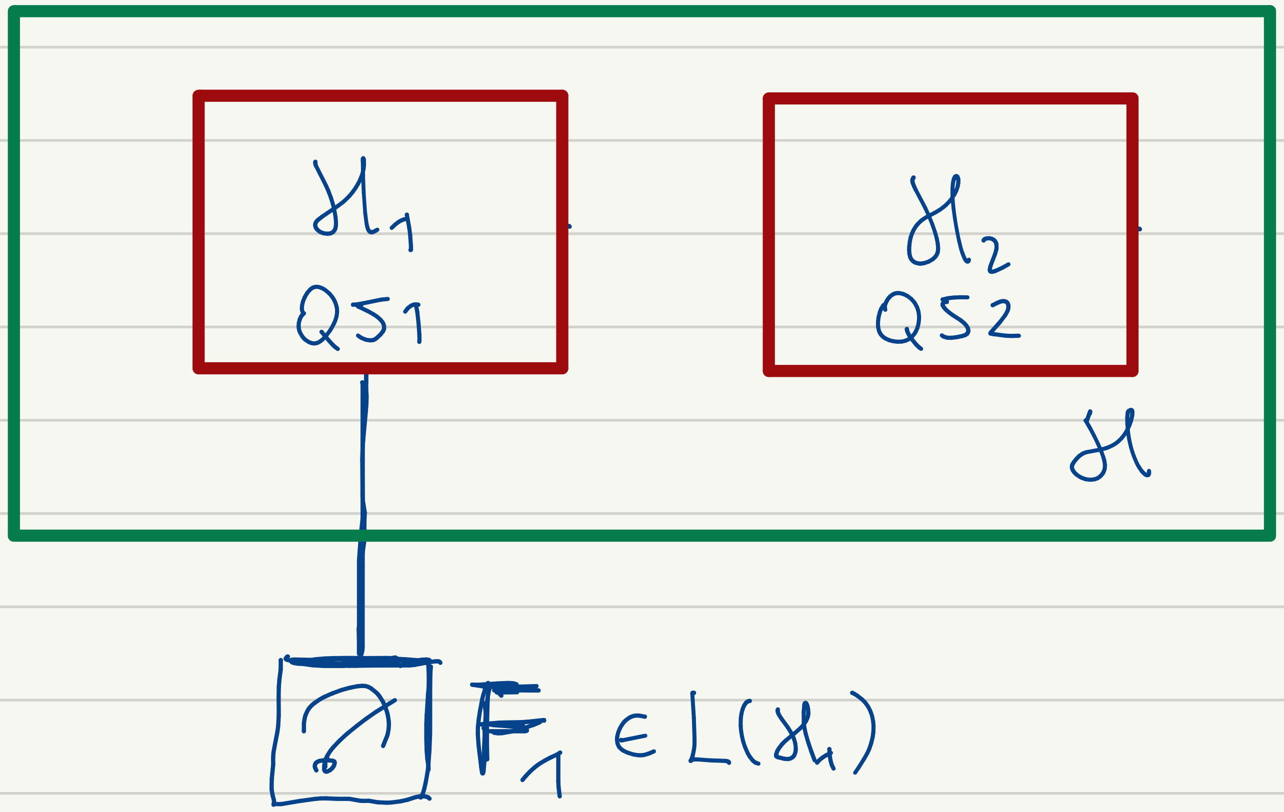
$\forall_k$ QS_k is in the state $|\psi_k\rangle$

