Power Series Method of Equivalence

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SCREAMs GRIEG Workshop on Cartan Geometries, Orsay, Paris, France

Thursday 6 March 2024

Prologue

- **Equivalence Methods SCREAMs:**

□ Lie
Tresse
Olver
Lychagin
Kruglikov . . .

□ Cartan
Sternberg
Gardner
Kamran
Olver
Anderson
Nurowski . . .

□ Poincaré
Moser
Beloshapka
Loboda
Ezhov
Eastwood . . .

- **Objective:** *Determine homogeneous models of geometric structures.*
- **Question:** *Who would term Cartan's method "straightforward"?*
- **Question:** *Who found a "straightforward" method?*

- **Difficulty:**

☐ Wide universe of geometric structures of a certain kind

versus

☐ Exceptionally small subset of symmetric ones.

- **Chern-Moser 1974:** Levi nondegenerate hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$:

  Zero CR curvature at any $p \in M$ $\overset{?}{\Longrightarrow}$ $M^{2n+1} \simeq S^{2n+1}$ **?**

Proof. Observe:

$$\text{Moser curvature at } p \equiv \text{Hachtroudi-Chern curvature at } p$$

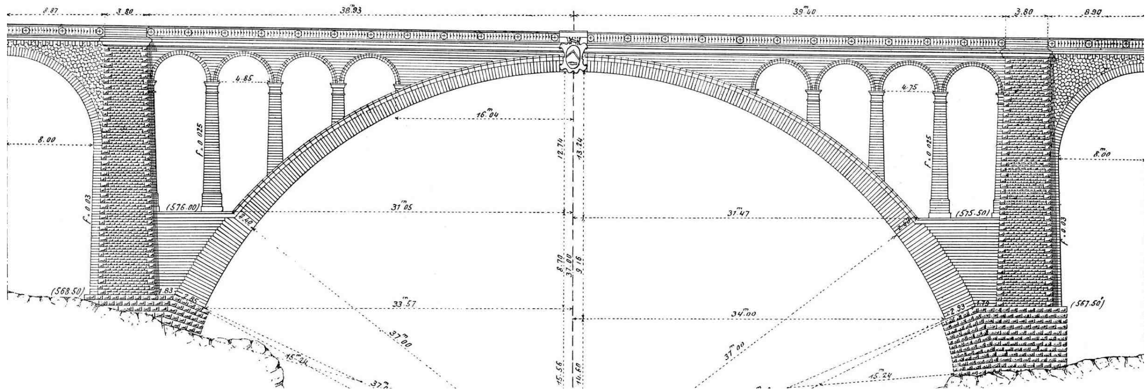
Conclude thanks to Cartan-Frobenius EDS flatness theorem. ☐

- **Fact:** [Xiaojun Huang] *A purely Poincaré-Moser proof is missing!*
- **Delicate:** Exist **2** distinct kinds of techniques

  $\Longrightarrow$ **2** distinct mathematical traditions/communities

• **Bridge:**

Poincaré
Moser



Cartan
Nurowski

• **Bridge objective:** *Transfer/Adapt some of the Concepts/Techniques of Cartan.*

Deep acknowledgments to Paweł Nurowski!

Results

- **Ph. D. under finalization (jww):** [Julien Heyd]

We determine all affinely homogeneous models for:

☐ Constant Hessian rank 1 hypersurfaces $H^n \subset \mathbb{R}^{n+1}$ with $n = 2, 3, 4$;

☐ Constant Hessian rank 2 hypersurfaces $S^3 \subset \mathbb{R}^4$;

☐ Surfaces $S^2 \subset \mathbb{R}^4$;

including the *simply transitive* models.

We also determine all multiply transitive homogeneous models for:

☐ 5D PDE systems under fiber-preserving diffeomorphisms.

We employ an improved *power series method of equivalence*, which captures invariants at the origin, creates branches, and infinitesimalizes calculations.

We find several inequivalent terminal branches yielding each to some nonempty moduli space of homogeneous models, sometimes parametrized by a certain invariant algebraic variety.

Three main features may be emphasized:

1. Iterated *single-pointed* jet bundles;

2. Cartan-enhanced power series method of equivalence;

3. Constant *ping-pong* between normal forms (nf) and vector fields (vf).

Differential Invariants and Homogeneous Models

Consider a Lie group G acting on a given type of geometric structure. Examples are: Euclidean, affine, conformal, projective, (pseudo-)Riemannian, symplectic, quaternionic, Cauchy-Riemann (CR), para-CR, $\ldots$, structures. Other examples are: ordinary differential equations; partial differential equations; integrability systems; Pfaffian systems, $\ldots$.

In his complete works, Élie Cartan often started by re-expressing the considered geometric structure as being a specific exterior differential system.

On the other hand, as explained in Peter Olver's monographs and articles, after transfer to an appropriate associated space (*e.g.* a jet bundle), several (local) geometric structures with a (local) Lie group G acting on them can be expressed as (local) *graphs* $\{u = F(x)\}$ in the associated space equipped with a G-action.

In this talk, we adopt the graph point of view. Although our considerations are valid for infinite-dimensional Lie groups, like the groups of diffeomorphisms, of biholomorphisms, of CR-equivalences, $\ldots$, we shall restrict ourselves to the finite-dimensional setting. We shall work over $\mathbb{R}$ or $\mathbb{C}$.

Consider therefore a Lie group G of finite dimension $1 \leqslant r < \infty$. Let $n \in \mathbb{N}_{\geqslant 1}$ and $c \in \mathbb{N}_{\geqslant 1}$. In $\mathbb{R}^{n+c}$ with coordinates $x = (x_1, \ldots, x_n)$ and $u = (u_1, \ldots, u_c)$, consider a c-codimensional graph:

$$u_j = F_j\big(x_1, \ldots, x_n\big) \qquad\qquad (1 \leqslant j \leqslant c).$$

Throughout, our point of view will be local, and the F_j will be assumed to be *analytic*. We will *not* introduce notations for open sets, subsets, sub-subsets.

Let the group G act on $\mathbb{R}^{n+c}$, by analytic diffeomorphisms. In this talk, G will consist of *affine* transformations. Also, an element g of the group G will always be explicitly given by *group parameters* $(g_1, \ldots, g_r) \in \mathbb{R}^r$.

Two general problems are of interest, about which we will be more specific later, *see* Problems on p. 36 and p. 60 *infra*.

Problem 1. *Describe algebras of differential invariants.*

Problem 2. *Determine homogeneous models.*

These two problems are tightly linked with each other, because most of the times, homogeneous models of a given geometric action are 'exceptional' objects in a wide universe of *non*symmetric objects. The 'exceptional' symmetric objects have *constant* differential invariants, while the 'general' *non*symmetric objects often have infinitely many *functional* differential invariants, which share *complicated* differential-algebraic relations.

The Lie-Fels-Olver *recurrence relations* between differential invariants constitute a natural '*bridge*' between these two general problems. Indeed, the effectiveness of Peter Olver's equivariant moving frame approach lies in the powerful *recurrence relations*, which produce complete and explicit *differential-algebraic* structures for the underlying algebras of differential invariants — this without requiring explicit coordinate expressions for either the moving frame or the invariants. Evidently, differential invariants of homogeneous structures are *constant*, and it is a fact that the *algebraic* relations between them retain major part of the recurrence relations.

Fibers Over Group Transversals Versus Full Jet Bundle

Abbreviate $z := (x, u)$. Denote the target coordinates as $\overline{z} := \big(\overline{x}, \overline{u}\big)$. An element $g \in G$ in some neighborhood of the identity sends the graph:

$$M := \{u = F(x)\}$$

to a similar graph:

$$\overline{M} := \big\{\overline{u} = \overline{F}\big(\overline{x}, g\big)\big\},$$

with certain analytic functions $\overline{F}_j$ which depend on the group parameters.

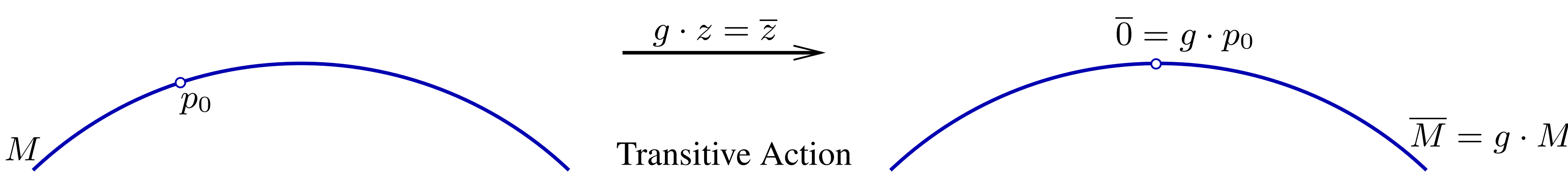
The expressions of these $\overline{F}_j(\overline{x}, g)$ are difficult to write down, highly nonlinear, often cumbersome. They in fact require the full strength of the implicit function theorem.

Such transformations of graphs appear regularly in the original complete works of Lie.

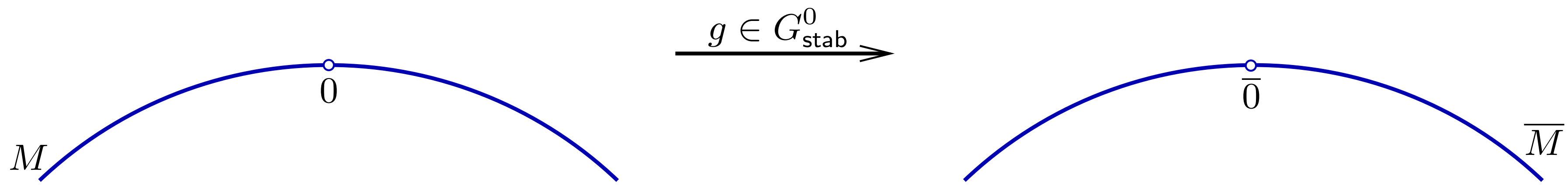
Let us write:

$$g \cdot z = g \cdot (x, u) =: \big(\overline{x}, \overline{u}\big) = \overline{z}.$$

We whall assume that the group G acts transitively on $\mathbb{R}^{n+c}$, and even, that G contains all translations. (Non-transitive group actions are sometimes considered in Peter Olver's articles.) 'Morally', the fact that G acts *transitively* implies that all points are somewhat 'equivalent'.



Therefore, any point $p_0 \in M$ can be 'moved by G' to some 'central' point, $\overline{0} \in \mathbb{R}^{n+c}$, the origin of the target coordinates $\overline{z}$. Next, coordinates z can be 're-centered' at p_0.



So both graphs M and $\overline{M}$ pass through the origin. And in fact, only the (isotropy) *subgroup* $G^0_{\mathsf{stab}} \subset G$ of transformations $g \in G$ sending 0 to $\overline{0}$ should be considered onward, as we will argue later.

To study invariants under G-actions and to classify G-homogeneous geometries, (roughly) two different (general) approaches exist:

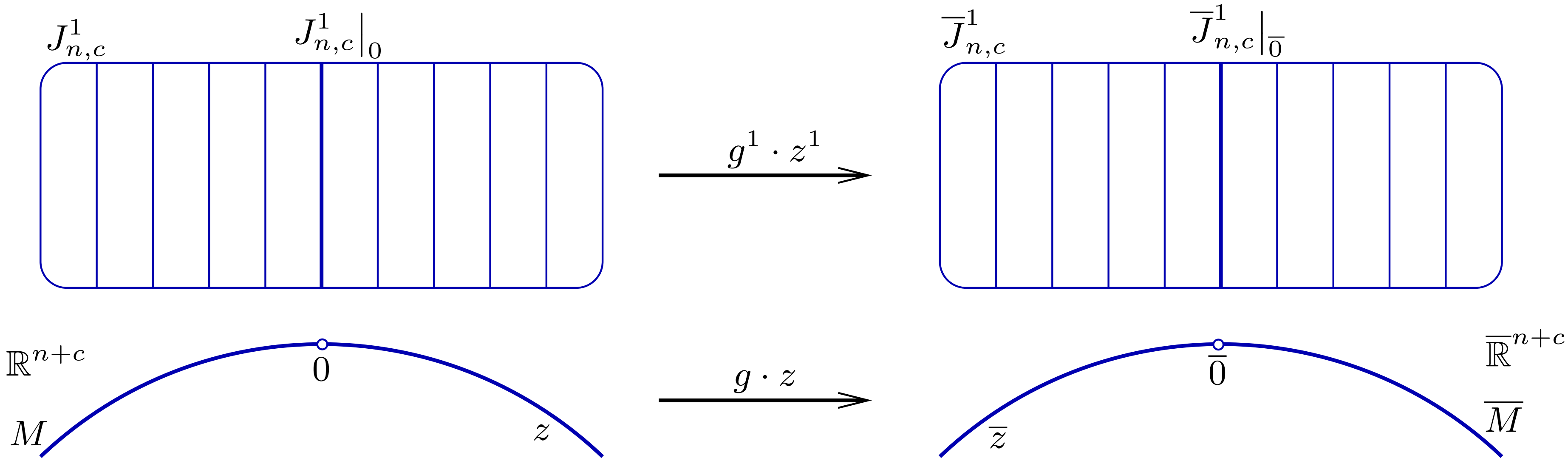
- Work within (full) jet bundles (Lie, Cartan, Olver, ...);

- Work with (truncated) power series centered at the origin (Lagrange, Poincaré, Moser, ...).

The second approach, less developed, has several defects. One obvious defect is that differential invariants of Lie type, which require differentiation with respect to $x_1, \ldots, x_n$, cannot be computed by manipulating power series only at $x_1 = \cdots = x_n = 0$! Other defects will be discussed later.

The first steps of Lie's theory of differential invariants consist in *prolongating* the G-action to jet bundles. Sketching only key aspects, we will not present the complete details.

For a jet order $\kappa \in \mathbb{N}$, let $J^{\kappa}_{n,c}$ be the bundle of κ-jets of c functions of n variables, at all base points $\big(x, u(x)\big) \in M$. For instance, $J^1_{n,c}$ has $n{+}c{+}n\,c$ independent coordinates corresponding to the x_i, and to the u_j together with all their first order derivatives u_{j,x_i}.



As is known, the G-action uniquely *lifts* as a G-action on *first jets* of graphs. This action is just the (differential) action on tangent spaces to the two graphs at corresponding points.

Interlude: Differential Invariants in Full Jet Bundles

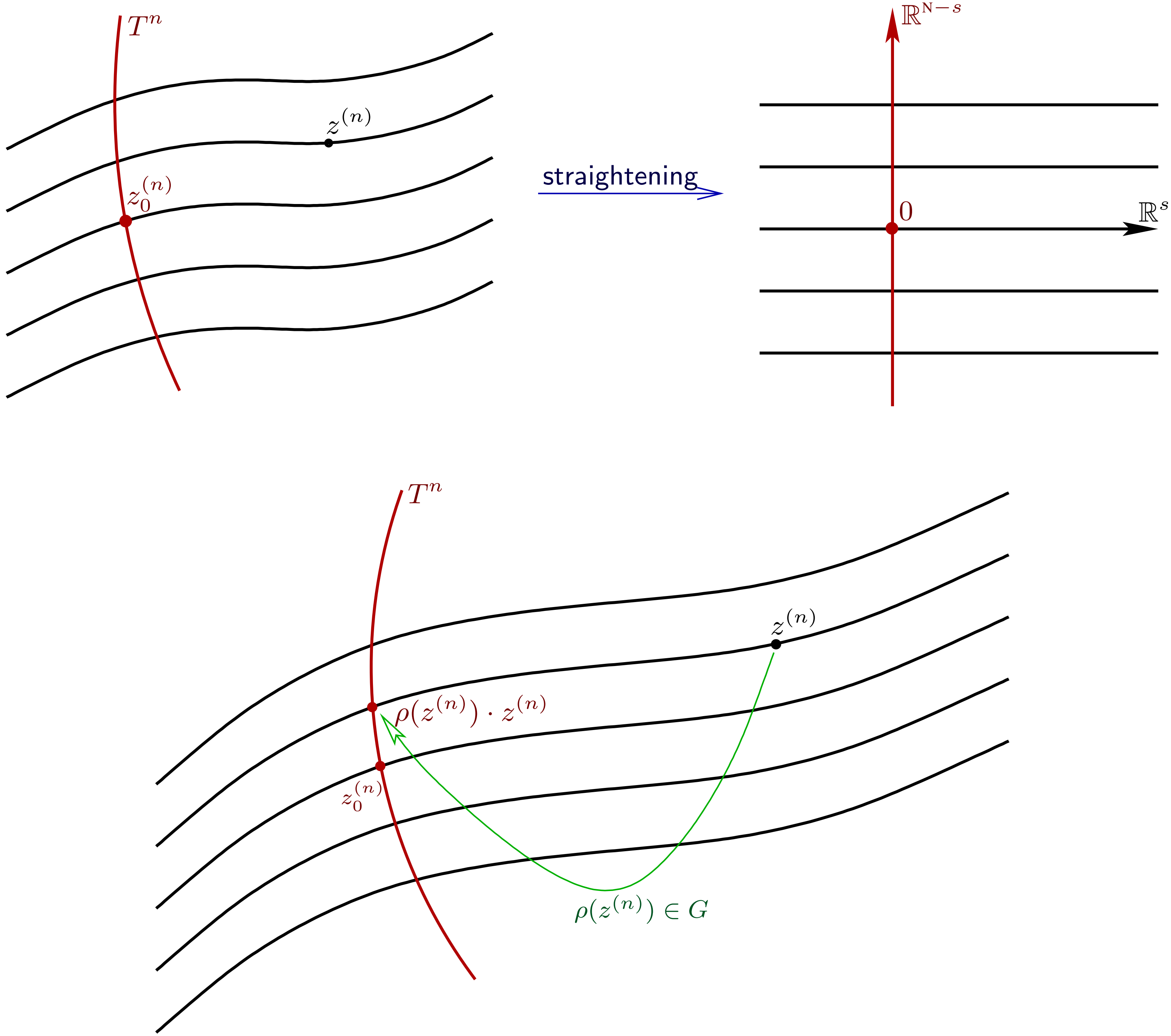
- **Theory in Full Jet Bundles:**

$$w \;=\; w\big(g,z\big) \;=\; g\cdot z.$$

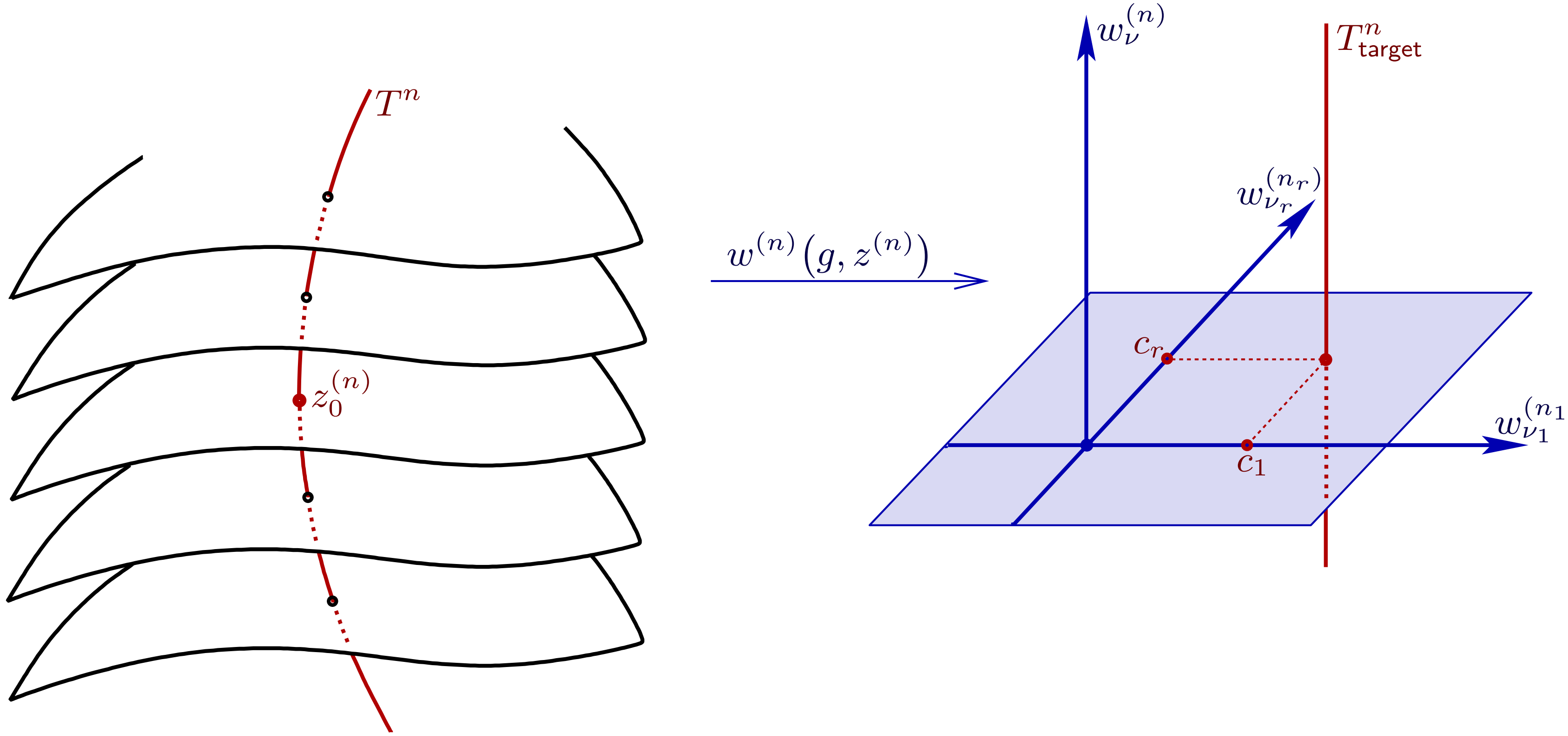
$$\begin{array}{ccc} J^n_z & \xrightarrow{\;g\cdot\;} & J^n_w \\ \downarrow & & \downarrow \\ \mathbb{R}^m_z & \xrightarrow[\;g\cdot\;]{} & \mathbb{R}^m_w, \end{array} \qquad\qquad \begin{array}{ccc} z^{(n)} & \xrightarrow{\;g\cdot\;} & w^{(n)}\big(g,z^{(n)}\big) \\ \downarrow & & \downarrow \\ z & \xrightarrow[\;g\cdot\;]{} & w(g,z), \end{array}$$

