On maximal totally real embeddings

Def A real sub-variety $M \subset X$ complex is called totally real (tR) if $T_M \cap J T_M = \mathrm{Im}\, O_M$. Moreover M tR is called maximal-tR (MtR) if $\dim M = \dim_{\mathbb{C}} X$.

TH (BRUHAT-WHITNEY). For any compact real analytic variety M there exist a real analytic embedding $M \hookrightarrow X$ complex such that M is MTR in X. Moreover if M is MtR for J_1, J_2 r.a ~~over X~~ complex structures over X then there exists $U_1, U_2 \subset X$ open neighborhoods of M such that

$$\begin{array}{ccc} (U_1, J_1) & \xrightarrow[\simeq]{\Phi} & (U_2, J_2) \\ \cup & & \cup \\ M & \xrightarrow{id} & M \end{array}$$

$M \subset T_M$ zero section

$T_{T_M|M} \simeq T_M \oplus T_M$ complex vector bundle, $J^{can} : (u,v) \mapsto (-v,u)$.

In B-W thm we can assume $X = \underset{\substack{\cup \\ M}}{U} \subset T_M$ and $J_{|M} = J^{can}$.

If we extend the definition of tR to the almost complex case, then any almost complex structure which is a continuous extension of J^{can} in a neighborhood of $M \subset T_M$ makes M - mtR in a small neighborhood of M and the complex distribution $T^{0,1}_{T_M}$ is π-horizontal, $\pi : T_M \to M$.

Def A π-horizontal complex distribution over T_M is a section $A \in C^\infty(T_M, \pi^*\mathbb{C}T_M^* \otimes_{\mathbb{C}} \mathbb{C}T_{T_M})$,

$d\pi \cdot A = \mathbb{I}_{\pi^*\mathbb{C}T_M}$.

$d\pi \cdot \alpha = \mathbb{I}_{\pi^* T_M}$

RMQ1 We can write $A = \alpha + i T B$, $\alpha \in C^\infty(T_M, \pi^* T_M \otimes T_{T_M})$

$B \in C^\infty(T_M, \pi^* End(T_M))$

$T_\eta : T_{M,\pi(\eta)} \xrightarrow[\simeq_{CAN}]{} T_{T_{M,\pi(\eta)},\eta}$

RMQ2 The section A determines an almost-complex structure J_A over T_M such that

$$T^{0,1}_{T_M, J_A, \eta} = \mathrm{Im}\, A_\eta \equiv A_\eta(\mathbb{C}T_{M,\pi(\eta)}) \subset \mathbb{C}T_{T_M,\eta} \qquad (*)$$

iff $\mathrm{Im}\, A_\eta \cap \overline{\mathrm{Im}\, A_\eta} = \{0_\eta\}$, iff $\mathrm{Ker}(A - \bar{A}) = 0$ i.e

$B \in C^\infty(T_M, \pi^* GL(T_M))$ Bundle of the group of linear transformations of the tangent bundle.

RMQ 3 Over $(T_{T_M|M}, J^{can})$, $(u,v)^{0,1} = (\xi, i\xi)$, $\xi = (u - iv)/2$

Thus (by taking $v=0$) J_A is an extension of J^{can} iff

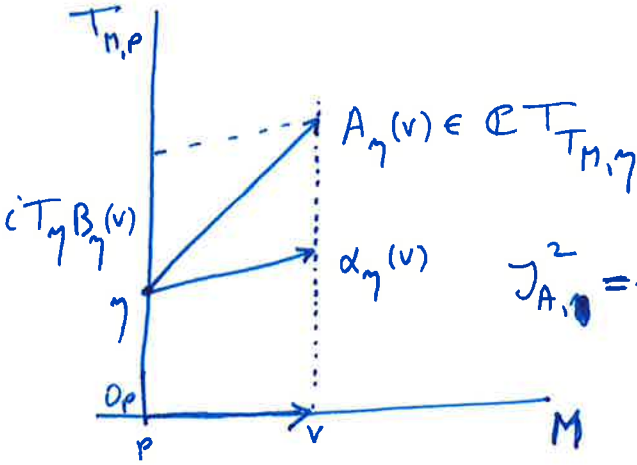
$\alpha_{0_p} = d_p O_M$ and $B_{0_p} = \mathbb{I}_{T_{M,p}}$, $p \in M$.