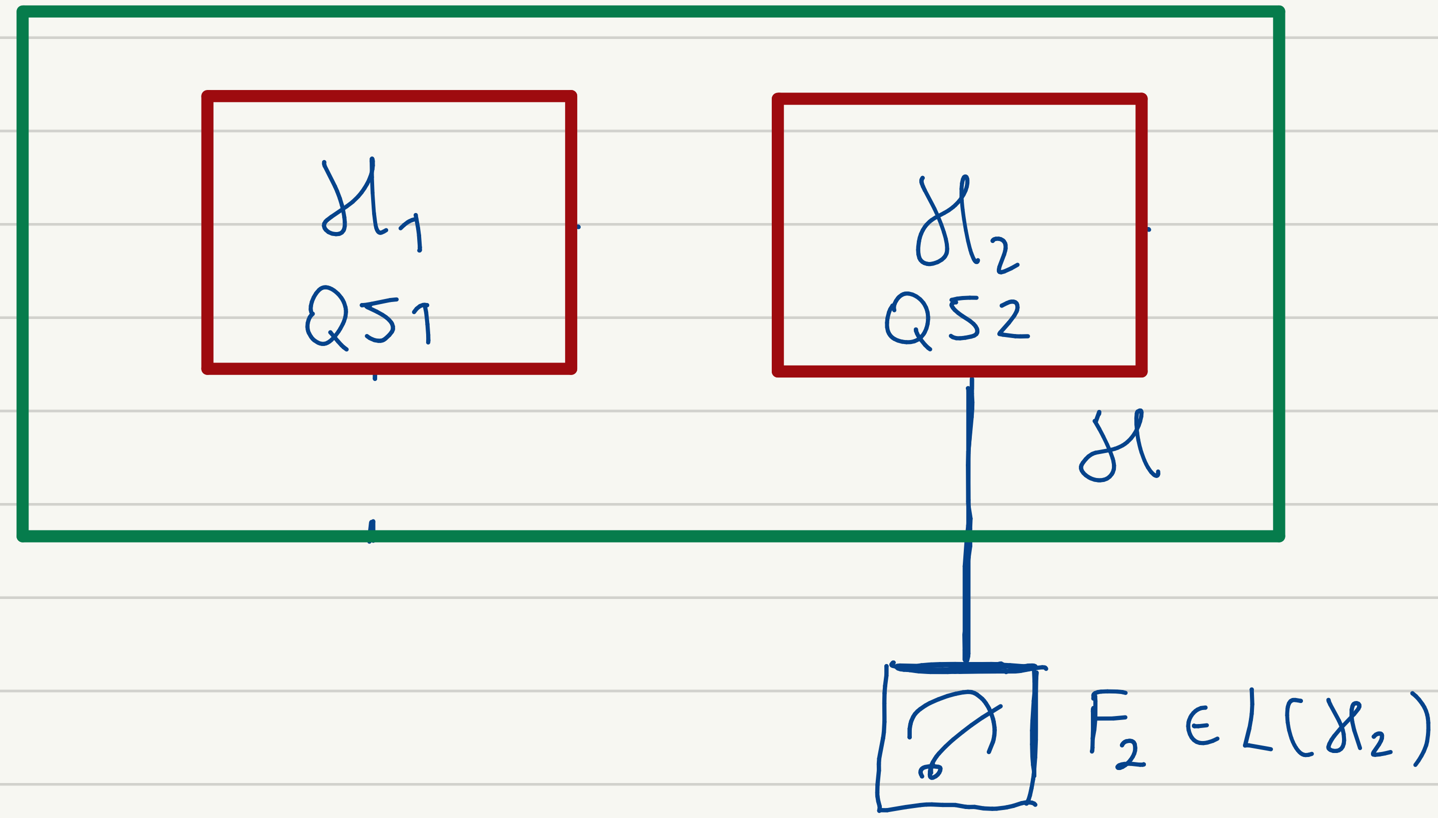
We say $|\psi\rangle$ is separable or not entangled

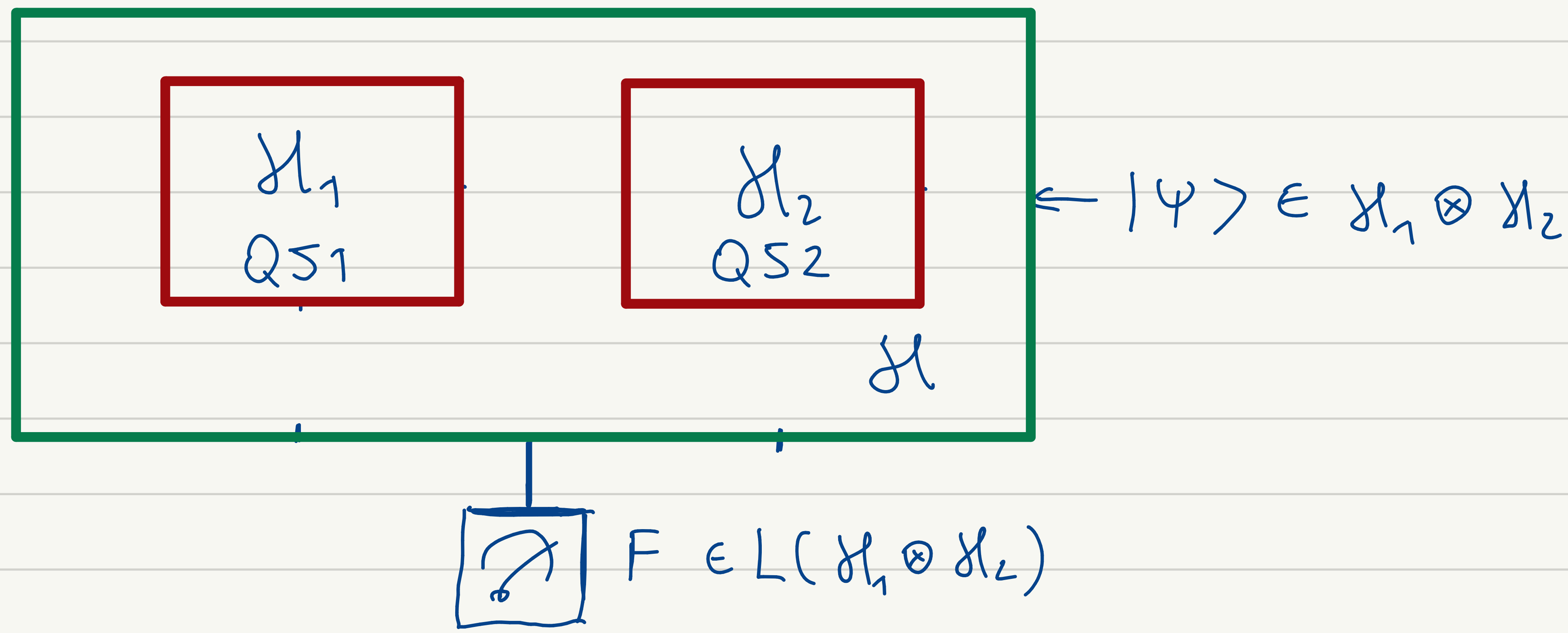
$\to$ $N=2$, $\mathcal{H} = \underbrace{\mathbb{C}^d \otimes \mathbb{C}^d}_{2 \text{ qudits}}$, $\mathcal{H}_1 = \text{Span}\{|e_1\rangle, \dots, |e_d\rangle\}$, $\mathcal{H}_2 = \text{Span}\{|f_1\rangle, \dots, |f_d\rangle\}$