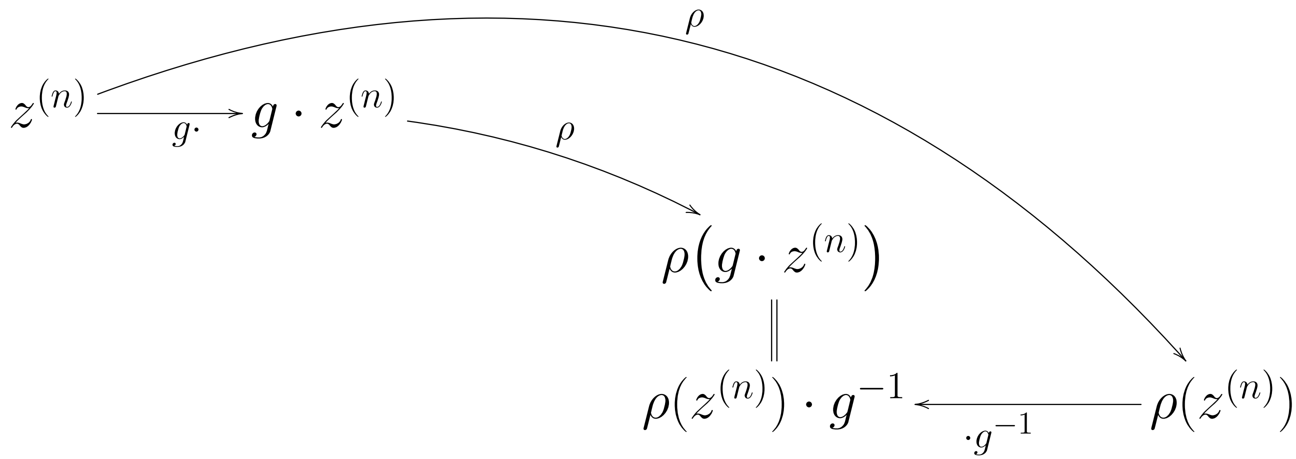
$$\dim J^n_{p,q} \;=\; p+q\binom{p+n}{n},$$

$$\begin{array}{ccc} J^{n'}_z & \xrightarrow{\;g\cdot\;} & J^{n'}_w \\ \downarrow & & \downarrow \\ J^n_z & \xrightarrow{\;g\cdot\;} & J^n_w \\ \downarrow & & \downarrow \\ \mathbb{R}^m_z & \xrightarrow[\;g\cdot\;]{} & \mathbb{R}^m_w, \end{array}$$



T^n
$z^{(n)}$
$z_0^{(n)}$
straightening
$\mathbb{R}^{N-s}$
0
$\mathbb{R}^s$
T^n
$z^{(n)}$
$\rho(z^{(n)}) \cdot z^{(n)}$
$z_0^{(n)}$
$\rho(z^{(n)}) \in G$

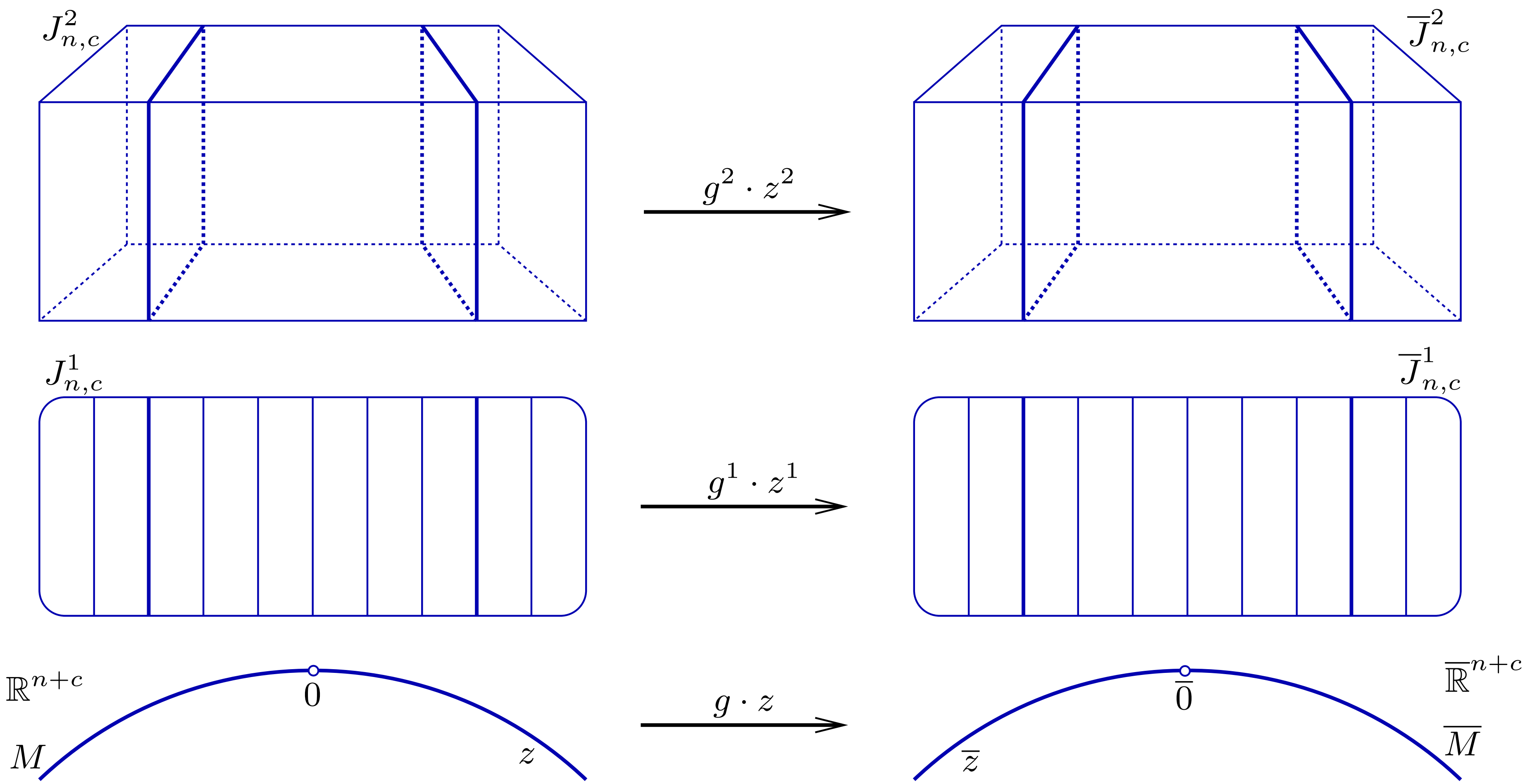

T^n
$z_0^{(n)}$
$w^{(n)}(g, z^{(n)})$
$w_\nu^{(n)}$
T^n_{target}
$w_{\nu_r}^{(n_r)}$
c_r
c_1
$w_{\nu_1}^{(n_1)}$

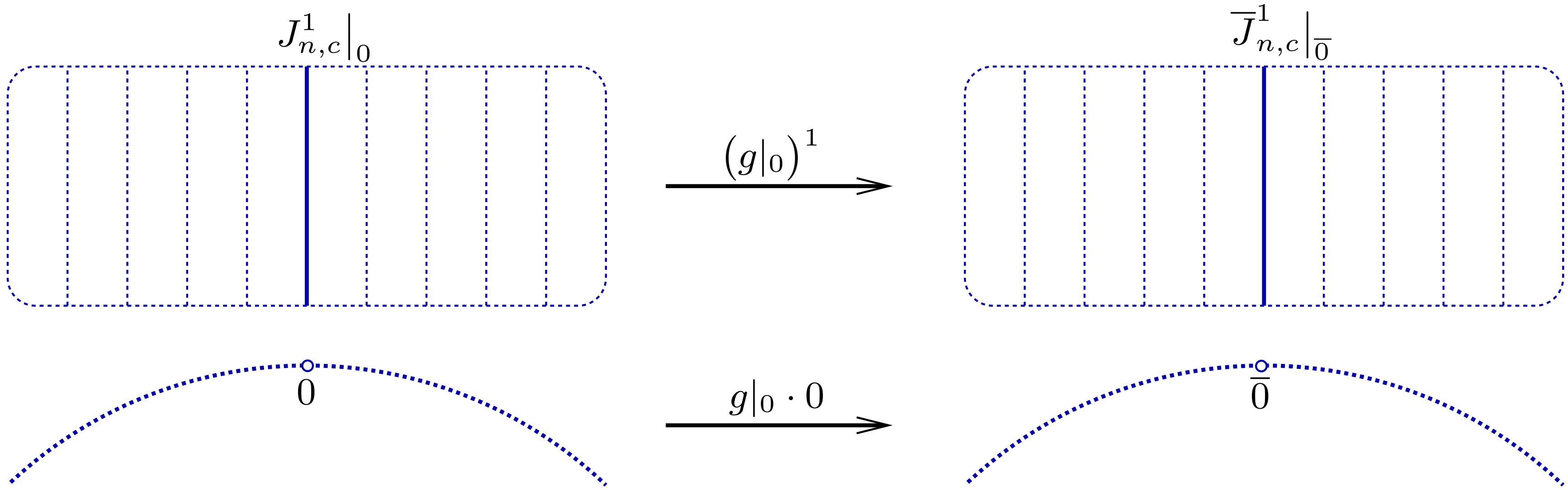

$z^{(n)}$
$g\cdot$
$g \cdot z^{(n)}$
ρ
ρ
$\rho\big(g \cdot z^{(n)}\big)$
$\rho\big(z^{(n)}\big) \cdot g^{-1}$
$\cdot g^{-1}$
$\rho\big(z^{(n)}\big)$

Back to Fibers Over Group Transversals Versus Full Jet Bundle

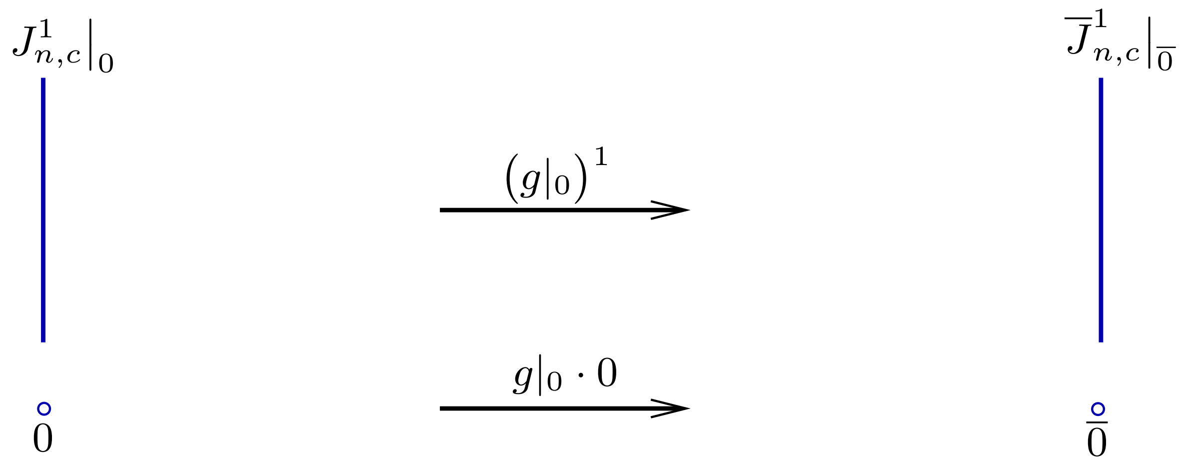
Denote $z^1 = (x, u, u^1)$ and similarly $\overline{z}^1 = \big(\overline{x}, \overline{u}, \overline{u}^1\big)$. Although it is the same group G that acts on $J^1_{n,c}$, denote its lifted action with the symbol g^1:

$$g^1 \cdot z^1 \; =: \; \overline{z}^1 \qquad\qquad (g \in G).$$

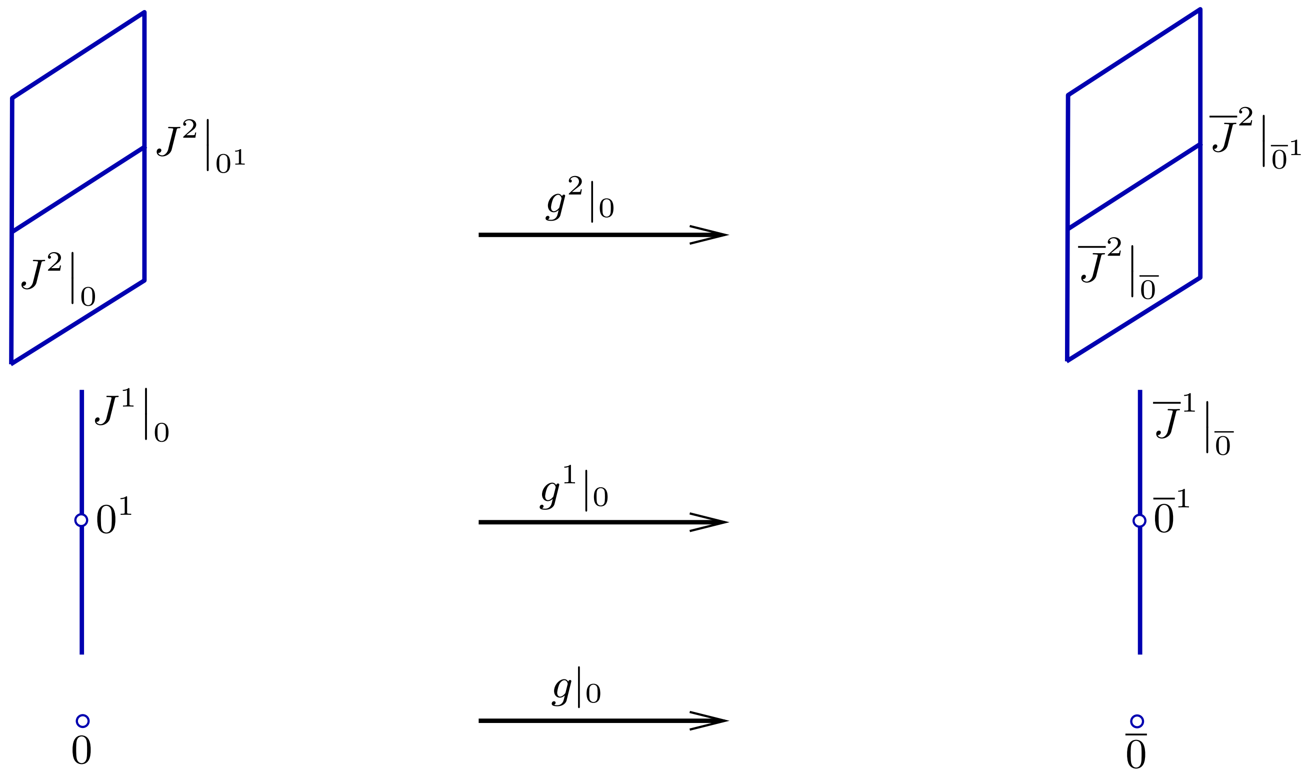




First of all, since the origin $0 \in M$ is sent to the origin $\overline{0} \in \overline{M}$, the group action sends the first jet fiber $J^1_{n,c}\big|_0$ over 0 to the first jet fiber $\overline{J}^1_{n,c}\big|_{\overline{0}}$ over $\overline{0}$. Of course, we are considering only group elements g of the *subgroup* $G^0_{\mathsf{stab}} \subset G$ fixing the origin, which we denote by $g\big|_0$.



As a key decision here, we decide to *forget* other jet fibers (!). Full bundles will not anymore be dealt with (!). When passing to higher jet orders, this decision of restricting to selected fibers will be iterated. Of course, there are prolongations $\big(g\vert_0\big)^1$, $\big(g\vert_0\big)^2$, $\dots$, to jet fibers $J^1\big\vert_0$, $J^2\big\vert_0$, $\dots$, and we will later show *formulas* for such prolongations, which are simpler than the formulas in the full jet bundle.



Several groups G, as *e.g.* the affine or projective groups, contain not only translations but also *transvections*, namely maps of the form:

$$v_j \;=\; u_j + \boldsymbol{q}_{j,1}\, x_1 + \cdots + \boldsymbol{q}_{j,x}\, x_n \qquad\qquad (1 \leqslant j \leqslant c),$$

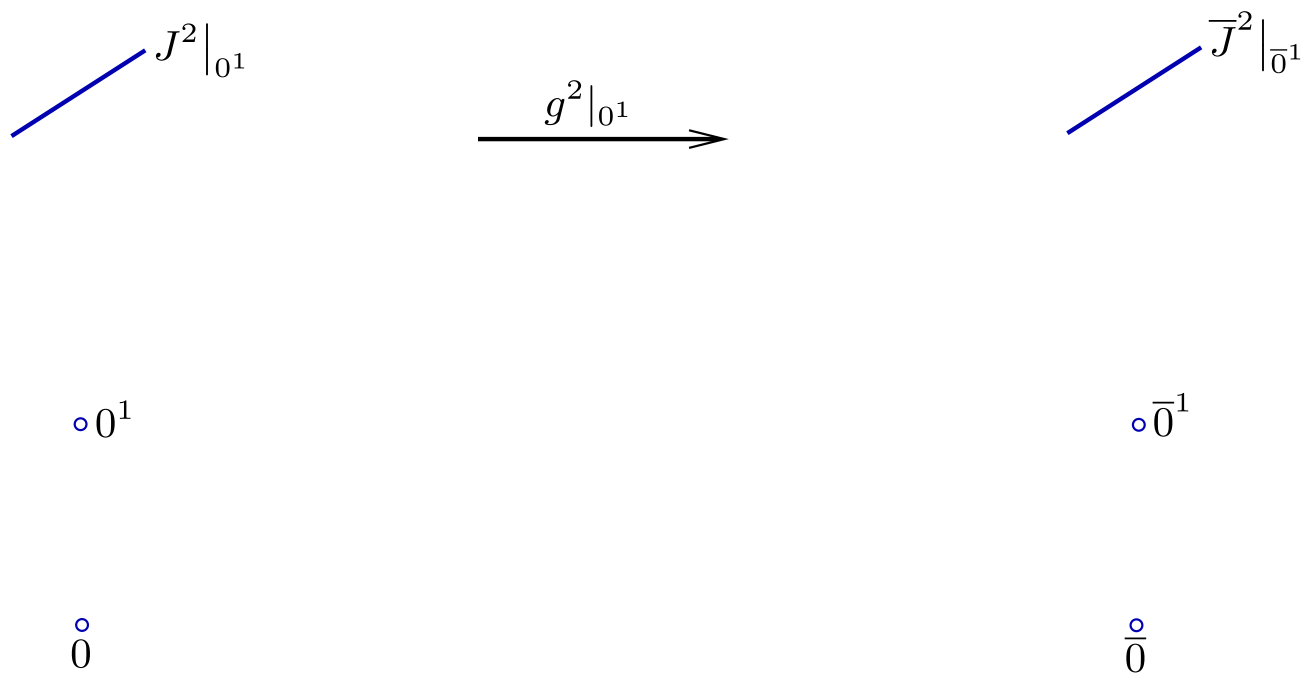
with arbitrary $q_{j,i} \in \mathbb{R}$. Such maps enable to 'strengthen' tangent spaces of both M at 0 and $\overline{M}$ at $\overline{0}$ to be 'horizontal', that is, to normalize to zero all first order terms in the power series expansions:

$$u = 0 + \mathrm{O}_{x_1,\dots,x_n}(2) \qquad \text{and} \qquad \overline{u} = 0 + \mathrm{O}_{\overline{x}_1,\dots,\overline{x}_n}(2),$$

where of course:

$$\mathrm{O}_{x_1,\dots,x_n}(2) = \sum_{i_1+\cdots+i_n \geqslant 2} x_1^{i_1} \cdots x_n^{i_n} \, F_{i_1,\dots,i_n}.$$

Precise formulas and normalization equations can easily be written, for $G = \mathsf{Aff}(\mathbb{R}^{n+1})$. Geometrically, this means that the G-action lifted to the first jet bundle J^1 and *restricted* to its fiber $J^1\big|_0$ over the origin 0 only, is *transitive*, and this means that the *origin* $0^1 \in J^1\big|_0$ is taken as a *transversal* to the unique G^0_{stab}-orbit in $J^1\big|_0$.