Def Let M be a smooth manifold. An M-totally real almost complex structure (M-tr) over an open neighborhood $U \subset T_M$ of $M \equiv \mathrm{Im}\, O_M$ is a couple (α, B),

$\alpha \in \mathcal{C}^\infty(U, \pi^* T_M^* \otimes T_{T_M})$, $d\pi \cdot \alpha = \mathbb{I}_{\pi^* T_M}$, $\alpha_{0_p} = d_p O_M$

$B \in \mathcal{C}^\infty(U, \pi^* GL(T_M))$, $B_{0_p} = \mathbb{I}_{T_{M,p}}$, $p \in M$. With $A := \alpha + iTB$, the associated almost-complex structure J_A is the one satisfying condition $(*)$.

Every almost complex structure, $\mathcal{C}^\infty$-extension of J^{can} over a neighborhood $M \subset T_M$ can be written over a small neighborhood of $M \subset T_M$ as the almost-complex structure associated to a <u>unique</u> M-tr structure.



$J_{A,\eta}^2 = -\mathbb{I}_{T_{T_M}}$

$$\begin{cases} J_{A,\eta}\,\alpha_\eta(v) = T_\eta B_\eta(v) \\ J_{A,\eta}\, T_\eta(w) = -\alpha_\eta B_\eta^{-1}(w) \end{cases}$$

$w \in T_{M,p}$

tensorial generalization of the canonical structure J^{can}.

For any connection operator ∇ acting on the sections of $\mathbb{C}T_M$ we define the <u>1-differential</u>

$$d_1^\nabla : \mathcal{C}^\infty(M, T_M^{*,\otimes k} \otimes \mathbb{C}T_M) \longrightarrow \mathcal{C}^\infty(M, \Lambda^2 T_M^* \otimes T_M^{*,\otimes(k-1)} \otimes \mathbb{C}T_M)$$

<u>$k \geq 1$</u> $\quad d_1^\nabla\theta(v_1, v_2, \cdot) := \nabla_{v_1}\theta(v_2, \cdot) - \nabla_{v_2}\theta(v_1, \cdot)$. <u>RMQ</u> $d_1^{\nabla,2} \neq 0$

k-<u>Tensor</u>

Let θ_j , $j = 1, 2$ $(k_j + 1)$-tensors , $k_1 \geq 1$. We define the

<u>1-exterior product</u> $\quad \theta_1 \wedge_1 \theta_2 \in \Lambda^2 T_M^* \otimes T_M^{*,\otimes(k_1+k_2-1)} \otimes \mathbb{C}T_M$

$$(\theta_1 \wedge_1 \theta_2)(u_1, u_2, v, \mu) := \theta_1(u_1, \theta_2(u_2, v), \mu) - \theta_1(u_2, \theta_2(u_1, v), \mu)$$

$u_j \in T_M$, $v \in T_M^{\oplus k_2}$, $\mu \in T_M^{\oplus(k_1-1)}$

We define the <u>circular operator</u>

$$(\mathrm{Circ}\,\theta)(v_1, v_2, v_3, \cdot) := \theta(v_1, v_2, v_3, \cdot) + \theta(v_2, v_3, v_1, \cdot) + \theta(v_3, v_1, v_2, \cdot) .$$

We define the <u>1-permutation</u> operator

$$\tilde{\theta}(v_1, v_2, \cdot) := \theta(v_2, v_1, \cdot) .$$

TH Let M be a smooth manifold equipped with a torsion free connection operator ∇ acting on the smooth sections of T_M, let $U \subseteq T_M$ be an open neighborhood of $M \equiv \operatorname{Im} 0_M$ with connected fibers, let J_A a M-tr almost complex structure over U, real analytic along the fibers of U. We consider the Taylor expansion along the fibers at the origin

$$T_\eta^{-1}(H^\nabla - \bar{A})_\eta \cdot \xi = i\xi + \sum_{k\geq 1} S_k(\xi, \eta^k), \quad \eta \in (U, M), \; \xi \in T_{M,\pi(\eta)}$$

Horizontal distrib of ∇

$S_k \in C^\infty(M, T_M^* \otimes S^k T_M^* \otimes \mathbb{C}T_M)$. Then the statements (1) and (2) are equivalent

(1) J_A is integrable and for any $\eta \in U$ the complex curve $(\mathbb{C}, 0) \ni t + is \longmapsto s\,\Phi_t^\nabla(\eta) \subset U$ is J_A-holomorphic.