$$|\psi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} |e_i\rangle \otimes |f_j\rangle$$

$\to$ To determine states of QS_1 and QS_2 we consider observables







$$L(\mathcal{H}_1) \hookrightarrow L(\mathcal{H}_1 \otimes \mathcal{H}_2) \qquad F_1 \longmapsto F_1 \otimes \mathbb{1}$$
$$L(\mathcal{H}_2) \hookrightarrow L(\mathcal{H}_1 \otimes \mathcal{H}_2) \qquad F_2 \longmapsto \mathbb{1} \otimes F_2$$

A state of a system is determined by all expected values of observables.

$$\forall F_1 \in L(\mathcal{H}_1) \quad \langle F_1 \rangle_{\rho_1} = \langle F_1 \otimes \mathbb{1} \rangle_{|\psi\rangle}, \quad tr(\rho_1 F_1) = \langle \psi | F_1 \otimes \mathbb{1}\, \psi \rangle \quad \forall F_1 \in L(\mathcal{H}_1)$$
$$\forall F_2 \in L(\mathcal{H}_2) \quad \langle F_2 \rangle_{\rho_2} = \langle \mathbb{1} \otimes F_2 \rangle_{|\psi\rangle}, \quad tr(\rho_2 F_2) = \langle \psi | \mathbb{1} \otimes F_2\, \psi \rangle \quad \forall F_2 \in L(\mathcal{H}_2)$$

- $\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d$

- In general $|\psi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} |e_i\rangle \otimes |f_j\rangle$

$$\rho_1 = tr_2(|\psi\rangle\langle\psi|) = \sum_{i,j,k} c_{ik}\overline{c_{jk}} \; |e_i\rangle\langle e_j|$$

$$\rho_2 = tr_1(|\psi\rangle\langle\psi|) = \sum_{i,j,k} c_{ki}\overline{c_{kj}} \; |f_i\rangle\langle f_j|$$

If $\overbrace{|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle}^{\text{simple tensor}}$ then

$$\rho_1 = |\psi_1\rangle\langle\psi_1|$$
$$\rho_2 = |\psi_2\rangle\langle\psi_2|$$

Pure states

- Entanglement entropy:

$$S_E(\psi) = S(\rho_1(\psi)) + S(\rho_2(\psi)) = -\sum_{i=1}^{d} (p_i \log(p_i) + q_i \log(q_i))$$

p_i's, q_i's ← eigenvalues of ρ_1 / ρ_2

→ $|\psi\rangle$ – Separable state, that is $|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \Rightarrow S_E([\psi]) = 0$

→ $|\psi\rangle = \frac{1}{\sqrt{d}}\left(|e_1\rangle \otimes |f_1\rangle + \ldots + |e_d\rangle \otimes |f_d\rangle\right) \Rightarrow \rho_1 = \rho_2 = \frac{1}{d}\mathbf{I}$, $S_E = 2\log(d)$ ← maximal

→ $S_E(\psi)$ is invariant w.r.t $K = SU(d) \times SU(d)$ action

Can we give any „reasonable" geometric meaning to entanglement?