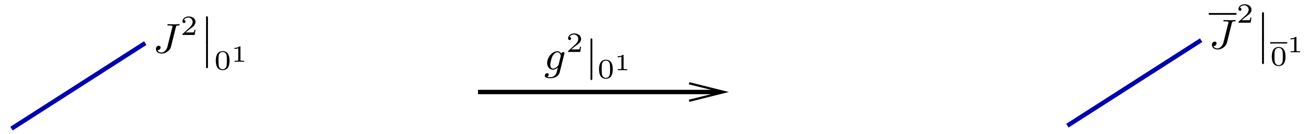
Therefore, not the whole second order jet fibers $J^2\big|_0$ and $\overline{J}^2\big|_{\overline{0}}$ over the origins $0 \in M$ and $\overline{0} \in \overline{M}$ should be dealt with. Instead, and precisely as it is drawn in the simplified diagram above, one should consider only:

- $J^2\big|_{0^1} :=$ the part of J^2 over the origin 0^1 of $J^1\big|_0$;
- $\overline{J}^2\big|_{\overline{0}^1} :=$ the part of $\overline{J}^2$ over the origin $\overline{0}^1$ of $\overline{J}^1\big|_{\overline{0}}$.

These two *smaller* subspaces are the respective two preimages of 0^1 and of $\overline{0}^1$ under the (unwritten) projections from the second floor to the first floor.

Furthermore, only the *subgroup* $G^1_{\mathsf{stab}} \subset G^0_{\mathsf{stab}} \subset G$ of transformations sending 0^1 to $\overline{0}^1$ (hence sending 0 to $\overline{0}$) should be dealt with. As in the figure above, let us denote by $g^2\big|_{0^1}$ the prolongation to $J^2\big|_{0^1}$ of group elements g belonging to G^1_{stab}.

Thus, exactly as in Cartan's method of equivalence, there are here successive *group reductions*.



So again, there is an action on selected (reduced) fibers. And again, the concerned fiber must be decomposed into group orbits. Theorem 42 of Lie — probably the most complicated statement of the whole Volume I of *Theorie der Transformationsgruppen* — explains in an algorithmic way how to decompose group actions into orbits, applying an infinitesimal technique.

Example: Parabolic Surfaces $S^2 \subset \mathbb{C}^3$

With $G := \mathsf{Aff}(\mathbb{C}^3)$, in the left space, let $S^2 \subset \mathbb{C}^3 \ni (x, y, u)$ be a graphed (analytic) surface:

$$u = F(x,y) = 0 + 0 + F_{2,0}\, x^2 + F_{1,1}\, x\, y + F_{0,2}\, y^2 + \mathrm{O}_{x,y}(3),$$

its constant term 0 and its first order term 0 being already normalized. Of course:

$$\mathrm{O}_{x,y}(3) = \sum_{i+j \geqslant 3} F_{i,j}\, x^i\, y^j.$$

Clearly, $J^2\big|_{0^1}$ is coordinatized by $\big(F_{2,0}, F_{1,1}, F_{0,2}\big)$.

In the right space, let the target surface in $\mathbb{C}^3 \ni (p, q, v)$ be similarly graphed as:

$$v = G(p,q) = 0 + 0 + G_{2,0}\, p^2 + G_{1,1}\, p\, q + G_{0,2}\, q^2 + \mathrm{O}_{p,q}(3),$$

with $\big(G_{2,0}, G_{1,1}, G_{0,2}\big)$ being coordinates on $\overline{J}^2\big|_{\overline{0}^1}$.

A general transformation of $\mathsf{Aff}(\mathbb{C}^3)$ writes:

$$\begin{aligned} p &:= a_{1,1}\, x + a_{1,2}\, y + b_1\, u + \tau_1, \\ q &:= a_{2,1}\, x + a_{2,2}\, y + b_2\, u + \tau_2, \\ v &:= c_1\, x + c_2\, y + d\, u + \sigma, \end{aligned} \qquad \text{with} \qquad 0 \neq \begin{vmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ c_1 & c_2 & d \end{vmatrix}.$$

But 0 should be sent to $\overline{0}$, which holds if and only if all translational parameters $\tau_1 = \tau_2 = \sigma = 0$ vanish, so that the transformation belongs to $\mathsf{GL}(\mathbb{C}^3)$:

$$\begin{bmatrix} p \\ q \\ v \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,1} & b_1 \\ c_1 & c_2 & d \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix}.$$

Thus $G^0_{\mathsf{stab}} = \mathsf{GL}(\mathbb{C}^3)$ here.

Furthermore, 0^1 should be sent to $\overline{0}^1$, and the reader can verify that this corresponds to the group reduction towards G^1_{stab}:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ c_1 & c_2 & d \end{bmatrix}^{\mathbf{0}} \quad \rightsquigarrow \quad \begin{bmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ \mathbf{0} & \mathbf{0} & d \end{bmatrix}^{\mathbf{1}}.$$

How? Simply by inspecting the *fundamental equation:*

$$\begin{aligned} 0 \equiv\ & -c_1\,x - c_2\,y - d\,F(x,y) \\ & + G\Big(a_{1,1}\,x + a_{1,2}\,y + b_1\,F(x,y),\ \ a_{2,1}\,x + a_{2,2}\,y + b_2\,F(x,y)\Big), \end{aligned} \tag{0.1}$$

which expresses that $\{u = F(x,y)\}$ is mapped to $\{v = G(p,q)\}$. This fundamental equation must hold *identically* in the ring $\mathbb{C}\{x,y\}$ of convergent power series. Thus, neglecting second and higher order terms:

$$0 \equiv -c_1\,x - c_2\,y + \mathrm{O}_{x,y}(2),$$

we see that $0 = c_1 = c_2$, necessarily. Visibly, in G^1_{stab}, there remain 7 (isotropy) parameters.

And now, what is the action of G^1_{stab} on $J^2\big|_{0^1}$? How to prolong G^1_{stab} to second order jets? Simply by looking at second order terms in the fundamental equation! By hand or using a computer, we find:

$$\begin{aligned}
0 \equiv\ & x^2\left[a_{2,1}^2\,G_{0,2}+a_{1,1}\,a_{2,1}\,G_{1,1}+a_{1,1}^2\,G_{2,0}-d\,F_{2,0}\right]\\
&+x\,y\left[2\,a_{2,1}\,a_{2,2}\,G_{0,2}+a_{1,1}\,a_{2,2}\,G_{1,1}+a_{1,2}\,a_{2,1}\,G_{1,1}+2\,a_{1,1}\,a_{1,2}\,G_{2,0}-d\,F_{1,1}\right]\\
&+y^2\left[a_{2,2}^2\,G_{0,2}+a_{1,2}\,a_{2,2}\,G_{1,1}+a_{1,2}^2\,G_{2,0}-d\,F_{0,2}\right]+\mathrm{O}_{x,y}(3).
\end{aligned}\tag{0.2}$$

(Another — less economic — way of doing would consist in applying Lie's prolongation formulas of diffeomorphisms to the *full bundle* of second order jets, before restricting these formulas to the considered fiber.)

Since G^1_{stab} is a subgroup of $\mathsf{GL}(\mathbb{C}^3)$, its determinant must be nonzero:

$$0\neq\big(a_{1,1}\,a_{2,2}-a_{2,1}\,a_{1,2}\big)\,d=\mathsf{det}\begin{bmatrix}a_{1,1}&a_{1,2}&b_1\\a_{2,1}&a_{2,2}&b_2\\\mathbf{0}&\mathbf{0}&d\end{bmatrix}.$$

Equating to zero the coefficients of x^2, of $x\,y$, of y^2, and solving for $G_{2,0}$, $G_{1,1}$, $G_{0,2}$ gives a *linear representation* on $\mathbb{C}^3$:

$$\begin{bmatrix}G_{2,0}\\G_{1,1}\\G_{0,2}\end{bmatrix}=\frac{1}{\big(a_{1,1}\,a_{2,2}-a_{2,1}\,a_{1,2}\big)^2}\begin{bmatrix}a_{2,2}^2\,d&-\,a_{2,1}\,a_{2,2}\,d&a_{2,1}^2\,d\\-\,2\,a_{1,2}\,a_{2,2}\,d&a_{1,1}\,a_{2,2}\,d+a_{2,1}\,a_{1,2}\,d&-\,2\,a_{1,1}\,a_{2,1}\,d\\a_{1,2}^2\,d&-\,a_{1,1}\,a_{1,2}\,d&a_{1,1}^2\,d\end{bmatrix}\begin{bmatrix}F_{2,0}\\F_{1,1}\\F_{0,2}\end{bmatrix}.$$

This is the action of G^1_{stab} on $J^2\big|_{0^1} = \mathbb{C}^3$, and in fact, the action of the block-diagonal subgroup $\mathsf{GL}(\mathbb{C}^2) \times \mathbb{C}^* \subset \mathbb{G}^1_{\mathsf{stab}}$, because b_1, b_2 are absent.

It is elementary to realize that this action is equivalent, up to dilation, to the action of $\mathsf{SL}(\mathbb{C}^2)$ on binary quadrics, and to deduce that there are exactly 3 possible inequivalent normal forms at order 2:

$$
\begin{aligned}
&\textbf{Branch 2a} && u = 0 \quad + \mathrm{O}_{x,y}(3),\\
&\textbf{Branch 2b} && u = x^2 \ + \mathrm{O}_{x,y}(3),\\
&\textbf{Branch 2c} && u = x\,y + \mathrm{O}_{x,y}(3).
\end{aligned}
$$

Indeed, over the comblex numbers, both $x^2 + y^2$ and $x^2 - y^2$ are equivalent to $x\,y$. Geometrically, there are 3 group-orbits, and there are 3 — point-like, zero-dimensional — transversals

A quick way to recover this fact is to realize by a direct computation that the Hessian at the origin is a relative invariant:

$$
4\,G_{2,0}\,G_{0,2} - G_{1,1}^2 \;=\; \frac{d^2}{a_{1,1}\,a_{2,2} - a_{2,1}\,a_{1,2}}\Big[4\,F_{2,0}\,F_{0,2} - F_{1,1}^2\Big].
$$

Higher-dimensional Hessian matrices are also known to be relatively invariant.

Observation. *In all affine structures classified in this talk, at every jet order, there will always appear* explicit linear representations *of subsequently reduced subgroups* $G^{\kappa-1}_{\mathsf{stab}}$ *on jet fibers* $J^\kappa\big|_{T^{\kappa-1}}$ *over certain group-transversals* $T^{\kappa-1} \subset J^{\kappa-1}$ *from the jet level beneath.*

Infinitesimal Counterpart

At the infinitesimal level, a general affine vector field:

$$\begin{aligned} L = & \big(T_1 + A_{1,1}\,x + A_{1,2}\,y + B_1\,u\big)\,\frac{\partial}{\partial x} \\ & + \big(T_2 + A_{2,1}\,x + A_{2,2}\,y + B_2\,u\big)\,\frac{\partial}{\partial y} \\ & + \big(U_0 + C_1\,x + C_2\,y + D\,u\big)\,\frac{\partial}{\partial u}, \end{aligned}$$

is tangent to $\big\{u = F(x,y)\big\}$ if and only if:

$$0 \equiv L\big(-\,u + F(x,y)\big)\Big|_{u=F(x,y)},$$

identically in $\mathbb{C}\{x,y\}$. With the normalization up to order 2 included:

$$u = 0 + 0 + F_{2,0}\,x^2 + F_{1,1}\,x\,y + F_{0,2}\,y^2 + \mathrm{O}_{x,y}(3),$$

these tangency equation reads:

$$0 \equiv -U_0 + x\,\big[F_{1,1}\,T_2 + 2\,F_{2,0}\,T_1 - C_1\big] + y\,\big[2\,F_{0,2}\,T_2 + F_{1,1}\,T_1 - C_2\big] + \mathrm{O}_{x,y}(2),$$

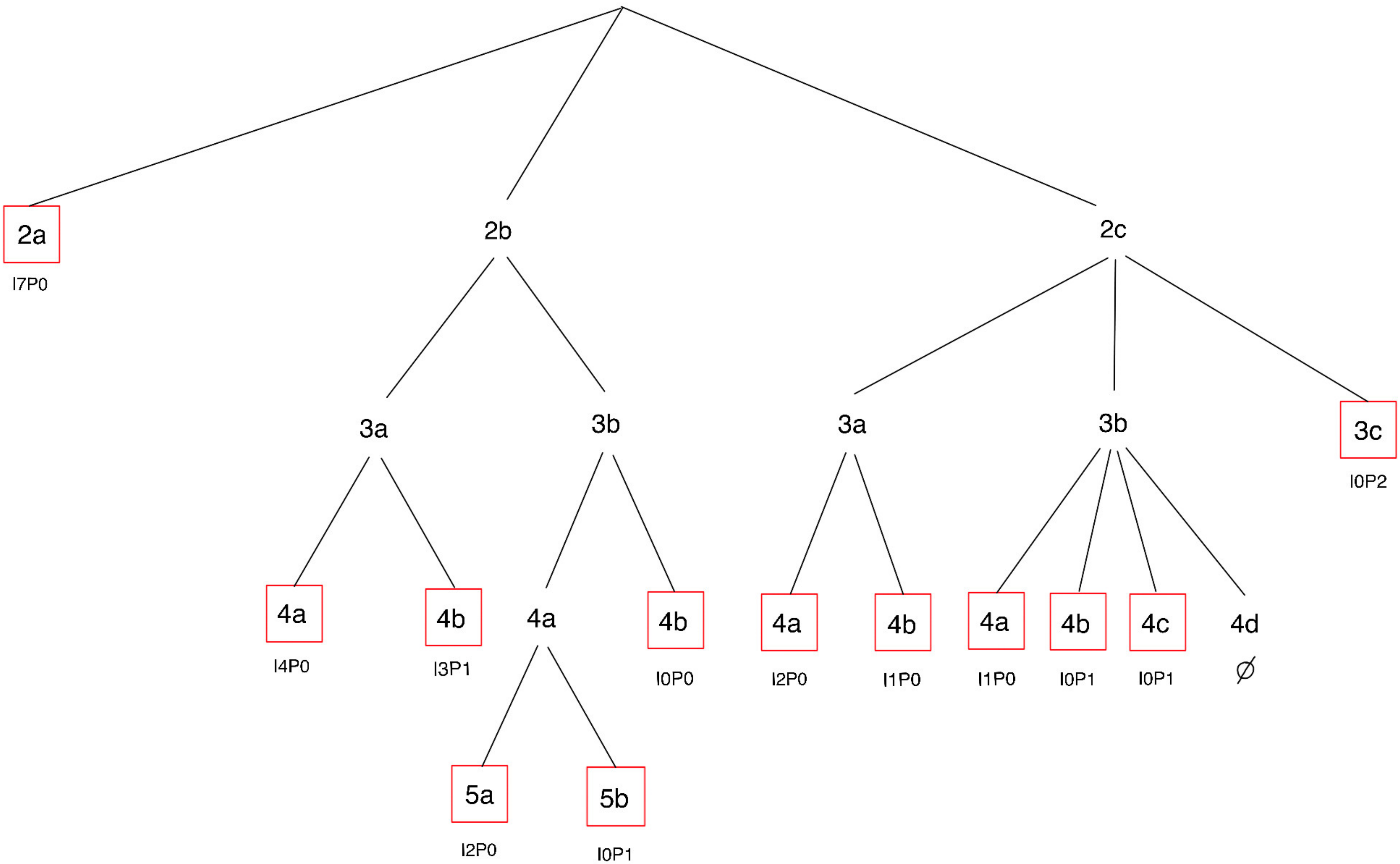
whence necessarily:

$$\begin{aligned} U_0 &:= 0, \\ C_1 &:= F_{1,1}\,T_2 + 2\,F_{2,0}\,T_1, \\ C_2 &:= 2\,F_{0,2}\,T_2 + F_{1,1}\,T_1. \end{aligned}$$

This corresponds to the group reduction to G^1_{stab} seen above, and this means that the general infinitesimal generator of G^1_{stab} writes:

$$\begin{aligned} L^1_{\mathsf{stab}} := & \Big(T_1 + A_{1,1}\,x + A_{1,2}\,y + B_1\,u\Big)\,\frac{\partial}{\partial x} \\ & + \Big(T_2 + A_{2,1}\,x + A_{2,2}\,y + B_2\,u\Big)\,\frac{\partial}{\partial y} \\ & + \Big(\big[F_{1,1}\,T_2 + 2\,F_{2,0}\,T_1\big]\,x + \big[2\,F_{0,2}\,T_2 + F_{1,1}\,T_1\big]\,y + D\,u\Big)\,\frac{\partial}{\partial u}. \end{aligned}$$

Branching Diagram for Surfaces $S^2 \subset \mathbb{C}^3$



General Setting: Induction on Jet Order

In the (x,u)-space and in the $(\overline{x},\overline{u})$-space as well, let the two normal forms be written as:

$$\begin{aligned} u_j &= \mathsf{N}_{j,\kappa-1}^{\mathsf{normal}}\big(\boldsymbol{I}_\bullet, x\big) + \sum_{i_1+\cdots+i_n\geqslant\kappa} F_{j,i_1,\dots,i_n}\, x_1^{i_1}\cdots x_n^{i_n}, \\ \overline{u}_j &= \mathsf{N}_{j,\kappa-1}^{\mathsf{normal}}\big(\boldsymbol{I}_\bullet, \overline{x}\big) + \sum_{i_1+\cdots+i_n\geqslant\kappa} \overline{F}_{j,i_1,\dots,i_n}\, \overline{x}_1^{i_1}\cdots \overline{x}_n^{i_n}, \end{aligned}$$

with $1\leqslant j\leqslant c$, where the following holds.

□ The $\mathsf{N}_{j,\kappa-1}^{\mathsf{normal}}$ represent all x-monomials in the left space and all $\overline{x}$-monomials in the right space, monomials which are *normalized and finalized* up to order $\leqslant \kappa-1$.

□ These normalized polynomials $\mathsf{N}_{j,\kappa-1}^{\mathsf{normal}}$ are *exactly the same functions* on both sides — only the argument x is changed to $\overline{x}$.

□ The supplementary argument $\boldsymbol{I}_\bullet$ (without indices, sometimes absent) indicates that in some branches, there might remain a certain number of *absolute invariants* found in preceding orders, namely function satisfying in this branch:

$$\boldsymbol{I}_\bullet\Big(J^{\kappa-1}F\Big) = \boldsymbol{I}_\bullet\Big(\overline{J}^{\kappa-1}\overline{F}\Big),$$

with on both sides *exactly the same functions* $\boldsymbol{I}_\bullet$ of the collection of order $\leqslant \kappa-1$ power series coefficients — plainly denoted here with the notation $J^{\kappa-1}$.

So now, how to determine $G_{\mathsf{stab}}^{\kappa-1}$? Just by requiring that the normal form is preserved by a transformation $g\in G$ up to order $\leqslant \kappa-1$. In the example of $S^2\subset\mathbb{C}^3$ under

$\mathsf{Aff}(\mathbb{C}^3)$, we saw the *fundamental equation* (0.1), and we truncated it at order 1 to get G^1_{stab} with $c_1 = c_2 = 0$.