geodesic flow of ∇ : $\Phi_t(\eta) = \dot{c}_t$, $\dot{c}_0 = \eta$, c geodesic

(2) $S_1 = 0$ and for $k \geq 2$

$$S_k = \frac{1}{(k+1)!\,k!} \operatorname{Sym}_{2,\dots,k+1} \Theta_k^\nabla \quad \text{with } \Theta_2^\nabla := 2iR^\nabla \text{ and for } k \geq 3 \qquad (**)$$

$$\Theta_k^\nabla := 2(i d_1^\nabla)^{k-3}(\widetilde{\nabla R^\nabla}) + i \sum_{r=3}^{k-1} (r+1)! \sum_{p=2}^{r-1} (i d_1^\nabla)^{k-r-1} (p S_p \wedge_1 S_{r-p+1})$$ and

$$I_k \quad \boxed{\operatorname{circ} \operatorname{Sym}_{3,\dots,k+1} \Theta_k^\nabla = 0 \text{ for } k \geq 4}$$

TH $I_k = 0$ $k = 4, 5$
$k = 6, 7$ by using Maple with SACVY

In the case (M, ∇) compact and real analytic Bielawsky-Lempert-Stöke show existence of J. In this case the $I_u = 0$ $\forall u$. We can conjecture that they are allways satisfied by the M-tv structures under the form (**) even in the case ∇ on $\mathbb{C}T_M$ (no geodesics) (Cauchi's existence does not apply).

(cases of diff op def by and ∇.)

Notations We consider a composition

$H \equiv (h_1, \ldots, h_\ell) \in \mathbb{Z}^\ell_{\geq 0}$, $\ell > 0$, $\ell_H := \ell$ (lenghth of H)

$|H| := \sum_{s=1}^{\ell} h_s$, $|H|_j := \sum_{s=1}^{j} h_s$, $1 \leq j \leq \ell$ and we def the pseudo-norms and

$\|H\|_j := |H|_j + j$, $\|H\| := \|H\|_\ell + \ell$ $(\|H\| = 0 \Longleftrightarrow H = \phi)$ ϕ is the neutral element in the space of compositions (they have an algebra).

$(T_j)_{j=1}^{p}$ family of Endomorphisms

$\prod_{j=1}^{p} T_j := T_1 \circ \ldots \circ T_p$. Let $R^{(h)} := \nabla^h R^\nabla$ and $(v_1, \ldots, v_p) \equiv (1, \ldots, p)$.

For any composition D we define the curvature monomial of multi-degree D

$$R^D(1, \ldots, \|D\| + 1) := \Big[\prod_{j=1}^{\ell_D} R^{(d_j)}(\|D\|_{j-1} + 2, \ldots, \|D\|_j + 1, \bullet, \|D\| - j + 2)\Big] \cdot 1$$

$\uparrow$ Endo of $\mathbb{C}T_M$.

$$(\ldots(\underbrace{\ldots}_{d_j+1}(\ldots\bullet))\ldots)$$

$H_j^- := (h_1, \ldots, h_j)$, $H'_{s,j} := (h_{s+1}, \ldots, h_j)$.
$s < j$

$$C_D := \sum_{0 \leq H \leq D} \frac{(-1)^{|H|} C(H)}{H!(D-H)!}$$

$l_H = l_D$ ↑ product of the factorials of the components

$$C(H) := \sum_{\substack{|\lambda| = l_H \\ \lambda \in \mathbb{Z}^{\bullet}_{\geq 1}}} (-1)^{l_\lambda} \| H_{\lambda_1} \| \prod_{j=1}^{l_\lambda} \prod_{s=|\lambda|_{j-1}}^{|\lambda|_j - 1} \frac{1}{\| H'_{s,|\lambda|_j} \| \left(\| H'_{s,|\lambda|_j} \| + 1 \right)} .$$

TH (P-Salvy) An M-tr almost complex structure admits a Taylor expansion along the fibers at the origin under the form (**) iff

$$S_k = \frac{1}{i^{k+1} k!} \sum_{\substack{\|D\| = k \\ D \geq 0}} C_D \, \mathrm{Sym}_{2,\ldots,k+1} R^D \quad , (k \geq 2).$$

It has been 64 years since the existence of complex structures on Grauert Tubes was proven for the first time by B-W. Still, up to now, the explicit form of the Taylor expansion has remained mysterious. This is finally clarified by the above rather simple and explicit global expression of the complex structure. The main application concerns global analytic micro-local analysis which needs Hardy-spaces over Grauert Tubes. We don't know what is a holomorphic function untill we don't know J.

h r.a $(M, \pi(\eta))$. H hol extension of h : $H_\eta(t+is) := H(s\,\Phi_t^\nabla(\eta))$ hol extension of $h_\eta(t) := h(\pi\,\Phi_t^\nabla(\eta)) \equiv h(\exp_{\pi(\eta)}^\nabla(t))$