Symmetries of symlectic spaces

- (M,ω) – a compact symplectic manifold, ω – non-degenerate 2-form on M which is closed $d\omega=0$

- ω – nondegenerate: $\omega^b : \mathfrak{X}(M) \to T^*M$ ← Isomorphism between vector fields and 1-forms on M
$$Z \mapsto \omega(Z,\cdot)$$

- A vector field $\xi \in \mathfrak{X}(M)$ is called Hamiltonian iff $\mathcal{L}_\xi \omega \equiv 0$

- We denote space of Hamiltonian vector fields by $\mathrm{Ham}(M)$

- Cartan formula $\mathcal{L}_\xi \omega = i_\xi \circ d\omega + d \circ i_\xi \omega$

- Assume $\xi \in \mathrm{Ham}(M)$. Then

$$d\omega(\xi,\cdot) = 0 \Rightarrow \omega(\xi,\cdot) \text{ – closed 1-form}$$

- We assume $H^1(M)$ is trivial. Then $\exists\, F \in C^\infty(M)$ s.t. $dF = \omega(\xi,\cdot)$

- So we get a map $F \mapsto \xi_F$, $dF = \omega(\xi_F,\cdot)$

- So we get a map $F \longmapsto \xi_F \qquad dF = \omega(\xi_F, \cdot)$

- If we define $\{F, G\} := \omega(\xi_F, \xi_G)$ then

$$\left(C^\infty(M), \{\cdot,\cdot\} \right) \longrightarrow \left(\mathrm{Ham}(M), [\cdot,\cdot] \right)$$

$$F \longmapsto \xi_F$$

$$[\xi_F, \xi_G] = \xi_{\{F,G\}}$$

← Homomorphism of Lie algebras (surjective)

- K - a compact semisimple Lie group

- We consider action of K on (M, ω), $K \curvearrowright M$ that preserves symplectic structure:

smooth map → $\Phi : K \times M \to M$, $\Phi(g, x) = \Phi_g(x)$

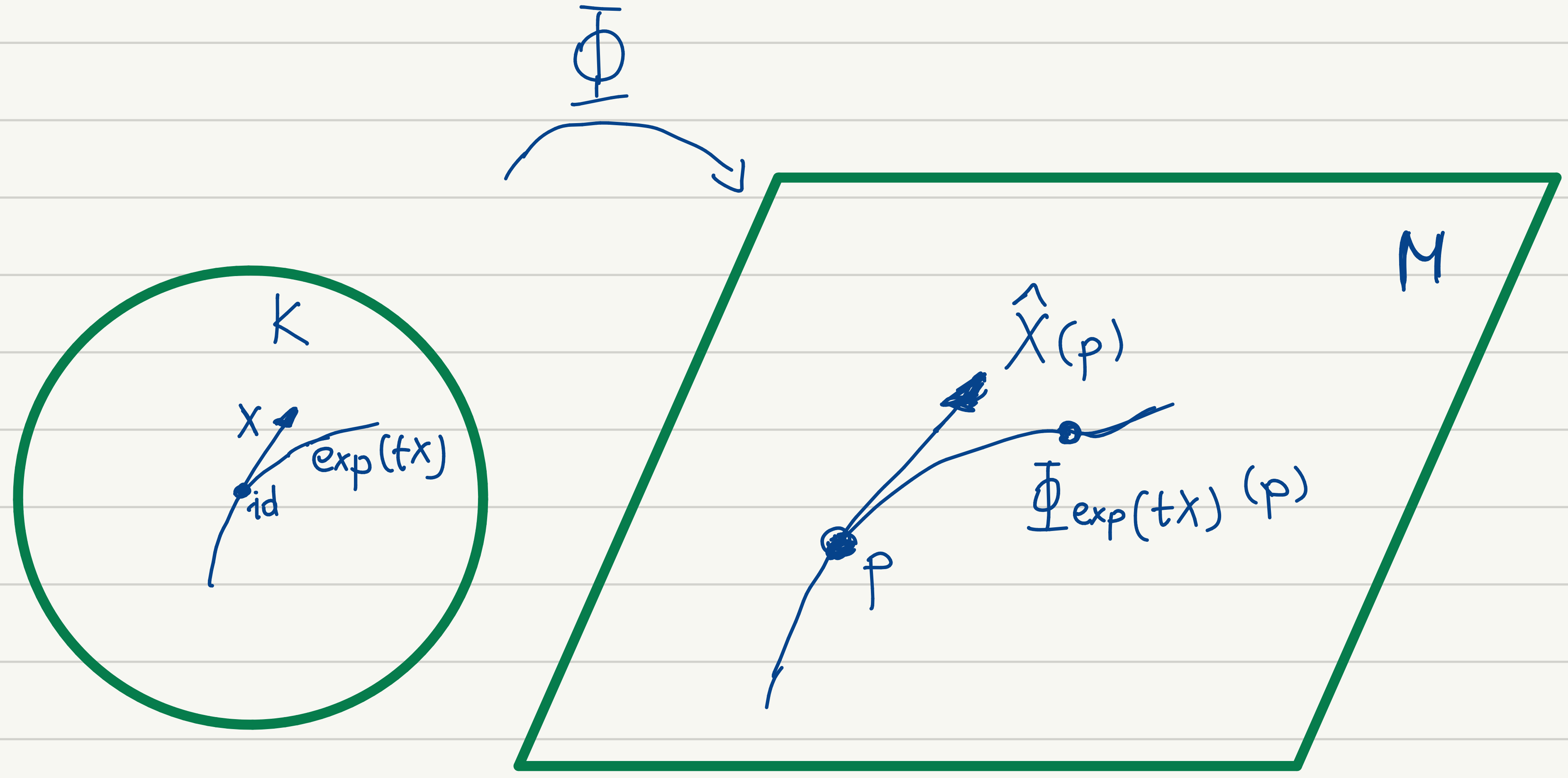
diffeomorphism of M → $\Phi_g : M \to M$, $\Phi_{g_1 g_2}(x) = \Phi_{g_1} \circ \Phi_{g_2}(x)$