In the general setting, the reduced group $G^{\kappa-1}_{\mathsf{stab}} \subset G^{\kappa-2}_{\mathsf{stab}} \subset \cdots \subset G$ can be determined, theoretically, as follows. At first, with $g \in G^{\kappa-2}_{\mathsf{stab}}$, let the group-dependent diffeomorphism $(x, u) \longmapsto \big(\overline{x}, \overline{u}\big)$ be written as:

$$\overline{x} = \overline{x}\big(x, u, g\big), \qquad\qquad \overline{u} = \overline{u}\big(x, u, g\big).$$

For $G = \mathsf{Aff}(\mathbb{C}^3)$, such formulas are explicit. Such a diffeomorphism maps $\big\{u = F(x)\big\}$ to $\big\{\overline{u} = \overline{F}(\overline{x})\big\}$ if and only if:

$$u = F(x) \qquad \Longrightarrow \qquad \overline{u} = \overline{F}\big(\overline{x}\big),$$

which yields the *fundamental equations*, in the current branch:

$$\begin{aligned} 0 &\equiv -\,\overline{u}_j\big(x, F(x), g\big) + \overline{F}_j\Big(\overline{x}\big(x, F(x), g\big)\Big) \\ &\equiv \sum_{i_1+\cdots+i_n \geqslant 0} \mathsf{E}^{\mathsf{nf}}_{j,i_1,\dots,i_n}\Big(\boldsymbol{I}_{\bullet}, F_{\bullet}, \overline{F}_{\bullet}, g\Big)\, x_1^{i_1} \cdots x_n^{i_n}. \end{aligned}$$

These c equations for $1 \leqslant j \leqslant c$ should be satisfied *identically* in $\mathbb{C}\{x_1, \dots, x_n\}$. The upper index $^{\mathsf{nf}}$ in $\mathsf{E}^{\mathsf{nf}}_{\bullet}$ indicates that these equations are involved in the production of n*ormal* f*orms*. *Infra*, we will introduce other kinds of equations $\mathsf{E}^{\mathsf{vf}}_{\bullet}$ with the upper index $^{\mathsf{vf}}$, indicating that they come from tangential v*ector* f*ields*.

So all these $E^{\mathsf{nf}}_{j,i_1,\dots,i_n} = 0$ should vanish. Above, the lightened notation $F_{\bullet}$ denotes a certain finite collections of power series coefficient $F_{j,i'_1,\dots,i'_n}$, always with $i'_1 + \cdots + i'_n \leqslant$

$i_1 + \cdots + i_n$, and the same for $\overline{F}_\bullet$. In practice, real formulas are challenging, even for powerful symbolic computers.

By the induction hypothesis, since $g \in G_{\mathsf{stab}}^{\kappa-2}$, all equations $E_{j,i_1,\dots,i_n}^{\mathsf{nf}} = 0$ with $1 \leqslant j \leqslant c$ and with $i_1 + \cdots + i_n \leqslant \kappa - 2$ are already fulfilled, and it remains:

$$0 \equiv \sum_{i_1+\cdots+i_n=\kappa-1} \mathsf{E}_{j,i_1,\dots,i_n}^{\mathsf{nf}}\Big(\boldsymbol{I}_\bullet, F_\bullet, \overline{F}_\bullet, g\Big)\, x_1^{i_1} \cdots x_n^{i_n} + \mathrm{O}_{x_1,\dots,x_n}(\kappa) \qquad (1 \leqslant j \leqslant c),$$

whence:

$$0 = \mathsf{E}_{j,i_1,\dots,i_n}^{\mathsf{nf}}\Big(\boldsymbol{I}_\bullet, F_\bullet, \overline{F}_\bullet, g\Big) \qquad (\forall\, 1 \leqslant j \leqslant c,\ \forall\, i_1+\cdots+i_n = \kappa-1).$$

Once $F_\bullet$ is chosen in a certain transversal $T^{\kappa-1}$ with (by invariancy) the same choice for $\overline{F}_\bullet$, these (algebraic) equations are used as supplementary constraints on $g \in G_{\mathsf{stab}}^{\kappa-2}$. These equations therefore force g to belong to a specific *reduced* subgroup $G_{\mathsf{stab}}^{\kappa-1} \subset G_{\mathsf{stab}}^{\kappa-2}$.

In this order $\kappa - 1$ preceding the working order κ, because we reason by induction, we have not yet explained how transversals $T^{\kappa-1}$ to $G_{\mathsf{stab}}^{\kappa-1}$-orbits were constructed/chosen. This aspect is more delicate. *Infra*, at the next (working) order κ, we will explain how to create transversals T^κ. At least for now, in our reasoning by induction, we have explained what we assume to be achieved at orders $\leqslant \kappa - 1$

Once $G_{\mathsf{stab}}^{\kappa-1}$ is known, the next step is to *prolong* its action to the space of κ-jets. Remember that we do *not* work in full jet bundles, which is a key trick to dominate the complexity of computations. We work only above successive transversals. This means that we work over the already normalized power series coefficients, at orders $\leqslant \kappa - 1$, namely 'over' $\mathsf{N}_{j,i_1,\dots,i_n}^{\mathsf{normal}}$, symetrically on both left and right sides.

Also, this means that the relative fiber of the projection from κ-jets to normalized jets of order $\leqslant \kappa - 1$ is represented just by letting appear order $= \kappa$ power series coefficients:

$$u_j = \mathsf{N}_{j,\kappa-1}^{\mathsf{normal}}\big(\boldsymbol{I}_\bullet, x\big) + \sum_{i_1+\cdots+i_n=\kappa} F_{j,i_1,\dots,i_n}\, x_1^{i_1}\cdots x_n^{i_n} + \mathrm{O}_{x_1,\dots,x_n}(\kappa+1),$$

and the same for:

$$\overline{u}_j = \mathsf{N}_{j,\kappa-1}^{\mathsf{normal}}\big(\boldsymbol{I}_\bullet, \overline{x}\big) + \sum_{i_1+\cdots+i_n=\kappa} \overline{F}_{j,i_1,\dots,i_n}\, \overline{x}_1^{i_1}\cdots \overline{x}_n^{i_n} + \mathrm{O}_{\overline{x}_1,\dots,\overline{x}_n}(\kappa+1).$$

Of course, the appearing $F_{j,i_1,\dots,i_n}$ and $\overline{F}_{j,i_1,\dots,i_n}$ are *a priori* different here (while at orders $\leqslant \kappa - 1$, they are equal by construction).

The goal is to *normalize* these $F_{j,i_1,\dots,i_n}$ and $\overline{F}_{j,i_1,\dots,i_n}$, *i.e.* to find appropriate orbit transversals. But for which group action? It is at this precise step that things often happen to become delicate.

Abbreviating:

$$\begin{aligned} J_*^\kappa F &:= \big\{F_{j,i_1,\dots,i_n}\big\}_{i_1+\cdots+i_n=\kappa}^{1\leqslant j\leqslant c}, \\ J_*^\kappa \overline{F} &:= \big\{\overline{F}_{j,i_1,\dots,i_n}\big\}_{i_1+\cdots+i_n=\kappa}^{1\leqslant j\leqslant c}, \end{aligned}$$

the fundamental equation, which is now identically satisfied up to all orders $\leqslant \kappa - 1$ when $g \in G_{\mathsf{stab}}^{\kappa-1}$, reads at order κ as:

$$0 = \mathsf{E}_{j,i_1,\dots,i_n}^{\mathsf{nf}}\Big(\boldsymbol{I}_\bullet, J_*^\kappa F, J_*^\kappa \overline{F}, g\Big) \qquad\qquad (\forall\, 1\leqslant j\leqslant c,\ \forall\, i_1+\cdots+i_n=\kappa).$$

Provided that $g \in G^{\kappa-1}_{\mathsf{stab}}$ lies in some neighborhood of the identity, these algebraic equations, of degree 1 with respect to $J^\kappa_* F$ and to $J^\kappa_* \overline{F}$, may always be solved under the form:

$$J^\kappa_* \overline{F} = \Lambda \Big(\boldsymbol{I}_\bullet, J^\kappa_* F, g \Big).$$

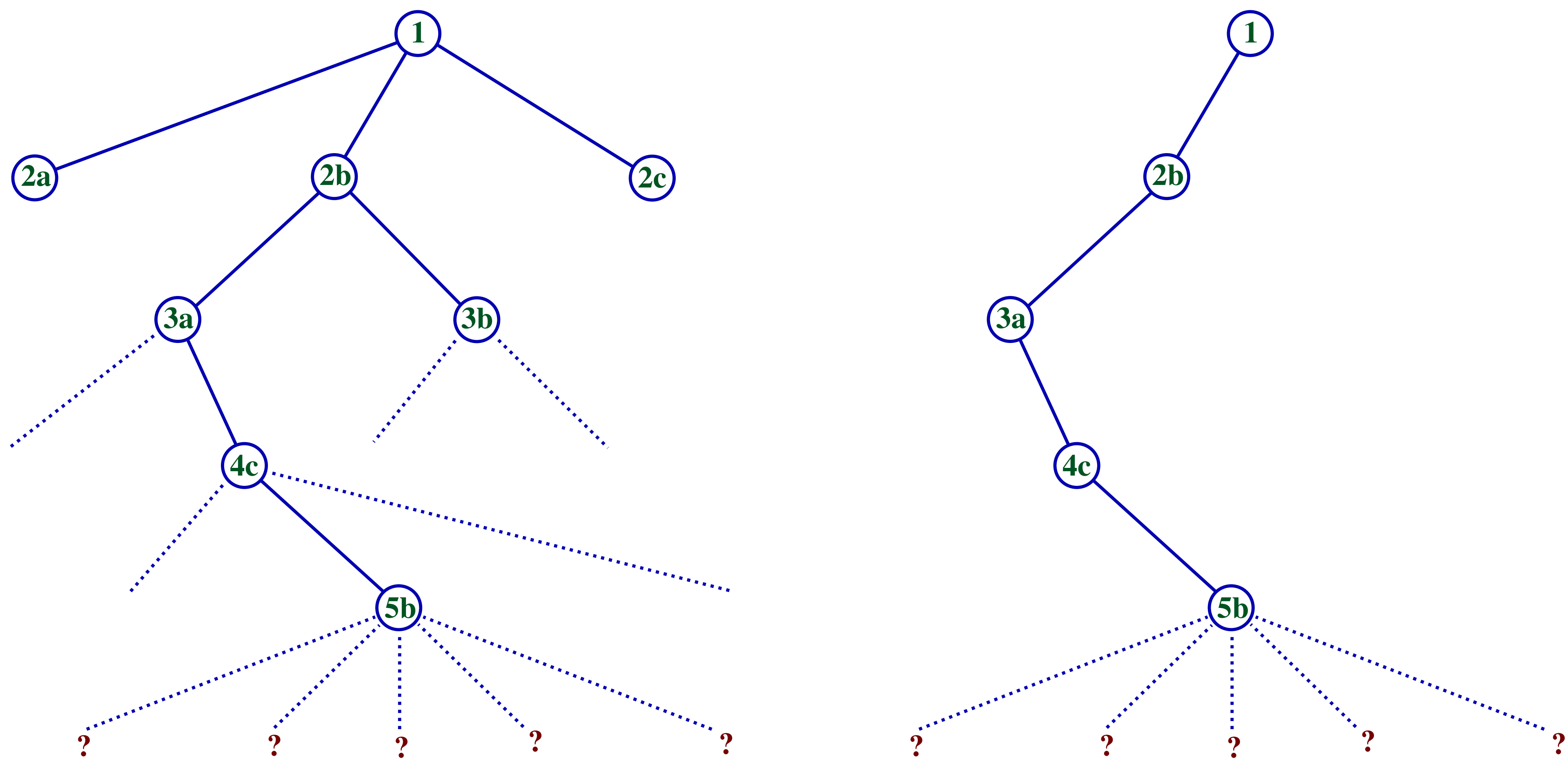
But some key information may be missing. This is a 'defect' of the normal form equations $0 = \mathsf{E}^{\mathsf{nf}}_\bullet$ which, by working only over the origin $(x, u) = (0, 0)$, are unable *per se* to capture differentialo-geometric information.

Reduced Linear Representation and Branch Creation

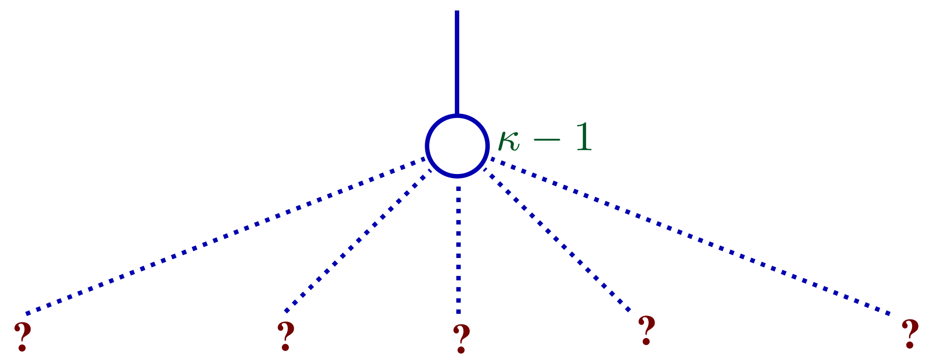
Back to the general setting, with $g \in G_{\mathsf{stab}}^{\kappa-1}$, in the order κ normal form equations:

$$0 \;=\; \mathsf{E}^{\mathsf{nf}}_{j,i_1,\dots,i_n}\Big(\boldsymbol{I}_\bullet,\, J^\kappa_* F,\, J^\kappa_* \overline{F},\, g\Big) \qquad\qquad (\forall\, 1 \leqslant j \leqslant c,\ \forall\, i_1 + \cdots + i_n = \kappa).$$

some jet coordinates in $J^\kappa_* F$ and, parallelly, in $J^\kappa_* \overline{F}$, should disappear due to the previous history within the branches created before.



In the illustrating figure above, **5b** would be the branch at order $\kappa - 1 = 5$ at which considerations hold (instead of $\kappa - 1 = 3$ above), with nearby branches, and with the whole history of preceding branches. Still, the creation of order $6 = \kappa$ subsequent branches is not yet done.



In the previous history, some relative differential invariants, say $\boldsymbol{K}_1, \dots, \boldsymbol{K}_t$, were encountered which were assumed to be $\equiv 0$. (Some other relative differential invariants may have been assumed to be nonzero and then normalized to $+1$ or to -1 with associated group reductions, but such kinds of normalizations have no differential consequences.) These *invariant* differential relations:

$$0 \equiv \boldsymbol{K}_1\big(J^{\kappa-1}F\big), \quad \dots\dots\dots, \quad 0 \equiv \boldsymbol{K}_t\big(J^{\kappa-1}F\big),$$

encountered at jet orders $\leqslant \kappa - 1$, do not depend on $J^*_\kappa F$.

But by differentiation with respect to $x_1, \dots, x_n$, these PDEs do (in general) provide resolutions of certain *dependent* $J^\kappa_{*,\mathsf{dep}} F$ in terms of some other *independent* $J^\kappa_{*,\mathsf{ind}} F$, possibly with discussion of determinantal loci, hence with creation of branches. For instance, from the parabolic surfaces differential relation $F_{yy} = \frac{F_{xy}^2}{F_{xx}}$ with $\kappa - 1 = 2$, it comes:

$$F_{xyy} = 2\,\frac{F_{xy}\,F_{xxy}}{F_{xx}} - \frac{F_{xy}^2\,F_{xxx}}{F_{xx}^2},$$
$$F_{yyy} = 3\,\frac{F_{xy}^2\,F_{xxy}}{F_{xx}^2} - 2\,\frac{F_{xy}^3\,F_{xxx}}{F_{xx}^3}.$$

In summary, coming back to our power series, let us admit that all the order κ differential consequences of the degeneracy assumptions encountered before in the current branch are computable in some 'external' way and have been inserted in the order κ normal form equations:

$$0 = \mathsf{E}^{\mathsf{nf}}_{j,i_1,\dots,i_n}\Big(\boldsymbol{I}_\bullet,\, J^\kappa_{*,\mathsf{ind}}F,\, J^\kappa_{*,\mathsf{ind}}\overline{F},\, g\Big) \qquad\qquad (\forall\, 1\leqslant j\leqslant c,\ \forall\, i_1+\cdots+i_n=\kappa),$$

with $g \in G^{\kappa-1}_{\mathsf{stab}}$.

In fact, since we will abandon the Differential Invariants Problem on p. 7, and focus only on the Homogenous Models Problem on p. 7, we will develop a precise, elementary, and unambiguous method for determining the explicit expressions of the *dependent* jets $J^\kappa_{*,\mathsf{dep}}F$, together with some extra jet constraints required to construct homogeneous geometries, *see* the explanations below. This method will only use power series at the origin.

It seems that now, the appropriate linear representation can be obtained by solving for $J^\kappa_{*,\mathsf{ind}}\overline{F}$. But using some of the group parameters $g \in G^{\kappa-1}_{\mathsf{stab}}$, some of the power series coefficients $J^\kappa_{*,\mathsf{ind}}\overline{F}$ may still be normalized, *e.g.* to 0, and then, associated group reductions must be set up.

Let us assume that such extra normalizations have been made, let us keep the same notation $J^{\kappa}_{*,\mathsf{ind}}F$ for the remaining independent jets, and let us keep the same notation $G^{\kappa-1}_{\mathsf{stab}}$ for the reduced group.

Once all these tasks are achieved, we can really solve:

$$J^{\kappa}_{*,\mathsf{ind}}\overline{F} \;=\; \Lambda\left(\boldsymbol{I}_{\bullet},\, J^{\kappa}_{*,\mathsf{ind}}F,\, g\right) \qquad\qquad (g \in G^{\kappa-1}_{\mathsf{stab}}).$$

Observation. *In all affine structures treated in this talk, and in other geometric structures as well, at every jet order κ, these Λ-formulas* always *were certain* explicit linear matrix representations *of a certain reduced Lie group $G^{\kappa-1}_{\mathsf{stab}} \subset G^{\kappa-2}_{\mathsf{stab}} \subset \cdots \subset G$, and even, always independent of the absolute invariants $\boldsymbol{I}_{\bullet}$ coming from the preceding jet orders.*

Consequently, to each node of the final branching tree is attached a *linear representation* of a Lie group!

This is very analogous to the existence of G-structures with their successive reductions, a central feature of Cartan's method of equivalence. But there is an important difference: G structures have *functional* entries, while our Λ-matrices always have *scalar* entries, even when G is a group of diffeomorphisms — is infinite-dimensional.

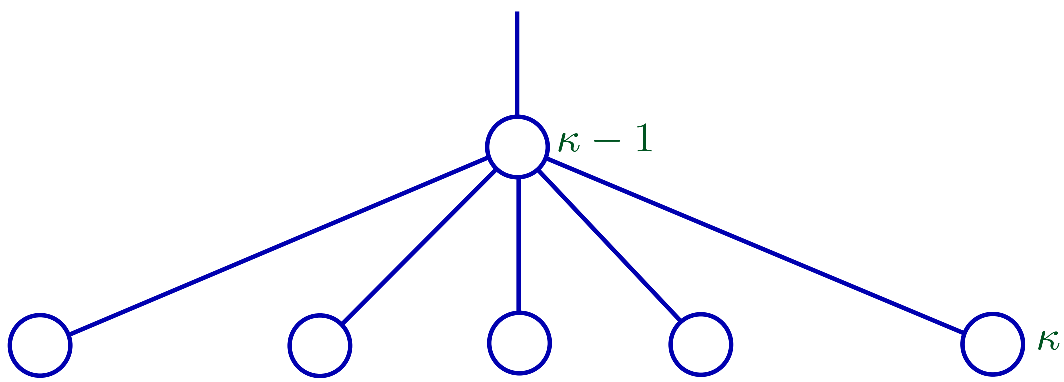
This is explained by our key decision not to work in full jet bundles, but only above successively selected points or transversals to group-orbits.

So quite unexpectedly for researchers like us who during several years worked out the *very nonlinear* and PDE-theoretic (parametric) Cartan equivalence method, *the theory* *of* linear *representations of Lie groups became very useful, very universal, and present at each step of the process, at every node of every branching tree!*

To our knowledge, the observation that linear representations of Lie groups are universally present has not been made in the literature.

We can now terminate our induction reasoning. The linear representation written above of $G_{\mathsf{stab}}^{\kappa-1}$ in the (finite-dimensional) vector space of the components of $J_{*,\mathsf{ind}}^{\kappa} F$ then decomposes this vector space into a finite number of group-orbits.

Transversals $T_{\bullet}^{\kappa}$ to all these group-orbits must then be appropriately chosen.



This is how we create the branches at the working order κ.

This terminates our description of the process, by induction on the jet order κ.

Of course, in the specific examples treated in papers, details are presented, especially, linear representations.

Thus, we focus our attention mainly on

Problem. *Given a finite-dimensional local Lie group acting G on graphed submanifolds $M^n = \{u = F(x)\}$ in $\mathbb{R}^{n+c}_{x,u}$, find and classify all possible M having a locally transitive local automorphisms group* $\mathsf{Sym}(M) \subset G$.

Here:

$$\mathsf{Sym}(M) = \big\{g \in G\colon\ g(M) \subset M\big\},$$

where we do not stipulate that open subsets $V \subset U \subset M$ should be chosen with $g(V) \subset U$ and that $g \in G$ should lie in some neighborhood of the identity.

Local Lie groups, not often considered in the modern literature, are easy to handle because they are well represented (in a one-to-one manner) by Lie algebras of vector fields.