Symplectomorphism → $\Phi_g^* \omega = \omega$, $\forall g \in K$

- Let $k = \mathrm{Lie}(K)$ – the Lie algebra of K

$\forall X \in k$ we define the fundamental vector field:

$$\hat{X}(p) := \frac{d}{dt}\Big|_{t=0} \Phi_{\exp(tX)}(p)$$



$$\hat{\ }: (k, [\cdot,\cdot]) \longrightarrow (\mathfrak{X}(M), [\cdot,\cdot])$$ ← Homomorphism of Lie algebras

$$k \ni X \longmapsto \hat{X} \in \mathfrak{X}(M)$$

$$[\hat{X}, \hat{Y}] = \widehat{[X,Y]}$$

- As $K \curvearrowright M$ preserves ω every fundamental vector field is Hamiltonian

$$\forall X \in k \quad \mathcal{L}_{\hat{X}}\,\omega = 0$$

$$\forall X \in k \ \exists \mu_X \quad d\mu_X = \omega(\hat{X}, \cdot)$$

- μ_X can be chosen to be linear in $X \in k$. $\mu_X(p) = \langle \mu(p), X \rangle$.

$$\mu : M \to k^* \quad \text{– a momentum map}$$

- Adjoint action of K on K: $A_g(h) = ghg^{-1}$
- Adjoint action of K on $\mathfrak{k} = Lie(K)$: $\quad Ad_g(X) = \frac{d}{dt}\Big|_{t=0} g\exp(tX)g^{-1}, \quad X \in \mathfrak{k}$
- K is compact hence $\mathfrak{k} = Lie(K)$ has Ad_K-invariant inner product (just take any and average)

$$(X|Y) = (Ad_g(X) | Ad_g(Y)) \quad \forall X, Y \in \mathfrak{k}, \; g \in K$$

- Ad_K-invariant inner product $(\cdot|\cdot)$ allows to identify $\mathfrak{k}$ and $\mathfrak{k}^*$
- For a compact semisimple K

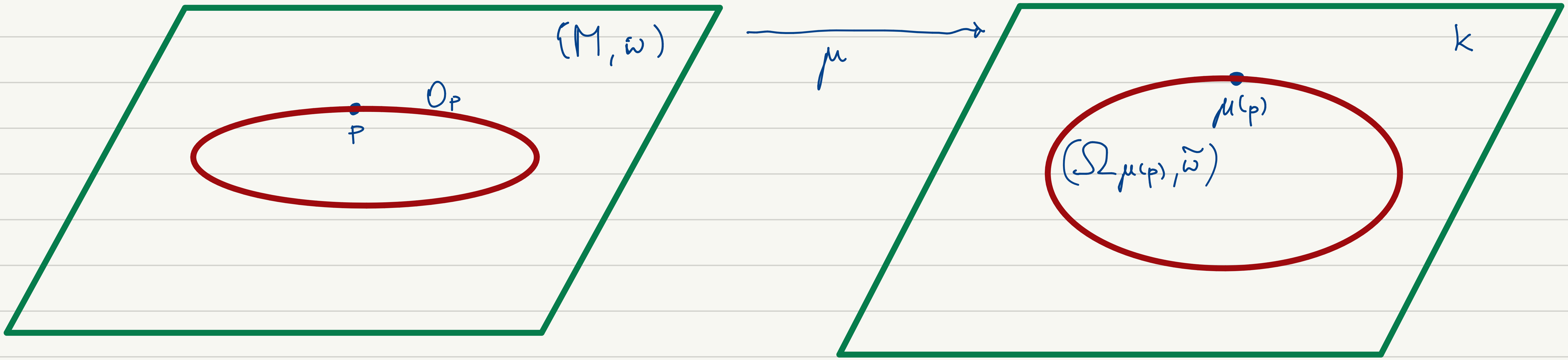
$$1)\quad \mu_X(p) = (\mu(p)|X)$$

$\mu : M \to \mathfrak{k}$ s.t. $\forall X, Y \in \mathfrak{k}$:

$$2)\quad d\mu_X = \omega(\hat{X}, \cdot)$$

equivariance – $3)\quad \mu(\Phi_g(p)) = Ad_g\, \mu(p) \iff \mu_{[X,Y]} = \{\mu_X, \mu_Y\}$

- $O_p = \{\Phi_g(p) \mid g \in K\}$ – K-orbit through $p \in M$
- $\Omega_{\mu(p)} = \{Ad_g\, \mu(p) \mid g \in K\}$ – adjoint orbit through $\mu(p)$ in $\mathfrak{k}$



- $\Omega_{\mu(p)}$ is symplectic manifold $\tilde{\omega}$ – Kirillov–Kostant–Souriau (KKS) symplectic form on $\Omega_{\mu(p)}$

$$\forall \alpha \in \Omega_{\mu(p)} \qquad \tilde{\omega}_\alpha(\tilde{X},\tilde{Y}) := (\alpha \,|\, [X,Y])$$

fundamental vector fields of $X,Y \in k$, $\tilde{X},\tilde{Y} \in \mathfrak{X}(\Omega_{\mu(p)})$

$$\tilde{X}(\alpha) = \frac{d}{dt}\Big|_{t=0} Ad_{\exp(tX)}(\underset{k}{\underset{\cap}{\alpha}})$$

- $\omega = \mu^* \tilde{\omega}$ that is:

$$\omega_p(\hat{X},\hat{Y}) = \tilde{\omega}_{\mu(p)}(\mu_* \hat{X}, \mu_* \hat{Y}) \overset{\text{equivariance}}{=} \tilde{\omega}_{\mu(p)}(\tilde{X},\tilde{Y})$$

- $(O_p, \omega|_{O_p})$ is symplectic iff $\mu|_{O_p}: O_p \to \Omega_{\mu(p)}$ is diffeomorphism

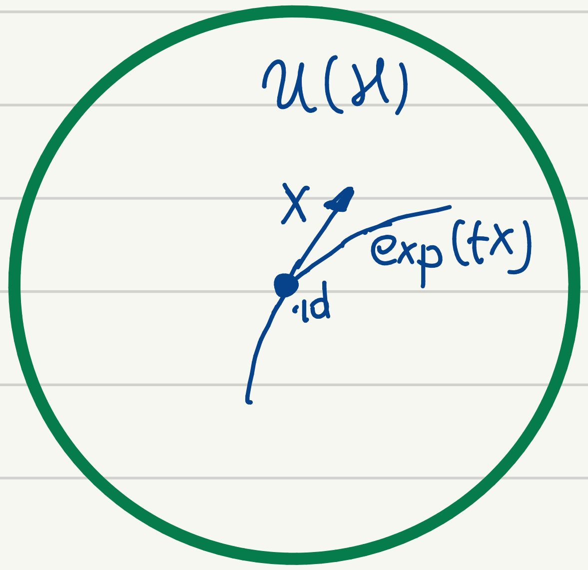
- Degeneracy of $\omega|_{O_p}$ has dimension $D_p = \dim O_p - \dim \Omega_{\mu(p)}$