In fact, $\mathsf{Sym}\, M$ has Lie algebra:

$$\mathsf{Lie}\,\mathsf{Sym}(M) = \mathfrak{sym}(M) := \big\{L \in \mathfrak{g}\colon\ L\big|_M \text{ is tangent to } M\big\},$$

where $\mathfrak{g}$ denotes the Lie algebra of vector fields *inside* $\mathbb{R}^{n+c}$ obtained by diffentiating at the identity the action of G on $\mathbb{R}^{n+c}$. For instance, when $G = \mathsf{Aff}(\mathbb{R}^{n+c})$:

$$\mathfrak{g} = \mathsf{Span}\Big(\partial_{x_i},\ \partial_{u_j},\ x_{i_1}\,\partial_{x_{i_2}},\ u_j\,\partial_{x_i},\ x_i\,\partial_{u_j},\ u_{j_1}\,\partial_{u_{j_2}}\Big).$$

Since all our considerations are *local*, after recentering the coordinates, we can assume that everything takes place in some neighborhood of the origin $0 \in M$.

Definition. A c-codimensional submanifold $M^n \subset \mathbb{R}^{n+c}$ is said to be *(locally) affinely homogeneous* if:

$$T_0 M \;=\; \mathsf{Span}_{\mathbb{R}} \left\{ L\big|_0 \colon \; L \in \mathfrak{sym}(M) \right\}.$$

According to basic Lie theory, the 1-parameter group $p \longmapsto \mathsf{exp}(t\,L)(p)$ stabilizes M, and $\mathsf{Sym}(M)$ is then locally transitive in a neighborhood of $0 \in M$.

As is known, the datum of the Lie algebra $\mathfrak{sym}(M)$ enables (by exponentiation) to reconstitute (a neighborhood of the identity in) $\mathsf{Sym}(M)$. But $\mathfrak{sym}(M)$ is much better handled than $\mathsf{Sym}(M)$, thanks to its *linear* and *infinitesimal* features. Lie himself insisted on the fact that *Lie algebras of vector fields* are the right objects of study when classifying continuous transformation group actions. And all of Lie's classifications consist in *lists* of Lie algebras of *infinitesimal transformations* (vector fields), *see e.g.* on pages 6, 17, 26, 57, 71, 106, 116, 139, 167, 203, 209, 214, 226, 246, 257, 271, 334, 370, 388, 384, 388, 391 of Engel-Lie 1893.

We will adopt Lie's way of classifying geometries, namely, by presenting explicit Lie algebras of vector fields.

Now, in continuation with what precedes, set:

$$\mathfrak{g}_{\mathsf{stab}}^{\kappa-1} \;:=\; \mathsf{Lie}\, G_{\mathsf{stab}}^{\kappa-1}.$$

Reasoning by induction on the jet order, assume that there are vector fields:

$$e_1, \dots, e_n \;\in\; \mathfrak{g}_{\mathsf{stab}}^{\kappa-1},$$

such that, at the origin $0 \in M$:

$$\mathsf{Span}\left(e_1\big|_0, \; \dots, \; e_n\big|_0 \right) \;=\; T_0 M.$$

Certainly, $n \leqslant \dim \mathfrak{g}_{\mathsf{stab}}^{\kappa-1} \leqslant \dim G$.

Together with $e_1, \dots, e_n$, there are a certain number $\nu \geqslant 0$ of *isotropy* vector fields $f_1, \dots, f_\nu \in \mathfrak{g}_{\mathsf{stab}}^{\kappa-1}$, *i.e.* vector fields vanishing at the origin $(x, u) = (0, 0)$, such that the general infinitesimal transformation $L \in \mathfrak{g}_{\mathsf{stab}}^{\kappa-1}$ writes for $1 \leqslant j \leqslant c$:

$$L = T_1 \, e_1 + \cdots + T_n \, e_n + A_1 \, f_1 + \cdots + A_\nu \, f_\nu,$$

with $n + \nu$ arbitrary parameters T_m and A_μ.

To guarantee local homogeneity (transitivity), no linear relation can ever exist between $T_1, \dots, T_n$.

The condition that L be tangent to M up to orders $\leqslant \kappa - 1$, writes:

$$\begin{aligned} 0 &\equiv L\Big(- u_j + F_j(x) \Big) \Big|_{u=F(x)} \\ &\equiv \sum_{i_1 + \cdots + i_n \leqslant \kappa - 2} x_1^{i_1} \cdots x_n^{i_n} \underbrace{\Big(\cdots \Big)}_{\text{vanish by induction}} \\ &\quad + \sum_{i_1 + \cdots + i_n = \kappa - 1} x_1^{i_1} \cdots x_n^{i_n} \, E_{j, i_1, \dots, i_n}^{\mathsf{vf}} \Big(\boldsymbol{I}_\bullet, J_*^\kappa F, T_1, \dots, T_n, A_1, \dots, A_\nu \Big) + \mathrm{O}_{x_1, \dots, x_n}(\kappa), \end{aligned}$$

that is, after reorganization:

$$0 \equiv \sum_{m=1}^{n} T_m \left(\Phi_{j, i_1, \dots, i_n, m}^{\mathsf{vf}} \big(\boldsymbol{I}_\bullet, J_*^\kappa F \big) \right) + \sum_{\mu=1}^{\nu} A_\mu \left(\Psi_{j, i_1, \dots, i_n, \mu}^{\mathsf{vf}} \big(\boldsymbol{I}_\bullet, J_*^\kappa F \big) \right)$$
$$(1 \leqslant j \leqslant c, \; i_1 + \cdots + i_n = \kappa - 1).$$

A few times below, we will abbreviate these equations as:

$$0 = \mathsf{E}_{\bullet}^{\mathsf{vf}}.$$

Whenever one of these equations, say for some indices $\underline{j}, \underline{i}_1, \dots, \underline{i}_n$, does not incorporate any of the *isotropy parameters* $A_1, \dots, A_\nu$, but incorporates only the *transitivity parameters* $T_1, \dots, T_n$, we receive n equations:

$$0 = \Phi^{\mathsf{vf}}_{\underline{j}, \underline{i}_1, \dots, \underline{i}_n, m}\big(\boldsymbol{I}_{\bullet}, J^{\kappa}_{*}F\big) \qquad (1 \leqslant m \leqslant n),$$

which are of degree 1 with respect to $J^{\kappa}_{*}F$, and which express constraints on certain 'dependent' jets $J^{\kappa}_{*,\mathsf{dep}}F$ to be resolved in terms of certain other 'independent' jets $J^{\kappa}_{*,\mathsf{ind}}F$.

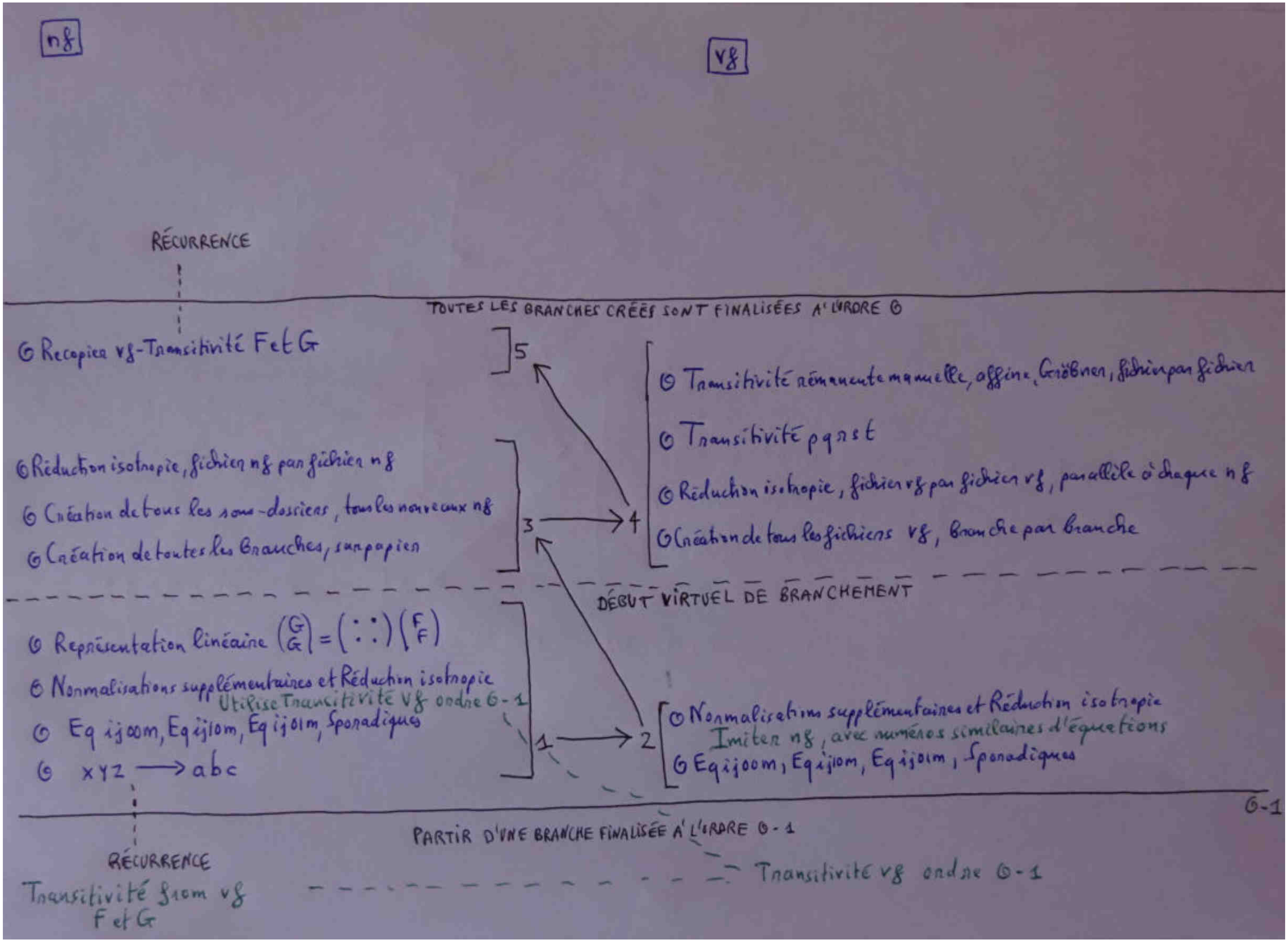
Some of these 'independent' jets may simultaneously become absolute invariants at order κ, hence join the current collection $\boldsymbol{I}_{\bullet}$ before passing to order $\kappa + 1$.

Sometimes even, some linear combinations between these equations must be performed in some tricky way in order to *eliminate* $A_1, \dots, A_\nu$, so as to 'discover' further *transitivity equations* which would reveal new constraints. In many branches of our classification of affinely homogeneous surfaces $S^2 \subset \mathbb{R}^4$, we were blocked for this reason.

This method based on transitivity equations has already been applied in in a paper of Foo-M.-Nurowski-Ta 2021, in a *degenerate* CR-geometric context, for the infinite-dimensional group of biholomorphisms of $\mathbb{C}^3$. However, no details of proof were given. A complete written proof would be about 50 pages long, due to subtle computational aspects in degenerate branches. Indeed, even for a reduction to an *explicit*, *parametric*, Cartan-type $\{e\}$-structure, which is a preliminary step to determine homogeneous models, the calculations are long. Similarly, in the degenerate para-CR context, by lack of space, several computations used to determine homogeneous geometries are not fully presented by M.-Nurowski 2020.

Lastly, and importantly, at the end of the process, we often obtain a collection of *algebraic* equations in the remaining absolute invariants $I_\bullet$, some *key equations* whose zero-set defines an *algebraic moduli space* of a collection of homogeneous models, represented by a *terminal leaf* of the tree.

Ping-Pong Method of Equivalence



Invariant Quartic for PDEs Under Fiber-Preserving Transformations

Proposition. Any equivalence fixing the origin Φ from:

$$
\begin{aligned}
F = \; & c + ax + by \\
& + F_{2,0,2,0,0}x^2a^2 + F_{2,0,1,1,0}x^2ab + F_{2,0,0,2,0}x^2b^2 \\
& + F_{1,1,2,0,0}xya^2 + 0 + F_{1,1,0,2,0}xyb^2 \\
& + F_{0,2,2,0,0}y^2a^2 + F_{0,2,1,1,0}y^2ab + F_{0,2,0,2,0}y^2b^2 + \mathrm{O}_{x,y,a,b,c}(5),
\end{aligned}
$$

to:

$$
\begin{aligned}
G = \; & c' + a'x' + b'y' \\
& + G_{2,0,2,0,0}x'^2a'^2 + G_{2,0,1,1,0}x'^2a'b' + G_{2,0,0,2,0}x'^2b'^2 + G_{1,1,2,0,0}x'y'a'^2 + G_{1,1,0,2,0}x'y'b'^2 \\
& + G_{0,2,2,0,0}y'^2a'^2 + G_{0,2,1,1,0}y'^2a'b' + G_{0,2,0,2,0}y'^2b'^2 + \mathrm{O}_{x',y',a',b',c'}(5),
\end{aligned}
$$

transforms the 8 order 4 coefficients as:

$$
\begin{pmatrix}
G_{2,0,2,0,0} \\ G_{2,0,1,1,0} \\ G_{2,0,0,2,0} \\ G_{1,1,2,0,0} \\ G_{1,1,0,2,0} \\ G_{0,2,2,0,0} \\ G_{0,2,1,1,0} \\ G_{0,2,0,2,0}
\end{pmatrix}
= A(\alpha, \beta, \gamma, \delta, \chi) \cdot
\begin{pmatrix}
F_{2,0,2,0,0} \\ F_{2,0,1,1,0} \\ F_{2,0,0,2,0} \\ F_{1,1,2,0,0} \\ F_{1,1,0,2,0} \\ F_{0,2,2,0,0} \\ F_{0,2,1,1,0} \\ F_{0,2,0,2,0}
\end{pmatrix},
$$

where:

$$A(\alpha,\beta,\gamma,\delta,\chi) = \frac{1}{\chi(\alpha\delta-\beta\gamma)^2}, \widetilde{A},$$

and where:

$$\widetilde{A} = \begin{pmatrix} \alpha\delta(\alpha\delta+2\beta\gamma) & \beta\delta(2\alpha\delta+\beta\gamma) & 3\beta^2\delta^2 & -\gamma\alpha(2\alpha\delta+\beta\gamma) & -\beta\delta(\alpha\delta+2\beta\gamma) & 3\alpha^2\gamma^2 & \gamma\alpha(\alpha\delta+2\beta\gamma) & \beta\gamma(2\alpha\delta+\beta\gamma) \\ 2\alpha\delta^2\gamma & \delta^2(\alpha\delta+\beta\gamma) & 2\beta\delta^3 & -2\gamma^2\alpha\delta & -2\beta\delta^2\gamma & 2\alpha\gamma^3 & \gamma^2(\alpha\delta+\beta\gamma) & 2\gamma^2\beta\delta \\ \delta^2\gamma^2 & \delta^3\gamma & \delta^4 & -\gamma^3\delta & -\delta^3\gamma & \gamma^4 & \gamma^3\delta & \gamma^2\delta^2 \\ -2\alpha^2\beta\delta & -2\alpha\beta^2\delta & -2\beta^3\delta & \alpha^2(\alpha\delta+\beta\gamma) & \beta^2(\alpha\delta+\beta\gamma) & -2\alpha^3\gamma & -2\alpha^2\beta\gamma & -2\alpha\beta^2\gamma \\ -2\gamma^2\beta\delta & -2\beta\delta^2\gamma & -2\beta\delta^3 & \gamma^2(\alpha\delta+\beta\gamma) & \delta^2(\alpha\delta+\beta\gamma) & -2\alpha\gamma^3 & -2\gamma^2\alpha\delta & -2\alpha\delta^2\gamma \\ \beta^2\alpha^2 & \alpha\beta^3 & \beta^4 & -\alpha^3\beta & -\alpha\beta^3 & \alpha^4 & \alpha^3\beta & \alpha^2\beta^2 \\ 2\alpha\beta^2\gamma & \beta^2(\alpha\delta+\beta\gamma) & 2\beta^3\delta & -2\alpha^2\beta\gamma & -2\alpha\beta^2\delta & 2\alpha^3\gamma & \alpha^2(\alpha\delta+\beta\gamma) & 2\alpha^2\beta\delta \\ \beta\gamma(2\alpha\delta+\beta\gamma) & \beta\delta(\alpha\delta+2\beta\gamma) & 3\beta^2\delta^2 & -\gamma\alpha(\alpha\delta+2\beta\gamma) & -\beta\delta(2\alpha\delta+\beta\gamma) & 3\alpha^2\gamma^2 & \gamma\alpha(2\alpha\delta+\beta\gamma) & \alpha\delta(\alpha\delta+2\beta\gamma) \end{pmatrix}.$$

set:

$$P := P_1\,P_2 = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \tag{0.3}$$

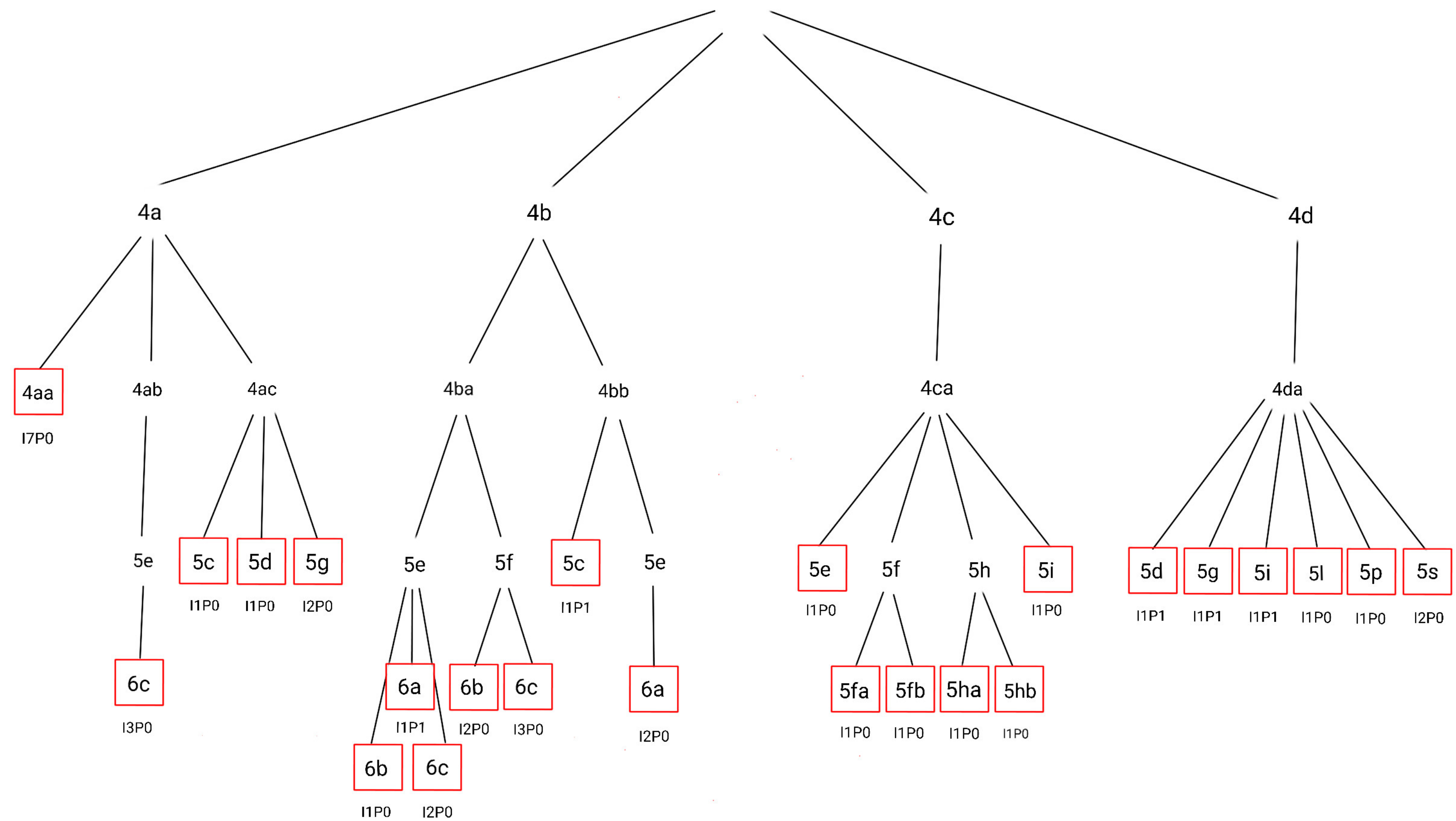
and set:

$$\widetilde{B} := \left(P_1\,P_2\right)^{-1}\widetilde{A}\,P_1\,P_2,$$

so that:

$$\widetilde{B} = \begin{pmatrix} \delta^4 & \delta^3\gamma & \delta^2\gamma^2 & \gamma^3\delta & \gamma^4 & 0 & 0 & 0 \\ 4\beta\delta^3 & \delta^2(\alpha\delta+3\beta\gamma) & 2\delta\gamma(\alpha\delta+\beta\gamma) & \gamma^2(3\alpha\delta+\beta\gamma) & 4\alpha\gamma^3 & 0 & 0 & 0 \\ 6\beta^2\delta^2 & 3\beta\delta(\alpha\delta+\beta\gamma) & \alpha^2\delta^2+4\alpha\delta\beta\gamma+\beta^2\gamma^2 & 3\alpha\gamma(\alpha\delta+\beta\gamma) & 6\alpha^2\gamma^2 & 0 & 0 & 0 \\ 4\beta^3\delta & \beta^2(3\alpha\delta+\beta\gamma) & 2\alpha\beta(\alpha\delta+\beta\gamma) & \alpha^2(\alpha\delta+3\beta\gamma) & 4\alpha^3\gamma & 0 & 0 & 0 \\ \beta^4 & \alpha\beta^3 & \beta^2\alpha & \alpha^3\beta & \alpha^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta^2(\alpha\delta-\beta\gamma) & \delta\gamma(\alpha\delta-\beta\gamma) & \gamma^2(\alpha\delta-\beta\gamma) \\ 0 & 0 & 0 & 0 & 0 & 2\beta\delta(\alpha\delta-\beta\gamma) & (\alpha\delta-\beta\gamma)(\alpha\delta+\beta\gamma) & 2\alpha\gamma(\alpha\delta-\beta\gamma) \\ 0 & 0 & 0 & 0 & 0 & \beta^2(\alpha\delta-\beta\gamma) & \alpha\beta(\alpha\delta-\beta\gamma) & \alpha^2(\alpha\delta-\beta\gamma) \end{pmatrix},$$