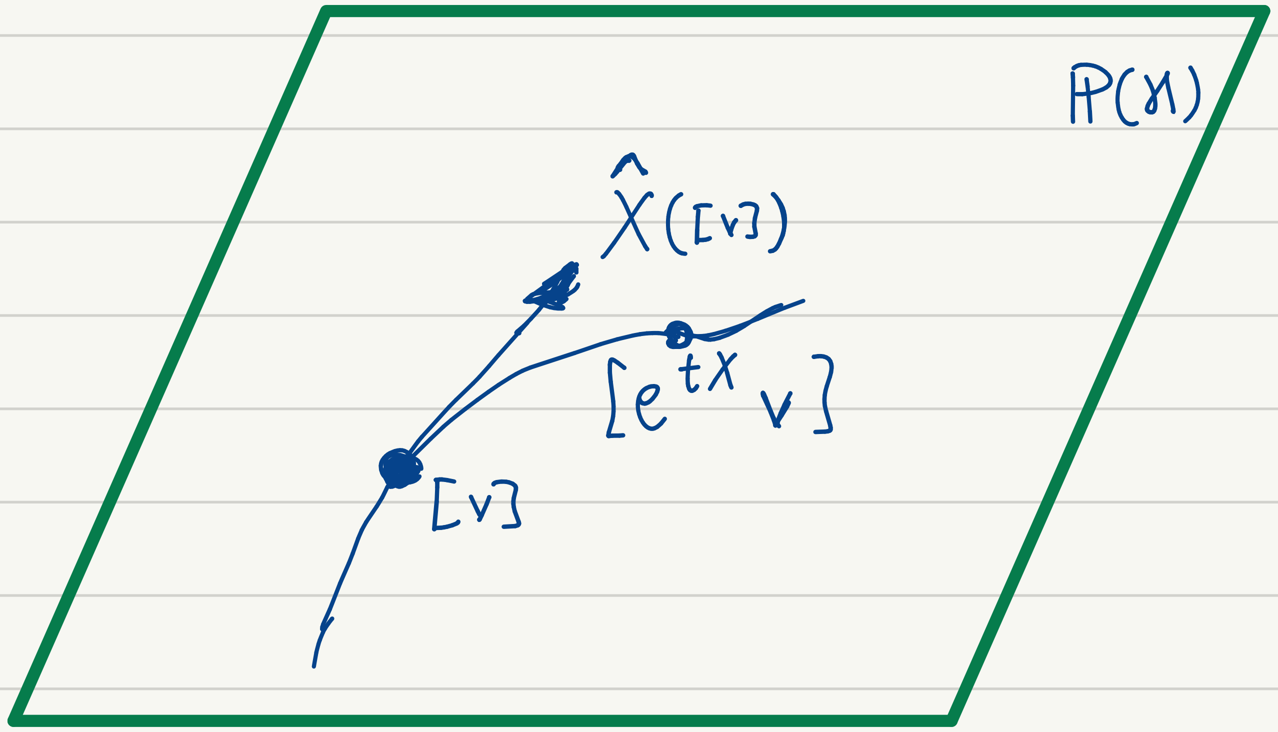
Quantum Example

- $M = \mathbb{P}(\mathcal{H})$ a complex projective space – symplectic (Kähler) manifold on which $SU(\mathcal{H})$ acts transitively

$v \in \mathcal{H}$, $[v] \in \mathbb{P}(\mathcal{H})$, $X \in su(\mathcal{H}) = \mathrm{Lie}(SU(\mathcal{H}))$

$$\hat{X}([v]) = \left[\frac{d}{dt}\Big|_{t=0} e^{tX} v\right] = [Xv]$$





- $T_{[v]}\mathbb{P}(\mathcal{H})$ is spanned by vectors $\hat{X}([v])$, $X \in su(\mathcal{H})$

$$\omega(\hat{X}([v]), \hat{Y}([v])) = i\,\frac{\langle v | [X,Y] v\rangle}{\langle v | v\rangle}, \quad X, Y \in su(\mathcal{H})$$

- ω is $SU(\mathcal{H})$-invariant

$$\omega(\hat{X}, \hat{Y}) = \omega(U_* \hat{X}, U_* \hat{Y})$$

- The momentum map is : $\mu : \mathbb{P}(\mathcal{H}) \to su(\mathcal{H})$

$$\mu([v]) = i\left(|v\rangle\langle v| - \frac{1}{\dim(\mathcal{H})} I\right)$$ ← shifted density matrix corresponding to state $|v\rangle$

$$(\mu([v]) | X) = tr(\mu([v]) \cdot X) = tr(|v\rangle\langle v| iX) = \langle iX \rangle_{|v\rangle}$$

expected value of observable iX in the state $|v\rangle$

- Assume now $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\mathcal{H}_i \cong \mathbb{C}^d$ and take $K = SU(d) \times SU(d) \hookrightarrow SU(d^2)$

$$K \ni (U_1, U_2) \mapsto U_1 \otimes U_2 \in SU(\mathcal{H})$$

$$K \ni U_1 \otimes U_2 [v_1 \otimes v_2] = [U_1 v_1 \otimes U_2 v_2]$$

- K does not act transitively on $\mathbb{P}(\mathcal{H})$

$$O_{[\psi]} = \{ [U_1 \otimes U_2 |\psi\rangle] \mid U_1, U_2 \in SU(d) \}$$ ← equally entangled states

- Tangent vectors to $O_{[\psi]}$:

$(X_1, X_2) \in su(d) \oplus su(d)$ gives $[(X_1 \otimes I + I \otimes X_2) v] \in T_{[v]} O_{[v]}$

- Momentum map of K-action on $\mathbb{P}(\mathcal{H})$

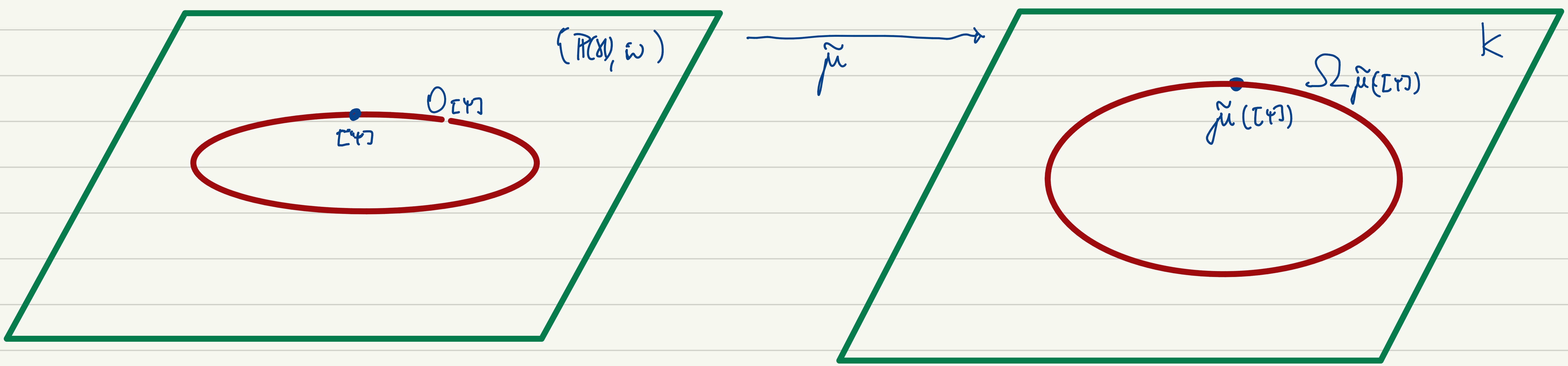
$$\tilde{\mu} : \mathbb{P}(\mathcal{H}) \longrightarrow su(d) \oplus su(d)$$