- **Branching Diagram for Multiply Transitive Models:**



- **Example:** In branch **4bb**, the found linear representation is:

$$\begin{pmatrix} G_{1,1,3,0,0} \\ G_{1,1,0,1,1} \\ G_{1,1,2,1,0} \\ G_{1,2,2,0,0} \\ G_{1,1,1,0,1} \\ G_{0,2,3,0,0} \\ G_{0,3,2,0,0} \end{pmatrix} = \begin{pmatrix} \delta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\delta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 & 0 \\ 0 & -\frac{16\beta}{\delta^2} & 0 & \frac{1}{\delta} & 0 & 0 & 0 \\ 0 & \frac{2\beta}{\delta^2} & 0 & 0 & \frac{1}{\delta} & 0 & 0 \\ \frac{\beta}{2} & 0 & 4\beta & 0 & 0 & \delta & 0 \\ 0 & \frac{4}{3}\frac{\beta^2}{\delta^3} + \frac{256}{9}\frac{\beta}{\delta^2} & 0 & -\frac{1}{3}\frac{\beta}{\delta^2} & -\frac{4}{3}\frac{\beta}{\delta^2} & 0 & \frac{1}{\delta} \end{pmatrix} \begin{pmatrix} F_{1,1,3,0,0} \\ F_{1,1,0,1,1} \\ F_{1,1,2,1,0} \\ F_{1,2,2,0,0} \\ F_{1,1,1,0,1} \\ F_{0,2,3,0,0} \\ F_{0,3,2,0,0} \end{pmatrix},$$

This leads to the creation of 5 branches:

4bb $\downarrow$	$F_{1,1,3,0,0}$	$F_{1,1,0,1,1}$	$F_{1,1,2,1,0}$	$F_{1,2,2,0,0}$	$F_{1,1,1,0,1}$	$F_{0,2,3,0,0}$	$F_{0,3,2,0,0}$
5a	1	0	$-\frac{1}{8}$	$-4F_{1,1,1,0,1}$	$F_{1,1,1,0,1}$	$F_{0,2,3,0,0}$	$F_{0,3,2,0,0}$
5b	0	0	0	1	$-\frac{1}{4}$	$F_{0,2,3,0,0}$	$F_{0,3,2,0,0}$
5c	0	0	0	0	0	1	$F_{0,3,2,0,0}$
5d	0	0	0	0	0	0	1
5e	0	0	0	0	0	0	0

Order 2 Branches for Surfaces $S^2 \subset \mathbb{R}^4$

In $\mathbb{R}^4 \ni (x, y, u, v)$, local analytic surfaces S^2 can be graphed, after an affine transformation, as:

$$u = F\big(x, y\big) = F_{2,0}\, x^2 + F_{1,1}\, x\, y + F_{0,2}\, y^2 + \mathrm{O}_{x,y}(3),$$
$$v = G(x, y) = G_{2,0}\, x^2 + G_{1,1}\, x\, y + G_{0,2}\, y^2 + \mathrm{O}_{x,y}(3),$$

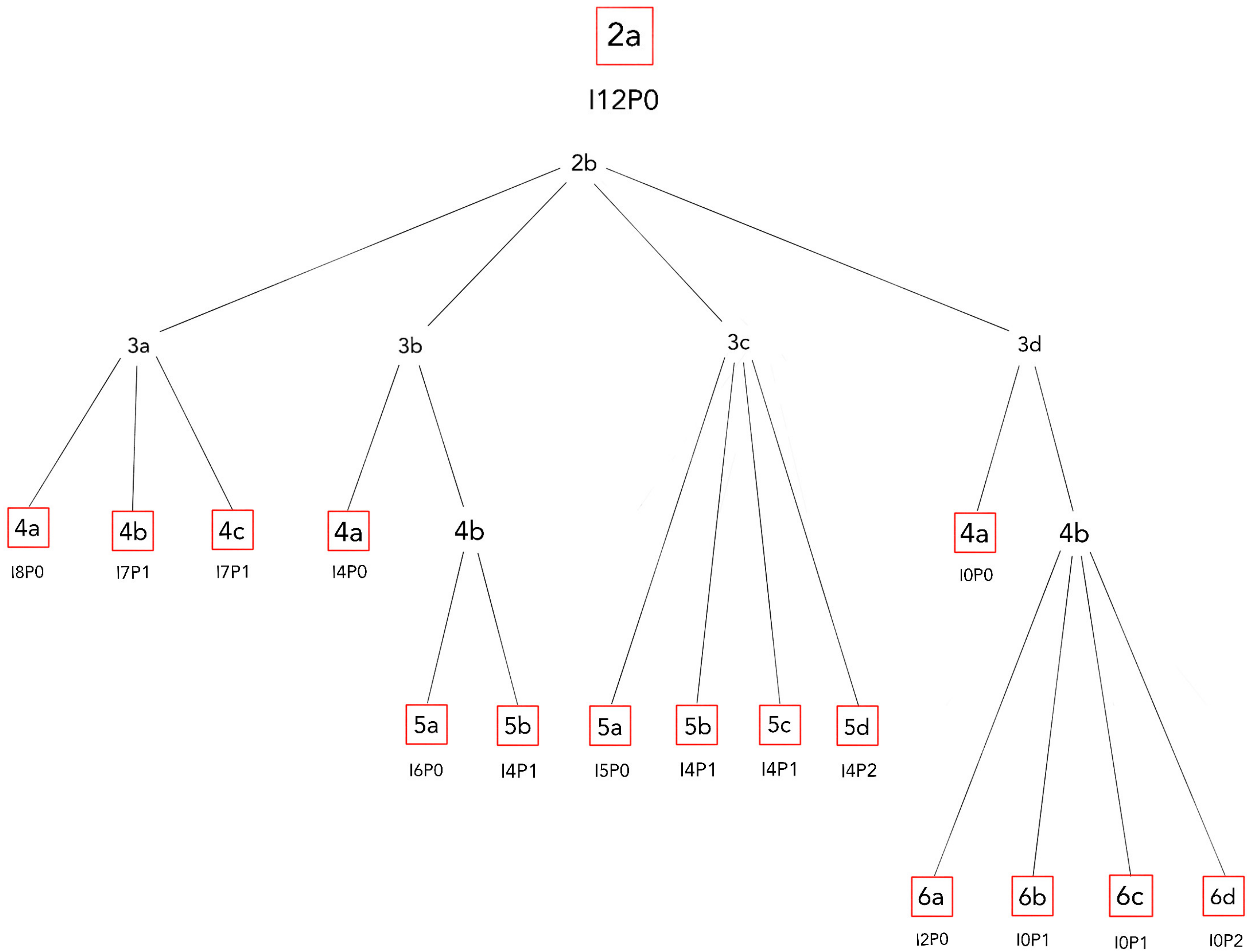
with F, G real-analytic at the origin.

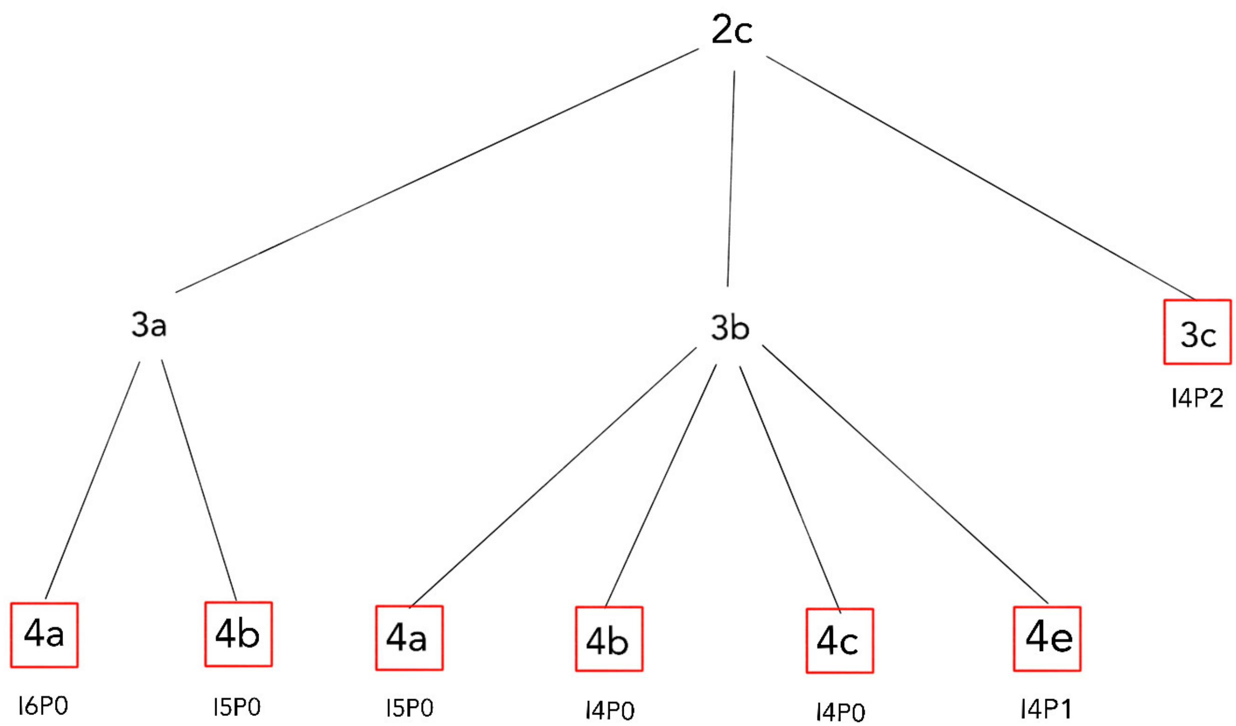
The property that the two quadratic forms F_2 and G_2 are *parallel* (colinear) is affinely invariant.

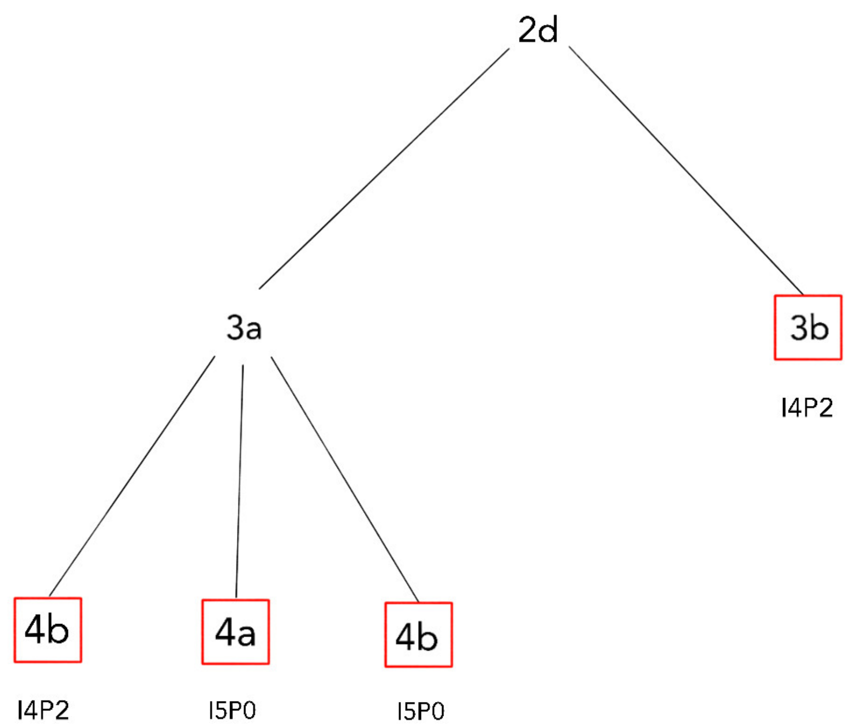
Then 7 inequivalent normalizations exist at order 2:

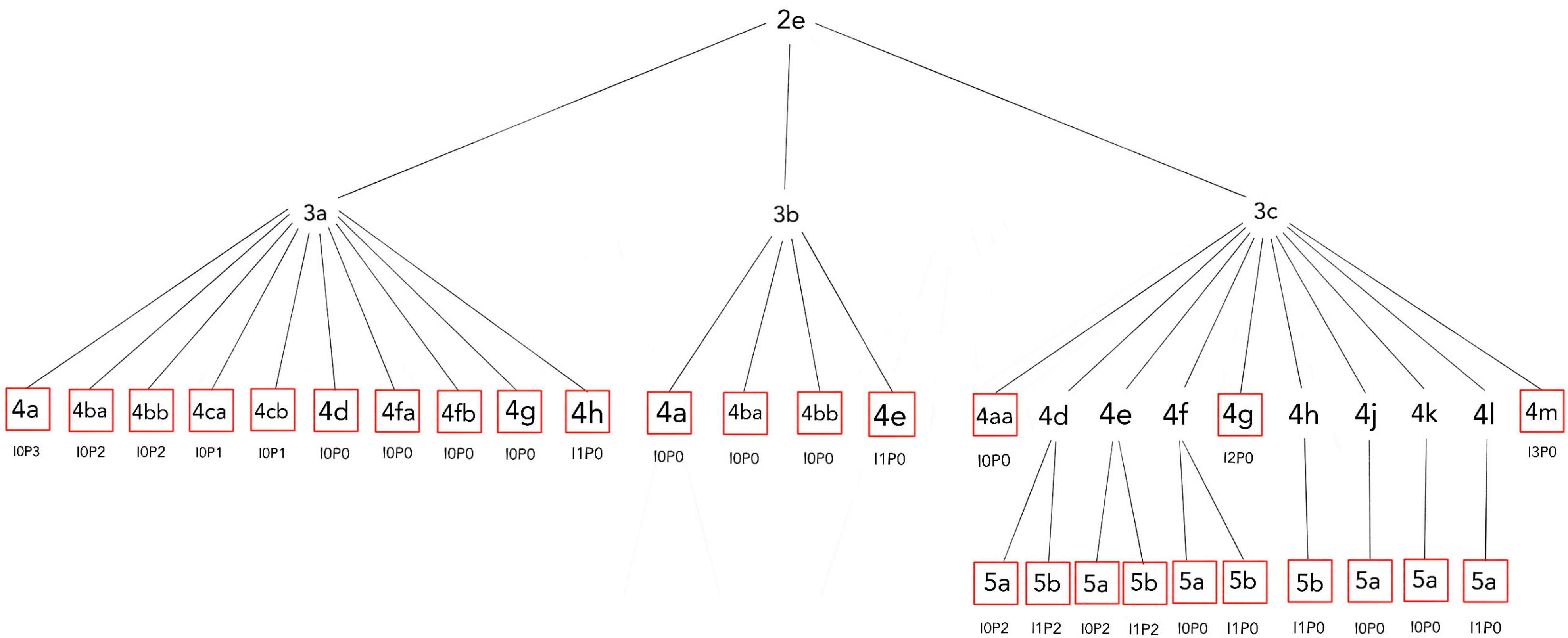
	F_2	G_2
2a	0	0
2b	x^2	0
2c	$x\,y$	0
2d	x^2+y^2	0
2e	$x\,y$	x^2
2f	$x\,y$	x^2+y^2
2g	$x\,y$	x^2-y^2

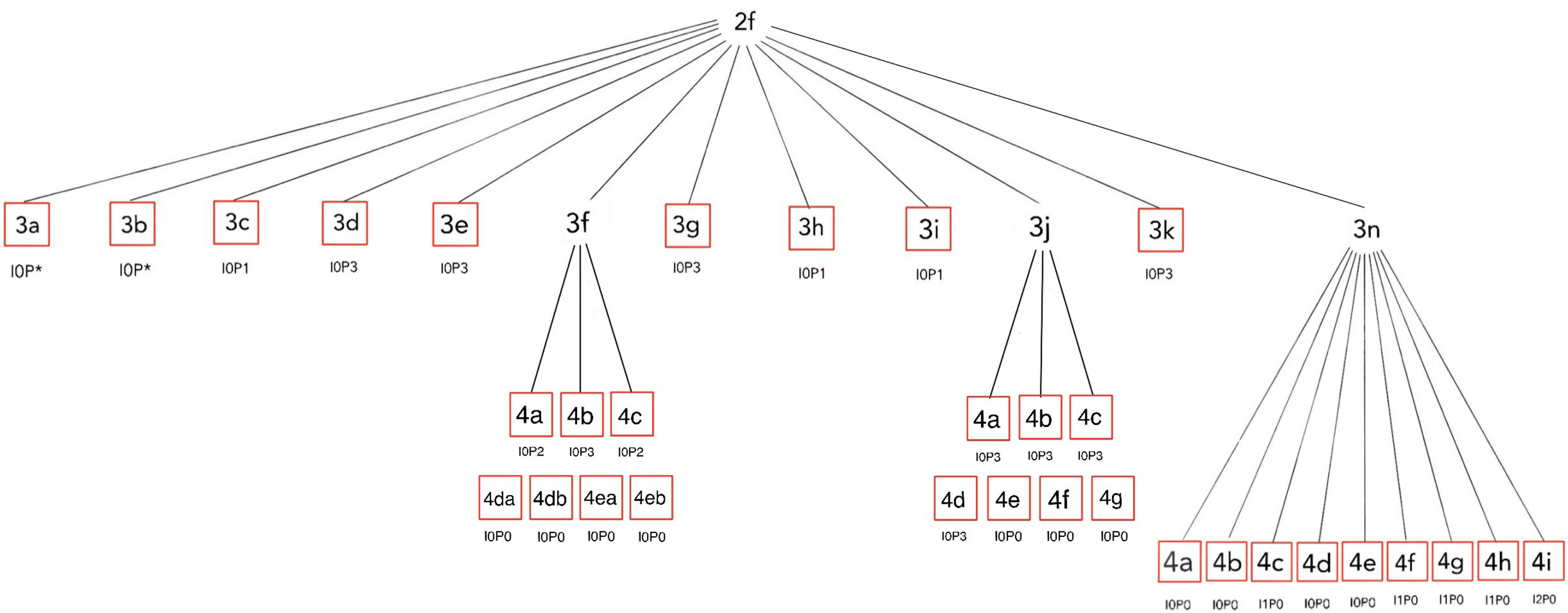
Branching Diagrams for Surfaces $S^2 \subset \mathbb{R}^4$

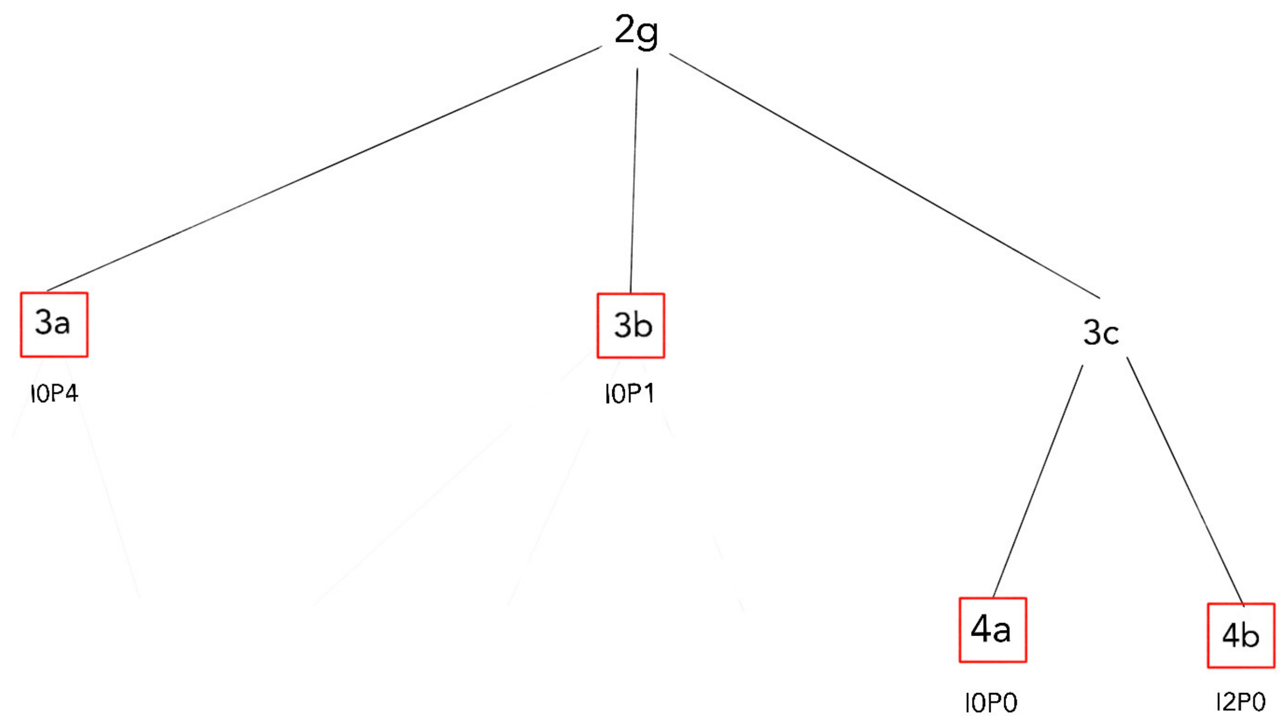



2c
3a
3b
3c
I4P2
4a
4b
4a
4b
4c
4e
I6P0
I5P0
I5P0
I4P0
I4P0
I4P1


2d
3a
3b
I4P2
4b
4a
4b
I4P2
I5P0
I5P0


2e
3a
3b
3c
4a
I0P3
4ba
I0P2
4bb
I0P2
4ca
I0P1
4cb
I0P1
4d
I0P0
4fa
I0P0
4fb
I0P0
4g
I0P0
4h
I1P0
4a
I0P0
4ba
I0P0
4bb
I0P0
4e
I1P0
4aa
I0P0
4d
4e
4f
4g
I2P0
4h
4j
4k
4l
4m
I3P0
5a
I0P2
5b
I1P2
5a
I0P2
5b
I1P2
5a
I0P0
5b
I1P0
5b
I1P0
5a
I0P0
5a
I0P0
5a
I1P0


2f
3a
I0P*
3b
I0P*
3c
I0P1
3d
I0P3
3e
I0P3
3f
3g
I0P3
3h
I0P1
3i
I0P1
3j
3k
I0P3
3n
4a
I0P2
4b
I0P3
4c
I0P2
4da
I0P0
4db
I0P0
4ea
I0P0
4eb
I0P0
4a
I0P3
4b
I0P3
4c
I0P3
4d
I0P3
4e
I0P0
4f
I0P0
4g
I0P0
4a
I0P0
4b
I0P0
4c
I1P0
4d
I0P0
4e
I0P0
4f
I1P0
4g
I1P0
4h
I1P0
4i
I2P0


2g
3a
I0P4
3b
I0P1
3c
4a
I0P0
4b
I2P0

Model 2e3a4a for Surfaces $S^2 \subset \mathbb{R}^4$

Model 2e3a4a

$$\begin{cases} u = xy + y^3 + F_{0,4}y^4 + F_{1,3}xy^3 + F_{2,2}x^2y^2 + F_{3,1}x^3y + x^4 + \\ \quad + (\frac{9}{10}F_{1,3} - \frac{9}{250}F_{0,4}F_{1,3}F_{3,1} + \frac{1}{25}F_{0,4}F_{1,3}G_{4,0} + \frac{32}{25}F_{0,4}^2 + \\ \quad - \frac{1}{250}F_{0,4}G_{4,0}F_{3,1}G_{3,1} + \frac{2}{375}F_{0,4}G_{4,0}F_{3,1}F_{2,2} + \frac{1}{200}F_{0,4}G_{3,1}^2 + \\ \quad + \frac{1}{75}F_{0,4}F_{2,2}^2 - \frac{1}{60}F_{0,4}G_{3,1}F_{2,2} + \frac{1}{25}F_{0,4}F_{3,1}F_{5,0} + \frac{1}{250}F_{0,4}F_{3,1}^2G_{3,1} + \\ \quad - \frac{2}{375}F_{0,4}F_{3,1}^2F_{2,2})y^5 + (\frac{3}{4}G_{3,1} + F_{2,2} + \frac{1}{10}F_{0,4}F_{3,1}G_{3,1} + \\ \quad - \frac{2}{15}F_{0,4}F_{3,1}F_{2,2} - \frac{1}{10}F_{0,4}G_{4,0}G_{3,1} + \frac{2}{15}F_{0,4}G_{4,0}F_{2,2} + \frac{8}{5}F_{0,4}F_{1,3} + \\ \quad + F_{0,4}F_{5,0})xy^4 + (6F_{3,1} - 4G_{4,0} + \frac{1}{10}G_{3,1}F_{1,3}F_{3,1} - \frac{2}{15}F_{2,2}F_{1,3}F_{3,1} + \\ \quad - \frac{1}{10}G_{3,1}F_{1,3}G_{4,0} + \frac{2}{15}F_{2,2}F_{1,3}G_{4,0} + \frac{3}{5}F_{1,3}^2 + F_{1,3}F_{5,0} - F_{0,4}G_{3,1} + \\ \quad + \frac{4}{3}F_{2,2}F_{0,4})x^2y^3 + (4 + \frac{1}{10}G_{3,1}F_{3,1}F_{2,2} - \frac{1}{10}G_{3,1}G_{4,0}F_{2,2} - \frac{1}{2}G_{3,1}F_{1,3} + \\ \quad + \frac{14}{15}F_{2,2}F_{1,3} + F_{2,2}F_{5,0} - \frac{2}{15}F_{3,1}F_{2,2}^2 + \frac{2}{15}G_{4,0}F_{2,2}^2)x^3y^2 + \\ \quad + (-\frac{1}{10}G_{4,0}F_{3,1}G_{3,1} + \frac{2}{15}G_{4,0}F_{3,1}F_{2,2} + \frac{1}{8}G_{3,1}^2 + \frac{1}{3}F_{2,2}^2 - \frac{5}{12}G_{3,1}F_{2,2} + \\ \quad + \frac{1}{10}F_{3,1}F_{1,3} + F_{3,1}F_{5,0} + \frac{1}{10}F_{3,1}^2G_{3,1} - \frac{2}{15}F_{3,1}^2F_{2,2})x^4y + F_{5,0}x^5 + \cdots, \\ v = x^2 - \frac{3}{2}y^4 + G_{3,1}x^3y + G_{4,0}x^4 - \frac{18}{5}F_{0,4}y^5 - 3F_{1,3}xy^4 + \\ \quad + (-2F_{2,2} + 3G_{3,1})x^2y^3 + (-\frac{1}{10}G_{3,1}F_{3,1}F_{2,2} + \frac{1}{10}G_{3,1}G_{4,0}F_{2,2} + \\ \quad + \frac{1}{5}G_{3,1}F_{1,3} + \frac{3}{4}G_{3,1}F_{5,0} + \frac{3}{40}F_{3,1}G_{3,1}^2 - \frac{3}{40}G_{4,0}G_{3,1}^2)x^4y + \\ \quad + (\frac{2}{25}G_{4,0}F_{3,1}G_{3,1} - \frac{8}{75}G_{4,0}F_{3,1}F_{2,2} - \frac{1}{10}G_{3,1}^2 + \frac{2}{25}G_{4,0}F_{1,3} + \\ \quad + \frac{4}{5}G_{4,0}F_{5,0} - \frac{2}{25}G_{4,0}^2G_{3,1} + \frac{8}{75}G_{4,0}^2F_{2,2} + \frac{2}{15}G_{3,1}F_{2,2})x^5 + \cdots \end{cases}$$

$$
\begin{aligned}
e_1 \;:=\; & -(-1+\tfrac{1}{5}F_{1,3}x+2xF_{5,0}+\tfrac{1}{5}xF_{3,1}G_{3,1}-\tfrac{4}{15}xF_{3,1}F_{2,2}-\tfrac{1}{5}xG_{4,0}G_{3,1}+\tfrac{4}{15}xG_{4,0}F_{2,2}+\\
&+\tfrac{3}{2}G_{3,1}u+2vG_{4,0})\partial_x-(4v-\tfrac{1}{2}xG_{3,1}+\tfrac{2}{3}xF_{2,2}+\tfrac{3}{5}yF_{1,3}+yF_{5,0}-2uG_{4,0}+3uF_{3,1}+\\
&+\tfrac{1}{10}yF_{3,1}G_{3,1}-\tfrac{2}{15}yF_{3,1}F_{2,2}-\tfrac{1}{10}yG_{4,0}G_{3,1}+\tfrac{2}{15}yG_{4,0}F_{2,2})\partial_y-(-y+\tfrac{4}{5}uF_{1,3}+3uF_{5,0}+\\
&+\tfrac{3}{10}uF_{3,1}G_{3,1}-\tfrac{2}{5}uF_{3,1}F_{2,2}-\tfrac{3}{10}uG_{4,0}G_{3,1}+\tfrac{2}{5}uG_{4,0}F_{2,2}-\tfrac{1}{2}vG_{3,1}+\tfrac{2}{3}vF_{2,2})\partial_u+\\
&-(-2x+\tfrac{2}{5}vF_{1,3}+4vF_{5,0}+\tfrac{2}{5}vF_{3,1}G_{3,1}-\tfrac{8}{15}vF_{3,1}F_{2,2}-\tfrac{2}{5}vG_{4,0}G_{3,1}+\tfrac{8}{15}vG_{4,0}F_{2,2})\partial_v,\\
e_2 \;:=\; & -(\tfrac{4}{5}xF_{0,4}-\tfrac{9}{25}F_{1,3}xF_{3,1}+\tfrac{2}{5}F_{1,3}xG_{4,0}-\tfrac{1}{25}xG_{4,0}F_{3,1}G_{3,1}+\tfrac{4}{75}xG_{4,0}F_{3,1}F_{2,2}+\tfrac{1}{20}xG_{3,1}^2+\\
&+\tfrac{2}{15}xF_{2,2}^2-\tfrac{1}{6}xG_{3,1}F_{2,2}+\tfrac{2}{5}xF_{3,1}F_{5,0}+\tfrac{1}{25}xF_{3,1}^2G_{3,1}-\tfrac{4}{75}xF_{3,1}^2F_{2,2}+3y+\tfrac{1}{2}vG_{3,1})\partial_x+\\
&-(F_{1,3}x-1+\tfrac{12}{5}yF_{0,4}+\tfrac{1}{40}yG_{3,1}^2+\tfrac{1}{15}yF_{2,2}^2-\tfrac{1}{2}G_{3,1}u+2uF_{2,2}+vF_{3,1}-\tfrac{1}{50}yG_{4,0}F_{3,1}G_{3,1}+\\
&+\tfrac{2}{75}yG_{4,0}F_{3,1}F_{2,2}-\tfrac{9}{50}yF_{1,3}F_{3,1}+\tfrac{1}{5}yF_{1,3}G_{4,0}-\tfrac{1}{12}yG_{3,1}F_{2,2}+\tfrac{1}{5}yF_{3,1}F_{5,0}+\tfrac{1}{50}yF_{3,1}^2G_{3,1}+\\
&-\tfrac{2}{75}yF_{3,1}^2F_{2,2})\partial_y-(-x+\tfrac{16}{5}uF_{0,4}-\tfrac{27}{50}uF_{1,3}F_{3,1}+\tfrac{3}{5}uF_{1,3}G_{4,0}-\tfrac{3}{50}uG_{4,0}F_{3,1}G_{3,1}+\\
&+\tfrac{2}{25}uG_{4,0}F_{3,1}F_{2,2}+\tfrac{3}{40}uG_{3,1}^2+\tfrac{1}{5}uF_{2,2}^2-\tfrac{1}{4}G_{3,1}uF_{2,2}+\tfrac{3}{5}uF_{3,1}F_{5,0}+\tfrac{3}{50}uF_{3,1}^2G_{3,1}+\\
&-\tfrac{2}{25}uF_{3,1}^2F_{2,2}+vF_{1,3})\partial_u-(6u+\tfrac{8}{5}vF_{0,4}-\tfrac{18}{25}vF_{1,3}F_{3,1}+\tfrac{4}{5}vF_{1,3}G_{4,0}-\tfrac{2}{25}vG_{4,0}F_{3,1}G_{3,1}+\\
&+\tfrac{8}{75}vG_{4,0}F_{3,1}F_{2,2}+\tfrac{1}{10}vG_{3,1}^2+\tfrac{4}{15}vF_{2,2}^2-\tfrac{1}{3}vG_{3,1}F_{2,2}+\tfrac{4}{5}vF_{3,1}F_{5,0}+\tfrac{2}{25}vF_{3,1}^2G_{3,1}-\tfrac{8}{75}vF_{3,1}^2F_{2,2})\partial_v.
\end{aligned}
$$

Gröbner basis generators of moduli space core algebraic variety in $\mathbb{R}^7 \ni F_{0,4}, F_{2,2}, F_{1,3}, F_{3,1}, G_{3,1}, G_{4,0}, F_{5,0}$:

$$\begin{aligned}
\mathbb{B}_1 :=\ & 16F_{0,4}^2F_{3,1}G_{3,1} - 48F_{0,4}F_{3,1}^2 + 96F_{0,4}F_{3,1}G_{4,0} + 12F_{1,3}F_{2,2}F_{3,1} - 24F_{1,3}F_{2,2}G_{4,0}+ \\
& + 30F_{2,2}^2G_{3,1} - 15F_{2,2}G_{3,1}^2 - 24F_{0,4}G_{3,1}, \\
\mathbb{B}_2 :=\ & 180F_{0,4}F_{1,3}^2G_{3,1} + 576F_{0,4}F_{2,2}F_{3,1} + 864F_{0,4}F_{2,2}G_{4,0} + 528F_{0,4}F_{3,1}G_{3,1}+ \\
& - 1448F_{0,4}G_{3,1}G_{4,0} + 60F_{1,3}F_{2,2}^2 - 60F_{1,3}F_{2,2}G_{3,1} + 375F_{1,3}G_{3,1}^2 - 432F_{0,4}F_{1,3}+ \\
& - 4320F_{0,4}F_{5,0} - 2880F_{3,1}^2 + 9240F_{3,1}G_{4,0} - 6960G_{4,0}^2 + 5040F_{2,2} - 6840G_{3,1}, \\
\mathbb{B}_3 :=\ & -288F_{0,4}F_{2,2}F_{3,1} + 1728F_{0,4}F_{2,2}G_{4,0} + 996F_{0,4}F_{3,1}G_{3,1} - 2216F_{0,4}G_{3,1}G_{4,0}+ \\
& + 540F_{1,3}^2F_{3,1} - 1080F_{1,3}^2G_{4,0} - 120F_{1,3}F_{2,2}^2 - 285F_{1,3}F_{2,2}G_{3,1} + 465F_{1,3}G_{3,1}^2+ \\
& - 3024F_{0,4}F_{1,3} + 2160F_{0,4}F_{5,0} - 2340F_{3,1}^2 + 8520F_{3,1}G_{4,0} - 7680G_{4,0}^2 - 10080F_{2,2} + 1530G_{3,1}, \\
\mathbb{B}_4 :=\ & 320F_{0,4}F_{1,3}G_{3,1}^2 + 480F_{1,3}^3G_{3,1} + 1488F_{1,3}F_{2,2}F_{3,1} + 1632F_{1,3}F_{2,2}G_{4,0} - 2676F_{1,3}F_{3,1}G_{3,1}+ \\
& + 696F_{1,3}G_{3,1}G_{4,0} + 480F_{2,2}^3 + 1440F_{2,2}^2G_{3,1} - 450F_{2,2}G_{3,1}^2 - 195G_{3,1}^3 - 1920F_{0,4}F_{2,2}+ \\
& + 1440F_{0,4}G_{3,1} - 576F_{1,3}^2 - 12960F_{1,3}F_{5,0} + 5040F_{3,1} + 12960G_{4,0}, \\
\mathbb{B}_5 :=\ & 240F_{0,4}^2F_{2,2}G_{3,1} - 180F_{0,4}^2G_{3,1}^2 - 72F_{0,4}F_{2,2}F_{3,1} + 432F_{0,4}F_{2,2}G_{4,0} + 204F_{0,4}F_{3,1}G_{3,1}+ \\
& - 464F_{0,4}G_{3,1}G_{4,0} + 60F_{1,3}F_{2,2}^2 + 525F_{1,3}F_{2,2}G_{3,1} - 255F_{1,3}G_{3,1}^2 + 864F_{0,4}F_{1,3}+ \\
& - 2160F_{0,4}F_{5,0} - 180F_{3,1}^2 + 240F_{3,1}G_{4,0} + 240G_{4,0}^2 + 5040F_{2,2} - 2790G_{3,1}, \\
\mathbb{B}_6 :=\ & 80F_{1,3}^2F_{2,2}G_{3,1} - 360F_{1,3}F_{3,1}^2 + 1440F_{1,3}F_{3,1}G_{4,0} - 1440F_{1,3}G_{4,0}^2 + 608F_{2,2}^2F_{3,1}+ \\
& - 208F_{2,2}^2G_{4,0} - 4F_{2,2}F_{3,1}G_{3,1} + 704F_{2,2}G_{3,1}G_{4,0} - 159F_{3,1}G_{3,1}^2 - 291G_{3,1}^2G_{4,0}+ \\
& + 1440F_{0,4}F_{3,1} - 2880F_{0,4}G_{4,0} - 456F_{1,3}F_{2,2} - 1344F_{1,3}G_{3,1} - 2760F_{2,2}F_{5,0}+ \\
& + 1110F_{5,0}G_{3,1} + 14400, \\
\mathbb{B}_7 :=\ & 32F_{0,4}F_{1,3}F_{3,1}G_{3,1} - 168F_{1,3}F_{3,1}^2 + 480F_{1,3}F_{3,1}G_{4,0} - 288F_{1,3}G_{4,0}^2 + 96F_{2,2}^2F_{3,1}+ \\
& - 144F_{2,2}^2G_{4,0} + 60F_{2,2}F_{3,1}G_{3,1} + 192F_{2,2}G_{3,1}G_{4,0} - 51F_{3,1}G_{3,1}^2 - 63G_{3,1}^2G_{4,0}+ \\
& + 288F_{0,4}F_{3,1} - 576F_{0,4}G_{4,0} + 24F_{1,3}F_{2,2} - 288F_{1,3}G_{3,1} - 360F_{2,2}F_{5,0} + 270F_{5,0}G_{3,1} + 2880, \\
\mathbb{B}_8 :=\ & 2560F_{0,4}F_{1,3}F_{2,2}G_{4,0} + 6840F_{1,3}F_{3,1}^2 - 15840F_{1,3}F_{3,1}G_{4,0} + 4320F_{1,3}G_{4,0}^2+ \\
& - 288F_{2,2}^2F_{3,1} + 25968F_{2,2}^2G_{4,0} - 10836F_{2,2}F_{3,1}G_{3,1} - 14784F_{2,2}G_{3,1}G_{4,0} + 5229F_{3,1}G_{3,1}^2+ \\
& + 1161G_{3,1}^2G_{4,0} - 27360F_{0,4}F_{3,1} - 6720F_{0,4}G_{4,0} + 216F_{1,3}F_{2,2} + 6624F_{1,3}G_{3,1}+ \\
& - 3240F_{2,2}F_{5,0} - 6210F_{5,0}G_{3,1} - 43200.
\end{aligned}$$

	e_1	e_2
e_1	0	$(-\frac{4}{5}F_{0,4} + \frac{9}{25}F_{3,1}F_{1,3} - \frac{2}{5}G_{4,0}F_{1,3} + \frac{1}{25}G_{4,0}F_{3,1}G_{3,1} - \frac{4}{75}G_{4,0}F_{3,1}F_{2,2} - \frac{1}{20}G_{3,1}^2 - \frac{2}{15}F_{2,2}^2 + \frac{1}{6}G_{3,1}F_{2,2} - \frac{2}{5}F_{3,1}F_{5,0} - \frac{1}{25}F_{3,1}^2G_{3,1} + \frac{4}{75}F_{3,1}^2F_{2,2})e_1 + (\frac{1}{10}F_{3,1}G_{3,1} - \frac{2}{15}F_{3,1}F_{2,2} - \frac{1}{10}G_{4,0}G_{3,1} + \frac{2}{15}G_{4,0}F_{2,2} - \frac{2}{5}F_{1,3} + F_{5,0})e_2$
e_2	$-(-\frac{4}{5}F_{0,4} + \frac{9}{25}F_{3,1}F_{1,3} - \frac{2}{5}G_{4,0}F_{1,3} + \frac{1}{25}G_{4,0}F_{3,1}G_{3,1} - \frac{4}{75}G_{4,0}F_{3,1}F_{2,2} - \frac{1}{20}G_{3,1}^2 - \frac{2}{15}F_{2,2}^2 + \frac{1}{6}G_{3,1}F_{2,2} - \frac{2}{5}F_{3,1}F_{5,0} - \frac{1}{25}F_{3,1}^2G_{3,1} + \frac{4}{75}F_{3,1}^2F_{2,2})e_1 - (\frac{1}{10}F_{3,1}G_{3,1} - \frac{2}{15}F_{3,1}F_{2,2} - \frac{1}{10}G_{4,0}G_{3,1} + \frac{2}{15}G_{4,0}F_{2,2} - \frac{2}{5}F_{1,3} + F_{5,0})e_2$	0

Creations of Geometries

By what precedes, the equations $0 = \mathsf{E}^{\mathsf{nf}}_{\bullet}$ together with the equations $0 = \mathsf{E}^{\mathsf{vf}}_{\bullet}$ are used to determine *homogeneous geometries*, namely submanifolds $M \subset \mathbb{R}^{n+c}$ having (locally) transitive symmetry group $\mathsf{Sym}(M)$, jet order after jet order. These equations are responsible for the creation of a certain *branching tree*. In principle, the *terminal leaves* of this tree correspond to (families of) homogeneous models.