$$\tilde{\mu}([\psi]) = \left(\tilde{\mu}_1([\psi]), \tilde{\mu}_2([\psi])\right) = i\left(\underset{\substack{\uparrow \\ tr_2(|\psi\rangle\langle\psi|)}}{\rho_1([\psi])} - \frac{1}{d} I,\ \underset{\substack{\uparrow \\ tr_1(|\psi\rangle\langle\psi|}}{\rho_2([\psi])} - \frac{1}{d} I\right)$$

$\tilde{\mu}([\psi]) \leftarrow$ information about one-qudit measurments (expected values)

- Momentum map $\tilde{\mu}$ maps $\mathcal{O}_{[\psi]}$ onto the adjoint orbit

$$\Omega_{\tilde{\mu}([\psi])} = \left\{ \left(U_1^{-1} \tilde{\mu}_1 U_1,\ U_2^{-1} \tilde{\mu}_2 U_2\right) \ \middle|\ U_1, U_2 \in SU(d) \right\}$$



- For which states $|\Psi\rangle$, $O_{[\Psi]}$ is symplectic?

- How the degeneracy of $\omega|_{O_{[\Psi]}}$ is related to entanglement properties of $|\Psi\rangle$?

$$|\Psi\rangle = \sum_{i,j=1}^{d} C_{ij}\, |e_i\rangle \otimes |f_j\rangle \,, \quad \rho_1([\Psi]) = C C^{\dagger} , \quad \rho_2([\Psi]) = C^T \bar{C}$$

- $U, V \in SU(d)$

$$U \otimes V |\Psi\rangle = \sum_{i,j=1}^{d} (U C V^T)_{ij}\, |e_i\rangle \otimes |f_j\rangle$$

- We can choose $U, V \in SU(d)$ s.t.

$$|\Psi_c\rangle = \sum_{i=1}^{d} \sqrt{p_i}\, |e_i\rangle \otimes |f_i\rangle$$

$$\rho_1([\Psi_c]) = \sum_{i=1}^{d} p_i\, |e_i\rangle\langle e_i| \qquad \rho_2([\Psi]) = \sum_{i=1}^{d} p_i\, |f_i\rangle\langle f_i|$$

- Degeneracy of $\omega|_{O_p}$ has dimension

$$D_{[\Psi_c]} = \dim O_{[\Psi_c]} - \dim \Omega_{\tilde{\mu}([\Psi_c])} = \dim \mathrm{Stab}(\tilde{\mu}([\Psi_c])) - \dim \mathrm{Stab}([\Psi_c])$$

• $\mathrm{Stab}(\tilde{\mu}([\psi]))$:

$$\mathcal{U}_1^{-1}\begin{pmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & p_3 & & & \\ & & & p_4 & & \\ & & & & \ddots & \\ & & & & & p_d \end{pmatrix}\mathcal{U}_1 = \begin{pmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & p_3 & & & \\ & & & p_4 & & \\ & & & & \ddots & \\ & & & & & p_d \end{pmatrix} \qquad \mathcal{U}_2^{-1}\begin{pmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & p_3 & & & \\ & & & p_4 & & \\ & & & & \ddots & \\ & & & & & p_d \end{pmatrix}\mathcal{U}_2 = \begin{pmatrix} p_1 & & & & & \\ & p_2 & & & & \\ & & p_3 & & & \\ & & & p_4 & & \\ & & & & \ddots & \\ & & & & & p_d \end{pmatrix}$$

$$D_{[\psi]} = \sum_{n=1}^{r} m_n^2 - 1$$

r: # of nonzero p_i's

m_n: multiplicities of p_i's

→ Separable state : $\begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$ $\Rightarrow$ $D_{[\psi]} = 0$ $\Rightarrow$ separable states are only symplectic K-orbit in $\mathbb{P}(\mathcal{H})$

→ Maximally entangled states $\frac{1}{d}\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ $\Rightarrow$ $D_{[\psi]} = d^2 - 1 = \dim \mathcal{O}_{[\psi]}$

$\Downarrow$

K-orbit of maximally entangled states is Lagrangian $\omega|_{\mathcal{O}_{[\psi]}} \equiv 0$

→ The more non-symplectic is a K-orbit the more entangled states it contains.