To each node of the branching tree, there is associated a certain linear representation of a certain subgroup $G' \subset G$ on a certain vector space V' coordinatized by certain (independent) jet coefficients of F. As we already explained, from this node are born as many edges towards the next jet order as there are transversals to G'-orbits in V'.

But instead of repeating in the next jet order the use of the equations $0 = \mathsf{E}^{\mathsf{nf}}_{\bullet}$ and $0 = \mathsf{E}^{\mathsf{vf}}_{\bullet}$ to continue to develop the branching tree of homogeneous models, we can *stop* the transitive analysis at this point. We can take each created edge as the departure for a *new subgeometry*, without continuing the tree, even without knowing what could happen next.

Indeed, in all the preceding jet orders, there were certain (relative) differential invariants which were assumed to be zero at the origin. And to each one of these punctual invariants there corresponded a (relative) differential invariant. Denote these differential invariants as $\boldsymbol{K}_1, \dots, \boldsymbol{K}_t$.

So these $\boldsymbol{K}_{\bullet}$ are assumed to vanish at the origin $0 \in M$. Of course, they can take nonzero values nearby, a situation that could be treated by Singularity Theory. But as we decided to study only *constant-type* geometries, adopting Lie's principle of thought, we are led to assume that:

$$0 \equiv \boldsymbol{K}_1\big(x, J^\bullet F(x)\big) \equiv \cdots \equiv \boldsymbol{K}_t\big(x, J^\bullet F(x)\big),$$

for x in some neighborhood of the origin.

Thus, we can stop the homogeneous geometries process $0 = \mathsf{E}_\bullet^{\mathsf{nf}} = \mathsf{E}_\bullet^{\mathsf{vf}}$ anywhere.

Principle. [Creation of constant-type (degenerate) geometries] *Given a group G acting transitively on graphs $\big\{u = F(x)\big\}$ in $\mathbb{R}^{n+c} \ni (x, u)$, with its prolonged actions to jet bundles $J^1_{n,c}$, $J^2_{n,c}$, $\dots$, $J^\kappa_{n,c}$, $\dots$, at each order $\kappa \geqslant 1$, at each node of the branching tree (even if incomplete) which is constructed to determine homogeneous models, create (introduce) new geometries, of constant type, degenerate in a certain sense, depending on the history of the node.*

Some of the nodes are such that all the power series coefficients of F are already uniquely determined, especially the final nodes, *i.e.* the terminal leaves.

Some other nodes are such that there still remain infinitely many power series F-coefficients which are free, not normalized, and then, the G'-action must be prolonged to the jet (sub)bundle of this (sub)geometry, in order to determine the corresponding algebras of differential invariants.

In conclusion, *many new geometries having algebras of differential invariants exist which should (can) be studied.*

Most of the times, the creation of constant-type geometries is well known at jet order 2. For instance, under the group $\mathsf{Aff}(\mathbb{C}^{n+1})$ of affine transformations of $\mathbb{C}^{n+1}$ — codimension $c = 1$ — since the punctual rank of the Hessian matrix is invariant, inequivalent graphed normal forms are:

$$u = x_1^2 + \cdots + x_m^2 + \mathrm{O}_{x_1,\dots,x_n}(3),$$

with an invariant integer $0 \leqslant m \leqslant n$, which produces $n + 1$ different (inequivalent!) geometric structures. Similarly, for hypersurfaces, the rank and the signature of the Levi form are invariant under CR equivalences, hence several order 2 geometries can be 'created'.

Applying his theory of moving frames, Olver studied algebras of differential invariants for elliptic and hyperbolic surfaces $S^2 \subset \mathbb{R}^3$ under Euclidean and Affine transformations, *i.e.* with Hessians of maximal rank 2, *see* Chen-M. 2019 for the Hessian rank 1 geometry. To study only a single one of these constant Hessian rank affine geometries from the point of view of differential invariants, for instance with $n = 5$ and $m = 3$ (a case probably never looked at), might already be a considerable task.

Constant type (degenerate) geometries at jet order $\geqslant 3$ are not much studied, but they are as legitimate as the order 2 (degenerate) geometries. The branching tree in M.-Nurowski 2020 shows certain degenerate para-CR geometries of jet order > 3, *i.e.* beyond Levi form (which is of order 2) and beyond 2-nondegeneracy (which of order 3).

Problem. [Algebras of differential invariants for degenerate geometries] *Describe algebras of differential invariants of constant-type degenerate geometries. Find minimal sets of (differential) generators.*

We insist on the fact that we formulate this general problem for *all possible constant-type degenerate (sub)geometries*. Even, the considered Lie group G can be an infinite-dimensional Lie pseudo-group.

At the opposite are the *generic* geometries, those for which it is allowed to assume that some functions, some determinants, are nonzero, some rank matrices are maximal, *etc*. For some generic geometries, under some classical groups, Peter Olver 2007, Hubert-Olver 2007, have established remarkable theorems that a *single* differential invariant is sufficient to (differentially) generate the whole algebra of differential invariants.

But certainly, the genericity of a geometry is a relative *concept! Genericity also concerns subgeometries!*

Indeed, in any node at which a constant-type (degenerate) geometry is created, by assuming that all higher order encountered (relative) differential invariants are non-vanishing (after restriction to open subsets), by assuming in addition if it is convenient that some functions, some determinants, *etc.*, are nonzero, then a certain '*generic*' (sub)geometry can be defined within the considered degenerate geometry.

Termination: Moduli Spaces of Homogeneous Models

Now, when, why, and how the Steps 1-2-3-4-5 'algorithm' *terminates*? What does its '*termination*' produce? Before answering these questions, let us present some aspects of the current state of the art.

First of all, as for Cartan's method of equivalence which is sometimes termed to be an 'algorithm', most of the times, as soon as the number n of independent variables $x_\bullet$ is $\geqslant 2$ or is $\geqslant 3$, any 'equivalence algorithm' can be 'blocked' by computational complexity, even with the help of powerful machines.

Indeed, the exploration of the branching tree of a given kind of homogeneous geometries requires in some circumstances to continue the computations until reaching *simply transitive* models, *i.e.* those with:

$$\mathsf{dim}\,\mathfrak{sym}\,M \;=\; \mathsf{dim}\,M,$$

and then in this case, the 'algorithm' necessarily terminates. The other homogeneous models M, those for which $\mathsf{dim}\,\mathfrak{sym}(M) > \mathsf{dim}\,M$, are termed *multiply transitive*.

As a matter of fact, it is (well) known in the literature that, for a number of famous geometric structures, either simply transitive models were never found yet, or were found by indirect methods, without discovering the complete branching tree created by invariants together with all the linear representations in the nodes. Let us give 5 examples.

• For $(2, 3, 5)$ distributions D^2 in a five-manifold M^5, Cartan classified multiply transitive models, *see* The-2022 (and the reference therein) for a recent synthesis based on Cartan (parabolic) geometries, and *see* also Doubrov-Govorov 2013 for a complete classification, including simply transitive models, which is based on Lie algebraic techniques. Beyond Cartan quartic types, it seems that no complete picture exists for the branching tree of order $\geqslant 5$ (punctual) invariants.

• For completely integrable second order PDE systems in 2 dependent complex variables and 1 independent complex variable, the multiply transitive models have been neatly classified in Doubrov-Medvedev-The-2019, but the complete branching tree of invariants is also missing, and simply transitive models have not been determined yet.

• For CR-homogeneous Levi nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$, the multiply transitive models have been neatly classified by Doubrov-Medvedev-The 2020, Loboda 2020, the complete branching tree of invariants is also missing, while the simply transitive models have been determined by abstract Lie algebraic method, *cf.* Loboda 2020, Doubrov-M.-The 2020.

• For 4^{th} order ODEs under point transformations, existing classifications are not complete, while classifications of homogeneous models 3^{th} order ODEs under fiber-preserving, point, contact, transformations have been achieved by Michal Godiński and Paweł Nurowski, *cf.* Godlinski 2008, Godlinski-Nurowski 2009.

• For affinely homogeneous hypersurfaces $H^3 \subset \mathbb{R}^4$, Eastwood-Ezhov 2001 do not show simply transitive models.

Now, let us come back to the Steps 1-2-3-4-5 'algorithm' in the general setting. Because:

$$n \leqslant \dim G < \infty,$$

it is clear that the dimensions of the isotropy subgroups $G_{\mathsf{stab}}^{\kappa} \subset G$ at orders $\kappa = 0, 1, 2, 3, \ldots$, can decrease (strictly) only a finite number of times, in all branches. So in each one of the branches constructed by induction, after a while, no more isotropy group reduction can occur. This is when and why the Steps 1-2-3-4-5 'algorithm' *terminates*.

And in fact, all boxed terminal leaves in the branching trees shown in this talk indicate *termination by end-of-isotropy-reduction*.

But from the computational point of view, how termination does occur, concretely? Namely, what really happens 'at the end' of the 'ping-pong' play between the equations $0 = \mathsf{E}_{\bullet}^{\mathsf{vf}}$ and $0 = \mathsf{E}_{\bullet}^{\mathsf{nf}}$?

First of all, after that Steps 1 and 2 have been passed, as soon as there is a nontrivial linear representation in Step 3, necessarily, there must be at least one further subbranch which is accompanied with a nontrivial group reduction — except for the mostly degenerate linear group-orbit: the origin in the vector space $\mathbb{R}^{\ell_\kappa}$, *cf.* for instance Branch **2b3a4a** below.

Consequently, termination holds if and only if no (nontrivial) linear representation occurs at Step 3, at all higher jet orders κ, wich requires to continue the 'ping-pong' between $0 = \mathsf{E}_{\bullet}^{\mathsf{vf}}$ (firstly) as Step 1 and $0 = \mathsf{E}_{\bullet}^{\mathsf{nf}}$ (secondly) as Step 2.

We did not attempt to prove or just state stabilization or pseudo-stabilization theorems as in Olver 1995, Olver 2007 — which, we believe, can be done —, because there is here a simple alternative *and direct* way of realizing that the process rigorously terminates.

Concretely, the process stops if, after having resolved the equations $0 = \mathsf{E}_\bullet^{\mathsf{vf}}$, all equations $0 = \mathsf{E}_\bullet^{\mathsf{nf}}$ only show constant power series coefficients or absolute invariants $G_\bullet = F_\bullet$, this, at every higher jet order.

Of course, it can happen that termination takes place with isotropy dimension being stably constant and > 0, whichever high is the jet order.

Thus, termination holds when the equations $0 = \mathsf{E}_\bullet^{\mathsf{nf}}$ no more bring any normalization of the $G_\bullet$ and $F_\bullet$ coefficients. However, the equations $0 = \mathsf{E}_\bullet^{\mathsf{vf}}$ still bring a lot of information!

Observation. *At all higher jet orders, when some absolute invariants $\boldsymbol{I}_\bullet$ which come from preceding jet orders are still present in computations, the equations $0 = \mathsf{E}_\bullet^{\mathsf{vf}}$ do bring more and more algebraic equations in terms of $\boldsymbol{I}_\bullet$ which coherently define a certain* algebraic moduli space of homogeneous models *— unless some algebraic contradiction occurs which indicates that no homogeneous model can exist in the considered terminal leaf.*

(Contradictory terminal leaves are indicated plainly with the $\emptyset$ symbol, or even sometimes, plainly erased.)

An example of such an *algebraic moduli space of homogeneous models* was already shown above, with the Branch **2c3c** for surfaces $S^2 \subset \mathbb{C}^3$. Certainly, the obtained algebraic equations are deeply related with the Lie-Fels-Olver *recurrence relations* between differential invariants.

Observation. *To each boxed terminal leaf, there corresponds a family of homogeneous models parametrized by a certain algebraic variety.*

Quite often, a terminal leaf of a branching tree is of the form IkP0, with k-dimensional isotropy Ik, where P0 means that zero Parameter is present, so that the concerned algebraic variety is just 1 point (or 2, 3, 4 points, never more in this talk).

As is known, every (complicated) algebraic variety may always be decomposed into a finite number of simpler disjoint smooth pieces, *e.g.*, by a process called *stratifification.*

However, *conceptionally*, group reduction in the spirit of Lie and Cartan is of *different nature*, compared with further explorations by stratifying algebraic moduli spaces of homogeneous models. In some papers, both group reduction — in fact not explicitly mentioned there — and moduli space stratification, seem to be treated on equal footing, *cf.* the flow diagrams on pages 67–69 there.

> Ma dernière remarque générale concerne un aspect de la Mathématique moderne en quelque sorte complémentaire de ses tendances unificatrices, à savoir sa capacité à dissocier ce qui était indûment confondu. Jean DIEUDONNÉ, 1964.

As Dieudonné writes, one must indeed:

'*dissociate what was unduly confused*'

Observation. *In this talk, we decided not to stratify the algebraic moduli spaces of homogeneous models that we obtained, after termination of group reduction, at any terminal leaf.*

Such a task could be endeavoured in a future publication. Of course, stratifying an algebraic moduli space of homogeneous models would bring further sub-branches (of a different nature), devloping and branching after the boxed terminal leaves.

Eastwood-Ezhov 1999 had only a single branch among all the branches which did not lead to a non-trivial algebraic variety, namely what we call here Branch **2c3c**, and what is called there '*Nonvanishing Pick Invariant*'.

In fact, if the system $E = F = G = H = 0$ is simply passed to the 'solve' routine of the computer algebra system MAPLE (Version V Release 3), then the program returns the correct solutions as a set of approximately 20 cases, in effect constructing its own flow diagram)!

These 4 equations $E = F = G = H = 0$ are precisely equivalent to the 3 equations appearing in Branch **2c3c**, and it is indeed already difficult to stratify their zero-locus.

All other branches treated there directly lead either to individual models with all coefficients $F_\bullet$ being numerical (hence uniquely prescribed), or to the existence of a single real or complex parameter (absolute invariant) $\boldsymbol{I} \in \mathbb{R}^1$ or $\boldsymbol{I} \in \mathbb{C}^1$, with no algebraic equation involved.

By contrast, in this talk, several terminal leaves, especially the *simply transitive* ones, led us to certain quite complicated algebraic moduli spaces of homogeneous models, far beyond what was handled above.

Even for just one terminal leaf like *e.g.*:

$$\mathbf{2f3a}, \qquad \text{or} \qquad \mathbf{2f3g}, \qquad \text{or} \qquad \mathbf{2g3a},$$

to set up a stratification could be a formidable task! We believe that a similar algebraic complexity lies behind the simply transitive affinely homogeneous hypersurfaces $H^3 \subset \mathbb{R}^4$, never attained in the literature.

In addition, no stratification in smooth neatly parametrized pieces would be 'canonical' in any sense — similarly as the choice of a group-transversal is never 'canonical'.

In conclusion, in this talk, our classification approach decides to stop (to terminate) once algebraic moduli spaces of homogeneous models have been reached.

And now, we know the reason why, in the existing literature, some classifications using the approach with (differential) invariants are missing, especially concerning the (difficult) simply transitive homogeneous models.

It is because the concerned algebraic varieties which parametrize the sought (simply transitive) homogeneous models happen to be very complicated.

Lie Algebras of Vector Fields Versus Closed Forms

In the literature, most of the times, classifications of affinely or projectively homogenous small-dimensional submanifolds in $\mathbb{R}^{n+c}$ (or in $\mathbb{C}^{n+c}$) attain *closed forms*, that is, equations $u = F(x)$ with F being expressed as a polynomial, or/and in terms of elementary transcendental functions: exponentials, logarithms, trigonometric functions.

However, often, Lie algebras $\mathfrak{sym}(M)$ of infinitesimal symmetries are not shown, simultaneously with the (nice) functions F. And it is then a non-immediate task to determine $\mathfrak{sym}(M)$ from a given closed graphed form $\{u = F(x)\}$, especially when some continuous parameters α, β, ..., are present in $F = F_{\alpha,\beta}$. Indeed, for various values of the parameters in the closed form, most probably, the (graphed) manifold:

$$M_{\alpha,\beta,\dots} = \{u = F_{\alpha,\beta,\dots}\},$$

crosses the branches of any invariant branching tree, so that the dimensions of $\mathfrak{sym}\big(M_{\alpha,\beta,\dots}\big)$ 'jump' in some way as the parameters $\alpha, \beta, \dots$ do vary. Mathematical life is complicated!

Not only the Lie algebras $\mathfrak{sym}\big(M_{\alpha,\beta,\dots}\big)$ are not always shown in the literature, but also, the branching trees created by invariants are almost never constructed until reaching all terminal leaves, except in some computationally simple cases, *e.g.* in the curve case $n = 1$.

Certainly, such invariant branching trees are often at least partly known, concerning relatively small order differential invariants, for instance: Cartan's quartic for $(2, 3, 5)$ distributions; or Chern-Moser's order 4 tensor for CR hypersurfaces $M^5 \subset \mathbb{C}^3$.

For the constant Hessian rank 1 affinely homogeneous hypersurfaces $H^2 \subset \mathbb{R}^3$ and $H^3 \subset \mathbb{R}^4$ that we treat in a forthcoming paper, we did not search for a closed form representation of the single (up to sign) model $H^4 \subset \mathbb{R}^5$ which we found.

And for higher-dimensional constant Hessian rank 1 hypersurfaces $H^n \subset \mathbb{R}^{n+1}$, a quite unexpected fact was established by M. 2022, namely that in any dimension $n \geqslant 5$, there are no nonproduct homogeneous models at all! *Passim*, let us raise a

Question. *Is it true, also, that in all high enough dimensions* $n \geqslant \mathrm{N}_2 \gg 1$, *there are, similarly, no nonproduct constant Hessian rank 2 hypersurfaces* $H^n \subset \mathbb{R}^{n+1}$ *(or* $\mathbb{C}^{n+1}$*) which are locally affinely homogeneous?*

A similar question may be formulated for any fixed constant Hessian rank $1 \leqslant r \leqslant n-1$. Also, the question may be considered with the *projective* group $\mathsf{Proj}(\mathbb{R}^{n+c})$ instead of $\mathsf{Aff}(\mathbb{R}^{n+c})$.

Back to closed forms (that we will not seek in this talk), experts to whom we asked whether there exist theoretical explanations — that could be read off from a given Lie algebra of vector fields — why, when, how, closed forms (may) exist, answered us that they ignore what could be such reason(s), and that they obtained closed forms with the help of Maple PDE integration programs.

It is therefore legitimate to raise a

Problem. *Find criteria, if not necessary and sufficient conditions, on given Lie algebras of vector fields that are symmetries of homogeneous models, in order that, after a change of coordinates belonging to the initial group* G, *the graphing function* $F(x)$ *is* *either polynomial, or is expressed in terms of usual transcendental functions: exponentials, logarithms, trigonometric functions.*

Of course, the theorem of Frobenius guarantees the existence of an analytic graph $M^n \subset \mathbb{R}^{n+c}$ which is simply the *orbit of the origin* under the action of the found transitive Lie algebra of vector fields. Such Lie algebras may be truncated, especially when dealing with an infinite-dimensional Lie group G acting on $\mathbb{R}^{n+c}$ (or $\mathbb{C}^{n+c}$), and again, the same problem appears to be meaningful.

In sum, there are 5 reasons why we did not seek closed forms (for the moment).

• No general theory seems to exist around Problem on p. 69, and probably, there might exist certain special homogeneous Lie algebras of vector fields which would *not* be elementarily integrable.

• Lie's original principle of classification, with which we agree, was to determine and to present *Lie algebras of vector fields*, only.

• Punctual invariants of homogeneous models are strongly related to algebras of differential invariants, a research field that we learned from Peter Olver's monographs and articles, and in this field, branching by invariants is a natural process.

• Successive group reductions leading to linear representations in all nodes seem to be universal, although they were not discovered in the existing normal forms articles we know.

• Branching trees of invariants lie at the heart matter, hence must be exhibited, even when quite ramified.